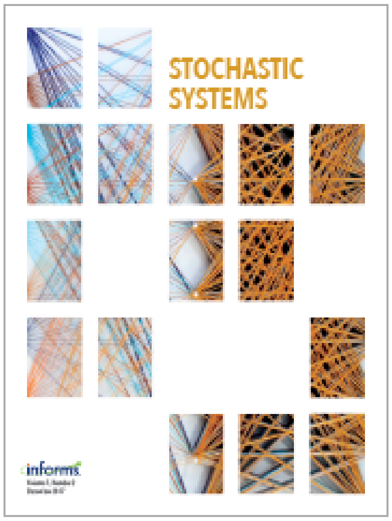

Stochastic Systems

Publication details, including instructions for authors and subscription information:
http://pubsonline.informs.org

Brownian Inventory Models with Convex Holding Cost, Part 2: Discount-Optimal Controls

J. G. Dai, Dacheng Yao

Stochastic Systems
2013, Vol. 3, No. 2, 500–573
DOI: 10.1214/11-SSY046

BROWNIAN INVENTORY MODELS WITH CONVEX HOLDING COST, PART 2: DISCOUNT-OPTIMAL CONTROLS*

BY J. G. DAI AND DACHENG YAO

Cornell University† *and Chinese Academy of Sciences*

We consider an inventory system in which inventory level fluctuates as a Brownian motion in the absence of control. The inventory continuously accumulates cost at a rate that is a general convex function of the inventory level, which can be negative when there is a backlog. At any time, the inventory level can be adjusted by a positive or negative amount, which incurs a fixed positive cost and a proportional cost. The challenge is to find an adjustment policy that balances the inventory cost and adjustment cost to minimize the expected total discounted cost. We provide a tutorial on using a three-step lower-bound approach to solving the optimal control problem under a discounted cost criterion. In addition, we prove that a four-parameter control band policy is optimal among all feasible policies. A key step is the constructive proof of the existence of a unique solution to the free boundary problem. The proof leads naturally to an algorithm to compute the four parameters of the optimal control band policy.

1. Introduction. Dai and Yao [6] studied the optimal control of Brownian inventory models under the *long-run average cost* criterion. This paper, which is a companion of [6], studies the same Brownian inventory models, but under the *discounted cost* criterion. Its main purpose is to provide a tutorial on the powerful, lower-bound approach to proving the optimality of a control band policy among all feasible policies. The tutorial is rigorous and self-contained with the exception of the standard Itô's formula. In addition, this paper contributes to the literature by proving the existence of a "smooth" solution to the free boundary problem with a general convex holding cost function. As a consequence, a four-parameter optimal control band policy is shown to be optimal. Our existence proof also leads naturally to an algorithm to compute the optimal control band parameters.

Received November 2011.

*Research supported in part by NSF grants CMMI-0727400, CMMI-0825840, CMMI-1030589, and CNS-1248117.

†On leave from Georgia Institute of Technology.

AMS 2000 subject classifications: Primary 60J70, 90B05, 93E20.

Keywords and phrases: Impulse control, singular control, control band, verification theorem, free boundary problem, smooth pasting, quasi-variational inequality.

The introduction in [6] describes the Brownian inventory models, control band policies, and the lower-bound approach in detail. That paper also includes an extensive review of the literature as well as an explanation of the motivation to study the non-linear holding cost function. Therefore, this introduction highlights the development specific to the discounted cost case.

As in [6], we assume an upward or downward adjustable inventory position, with all adjustments realized immediately without any leadtime delay. Each upward adjustment with amount $\xi > 0$ incurs a cost $K + k\xi$, where $K \geq 0$ and $k > 0$ are the fixed cost and the variable cost rate, respectively, for each upward adjustment. Similarly, each downward adjustment with amount ξ incurs a cost of $L + \ell\xi$ with fixed cost $L \geq 0$ and variable cost rate $\ell > 0$. In addition, we assume that the holding cost function $h : \mathbb{R} \to \mathbb{R}_+$ is a *general* convex function that satisfies some minimal assumptions in Assumption 1 in Section 2. Our objective is to find a control policy that balances the inventory cost and the adjustment cost so that, starting from any initial inventory level x, the (infinite-horizon) expected total discounted cost is minimized. When both upward and downward fixed costs are positive, the model is an *impulse control* problem. When both fixed costs are zero, the corresponding Brownian control problem is a *singular control* or *instantaneous control* problem. As in [6], we treat both of these controls within one framework. We will also consider the case when the inventory backlog is not allowed.

When the inventory holding cost function is linear, namely,

$$h(x) = \begin{cases} h_1 x & \text{if } x \geq 0, \\ -p_1 x & \text{if } x < 0 \end{cases} \tag{1.1}$$

for some constants $p_1 > 0$ and $h_1 > 0$, Constantinides and Richard [5] proved that a four-parameter control band policy is optimal under the condition

$$h_1 - \beta k > 0 \quad \text{and} \quad p_1 - \beta\ell > 0, \tag{1.2}$$

where $\beta > 0$ is a discount rate. As explained in [5], h_1/β is the present value of the holding cost of keeping one unit of inventory now to infinity. If $h_1/\beta \leq \ell$, it will never be optimal to reduce the inventory level as long as $L > 0$. Similarly, if $p_1/\beta \leq k$, it will never be optimal to increase the inventory as long as $K > 0$. Thus, condition (1.2) is also necessary for a four-parameter control band policy to be optimal. Baccarin [1] sketched a proof that a four-parameter control band policy is also optimal when the holding cost is quadratic as given by

$$h(x) = \begin{cases} h_1 x + h_2 x^2 & \text{if } x \geq 0, \\ -p_1 x + p_2 x^2 & \text{if } x < 0, \end{cases} \tag{1.3}$$

where $h_1 \geq 0$, $p_1 \geq 0$, $h_2 > 0$ and $p_2 > 0$. In his proof, condition (1.2) is not needed as long as $h_2 > 0$ and $p_2 > 0$. Baccarin [1] deferred the detailed proof for the existence of a solution to the four-parameter free boundary problem to an online supplement. Unfortunately, this document can no longer be located over the Internet. Assuming $K = L > 0$ and $k = \ell = 0$, Plehn-Dujowich [10] proved that a three-parameter control band policy is optimal when the holding cost function h satisfies

$$
\begin{aligned}
&h \text{ and } h' \text{ are continuous;}\\
&h \text{ is strictly concave and single-peaked;}\\
&|h|,\ |h'| \text{ and } |h''| \text{ are bounded by a polynomial.}
\end{aligned}
\tag{1.4}
$$

Both the linear cost in (1.1) and the quadratic cost in (1.3) do not satisfy the smoothness condition in (1.4).

In this paper, when the holding cost function is assumed to be general, satisfying Assumption 1 in Section 2, we prove that a four-parameter control band policy is optimal. Assumption 1 on the convex holding cost function h is considerably weaker than those found in the literature. The cost functions in [1, 5, 10] all satisfy Assumption 1. Condition (2.5) in Assumption 1 is analogous to (1.2) and is automatically satisfied for h in (1.3). Similar to the companion paper [6], we adopt the three-step lower-bound approach in our proof. In the first step, we prove that if there exists a "smooth" test function f that satisfies a set of differential inequalities, then the function f dominates the value function at every initial inventory level x. In the second step, given a control band policy, we show how to obtain the value function within the band as the unique solution to a second order differential equation. In the third step, we show the existence of a solution to a free boundary problem that satisfies the conditions for f in the first step.

The result in step 1 is known as the "verification theorem". All three prior papers [1, 5, 10] invoked the verification theorem in Richard [12], which in turn generalized the pioneering work of Bensoussan and Lions [2, 3]. This tutorial advocates the lower bound approach that was also adopted by Harrison et. al [8] and Harrison and Taksar [9]. Again, this approach's self-contained character, with the exception of applying the standard Itô's formula, makes it suitable for adoption in other related settings.

We specify the free boundary problem, the most difficult task in Step 3, by using the well known "smooth pasting" method (see, e.g., [4]). We prove the existence of a C^1 solution to the free boundary problem that has four free parameters. Though our proof is similar to the one in [5], where a linear holding cost function is used, our proof is considerably more difficult. Unlike the proof in [5], our proof is also constructive so that it leads naturally an algorithm to compute the four parameters of the optimal control band. Recently, Feng and Muthuraman [7] developed an algorithm to compute the parameters of an optimal control band policy for the discounted Brownian control problem. They illustrated the convergence of their algorithm through some numerical examples.

The remainder of this paper is organized as follows. Section 2 defines our Brownian control problem. Section 3 presents a version of the Itô's formula that does not require the test function f be C^2 function. A lower bound for all feasible policies is established in Section 4. Section 5.1 shows that under a control band policy, the value function within the band can be obtained as a solution to a second order ordinary differential equation (ODE). Under the assumption that a free-boundary problem has a unique solution that has desired regularity properties, Section 5.2 proves that there is a control band policy whose discounted cost achieves the lower bound. Thus, the control band policy is optimal among all feasible policies. Section 5.3 explains how to construct the solution to the free-boundary problem and characterizes the parameters for the optimal control band policy. Section 5.3 constitutes the main technical contribution of this paper. Section 6 solves the singular control problem. This section is short, essentially becoming a special case of Section 5 when both $K = 0$ and $L = 0$. Section 7 deals with impulse control problems when inventory is not allowed to be backlogged.

2. Brownian control models. We now introduce the Brownian control models that accommodate both the impulse and instantaneous controls. These models are identical to the ones in [6], but in this paper all costs are discounted.

Let $X = \{X(t), t \geq 0\}$ be a Brownian motion with drift μ and variance σ^2, starting from x. Then, X has the following representation

$$X(t) = x + \mu t + \sigma W(t), \quad t \geq 0,$$

where $W = \{W(t), t \geq 0\}$ is a standard Brownian motion that has drift 0, variance 1, starting from 0. We assume W is defined on some filtered probability space $(\Omega, \{\mathcal{F}_t\}, \mathcal{F}, \mathbb{P})$ and W is an $\{\mathcal{F}_t\}$-martingale. Thus, W is also known as an $\{\mathcal{F}_t\}$-standard Brownian motion. We use X to model the *netflow process* of an inventory system. For each $t \geq 0$, $X(t)$ represents the inventory level at time t if no control has been exercised by time t. The netflow process will be controlled and the actual inventory level at time t, after controls has been exercised, is denoted by $Z(t)$. The controlled process is denoted by $Z = \{Z(t), t \geq 0\}$. With a slight abuse of terminology, we call $Z(t)$ the inventory level at time t, although when $Z(t) < 0$, $|Z(t)|$ is the backorder level at time t.

Controls are dictated by a policy. A policy φ is a pair of stochastic processes (Y_1, Y_2) that satisfies the following three properties: (a) for each sample path $\omega \in \Omega$, $Y_i(\omega, \cdot) \in \mathbb{D}$, where $\mathbb{D}$ is the set of functions on $\mathbb{R}_+ = [0, \infty)$ that are right continuous on $[0, \infty)$ and have left limits in $(0, \infty)$; (b) for each ω, $Y_i(\omega, \cdot)$ is a nondecreasing function; and (c) Y_i is adapted to the filtration $\{\mathcal{F}_t\}$, namely, $Y_i(t)$ is $\mathcal{F}_t$-measurable for each $t \geq 0$, $i = 1, 2$. We call $Y_1(t)$ and $Y_2(t)$ the cumulative *upward* and *downward* adjustment, respectively, of the inventory in $[0, t]$. Under a given policy (Y_1, Y_2), the inventory level at time t is given by

$$(2.1) \quad Z(t) = X(t) + Y_1(t) - Y_2(t) = x + \sigma W(t) + \mu t + Y_1(t) - Y_2(t), \quad t \geq 0.$$

Therefore, Z is a semimartingale, namely, a martingale σW plus a process that is of bounded variation.

A point $t \geq 0$ is said to be an *increasing point* of Y_1 if $Y_1(s) - Y_1(t-) > 0$ for each $s > t$, where $Y_1(t-)$ is the left limit of Y_1 at t with convention that $Y_1(0-) = 0$. When t is an increasing point of Y_1, we call it an upward adjustment time. Similarly, we define an increasing point of Y_2 and call it a downward adjustment time. Let $N_i(t)$ be the cardinality of the set

$$\{s \in [0, t] : Y_i \text{ increases at } s\}, \quad i = 1, 2.$$

In general, we allow an upward or downward adjustment at time $t = 0$. By convention, we set $Z(0-) = x$ and call $Z(0-)$ the *initial inventory level*. By (2.1),

$$Z(0) = x + Y_1(0) - Y_2(0),$$

which can be different from the initial inventory level $Z(0-)$.

The two types of costs associated with a control are *fixed costs* and *proportional costs*. We assume that each upward adjustment incurs a fixed cost of $K \geq 0$ and each downward adjustment incurs a fixed cost of $L \geq 0$. In addition, each unit of upward adjustment incurs a proportional cost of $k > 0$ and each unit of downward adjustment incurs a proportional cost of $\ell > 0$. Thus, by time t, the system incurs the cumulative proportional cost $kY_1(t)$ for upward adjustment and the cumulative proportional cost $\ell Y_2(t)$ for downward adjustment. When $K > 0$, we are only interested in policies such that $N_1(t) < \infty$ for each $t > 0$; otherwise, the total cost would be infinite in the time interval $[0, t]$. Thus, when $K > 0$, we restrict upward controls that have a finitely many upward adjustment in a finite interval. This is equivalent to requiring Y_1 to be a piecewise constant function on each sample path. Under such an upward control, we can list the upward adjustment times as a discrete sequence $\{T_1(n) : n \geq 0\}$, where the nth upward adjustment time can be defined recursively via

$$T_1(n) = \inf\{t > T_1(n-1) : \Delta Y_1(t) > 0\},$$

where, by convention, $T_1(0) = 0$ and $\Delta Y_1(t) = Y_1(t) - Y_1(t-)$. The amount of the nth upward adjustment is denoted by

$$\xi_1(n) = Y_1(T_1(n)) - Y_1(T_1(n)-) \quad n = 0, 1, \ldots.$$

It is clear that specifying such an upward adjustment policy $Y_1 = \{Y_1(t), t \geq 0\}$ is equivalent to specifying a sequence of $\{(T_1(n), \xi_1(n)) : n \geq 0\}$. In particular, given the sequence, we have

$$Y_1(t) = \sum_{i=0}^{N_1(t)} \xi_1(i), \tag{2.2}$$

and $N_1(t) = \max\{n \geq 0 : T_1(n) \leq t\}$. Thus, when $K > 0$, it is sufficient to specify the sequence $\{(T_1(n), \xi_1(n)) : n \geq 0\}$ to describe an upward adjustment policy. Similarly, when $L > 0$, it is sufficient to specify the sequence $\{(T_2(n), \xi_2(n)) : n \geq 0\}$ to describe a downward adjustment policy and

$$Y_2(t) = -\sum_{i=0}^{N_2(t)} \xi_2(i). \tag{2.3}$$

Merging these two sequences gives the sequence $\{(T_n, \xi_n), n \geq 0\}$, where T_n is the nth adjustment time of the inventory and ξ_n is the amount of adjustment at time T_n. When $\xi_n > 0$, the nth adjustment is an upward adjustment and when $\xi_n < 0$, the nth adjustment is a downward adjustment. The policy (Y_1, Y_2) is adapted if T_n is an $\{\mathcal{F}_t\}$-stopping time and each adjustment ξ_n is $\mathscr{F}_{T_n-}$ measurable.

In addition to the adjustment cost, we assume that the system incurs a holding cost at rate $h(x)$: when the inventory level is at $Z(t) = x$, the system incurs a cost of $h(x)$ per unit of time. Therefore, the cumulative discounted holding cost in $[0, t]$ is

$$\int_0^t e^{-\beta s} h(Z(s)) ds,$$

where $\beta > 0$ is the discount rate.

Under a feasible policy $\varphi = \{(Y_1(t), Y_2(t))\}$ with initial inventory level $Z(0-) = x$ and a discount rate $\beta > 0$, the expected total discounted cost

$\mathrm{DC}(x,\varphi)$ is

$$\mathrm{DC}(x,\varphi) = \mathbb{E}_x\Big[\int_0^\infty e^{-\beta t} h(Z(t))dt + \int_0^\infty e^{-\beta t}\big(K dN_1(t) + L dN_2(t) + k dY_1(t) + \ell dY_2(t)\big)\Big].$$

where $\mathbb{E}_x$ is the expectation operator conditioning on the initial inventory level being $Z(0-) = x$. As mentioned earlier, when $K > 0$ and $L > 0$, it is sufficient to restrict feasible policies to be the impulse type given in (2.2) and (2.3). A Brownian control model with controls limited to impulse type is called the *impulse Brownian control model*. When $K = 0$ and $L = 0$, it turns out that under an optimal policy, $N_1(t) = \infty$ and $N_2(t) = \infty$ with positive probability for each $t > 0$. We call the corresponding control problem the *instantaneous Brownian control model* or *singular Brownian control model*. In the impulse control model, we need to restrict our feasible policies to satisfy

$$\mathbb{E}_x\Big[\sum_{n=0}^{\infty} e^{-\beta T_n}\big(1 + |\xi_n|\big)\Big] < \infty. \tag{2.4}$$

Otherwise, $\mathrm{DC}(x,\varphi) = \infty$. Note that some applications may always require a nonnegative inventory level, namely,

$$Z(t) \geq 0, \quad \text{for } t \geq 0.$$

We assume the inventory cost function $h : \mathbb{R} \to \mathbb{R}_+$ satisfies the following assumption.

ASSUMPTION 1. Assume that the cost function h satisfies the following conditions: (a) it is continuous and convex; (b) there exists an a such that $h \in C^2(\mathbb{R})$ except at a and $h(a) = 0$; (c) $h'(x) \leq 0$ for $x < a$ and $h'(x) \geq 0$ for $x > a$; (d)

$$\lim_{x\uparrow\infty} h'(x) > \ell\beta \quad \text{and} \quad \lim_{x\downarrow-\infty} h'(x) < -k\beta; \tag{2.5}$$

(e) $h''(x)$ has a smaller order than $e^{\lambda_1 x}$ as $x \uparrow \infty$, that is

$$\lim_{x\uparrow\infty} \frac{h''(x)}{e^{\lambda_1 x}} = 0, \tag{2.6}$$

where $\lambda_1 = \big[(\mu^2 + 2\beta\sigma^2)^{1/2} - \mu\big]/\sigma^2 > 0$; and

(f) $h''(x)$ has a smaller order than $e^{-\lambda_2 x}$ as $x \downarrow -\infty$, that is

$$\lim_{x\downarrow -\infty} \frac{h''(x)}{e^{-\lambda_2 x}} = 0, \tag{2.7}$$

where $\lambda_2 = \left[(\mu^2 + 2\beta\sigma^2)^{1/2} + \mu\right]/\sigma^2 > 0$.

Remark. (i) Conditions (a)-(b) of Assumption 1 are identical to Conditions (a)-(b) of Assumption 1 in [6]. Condition (d) here is applicable only to the discounted case. Condition (c) here is apparently weaker than the corresponding condition in [6]; without some additional conditions such as (2.5) in this paper, Condition (c) in Assumption 1 of [6] cannot be weakened. (ii) If h is given by (1.1), (2.5) becomes (1.2), which is consistent with (13) in [5]. When $\lim_{x\to\infty} h'(x) \le \ell\beta$, it follows the same reasoning as in [5] that it will never be optimal to reduce the inventory level as long as $L > 0$. Similarly, when $\lim_{x\to-\infty} h'(x) \ge -k\beta$, it will never be optimal to increase the inventory level as long as $K > 0$. (iii) All polynomial functions satisfy the conditions (2.6)-(2.7). (iv) The smoothness conditions on the convex holding cost function h can be relaxed to be continuously differentiable once and twice at all but a finitely many points.

The following elementary lemma on the holding cost function also will be useful in later development.

LEMMA 2.1. *(a) Under Assumption 1,*

$$\lim_{x\downarrow -\infty} \frac{\int_x^a e^{-\lambda_1(y-a)} h''(y)dy}{e^{-(\lambda_1+\lambda_2)(x-a)}} = 0, \tag{2.8}$$

$$\lim_{x\uparrow \infty} \frac{\int_a^x e^{\lambda_2(y-a)} h''(y)dy}{e^{(\lambda_1+\lambda_2)(x-a)}} = 0. \tag{2.9}$$

(b) Under Assumption 1,

$$\lim_{x\uparrow \infty} \frac{\lambda_2 \int_a^x e^{\lambda_2(y-a)} h'(y)dy}{e^{\lambda_2(x-a)}} = \lim_{x\uparrow\infty} h'(x), \tag{2.10}$$

$$\lim_{x\downarrow -\infty} \frac{\lambda_1 \int_x^a e^{-\lambda_1(y-a)} h'(y)dy}{e^{-\lambda_1(x-a)}} = \lim_{x\downarrow-\infty} h'(x). \tag{2.11}$$

PROOF. (a) We prove (2.8). The proof of (2.9) is similar and is omitted. If

$$\lim_{x\downarrow -\infty} \int_x^a e^{-\lambda_1(y-a)} h''(y)dy < \infty,$$

(2.8) clearly holds. Now assume that

$$\lim_{x\downarrow-\infty}\int_x^a e^{-\lambda_1(y-a)}h''(y)dy=\infty.$$

Using the *L' Hôpital rule* gives

$$\begin{aligned}\lim_{x\downarrow-\infty}\frac{\int_x^a e^{-\lambda_1(y-a)}h''(y)dy}{e^{-(\lambda_1+\lambda_2)(x-a)}} &= \lim_{x\downarrow-\infty}\frac{-e^{-\lambda_1(x-a)}h''(x)}{-(\lambda_1+\lambda_2)e^{-(\lambda_1+\lambda_2)(x-a)}}\\ &= \lim_{x\downarrow-\infty}\frac{h''(x)}{(\lambda_1+\lambda_2)e^{-\lambda_2(x-a)}}\\ &= 0,\end{aligned}$$

where the last equality is due to (2.7).

(b) We prove (2.10). The proof of (2.11) is similar and is omitted.

The first part of (2.5) implies that there exist a constant $c_1>0$ and an $x''\in(a,\infty)$ such that for any $x\ge x''$,

$$h'(x)\ge c_1,$$

which yields

$$\begin{aligned}\lim_{x\uparrow\infty}\int_a^x e^{\lambda_2(y-a)}h'(y)dy &\ge \lim_{x\uparrow\infty}\int_{x''}^x e^{\lambda_2(y-a)}h'(y)dy\\ &\ge c_1\cdot\lim_{x\uparrow\infty}\int_{x''}^x e^{\lambda_2(y-a)}dy\\ &= \infty,\end{aligned}$$

where the first inequality is due to the assumption $h'(x)\ge 0$ for $x>a$. Using the *L' Hôpital rule* gives

$$\begin{aligned}\lim_{x\uparrow\infty}\frac{\lambda_2\int_a^x e^{\lambda_2(y-a)}h'(y)dy}{e^{\lambda_2(x-a)}} &= \lim_{x\uparrow\infty}\frac{\lambda_2 e^{\lambda_2(x-a)}h'(x)}{\lambda_2 e^{\lambda_2(x-a)}}\\ &= \lim_{x\uparrow\infty}h'(x).\end{aligned}$$

□

3. The Itô's formula. In this section, we present the Itô's formula, tailored to the discounted setting. Recall that for a function $g\in\mathbb{D}$, it is right continuous on $[0,\infty)$ and has left limits in $(0,\infty)$. We use g^c to denote the continuous part of g, namely,

$$g^c(t)=g(t)-\sum_{0\le s\le t}\Delta g(s)\qquad\text{for } t\ge 0.$$

Here, we assume $g(0-)$ is well defined. Recall that under any feasible policy $\varphi = (Y_1, Y_2)$, the inventory process $Z = \{Z(t) : t \geq 0\}$ has the semimartingale representation (2.1). Because Brownian motion has continuous sample paths, we have

$$Z^c(t) = X(t) + Y_1^c(t) - Y_2^c(t) \quad \text{for } t \geq 0.$$

LEMMA 3.1. *Assume that $f \in C^1(\mathbb{R})$ and f' is absolutely continuous such that $f'(b) - f'(a) = \int_a^b f''(u)du$ for any $a < b$ with f'' being locally in $L^1(\mathbb{R})$. Then*

$$\begin{aligned} (3.1)\quad e^{-\beta t} f(Z(t)) = f(Z(0)) &+ \int_0^t e^{-\beta s}\big(\Gamma f(Z(s)) - \beta f(Z(s))\big)ds \\ &+ \sigma \int_0^t e^{-\beta s} f'(Z(s))dW(s) + \int_0^t e^{-\beta s} f'(Z(s-))dY_1^c(s) \\ &- \int_0^t e^{-\beta s} f'(Z(s-))dY_2^c(s) + \sum_{0<s\leq t} e^{-\beta s}\Delta f(Z(s)), \end{aligned}$$

where

$$\Gamma f(x) = \frac{1}{2}\sigma^2 f''(x) + \mu f'(x) \quad \textit{for each } x \in \mathbb{R} \textit{ such that } f''(x) \textit{ exists},$$

is the generator of the (μ, σ^2)-Brownian motion X, and $\int_0^t e^{-\beta t} f'(Z(s))dW(s)$ is interpreted as the Itô integral.

PROOF. Using (3.2) in [6] and the integration by parts formula for semimartingales (see, for example, Page 83 of [11]), we have (3.1). □

4. Lower bound. In this section, we state and prove a theorem that establishes a lower bound for the optimal expected total discounted cost. This theorem is closely related to the verification theorem in the literature. Its proof is self-contained, using the Itô formula in Section 3.

THEOREM 4.1. *Suppose that $f \in C^1(\mathbb{R})$ and f' is absolutely continuous with f'' being locally in $L^1(\mathbb{R})$. Suppose that there exists a constant $M > 0$ such that $|f'(x)| \leq M$ for all $x \in \mathbb{R}$. Assume further that*

$$(4.1)\qquad \Gamma f(x) - \beta f(x) + h(x) \geq 0 \quad \textit{for almost all } x \in \mathbb{R},$$

$$(4.2)\qquad f(y) - f(x) \leq K + k(x - y) \quad \textit{for } y < x,$$

$$(4.3)\qquad f(y) - f(x) \leq L + \ell(y - x) \quad \textit{for } x < y.$$

Then $DC(x, \varphi) \geq f(x)$ for each feasible policy φ and each initial state $Z(0-) = x \in \mathbb{R}$.

Remark. (i) When $K = 0$, condition (4.2) is equivalent to $f'(x) \geq -k$ for each $x \in \mathbb{R}$. When $L = 0$, condition (4.3) is equivalent to $f'(x) \leq \ell$ for each $x \in \mathbb{R}$. (ii) Under an arbitrary control policy, the inventory level Z can potentially reach any level. Thus, we require function f to be smoothly defined on the entire real line $\mathbb{R}$. It is *not* enough to have f smoothly defined on a certain interval $[d, u]$.

PROOF. By Itô formula (3.1),

$$
\begin{aligned}
(4.4)\quad e^{-\beta t} f(Z(t)) &= f(Z(0)) + \int_0^t e^{-\beta s}\big(\Gamma f(Z(s)) - \beta f(Z(s))\big) ds \\
&\quad + \sigma \int_0^t e^{-\beta s} f'(Z(s)) dW(s) + \int_0^t e^{-\beta s} f'(Z(s-)) dY_1^c(s) \\
&\quad - \int_0^t e^{-\beta s} f'(Z(s-)) dY_2^c(s) + \sum_{0<s\leq t} e^{-\beta s} \Delta f(Z(s)) \\
&\geq f(Z(0-)) - \int_0^t e^{-\beta s} h(Z(s)) ds \\
&\quad + \sigma \int_0^t e^{-\beta s} f'(Z(s)) dW(s) + \int_0^t e^{-\beta s} f'(Z(s-)) dY_1^c(s) \\
&\quad - \int_0^t e^{-\beta s} f'(Z(s-)) dY_2^c(s) + \sum_{0\leq s\leq t} e^{-\beta s} \Delta f(Z(s)),
\end{aligned}
$$

where the inequality is due to (4.1). In the remainder of the proof, we separate into different cases depending on the positivity of K and L. We will provide a complete proof for the case when $K > 0$ and $L > 0$. Sketches will be provided for proofs in other cases.

Case I: Assume that $K > 0$ **and** $L > 0$. In this case, it is sufficient to restrict feasible policies to impulse control policies $\{(T_n, \xi_n) : n = 0, 1, \ldots\}$. In this case, $Y_1^c = 0$ and $Y_2^c = 0$. Conditions (4.2) and (4.3) imply that $\Delta f(Z(T_n)) \geq -\phi(\xi_n)$ for $n = 0, 1, \ldots$, where

$$
(4.5)\qquad \phi(\xi) = \begin{cases} K + k\xi & \text{if } \xi > 0, \\ 0 & \text{if } \xi = 0, \\ L - l\xi & \text{if } \xi < 0. \end{cases}
$$

Therefore, (4.4) leads to

$$
\begin{aligned}
(4.6)\quad e^{-\beta t} f(Z(t)) &\geq f(Z(0-)) - \int_0^t e^{-\beta s} h(Z(s)) ds \\
&\quad + \sigma \int_0^t e^{-\beta s} f'(Z(s)) dW(s) - \sum_{n=0}^{N(t)} e^{-\beta T_n} \phi(\xi_n)
\end{aligned}
$$

for each $t \geq 0$. Fix an $x \in \mathbb{R}$. We assume that

$$\mathbb{E}_x\bigg(\int_0^t e^{-\beta s} h(Z(s))ds + \sum_{n=0}^{N(t)} e^{-\beta T_n}\phi(\xi_n)\bigg) < \infty$$

for each $t > 0$. Otherwise, $\mathrm{DC}(x,\varphi) = \infty$ and $\mathrm{DC}(x,\varphi) \geq f(x)$ is trivially satisfied. Because $|f'(x)| \leq M$, we have $\mathbb{E}_x\int_0^t e^{-\beta s} f'(Z(s))dW(s) = 0$. Meanwhile

$$f(Z(t)) \leq \big(f(Z(t))\big)^+$$

and $\mathbb{E}_x\big[e^{-\beta t}\big(f(Z(t))\big)^+\big]$ is well defined, though it can be ∞, where, for a $b \in \mathbb{R}$, $b^+ = \max(b,0)$. Taking $\mathbb{E}_x$ on the both sides of (4.6) and noting $f(Z(0-)) = f(x)$, we have

$$\mathbb{E}_x\big[e^{-\beta t}\big(f(Z(t))\big)^+\big] \geq f(x) - \mathbb{E}_x\bigg(\int_0^t e^{-\beta s} h(Z(s))ds + \sum_{n=0}^{N(t)} e^{-\beta T_n}\phi(\xi_n)\bigg).$$

Taking limit as $t \to \infty$, we have

$$\liminf_{t\to\infty}\bigg[\mathbb{E}_x\bigg(\int_0^t e^{-\beta s} h(Z(s))ds + \sum_{n=0}^{N(t)} e^{-\beta T_n}\phi(\xi_n)\bigg) + \mathbb{E}_x\big[e^{-\beta t}\big(f(Z(t))\big)^+\big]\bigg] \geq f(x).$$

The boundedness of f' implies

$$\big(f(x)\big)^+ \leq M(1+|x|),$$

which further implies

$$\big(f(Z(t))\big)^+ \leq M(1+|Z(t)|) \leq M(1+|x|+|\mu|t+\sigma|W(t)|+\sum_{n=0}^{N(t)}|\xi_n|). \tag{4.7}$$

The following arguments follow those on Page 842 of [7]. Let $\nu(t) = \sum_{n=0}^{N(t)}|\xi_n|$. Then (2.4) implies

$$\mathbb{E}_x\Big[\int_0^\infty e^{-\beta t} d\nu(t)\Big] < \infty.$$

From (7.5) of Taksar [13], we have

$$\mathbb{E}_x\Big[\int_0^\infty e^{-\beta t}\nu(t)dt\Big] \leq \frac{1}{\beta}\mathbb{E}_x\Big[\int_0^\infty e^{-\beta t} d\nu(t)\Big] < \infty.$$

Applying Fubini's theorem, we have

$$\int_0^\infty e^{-\beta t}\mathbb{E}_x\big[\nu(t)\big]\,dt < \infty,$$

which, together with Lemma 4.1 of [7], implies

$$\liminf_{t\to\infty} e^{-\beta t}\mathbb{E}_x[\nu(t)] = 0.$$

Therefore, (4.7) implies

$$\begin{aligned}&\liminf_{t\to\infty} \mathbb{E}_x\big[e^{-\beta t}\big(f(Z(t))\big)^+\big]\\ &\quad\le \liminf_{t\to\infty} \mathbb{E}_x\big[e^{-\beta t}\big(M(1+|x|+|\mu|t+\sigma|W(t)|+\nu(t)\big)\big]\\ &\quad= 0.\end{aligned}$$

Hence, the theorem is proved for $K>0$ and $L>0$.

Case II: Assume that $K=0$ **and** $L=0$. Condition (4.2) leads to $f'(u)\ge -k$ for all $u\in\mathbb{R}$ and Condition (4.3) leads to $f'(u)\le \ell$ for all $u\in\mathbb{R}$. Because f is continuous, $\Delta f(Z(s))\neq 0$ implies $\Delta Z(s)\neq 0$. If $\Delta Z(s)>0$, (4.2) implies

$$\Delta f(Z(s)) \ge -k\Delta Z(s).$$

If $\Delta Z(s)<0$, (4.3) implies

$$\Delta f(Z(s)) \ge \ell\Delta Z(s).$$

Thus, the last three terms in (4.4) are at least

$$\begin{aligned}&\int_0^t e^{-\beta s}f'(Z(s-))dY_1^c(s) - \int_0^t e^{-\beta s}f'(Z(s-))dY_2^c(s) + \sum_{0\le s\le t} e^{-\beta s}\Delta f(Z(s))\\ &\quad\ge -k\int_0^t e^{-\beta s}dY_1^c(s) - \ell\int_0^t e^{-\beta s}dY_2^c(s)\\ &\qquad -k\sum_{\substack{0\le s\le t\\ \Delta Z(s)>0}} e^{-\beta s}\Delta Z(s) + \ell\sum_{\substack{0\le s\le t\\ \Delta Z(s)<0}} e^{-\beta s}\Delta Z(s)\\ &\quad= -k\int_0^t e^{-\beta s}dY_1^c(s) - \ell\int_0^t e^{-\beta s}dY_2^c(s)\\ &\qquad -k\sum_{0\le s\le t} e^{-\beta s}\Delta Y_1(s) - \ell\sum_{0\le s\le t} e^{-\beta s}\Delta Y_2(s)\\ &\quad= -k\int_0^t e^{-\beta s}dY_1(s) - \ell\int_0^t e^{-\beta s}dY_2(s)\end{aligned}$$

Therefore, (4.4) leads to

$$\begin{aligned} e^{-\beta t} f(Z(t)) \quad &\geq \quad f(Z(0-)) - \int_0^t e^{-\beta s} h(Z(s)) ds + \sigma \int_0^t e^{-\beta s} f'(Z(s)) dW(s) \\ &\qquad - k \int_0^t e^{-\beta s} dY_1(s) - \ell \int_0^t e^{-\beta s} dY_2(s) \end{aligned}$$

for $t \geq 0$. The rest of the proof is identical to the case when $K > 0$ and $L > 0$.

Case III: Assume that $K > 0$ **and** $L = 0$. Consider a feasible policy (Y_1, Y_2) with a finite cost. The upward controls must be impulse controls and $Y_1(t) = \sum_{n=0}^{N_1(t)} \xi_1(n)$. Condition (4.2) implies

$$\sum_{\substack{0 \leq s \leq t \\ \Delta Z(s) > 0}} e^{-\beta s} \Delta f(Z(s)) \geq - \sum_{n=0}^{N_1(t)} e^{-\beta T_1(n)} (K + k\xi_1(n))$$

and condition (4.3) implies

$$-\ell \int_0^t e^{-\beta s} dY_2^c(s) + \sum_{\substack{0 \leq s \leq t \\ \Delta Z(s) < 0}} e^{-\beta s} \Delta f(Z(s)) \geq -\ell \int_0^t e^{-\beta s} dY_2(s).$$

Therefore, (4.4) leads to

$$\begin{aligned} e^{-\beta t} f(Z(t)) \quad &\geq \quad f(Z(0-)) - \int_0^t e^{-\beta s} h(Z(s)) ds + \sigma \int_0^t e^{-\beta s} f'(Z(s)) dW(s) \\ &\qquad - \sum_{n=0}^{N_1(t)} e^{-\beta T_1(n)} (K + k\xi_1(n)) - \ell \int_0^t e^{-\beta s} dY_2(s) \end{aligned}$$

for $t \geq 0$. The remainder of the proof is identical to the case when $K > 0$ and $L > 0$.

Case III: Assume that $K = 0$ **and** $L > 0$. This case is analogous to the case when $K > 0$ and $L = 0$. Thus, the proof is omitted. □

5. Impulse controls. We assume that $K > 0$ and $L > 0$. Therefore, we restrict our feasible policies to impulse controls as in (2.2) and (2.3). An impulse control band policy is defined by four parameters d, D, U, u, where $d < D < U < u$. Under the policy, when the inventory falls to d, the system instantaneously orders items to bring it up to level D; when the inventory rises to u, the system adjusts its inventory to bring it down to

U. Given a control band policy φ, in Section 5.1 we provide a method for performance evaluation. In Section 5.2, we first claim in Theorem 5.2 the existence of a solution to the free boundary problem with associated parameters (d^*, D^*, U^*, u^*). Assuming Theorem 5.2, we prove in Theorem 5.3 that the control band policy associated with (d^*, D^*, U^*, u^*) is indeed optimal among all feasible policies. Section 5.3 describes the proof of Theorem 5.2.

5.1. *Control band policies.* Use $\{d, D, U, u\}$ to denote the control band policy associated with parameters d, D, U and u with $d < D < U < u$. Fix a control band policy $\varphi = \{d, D, U, u\}$ and an initial inventory level $Z(0-) = x$. The adjustment amount ξ_n of the control band policy is given by

$$\xi_0 = \begin{cases} D - x & \text{if } x \le d, \\ 0 & \text{if } d < x < u, \\ U - x & \text{if } x \ge u, \end{cases}$$

and for $n = 1, 2, \ldots,$

$$\xi_n = \begin{cases} D - d & \text{if } Z(T_n-) = d, \\ U - u & \text{if } Z(T_n-) = u, \end{cases}$$

where again $Z(t-)$ denotes the left limit at time t, $T_0 = 0$ and

$$T_n = \inf\{t > T_{n-1} : Z(t) \in \{d, u\}\}$$

is the nth adjustment time. (By convention, we assume Z is right continuous having left limits.) Our first task is to obtain an expression for the *value function* $\mathrm{DC}(x, \varphi)$, the expected total discounted cost under a control band policy φ when the initial inventory level is x. We first present the following theorem.

THEOREM 5.1. *Assume that we fix a control band policy $\varphi = \{d, D, U, u\}$. If there exists a twice continuously differentiable function $V : [d, u] \to \mathbb{R}$ that satisfies*

$$\Gamma V(x) - \beta V(x) + h(x) = 0 \quad d \le x \le u, \tag{5.1}$$

with boundary conditions

$$V(d) - V(D) = K + k(D - d), \tag{5.2}$$

$$V(u) - V(U) = L + l(u - U), \tag{5.3}$$

then for each starting point $x \in \mathbb{R}$*, the expected total discounted cost* $DC(x,\varphi)$ *is given by*

$$DC(x,\varphi) = \begin{cases} V(D) + K + k(D-x) & \text{for } x \in (-\infty, d], \\ V(x) & \text{for } x \in (d,u), \\ V(U) + L - \ell(U-x) & \text{for } x \in [u,\infty), \end{cases}$$

where $V(x)$ *is in* (5.1)*.*

Remark. (5.2) and (5.3) imply that $\bar{V}$ is continuous at d and u.

Proof. Consider the control band policy $\varphi = \{d, D, U, u\}$. Let V be a twice continuously differentiable function on $[d,u]$ that satisfies (5.1)-(5.3). Because $d \le Z(t) \le u$, by Lemma 3.1, we have

$$\mathbb{E}_x[e^{-\beta t}V(Z(t))] \;=\; \mathbb{E}_x[V(Z(0))] + \mathbb{E}_x\Big[\int_0^t e^{-\beta s}\big(\Gamma V(Z(s)) - \beta V(Z(s))\big)ds\Big] + \mathbb{E}_x\Big[\sum_{n=1}^{N(t)} e^{-\beta T_n}\theta_n\Big],$$

where $\theta_n = V(Z(T_n)) - V(Z(T_n-))$. Boundary conditions (5.2) and (5.3) imply $\theta_n = V(Z(T_n)) - V(Z(T_n-)) = -\phi(\xi_n)$ for $n \ge 1$, where ϕ is as defined in (4.5). Therefore,

$$\begin{aligned} &\mathbb{E}_x[e^{-\beta t}V(Z(t))] - \mathbb{E}_x[V(Z(0))] \\ &\quad = \mathbb{E}_x\Big[\int_0^t e^{-\beta s}\big(\Gamma V(Z(s)) - \beta V(Z(s))\big)ds\Big] + \mathbb{E}_x\Big[\sum_{n=1}^{N(t)} e^{-\beta T_n}\theta_n\Big] \\ &\quad = -\mathbb{E}_x\Big[\int_0^t e^{-\beta s}h(Z(s))ds\Big] - \mathbb{E}_x\Big[\sum_{n=1}^{N(t)} e^{-\beta T_n}\phi(\xi_n)\Big] \\ &\quad = -\mathbb{E}_x\Big[\int_0^t e^{-\beta s}h(Z(s))ds\Big] - \mathbb{E}_x\Big[\sum_{n=0}^{N(t)} e^{-\beta T_n}\phi(\xi_n)\Big] + \mathbb{E}_x[\phi(\xi_0)]. \end{aligned}$$

Letting $t \to \infty$, we have

$$\text{DC}(x,\varphi) = \mathbb{E}_x[V(Z(0))] + \mathbb{E}_x[\phi(\xi_0)] \tag{5.4}$$

because

$$\lim_{t\to\infty} \mathbb{E}_x[e^{-\beta t}V(Z(t))] = 0.$$

If $Z(0-) = x \in (d, u)$, we have $Z(0) = Z(0-) = x$ and $\xi_0 = 0$, then

$$\mathrm{DC}(x, \varphi) = V(x).$$

If $x \leq d$, under control band policy $\varphi = \{d, D, U, u\}$, Z immediately jumps up to D. Therefore, $Z(0) = D$ and $\xi_0 = D - x$, then

$$\mathbb{E}_x[V(Z(0))] = V(D), \quad \mathbb{E}_x[\phi(\xi_0)] = \phi(D - x) = K + k(D - x),$$

which, together with (5.4), implies

$$\mathrm{DC}(x, \varphi) = V(D) + K + k(D - x).$$

The analysis for the case $x \geq u$ is analogous and is omitted. $\square$

We end this section by explicitly finding a solution V to (5.1)-(5.3). Define

$$\lambda_1 = \Big[(\mu^2 + 2\beta\sigma^2)^{1/2} - \mu\Big]/\sigma^2 > 0, \tag{5.5}$$

$$\lambda_2 = \Big[(\mu^2 + 2\beta\sigma^2)^{1/2} + \mu\Big]/\sigma^2 > 0. \tag{5.6}$$

Proposition 1. *Let $\varphi = \{d, D, U, u\}$ be a control band policy with*

$$d < D < U < u.$$

Define

$$V(x) = A_1 e^{\lambda_1 x} + B_1 e^{-\lambda_2 x} + V_0(x), \tag{5.7}$$

where

$$V_0(x) = \frac{2}{\sigma^2}\frac{1}{\lambda_1 + \lambda_2}\Big[\int_a^x e^{-\lambda_2(x-y)} h(y)dy - \int_a^x e^{\lambda_1(x-y)} h(y)dy\Big], \tag{5.8}$$

$$\begin{aligned} A_1 = {} & \frac{1}{a_1 b_2 - a_2 b_1}\Big[b_2\big(V_0(D) - V_0(d) + K + k(D - d)\big) \\ & - b_1\big(V_0(U) - V_0(u) + L + \ell(u - U)\big)\Big], \end{aligned} \tag{5.9}$$

$$\begin{aligned} B_1 = {} & \frac{1}{a_2 b_1 - a_1 b_2}\Big[a_2\big(V_0(D) - V_0(d) + K + k(D - d)\big) \\ & - a_1\big(V_0(U) - V_0(u) + L + \ell(u - U)\big)\Big]. \end{aligned} \tag{5.10}$$

Then V is a solution to (5.1)-(5.3). In (5.9) and (5.10), we set

$$a_1 = e^{\lambda_1 d} - e^{\lambda_1 D}, \quad a_2 = e^{\lambda_1 u} - e^{\lambda_1 U}, \tag{5.11}$$

$$b_1 = e^{-\lambda_2 d} - e^{-\lambda_2 D}, \quad b_2 = e^{-\lambda_2 u} - e^{-\lambda_2 U}. \tag{5.12}$$

Proof. Observe that $z = \lambda_1$ and $z = -\lambda_2$ are two solutions of the quadratic equation

$$\frac{1}{2}\sigma^2 z^2 + \mu z - \beta = 0.$$

The homogenous ordinary differential equation (ODE)

$$\Gamma g - \beta g = 0$$

has two independent solutions $g_1(x)$ and $g_2(x)$, where

$$g_1(x) = e^{\lambda_1 x} \quad \text{and} \quad g_2(x) = e^{-\lambda_2 x}.$$

Let

$$w(x) = \det \begin{pmatrix} g_1(x) & g_2(x) \\ g_1'(x) & g_2'(x) \end{pmatrix} = -(\lambda_1 + \lambda_2)e^{(\lambda_1 - \lambda_2)x} \neq 0$$

and

$$a_1(x) = \int_a^x \frac{1}{w(y)} g_2(y) \frac{2}{\sigma^2} h(y)dy = -\frac{1}{\lambda_1 + \lambda_2}\frac{2}{\sigma^2}\int_a^x e^{-\lambda_1 y} h(y)dy,$$

$$a_2(x) = -\int_a^x \frac{1}{w(y)} g_1(y) \frac{2}{\sigma^2} h(y)dy = \frac{1}{\lambda_1 + \lambda_2}\frac{2}{\sigma^2}\int_a^x e^{\lambda_2 y} h(y)dy,$$

where a is the minimum point of the convex inventory cost function h. Then the non-homogenous ODE (5.1) has a particular solution $V_0(x)$ given by $V_0(x) = a_1(x)g_1(x) + a_2(x)g_2(x)$, which equals to the expression in (5.8). A general solution $V(x)$ to (5.1) is given by (5.7), namely,

$$V(x) = A_1 e^{\lambda_1 x} + B_1 e^{-\lambda_2 x} + V_0(x).$$

Boundary conditions (5.2) and (5.3) become

$$\big(A_1 e^{\lambda_1 d} + B_1 e^{-\lambda_2 d} + V_0(d)\big) - \big(A_1 e^{\lambda_1 D} + B_1 e^{-\lambda_2 D} + V_0(D)\big) = K + k(D - d), \tag{5.13}$$

$$\big(A_1 e^{\lambda_1 u} + B_1 e^{-\lambda_2 u} + V_0(u)\big) - \big(A_1 e^{\lambda_1 U} + B_1 e^{-\lambda_2 U} + V_0(U)\big) = L + \ell(u - U). \tag{5.14}$$

Using the coefficients defined in (5.11)-(5.12), the boundary conditions (5.13) and (5.14) become

$$A_1 a_1 + B_1 b_1 + V_0(d) - V_0(D) = K + k(D - d),$$
$$A_1 a_2 + B_1 b_2 + V_0(u) - V_0(U) = L + \ell(u - U),$$

from which we have a unique solution for A_1 and B_1 given in (5.9) and (5.10). $\square$

5.2. *Optimal policy and optimal parameters.* Theorem 4.1 suggests the following strategy to obtain an optimal policy. We hope that a control band policy is optimal. Therefore, the first task is to find an optimal policy among all control band policies. We denote this optimal control band policy by $\varphi^* = \{d^*, D^*, U^*, u^*\}$ with the expected total discounted cost

$$\bar{V}(x) = \begin{cases} V(D^*) + K + k(D^* - x) & \text{for } x \in (-\infty, d^*], \\ V(x) & \text{for } x \in (d^*, u^*), \\ V(U^*) + L - \ell(U^* - x) & \text{for } x \in [u^*, \infty), \end{cases} \tag{5.15}$$

for any starting point $x \in \mathbb{R}$. We hope that $\bar{V}$ can be used as the function f in Theorem 4.1. To find the corresponding f that satisfies all of the conditions in Theorem 4.1, we provide the conditions that should be imposed on the optimal parameters

$$V'(D^*) = -k, \quad V'(U^*) = \ell, \tag{5.16}$$

$$V'(d^*) = -k, \quad V'(u^*) = \ell. \tag{5.17}$$

See Section 5.2 of [6] for an intuitive derivation of these conditions. Under condition (5.17), $\bar{V}$ is a C^1 function on $\mathbb{R}$. Therefore, (5.17) is also known as the "smooth-pasting" condition.

In this section, we first state Theorem 5.2, which claims the existence of parameters d^*, D^*, U^* and u^* such that the value function V, defined on $[d^*, u^*]$, corresponding to the control band policy $\varphi^* = \{d^*, D^*, U^*, u^*\}$ satisfies (5.1)-(5.3) and (5.16)-(5.17). As part of the solution, we need to find the boundary points d^*, D^*, U^* and u^* from (5.1)-(5.3) and (5.16)-(5.17). These equations define a *free boundary problem*. We then prove in Theorem 5.3 that the function $\bar{V}$ in (5.15) with parameters d^*, D^*, U^* and u^* satisfies all the conditions in Theorem 4.1; therefore, the control band policy φ^* is optimal among all feasible policies.

To facilitate the presentation of Theorem 5.2, we first find a general solution without worrying about boundary conditions (5.2) and (5.3). Proposition 1 shows that V is given by the expression in (5.7), namely

$$V(x) = A_1 e^{\lambda_1 x} + B_1 e^{-\lambda_2 x} + V_0(x) \quad \text{for } x \in \mathbb{R}, \tag{5.18}$$

where V_0 is given in (5.8). Since A_1 and B_1 are yet to be determined (both of them depend on d^*, D^*, U^* and u^*), V is also yet to be determined. Differentiating both sides of (5.1) with respect to x gives

$$\begin{aligned} g(x) &= V'(x) \\ &= \lambda_1 A_1 e^{\lambda_1 x} - \lambda_2 B_1 e^{-\lambda_2 x} \\ &\quad - \frac{2}{\sigma^2} \frac{1}{\lambda_1 + \lambda_2} \left[\lambda_2 \int_a^x e^{-\lambda_2 (x-y)} h(y) dy + \lambda_1 \int_a^x e^{\lambda_1 (x-y)} h(y) dy \right] \end{aligned} \tag{5.19}$$

is a solution to

$$\Gamma g(x) - \beta g(x) + h'(x) = 0 \quad \text{for all } x \in \mathbb{R} \setminus \{a\}. \tag{5.20}$$

We rewrite $g(x)$ in (5.19) as

$$\begin{aligned} g(x) &= V'(x) \\ &= \lambda_1 A_1 e^{\lambda_1 x} - \lambda_2 B_1 e^{-\lambda_2 x} \\ &\quad - \frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2}\left[\lambda_2 \int_a^x e^{-\lambda_2(x-y)} h(y)dy + \lambda_1 \int_a^x e^{\lambda_1(x-y)} h(y)dy\right] \\ &= \lambda_1 A_1 e^{\lambda_1 x} - \lambda_2 B_1 e^{-\lambda_2 x} \\ &\quad - \frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2}\left[-\int_a^x e^{-\lambda_2(x-y)} h'(y)dy + \int_a^x e^{\lambda_1(x-y)} h'(y)dy\right] \\ &= \frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2}\left[\frac{1}{\lambda_1}\big(A - \lambda_1 \int_a^x e^{-\lambda_1(y-a)} h'(y)dy\big) e^{\lambda_1(x-a)} \right. \\ &\quad \left. + \frac{1}{\lambda_2}\big(B + \lambda_2 \int_a^x e^{\lambda_2(y-a)} h'(y)dy\big) e^{-\lambda_2(x-a)}\right] \\ &\equiv g_{A,B}(x), \end{aligned} \tag{5.21}$$

where the third equality uses the assumption that $h(a) = 0$ and in the last equality A and B satisfy

$$\frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2}\frac{1}{\lambda_1}e^{-\lambda_1 a}A = \lambda_1 A_1, \quad \frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2}\frac{1}{\lambda_2}e^{\lambda_2 a}B = -\lambda_2 B_1.$$

The following theorem characterizes optimal parameters (d^*, D^*, U^*, u^*) and parameters A^* and B^* in (5.21) via solution $g = g_{A,B}$.

THEOREM 5.2. *Assume that holding cost function h satisfies Assumption 1. There exist unique A^*, B^*, x_1^*, x_2^*, d^*, D^*, U^*, u^* with*

$$d^* < x_1^* < D^* < U^* < x_2^* < u^*, \tag{5.22}$$

such that $g(x) = g_{A^*,B^*}(x)$ in (5.21) satisfies

$$\int_{d^*}^{D^*} \Big[g(x) + k\Big] dx = -K, \tag{5.23}$$

$$\int_{U^*}^{u^*} \Big[g(x) - \ell\Big] dx = L, \tag{5.24}$$

$$g(d^*) = -k, \tag{5.25}$$

$$g(D^*) = -k, \tag{5.26}$$

$$g(U^*) = \ell, \tag{5.27}$$

$$g(u^*) = \ell. \tag{5.28}$$

Furthermore,

$$h'(x_1^*) \le -\beta k \quad \text{and} \quad h'(x_2^*) \ge \beta \ell. \tag{5.29}$$

Function g has a local minimum at $x_1^* < a$ and a local maximum at $x_2^* > a$. This function strictly decreases on $(-\infty, x_1^*)$, strictly increases on (x_1^*, x_2^*), and strictly decreases again on (x_2^*, ∞).

If g satisfies all conditions (5.20), (5.23)-(5.28) in Theorem 5.2, $V(x)$ in (5.18) clearly satisfies all conditions (5.1)-(5.3) and (5.16)-(5.17). See Section 5.3 for an explanation of Theorem 5.2.

THEOREM 5.3. *Assume that holding cost function h satisfies Assumption 1. Let $d^* < D^* < U^* < u^*$, along with constants A^* and B^*, be the unique solution in Theorem 5.2. Then, control band policy $\varphi^* = \{d^*, D^*, U^*, u^*\}$ is optimal among all non-anticipating policies.*

PROOF. Let

$$\bar{g}(x) = \begin{cases} -k & \text{for } x \in (-\infty, d^*], \\ g_{A^*,B^*}(x) & \text{for } x \in (d^*, u^*), \\ \ell & \text{for } x \in [u^*, \infty) \end{cases}$$

be the function representing the red curve in Figure 1 and

$$\bar{V}(x) = \begin{cases} V(D^*) + K + k(D^* - x) & \text{for } x \in (-\infty, d^*], \\ V(x) & \text{for } x \in (d^*, u^*), \\ V(U^*) + L + \ell(x - U^*) & \text{for } x \in [u^*, \infty), \end{cases} \tag{5.30}$$

with

$$V(x) = A_1^* e^{\lambda_1 x} + B_1^* e^{-\lambda_2 x} + V_0(x), \tag{5.31}$$

where

$$\frac{2}{\sigma^2} \frac{1}{\lambda_1 + \lambda_2} \frac{1}{\lambda_1} e^{-\lambda_1 a} A^* = \lambda_1 A_1^*, \qquad \frac{2}{\sigma^2} \frac{1}{\lambda_1 + \lambda_2} \frac{1}{\lambda_2} e^{\lambda_2 a} B^* = -\lambda_2 B_1^*,$$

and $V_0(x)$ is given by (5.8). Therefore,

$$\bar{V}'(x) = \bar{g}_{A^*,B^*}(x) \quad \text{for } x \in \mathbb{R}. \tag{5.32}$$

We now show that $\bar{V}$ satisfies all of the conditions in Theorem 4.1. Thus, Theorem 4.1 shows that the expected total discounted cost under any feasible policy is at least $\bar{V}(x)$. Since $\bar{V}(x)$ is the expected total discounted cost

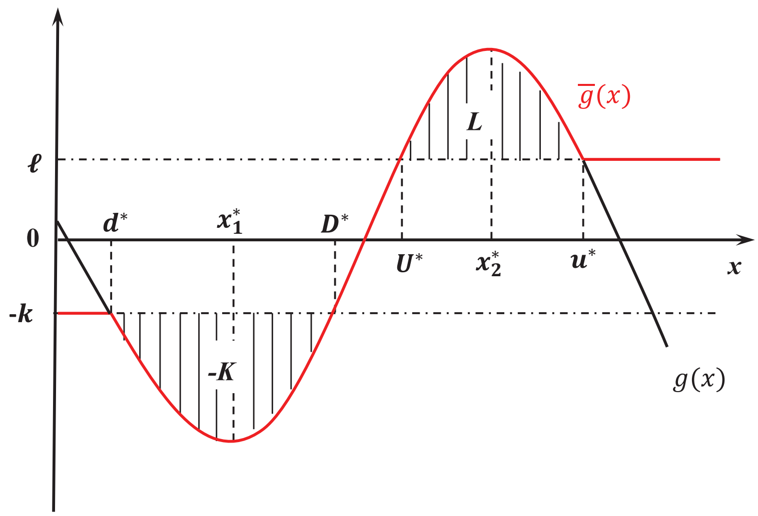

Fig 1. *(a) There exist $x_1^* < x_2^*$ such that function g decreases in $(-\infty, x_1^*)$, increases in (x_1^*, x_2^*), and deceases again in (x_2^*, ∞). Parameters d^*, D^*, U^* and u^* are determined by $g(d^*) = g(D^*) = -k$, $g(U^*) = g(u^*) = \ell$, the shaded area between U^* and u^* is L, and the shaded area between d^* and D^* is K. In the interval $[d^*, u^*]$, g is the derivative of the value function V associated with control band policy $\{d^*, D^*, U^*, u^*\}$.*
(b) The red curve plots function $\bar{g}$, where $\bar{g}(x) = -k$ for $x < d^*$, $\bar{g}(x) = g(x)$ for $d^* \le x \le u^*$ and $\bar{g}(x) = \ell$ for $x > u^*$.

under the control band policy φ^* with starting point x, $\bar{V}(x)$ is the optimal cost and this control band policy is optimal among all feasible policies.

First, $\bar{V}(x)$ is in $C^2((d^*, u^*))$. Condition (5.23) implies

$$V(d^*) - V(D^*) = -\int_{d^*}^{D^*} g_{A^*,B^*}(x)dx = K + k(D^* - d^*)$$

and (5.24) implies

$$V(u^*) - V(U^*) = \int_{U^*}^{u^*} g_{A^*,B^*}(x)dx = L + \ell(u^* - U^*).$$

(5.31) implies that V satisfies

$$\Gamma V(x) - \beta V(x) + h(x) = 0, \text{ for } x \in [d^*, u^*].$$

By Theorem 5.1, $\bar{V}$ defined in (5.30) must be the total discounted cost under control band policy φ^*.

Now, we show that $\bar{V}(x)$ satisfies the remainder of the conditions in Theorem 4.1. Conditions (5.25) and (5.28) imply that truncated function $\bar{V}'(x)$ is continuous in $\mathbb{R}$. Therefore, $\bar{V} \in C^1(\mathbb{R})$. Clearly, $\bar{V}'(x) = -k$ for $x \in (-\infty, d^*]$ and $\bar{V}'(x) = \ell$ for $x \in [u^*, \infty)$. Let

$$M = \sup_{x \in [d^*, u^*]} |g_{A^*,B^*}(x)|.$$

We have $|\bar{V}'(x)| \le M$ for all $x \in \mathbb{R}$. Furthermore,

$$\Gamma\bar{V} - \beta\bar{V}(x) + h(x) = \Gamma V - \beta V(x) + h(x) = 0 \quad \text{for } x \in [d^*, u^*].$$

In particular

$$\Gamma\bar{V}(d^*) - \beta\bar{V}(d^*) + h(d^*) = 0$$

and

$$\Gamma\bar{V}(u^*) - \beta\bar{V}(u^*) + h(u^*) = 0.$$

It follows from Theorem 5.2 that $d^* < x_1^* < a < x_2^* < u^*$, $\bar{V}''(d^*) = V''(d^*) = g'(d^*) \le 0$ and $\bar{V}''(u^*) = V''(u^*) = g'(u^*) \le 0$ (see Figure 1). Thus, we have

$$\mu\bar{V}'(d^*) - \beta\bar{V}(d^*) + h(d^*) \ge 0 \tag{5.33}$$

and

$$\mu\bar{V}'(u^*) - \beta\bar{V}(u^*) + h(u^*) \ge 0. \tag{5.34}$$

For $x < d^*$,

$$\begin{aligned}
&\Gamma\bar{V}(x) - \beta\bar{V}(x) + h(x) \\
&\qquad = \mu(-k) - \beta(\bar{V}(d^*) - k(x - d^*)) + h(x) \\
&\qquad = \mu\bar{V}'(d^*) - \beta\bar{V}(d^*) + h(d^*) + h(x) - h(d^*) + \beta k(x - d^*) \\
&\qquad \ge h(x) - h(d^*) + \beta k(x - d^*) \\
&\qquad \ge 0,
\end{aligned}$$

where the first inequality follows from (5.33) and the second inequality follows from (5.29) and the convexity of h. Similarly, for $x > u^*$,

$$\begin{aligned}
&\Gamma\bar{V}(x) - \beta\bar{V}(x) + h(x) \\
&\qquad = \mu(\ell) - \beta(\bar{V}(u^*) + \ell(x - u^*)) + h(x) \\
&\qquad = \mu\bar{V}'(u^*) - \beta\bar{V}(u^*) + h(u^*) + h(x) - h(u^*) - \beta\ell(x - u^*) \\
&\qquad \ge h(x) - h(u^*) - \beta\ell(x - u^*) \\
&\qquad \ge 0,
\end{aligned}$$

where the first inequality follows from (5.34) and the second inequality again follows from (5.29) and the convexity of h.

Now we verify that $\bar{V}$ satisfies (4.2). Let $x, y \in \mathbb{R}$ with $y < x$. Then,

$$\begin{aligned}
\bar{V}(x) - \bar{V}(y) + k(x-y) &= \int_y^x [\bar{g}(z) + k]dz \\
&\geq \int_{(y\vee d^*)\wedge D^*}^{(x\wedge D^*)\vee d^*} [\bar{g}(z) + k]dz \\
&\geq \int_{d^*}^{D^*} [\bar{g}(z) + k]dz \\
&= -K,
\end{aligned}$$

where the first inequality follows from $\bar{g}(z) = -k$ for $z \leq d^*$ and $\bar{g}(z) = g(z) \geq -k$ for $D^* < z < u^*$ and $\bar{g}(z) = \ell \geq -k$ for $z \geq u^*$, and the second inequality follows from the fact that $\bar{g}(z) = g(z) \leq -k$ for $z \in [d^*, D^*]$; see Figure 1. Thus (4.2) is proved.

It remains to verify that $\bar{V}$ satisfies (4.3). For $x, y \in \mathbb{R}$ with $y > x$.

$$\begin{aligned}
\bar{V}(y) - \bar{V}(x) - \ell(y-x) &= \int_x^y [\bar{g}(z) - \ell]dz \\
&\leq \int_{(x\vee U^*)\wedge u^*}^{(y\wedge u^*)\vee U^*} [\bar{g}(z) - \ell]dz \\
&\leq \int_{U^*}^{u^*} [\bar{g}(z) - \ell]dz \\
&= L,
\end{aligned}$$

proving (4.3). □

5.3. *Optimal control band parameters.* This section is devoted to the proof of Theorem 5.2. Recall that the function $g_{A,B}(x)$ in (5.21) depends on two parameters A and B. To prove the theorem, it suffices to prove that there exists (A^*, B^*) so that $g(x) = g_{A^*,B^*}(x)$, together with some constants x_1^*, x_2^*, d^*, D^*, U^*, u^*, satisfies (5.22)-(5.28) and the desired monotonicity properties of g as stated in the theorem. Figure 1 illustrates the conditions and properties that $g_{A^*,B^*}(x)$ must satisfy.

To prove the existence of (A^*, B^*), we start with a series of lemmas. Each lemma progressively narrows down the search range of (A, B) so that $g_{A,B}(x)$ satisfies a subset of the conditions in Theorem 5.2. Applying the integration by parts in (5.21), we have

$$g_{A,B}(x) = \begin{cases} \frac{1}{\beta}h'(x) + \frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2} \\ \quad \times\Big[\frac{1}{\lambda_1}\Big(A - h'(a-) + \int_x^a e^{-\lambda_1(y-a)}h''(y)dy\Big)e^{\lambda_1(x-a)} \\ \quad +\frac{1}{\lambda_2}\Big(B - h'(a-) + \int_x^a e^{\lambda_2(y-a)}h''(y)dy\Big)e^{-\lambda_2(x-a)}\Big] & \text{for } x < a, \\ \frac{1}{\beta}h'(x) + \frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2} \\ \quad \times\Big[\frac{1}{\lambda_1}\Big(A - h'(a+) - \int_a^x e^{-\lambda_1(y-a)}h''(y)dy\Big)e^{\lambda_1(x-a)} \\ \quad +\frac{1}{\lambda_2}\Big(B - h'(a+) - \int_a^x e^{\lambda_2(y-a)}h''(y)dy\Big)e^{-\lambda_2(x-a)}\Big] & \text{for } x > a, \end{cases} \tag{5.35}$$

where

$$h'(a-) = \lim_{x\uparrow a} h'(x) \quad \text{and} \quad h'(a+) = \lim_{x\downarrow a} h'(x)$$

are well defined due to the monotonicity of $h'(x)$. Clearly,

$$\begin{aligned} \overline{A} &= h'(a+) + \int_a^{+\infty} e^{-\lambda_1(y-a)}h''(y)dy \quad \text{and} \\ \underline{B} &= h'(a-) - \int_{-\infty}^a e^{\lambda_2(y-a)}h''(y)dy \end{aligned} \tag{5.36}$$

are two important constants. The search range of (A^*, B^*) is closely related to these two constants.

Before stating and proving these lemmas in this section, we briefly describe them. In Lemma 5.1, we show that the constant $\overline{A} > 0$ and the constant $\underline{B} < 0$. In Lemma 5.2, we show that as long as (A, B) satisfies $\underline{B} < B < A < \overline{A}$, there exist unique points $x_1 = x_1(A, B)$ and $x_2 = x_2(A, B)$ so that the function $g_{A,B}(x)$ has the desired monotonicity properties in various intervals; see Figure 1. In Lemma 5.3, we show that $x_1(A, B)$ and $x_2(A, B)$ each is continuous and monotone in A and B, respectively.

In Lemma 5.4, we identify a region G of (A, B). The region is defined in (5.85) and is the shaded area in Figure 4; when $(A, B) \in G$, we have $g_{A,B}(x_2) > \ell$ and $g_{A,B}(x_1) < -k$. Therefore, as long as $(A, B) \in G$, the curve $y = g_{A,B}(x)$ has exactly two intersections, denoted by $U(A, B)$ and $u(A, B)$, with line $y = \ell$ and exactly two intersections, denoted by $d(A, B)$ and $D(A, B)$, with line $y = -k$; see Figure 1. In other words, when $(A, B) \in G$, the areas above line $y = \ell$ and below line $y = -k$ in Figure 1 are both positive. When $(A, B) \in G$, conditions (5.25)-(5.28) are always satisfied for $d(A, B)$, $D(A, B)$, $U(A, B)$, and $u(A, B)$.

In Lemmas 5.5 and 5.6, we show that there is a curve $A = A^*(B)$, the solid red curve in Figure 6, of (A, B) inside the region G such that for any (A, B) on the curve, the area in Figure 1 above line $y = \ell$ is exactly equal to L. Thus, $g_{A,B}(x)$ satisfies condition (5.24) in Theorem 5.2 when (A, B) stays on the solid red curve inside G in Figure 6. In Lemma 5.7, we show that there exists a point (A^*, B^*) on the solid red curve such that the area below line $y = -k$ in Figure 1 is exactly equal to K, showing that (5.23) is satisfied when $(A, B) = (A^*, B^*)$. Finally, in Lemma 5.8, we prove (5.29). These lemmas show that by choosing $x_1^* = x_1(A^*, B^*)$, $x_2^* = x_2(A^*, B^*)$, $d^* = d(A^*, B^*)$, $D^* = D(A^*, B^*)$, $U^* = U(A^*, B^*)$, $u^* = u(A^*, B^*)$, function $g_{A^*,B^*}(x)$, together with constants x_1^*, x_2^*, d^*, D^*, U^*, u^*, satisfies conditions (5.22)-(5.29) and the associated monotonicity properties. Therefore, Theorem 5.2 is proved.

LEMMA 5.1. *Assume that h satisfies Assumption 1. Then*

$$\overline{A} > 0, \tag{5.37}$$

$$\underline{B} < 0. \tag{5.38}$$

PROOF. Assumption 1 (c) says that $h'(x) \le 0$ for $x < a$ and $h'(x) \ge 0$ for $x > a$, therefore we have

$$0 \le \lim_{x \downarrow a} h'(x) = h'(a+).$$

If $\int_a^{+\infty} e^{-\lambda_1 (y-a)} h''(y) dy > 0$, then (5.37) clearly holds. Now assume that

$$\int_a^{+\infty} e^{-\lambda_1 (y-a)} h''(y) dy = 0.$$

Because $h''(x) \ge 0$ and $h''(x)$ is assumed to be continuous on (a, ∞), $h''(x) = 0$ for $x > a$. Therefore, h must be linear in $x > a$. This fact and (2.5) imply

$$h'(a+) = \lim_{x \downarrow a} h'(x) > 0,$$

which proves (5.37). Similarly we can prove (5.38). □

LEMMA 5.2. *(a) For each (A, B) satisfying*

$$\underline{B} < B < A, \tag{5.39}$$

$g_{A,B}(x)$ attains a unique minimum in $(-\infty, a)$ at $x_1 = x_1(A, B) \in (-\infty, a)$.

For each (A, B) satisfying

$$B < A < \overline{A}, \tag{5.40}$$

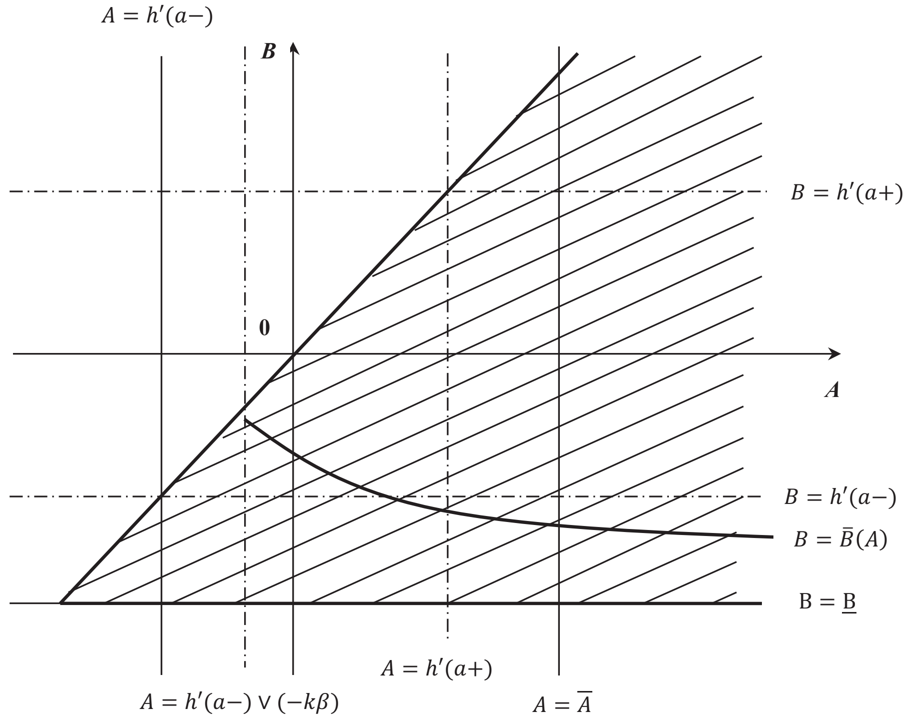

FIG 2. *(a) The shaded region is the set of (A, B) that satisfies (5.39). The unique minimum $x_1 = x_1(A, B) \in (-\infty, a)$ is well defined for all (A, B) in this region.*
(b) For each $A \in \big(h'(a-) \vee (-k\beta), +\infty\big)$, there exists a unique $\overline{B}(A) \in \big(\underline{B}, A \wedge 0\big)$ such that $g_{A,\overline{B}(A)}(x_1(A, \overline{B}(A))) = -k$. Thus, on the curve $B = \overline{B}(A)$, $g_{A,B}(x_1(A, B)) = -k$. The curve $B = \overline{B}(A)$ is decreasing.

$g_{A,B}(x)$ attains a unique maximum in (a, ∞) at $x_2 = x_2(A, B) \in (a, \infty)$.
(b) For each fixed (A, B) satisfying (5.39), the local minimizer $x_1 = x_1(A, B)$ is the unique solution in $(-\infty, a)$ to

$$(5.41) \qquad \Big(A - h'(a-) + \int_{x_1}^{a} e^{-\lambda_1(y-a)} h''(y)dy\Big) e^{\lambda_1(x_1-a)} = \Big(B - h'(a-) + \int_{x_1}^{a} e^{\lambda_2(y-a)} h''(y)dy\Big) e^{-\lambda_2(x_1-a)}.$$

Furthermore, $g'_{A,B}(x) < 0$ for $x \in (-\infty, x_1(A, B))$, $g'_{A,B}(x) > 0$ for $x \in (x_1(A, B), a)$, and

$$(5.42) \qquad \lim_{x \downarrow -\infty} g_{A,B}(x) = +\infty,$$

$$(5.43) \qquad g''_{A,B}(x_1(A, B)) > 0.$$

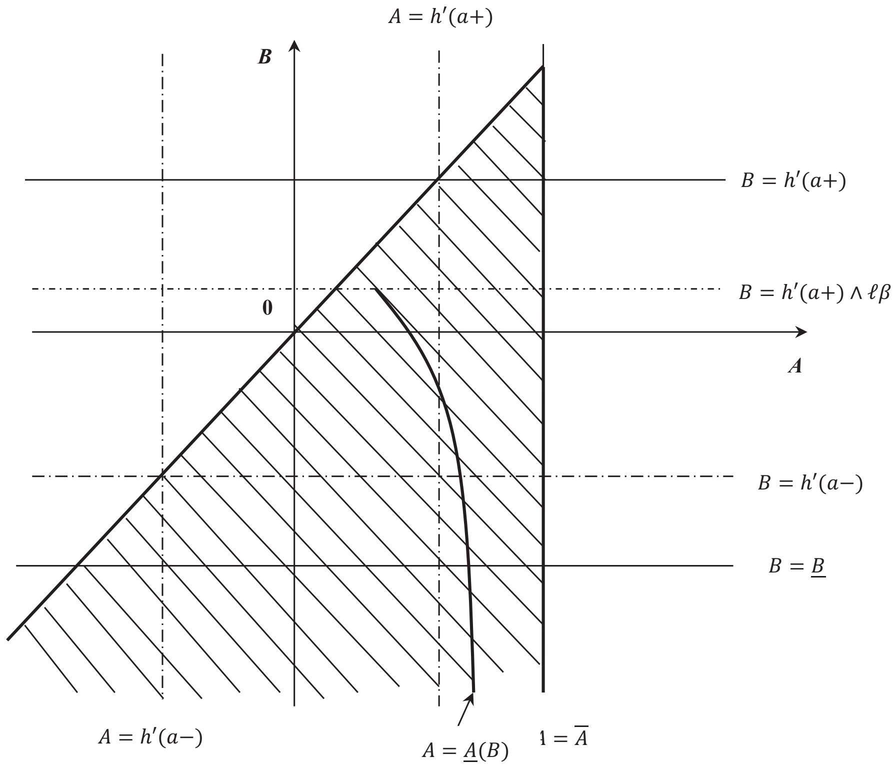

FIG 3. *(a) The shaded region is the set of (A, B) that satisfies (5.40). The unique maximum $x_2 = x_2(A, B) \in (a, \infty)$ is well defined for all (A, B) in the region.*
(b) For each $B \in \big(-\infty, h'(a+) \wedge \ell\beta\big)$, there exists a unique $\underline{A}(B) \in \big(B \vee 0, \overline{A}\big)$ such that $g_{\underline{A}(B),B}(x_2(\underline{A}(B), B)) = \ell$. Thus, on the curve $A = \underline{A}(B)$, $g_{A,B}(x_1(A, B)) = \ell$. The curve $A = \underline{A}(B)$ is decreasing.

For each fixed (A, B) satisfying (5.40), the local maximizer $x_2 = x_2(A, B)$ is the unique solution in (a, ∞) to

$$\left(A - h'(a+) - \int_a^{x_2} e^{-\lambda_1(y-a)} h''(y)dy\right) e^{\lambda_1(x_2-a)} = \left(B - h'(a+) - \int_a^{x_2} e^{\lambda_2(y-a)} h''(y)dy\right) e^{-\lambda_2(x_2-a)}. \tag{5.44}$$

Furthermore, $g'_{A,B}(x) > 0$ for $x \in (a, x_2(A, B))$, $g'_{A,B}(x) < 0$ for $x \in (x_2(A, B), \infty)$, and

$$\lim_{x\uparrow\infty} g_{A,B}(x) = -\infty,$$
$$g''_{A,B}(x_2(A, B)) < 0. \tag{5.45}$$

Remark. (a) The set of (A, B) that satisfies (5.39) is the shaded region in Figure 2. The set of (A, B) that satisfies (5.40) is the shaded region in Figure 3.
(b) Note that

$$
\begin{aligned}
(5.46)\quad & \big(A - \lambda_1 \int_a^{x_2(A,B)} e^{-\lambda_1(y-a)} h'(y)dy\big) e^{\lambda_1(x_2(A,B)-a)} \\
&= \big(A - h'(a+) - \int_a^{x_2(A,B)} e^{-\lambda_1(y-a)} h''(y)dy\big) e^{\lambda_1(x_2(A,B)-a)} \\
&\quad + h'(x_2(A,B)) \\
&= \big(B - h'(a+) - \int_a^{x_2(A,B)} e^{\lambda_2(y-a)} h''(y)dy\big) e^{-\lambda_2(x_2(A,B)-a)} \\
&\quad + h'(x_2(A,B)) \\
&= \big(B + \lambda_2 \int_a^{x_2(A,B)} e^{\lambda_2(y-a)} h'(y)dy\big) e^{-\lambda_2(x_2(A,B)-a)},
\end{aligned}
$$

where the first and third equalities follow from integration by parts, and the second is due to the definition of $x_2(A, B)$ in (5.44). This provide an alternative characterization of $x_2(A, B)$ in (5.44). Similarly, $x_1(A, B)$ has an alternative characterization. In the proof of the lemma, we use the following expressions for $g'(x)$ and $g''(x)$.

$$
(5.47)\quad g'(x) =
\begin{cases}
\frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2} \\
\quad \times\Big[\Big(A - h'(a-) + \int_x^a e^{-\lambda_1(y-a)} h''(y)dy\Big) e^{\lambda_1(x-a)} \\
\quad -\Big(B - h'(a-) + \int_x^a e^{\lambda_2(y-a)} h''(y)dy\Big) e^{-\lambda_2(x-a)}\Big] & \text{for } x < a, \\
\frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2} \\
\quad \times\Big[\Big(A - h'(a+) - \int_a^x e^{-\lambda_1(y-a)} h''(y)dy\Big) e^{\lambda_1(x-a)} \\
\quad -\Big(B - h'(a+) - \int_a^x e^{\lambda_2(y-a)} h''(y)dy\Big) e^{-\lambda_2(x-a)}\Big] & \text{for } x > a
\end{cases}
$$

and

$$
(5.48)\quad g''(x) =
\begin{cases}
\frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2} \\
\quad \times\Big[\big(A - h'(a-) + \int_x^a e^{-\lambda_1(y-a)} h''(y)dy\big) \lambda_1 e^{\lambda_1(x-a)} \\
\quad +\big(B - h'(a-) + \int_x^a e^{\lambda_2(y-a)} h''(y)dy\big) \lambda_2 e^{-\lambda_2(x-a)}\Big] & \text{for } x < a, \\
\frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2} \\
\quad \times\Big[\big(A - h'(a+) - \int_a^x e^{-\lambda_1(y-a)} h''(y)dy\big) \lambda_1 e^{\lambda_1(x-a)} \\
\quad +\big(B - h'(a+) - \int_a^x e^{\lambda_2(y-a)} h''(y)dy\big) \lambda_2 e^{-\lambda_2(x-a)}\Big] & \text{for } x > a.
\end{cases}
$$

Proof. We only prove the existence of x_1 and the properties of $g(x)$ in $x \in (-\infty, a)$. The proof for the existence of x_2 and the properties of $g(x)$ in $x \in (a, \infty)$ is similar, and it is omitted.

In order to prove the existence of $x_1 = x_1(A, B)$, we divide the domain in (5.39) for (A, B) into two subdomains: $B \in \big(\underline{B}, h'(a-)\big]$ and $A \in (B, \infty)$, and $B \in \big(h'(a-), \infty\big)$ and $A \in (B, \infty)$. We consider the two subdomains separately.

Case 1. $B \in \big(\underline{B}, h'(a-)\big]$ and $A \in (B, \infty)$.

Since h is convex, we have $h''(x) \geq 0$ for all $x \in \mathbb{R}$ except $x = a$. Therefore, $\int_x^a e^{\lambda_2(y-a)} h''(y)dy \geq 0$ is decreasing in $x \in (-\infty, a)$. Then for fixed $B \in \big(\underline{B}, h'(a-)\big]$, there exists an x' with $x' \in (-\infty, a]$ such that

$$B = h'(a-) - \int_{x'}^a e^{\lambda_2(y-a)} h''(y)dy. \tag{5.49}$$

Now we can prove that $g'(x)$ is strictly increasing in $x \in (-\infty, x')$ and

$$\lim_{x \downarrow -\infty} g'(x) = -\infty, \tag{5.50}$$

$$\lim_{x \uparrow x'} g'(x) > 0, \tag{5.51}$$

$$g'(x) > 0 \quad \text{ for } x \in (x', a). \tag{5.52}$$

Since $g'(x)$ is continuous and strictly increasing in $x \in (-\infty, x')$, (5.50) and (5.51) imply that there exists a unique x_1 with $x_1 \in (-\infty, x')$ such that

$$g'(x) \begin{cases} < 0, & x < x_1, \\ = 0, & x = x_1, \\ > 0, & x_1 < x < x'. \end{cases}$$

Combining this with (5.52) gives

$$g'(x) \begin{cases} < 0, & x < x_1, \\ = 0, & x = x_1, \\ > 0, & x_1 < x < a, \end{cases}$$

which proves the existence of x_1 and properties of $g(x)$ in $(-\infty, a)$.

It remains to prove that $g'(x)$ is strictly increasing in $x \in (-\infty, x')$, and that (5.43), and (5.50)-(5.52) hold. We first prove that $g'(x)$ is strictly increasing in $x \in (-\infty, x')$. For $x \in (-\infty, x')$,

$$B - h'(a-) + \int_x^a e^{\lambda_2(y-a)} h''(y)dy \geq 0$$

and

$$\begin{aligned} A - h'(a-) + \int_x^a e^{-\lambda_1(y-a)} h''(y)dy &> B - h'(a-) + \int_x^a e^{-\lambda_1(y-a)} h''(y)dy \\ &\geq B - h'(a-) + \int_x^a e^{\lambda_2(y-a)} h''(y)dy \\ &\geq 0, \end{aligned}$$

where the first inequality is due to $B < A$. Using (5.48), for $x \in (-\infty, x')$, we have

$$\begin{aligned} g''(x) = \frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2}\Big[&\Big(A - h'(a-) + \int_x^a e^{-\lambda_1(y-a)} h''(y)dy\Big)\lambda_1 e^{\lambda_1(x-a)} \\ &+\Big(B - h'(a-) + \int_x^a e^{\lambda_2(y-a)} h''(y)dy\Big)\lambda_2 e^{-\lambda_2(x-a)}\Big] \\ > 0.& \end{aligned} \tag{5.53}$$

This proves $g'(x)$ is strictly increasing in $(-\infty, x')$.

For (5.50), from (5.47) it follows that

$$\begin{aligned} \lim_{x\downarrow-\infty} \frac{g'(x)}{e^{-\lambda_2(x-a)}} = \lim_{x\downarrow-\infty} \frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2}\Big[&-\Big(B - h'(a-) + \int_x^a e^{\lambda_2(y-a)} h''(y)dy\Big) \\ &+\Big(A - h'(a-) + \int_x^a e^{-\lambda_1(y-a)} h''(y)dy\Big)e^{(\lambda_1+\lambda_2)(x-a)}\Big]. \end{aligned}$$

To evaluate this limit, we first have

$$\begin{aligned} &\lim_{x\downarrow-\infty}\Big(A - h'(a-) + \int_x^a e^{-\lambda_1(y-a)} h''(y)dy\Big)e^{(\lambda_1+\lambda_2)(x-a)} \\ &\quad= \lim_{x\downarrow-\infty} \int_x^a e^{-\lambda_1(y-a)} h''(y)dy \cdot e^{(\lambda_1+\lambda_2)(x-a)} \\ &\quad= 0, \end{aligned} \tag{5.54}$$

where the last equality follows from (2.8). Next,

$$\lim_{x\downarrow-\infty}\Big(B - h'(a-) + \int_x^a e^{\lambda_2(y-a)} h''(y)dy\Big) = B - \underline{B}. \tag{5.55}$$

Because $B - \underline{B} > 0$, (5.54) and (5.55) imply

$$\lim_{x\downarrow-\infty} g'(x) = -\infty. \tag{5.56}$$

For (5.51), from (5.47) it follows that

$$
\begin{aligned}
(5.57)\quad & \lim_{x\uparrow x'} g'(x) \\
&= \frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2}\Big[\big(A-h'(a-)+\int_{x'}^{a} e^{-\lambda_1(y-a)}h''(y)dy\big)e^{\lambda_1(x'-a)} \\
&\quad -\big(B-h'(a-)+\int_{x'}^{a} e^{\lambda_2(y-a)}h''(y)dy\big)e^{-\lambda_2(x'-a)}\Big] \\
&= \frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2}\Big[\big(A-h'(a-)+\int_{x'}^{a} e^{-\lambda_1(y-a)}h''(y)dy\big)e^{\lambda_1(x'-a)}\Big] \\
&\ge \frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2}\Big[\big(A-h'(a-)+\int_{x'}^{a} e^{\lambda_2(y-a)}h''(y)dy\big)e^{\lambda_1(x'-a)}\Big] \\
&= \frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2}(A-B)e^{\lambda_1(x'-a)} \\
&> 0,
\end{aligned}
$$

where the second and last equalities are due to (5.49) and the last inequality is due to $B < A$.

To see (5.52), for $x \in [x', a)$, (5.49) implies $B - h'(a-) + \int_x^a e^{\lambda_2(y-a)}h''(y)dy \le 0$, which, together with

$$
\begin{aligned}
A-h'(a-)+\int_x^a e^{-\lambda_1(y-a)}h''(y)dy &> B-h'(a-)+\int_x^a e^{-\lambda_1(y-a)}h''(y)dy \\
&\ge B-h'(a-)+\int_x^a e^{\lambda_2(y-a)}h''(y)dy
\end{aligned}
$$

implies

$$
\begin{aligned}
g'(x) &= \frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2}\Big[\big(A-h'(a-)+\int_x^a e^{-\lambda_1(y-a)}h''(y)dy\big)e^{\lambda_1(x-a)} \\
&\quad -\big(B-h'(a-)+\int_x^a e^{\lambda_2(y-a)}h''(y)dy\big)e^{-\lambda_2(x-a)}\Big] \\
&> \frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2}\Big[\big(B-h'(a-) \\
&\quad +\int_x^a e^{\lambda_2(y-a)}h''(y)dy\big)\big(e^{\lambda_1(x-a)}-e^{-\lambda_2(x-a)}\big)\Big] \\
&\ge 0.
\end{aligned}
$$

Case 2. $B \in \big(h'(a-), \infty\big)$ and $A \in (B, \infty)$.

Similar to proving (5.53), (5.56) and (5.57), we have

$$g''(x) > 0 \quad \text{for } x \in (-\infty, a), \tag{5.58}$$

$$\lim_{x\downarrow-\infty} g'(x) = -\infty \tag{5.59}$$

and

$$\lim_{x\uparrow a} g'(x) = \frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2}(A-B) > 0.$$

Therefore, there exists a unique x_1 such that

$$g'(x)\begin{cases} <0, & x<x_1,\\ =0, & x=x_1,\\ >0, & x_1<x<a.\end{cases}$$

Limit (5.42) can immediately be obtained by (5.56) and (5.59). Inequalities (5.53) and (5.58) and the definition of x_1 easily imply (5.43). □

LEMMA 5.3. *Suppose (A,B) satisfies (5.39); for fixed B, local minimizer $x_1(A,B)$ is continuous and strictly decreasing in A; for fixed A, local minimizer $x_1(A,B)$ is continuous and strictly increasing in B. Suppose (A,B) satisfies (5.40); for fixed B, local maximizer $x_2(A,B)$ is continuous and strictly increasing in A; for fixed A, local maximizer $x_2(A,B)$ is continuous and strictly decreasing in B.*

Furthermore,

$$\lim_{B\downarrow \underline{B}} x_1(A,B) = -\infty \quad \text{for } A>\underline{B}, \tag{5.60}$$

$$\lim_{B\uparrow A} x_1(A,B) = a \quad \text{for } A>h'(a-), \tag{5.61}$$

$$\lim_{A\downarrow B} x_2(A,B) = a \quad \text{for } B<h'(a+), \tag{5.62}$$

$$\lim_{A\uparrow \overline{A}} x_2(A,B) = \infty \quad \text{for } B<\overline{A}. \tag{5.63}$$

PROOF. The *Implicit Function Theorem* implies the continuity of $x_i(A,B)$, $i=1,2$. Applying the *Implicit Function Theorem* to (5.41) and (5.44), for any $\underline{B}<B<A$,

$$\frac{\partial x_1(A,B)}{\partial A} = -\frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2}\frac{e^{\lambda_1(x_1(A,B)-a)}}{g''(x_1(A,B))} < 0, \tag{5.64}$$

$$\frac{\partial x_1(A,B)}{\partial B} = \frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2}\frac{e^{-\lambda_2(x_1(A,B)-a)}}{g''(x_1(A,B))} > 0, \tag{5.65}$$

and for any $B<A<\overline{A}$,

$$\frac{\partial x_2(A,B)}{\partial A} = -\frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2}\frac{e^{\lambda_1(x_2(A,B)-a)}}{g''(x_2(A,B))} > 0, \tag{5.66}$$

$$\frac{\partial x_2(A,B)}{\partial B} = \frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2}\frac{e^{-\lambda_2(x_2(A,B)-a)}}{g''(x_2(A,B))} < 0, \tag{5.67}$$

where in obtaining (5.64) and (5.65) we have used $g''(x_1(A,B)) > 0$ in (5.43), and in obtaining (5.66) and (5.67) we have used $g''(x_2(A,B)) < 0$ in (5.45).

We first prove (5.61). Fix A satisfying $A > h'(a-)$. Taking limits as $B \uparrow A$ on both sides of (5.41) gives

$$\Big(A - h'(a-) + \int_{x_1(A)}^{a} e^{-\lambda_1(y-a)} h''(y)dy\Big) e^{\lambda_1(x_1(A)-a)}$$
$$= \Big(A - h'(a-) + \int_{x_1(A)}^{a} e^{\lambda_2(y-a)} h''(y)dy\Big) e^{-\lambda_2(x_1(A)-a)},$$

where $x_1(A) = \lim_{B\uparrow A} x_1(A,B)$. It follows that

$$(5.68) \qquad (A - h'(a-))\big(e^{\lambda_1(x_1(A)-a)} - e^{-\lambda_2(x_1(A)-a)}\big)$$
$$= \int_{x_1(A)}^{a} \big(e^{\lambda_2(y-x_1(A))} - e^{-\lambda_1(y-x_1(A))}\big) h''(y)dy.$$

If $x_1(A) < a$, we must have

$$(A - h'(a-))\big(e^{\lambda_1(x_1(A)-a)} - e^{-\lambda_2(x_1(A)-a)}\big) < 0$$

and

$$\int_{x_1(A)}^{a} \big(e^{\lambda_2(y-x_1(A))} - e^{-\lambda_1(y-x_1(A))}\big) h''(y)dy \geq 0.$$

These two inequalities contradict (5.68). Therefore, we must have (5.61).

We next prove (5.60). Fix $A > \underline{B}$. By the monotonicity of $x_1(A,B)$ in B, the limit $\lim_{B\downarrow \underline{B}} x_1(A,B)$ exists. We use $x_1(A,\underline{B})$ to denote the limit. We now prove that $x_1(A,\underline{B}) = -\infty$. Assume on the contrary that $x_1(A,\underline{B}) > -\infty$. Taking limits as $B \downarrow \underline{B}$ on both sides of (5.41) gives

$$(5.69) \qquad \Big(A - h'(a-) + \int_{x_1(A,\underline{B})}^{a} e^{-\lambda_1(y-a)} h''(y)dy\Big) e^{\lambda_1(x_1(A,\underline{B})-a)}$$
$$= \Big(\underline{B} - h'(a-) + \int_{x_1(A,\underline{B})}^{a} e^{\lambda_2(y-a)} h''(y)dy\Big) e^{-\lambda_2(x_1(A,\underline{B})-a)}$$
$$\leq 0,$$

where the inequality follows from the definition of $\underline{B}$. From (5.41) and (5.43), we have

$$A - h'(a-) + \int_{x_1(A,B)}^{a} e^{-\lambda_1(y-a)} h''(y)dy > 0 \quad \text{for any } \underline{B} < B < A$$

From (5.65), it follows that $A - h'(a-) + \int_{x_1(A,B)}^{a} e^{-\lambda_1(y-a)} h''(y)dy$ is increasing when $B \downarrow \underline{B}$. Thus, $A - h'(a-) + \int_{x_1(A,\underline{B})}^{a} e^{-\lambda_1(y-a)} h''(y)dy > 0$, contradicting (5.69) due to the finiteness assumption of $x_1(A, \underline{B})$. Therefore, we have proved $x_1(A, \underline{B}) = -\infty$.

The proofs for (5.62) and (5.63) are similar. □

LEMMA 5.4. *(a) For each*

$$B \in \big(-\infty, h'(a+) \wedge \ell\beta\big), \tag{5.70}$$

there exists a unique

$$\underline{A}(B) \in \big(B \vee 0, \overline{A}\big)$$

such that

$$g_{\underline{A}(B),B}(x_2(\underline{A}(B), B)) = \ell. \tag{5.71}$$

Furthermore, for $B \in (-\infty, h'(a+) \wedge \ell\beta)$,

$$\frac{d\underline{A}(B)}{dB} \;=\; -\frac{\lambda_1}{\lambda_2} e^{-(\lambda_1+\lambda_2)(x_2(\underline{A}(B),B)-a)} < 0. \tag{5.72}$$

Therefore, function $A = \underline{A}(B)$ *is strictly decreasing in* $B \in (-\infty, h'(a+) \wedge \ell\beta)$*; see Figure 3 for an illustration. For* $A \in (\underline{A}(B), \overline{A})$,

$$g_{A,B}(x_2(A, B)) > \ell.$$

(b) For each

$$A \in \big(h'(a-) \vee (-k\beta), +\infty\big), \tag{5.73}$$

there exists a unique

$$\overline{B}(A) \in \big(\underline{B}, A \wedge 0\big)$$

such that

$$g_{A,\overline{B}(A)}(x_1(A, \overline{B}(A))) = -k. \tag{5.74}$$

Furthermore, for $A \in (h'(a-) \vee (-k\beta), \infty)$,

$$\frac{d\overline{B}(A)}{dA} \;=\; -\frac{\lambda_2}{\lambda_1} e^{(\lambda_1+\lambda_2)(x_1(A,\overline{B}(A))-a)} < 0. \tag{5.75}$$

Therefore, function $B = \overline{B}(A)$ *is strictly decreasing in* $A \in (h'(a-) \vee (-k\beta), \infty)$*; see Figure 2 for an illustration. For* $B \in (\underline{B}, \overline{B}(A))$,

$$g_{A,B}(x_1(A, B)) < -k.$$

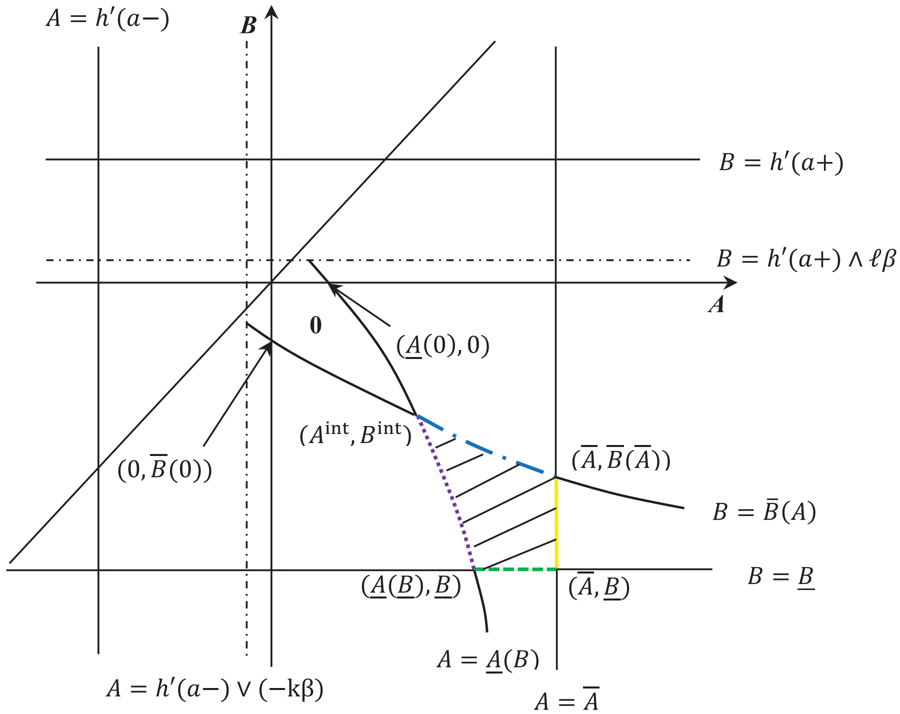

FIG 4. *(a) On the curve* $B = \overline{B}(A)$, $g_{A,B}(x_1(A,B)) = -k$ *and on the curve* $A = \underline{A}(B)$, $g_{A,B}(x_2(A,B)) = \ell$.
(b) The two curves $\{(\underline{A}(B), B) : B \in [\underline{B}, 0]\}$ *and* $\{(A, \overline{B}(A)) : A \in [0, \overline{A}]\}$ *have a unique intersection point* $(A^{\text{int}}, B^{\text{int}})$ *that satisfies* $0 < \underline{A}(0) < A^{\text{int}} < \underline{A}(\underline{B}) < \overline{A}$ *and* $0 > \overline{B}(0) > B^{\text{int}} > \overline{B}(\overline{A}) > \underline{B}$. *For any* (A, B) *in the shaded region* G *defined in (5.85),* $g_{A,B}(x_1(A,B)) < -k$ *and* $g_{A,B}(x_2(A,B)) > \ell$.

(c) The two curves $\{(\underline{A}(B), B) : B \in (-\infty, 0]\}$ *and* $\{(A, \overline{B}(A)) : A \in [0, \infty)\}$ *have a unique intersection point* $(A^{\text{int}}, B^{\text{int}})$ *that satisfies*

$$\overline{B}(A^{\text{int}}) = B^{\text{int}} \quad \textit{and } \underline{A}(B^{\text{int}}) = A^{\text{int}} \tag{5.76}$$

with

$$0 < \underline{A}(0) < A^{\text{int}} < \underline{A}(\underline{B}) < \overline{A}, \tag{5.77}$$

$$0 > \overline{B}(0) > B^{\text{int}} > \overline{B}(\overline{A}) > \underline{B}. \tag{5.78}$$

See Figure 4 for an illustration.

PROOF. (a) First, fix a B that satisfies (5.70). We consider the value of $g_{A,B}(x_2(A,B))$ for $A \in (B \vee 0, \overline{A})$.

$$\begin{aligned}
&\frac{\partial g_{A,B}(x_2(A,B))}{\partial A} \\
&\quad = g'_{A,B}(x_2(A,B))\frac{\partial x_2(A,B)}{\partial A} + \frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2}\frac{1}{\lambda_1}e^{\lambda_1(x_2(A,B)-a)} \\
&\quad = \frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2}\frac{1}{\lambda_1}e^{\lambda_1(x_2(A,B)-a)} \\
&\quad > 0.
\end{aligned}$$

Next we will prove that

$$\lim_{A\uparrow\overline{A}} g_{A,B}(x_2(A,B)) > \ell \tag{5.79}$$

and

$$\lim_{A\downarrow(B\vee 0)} g_{A,B}(x_2(A,B)) < \ell, \tag{5.80}$$

from which there exists a unique $\underline{A}(B) \in (B\vee 0, \overline{A})$ such that

$$g_{\underline{A}(B),B}(x_2(\underline{A}(B),B)) = \ell$$

and for $A \in (\underline{A}(B), \overline{A})$

$$g_{A,B}(x_2(A,B)) > \ell.$$

The derivative (5.72) follows from applying the Implicit Function Theorem to (5.71).

First we prove (5.79). (5.63) implies

$$\begin{aligned}
&\lim_{A\uparrow\overline{A}}\big(B + \lambda_2\int_a^{x_2(A,B)} e^{\lambda_2(y-a)}h'(y)dy\big)e^{-\lambda_2(x_2(A,B)-a)} \\
&\qquad = \lim_{A\uparrow\overline{A}} \lambda_2\int_a^{x_2(A,B)} e^{\lambda_2(y-a)}h'(y)dy\cdot e^{-\lambda_2(x_2(A,B)-a)} \\
&\qquad = \lim_{x\uparrow\infty}\frac{\lambda_2\int_a^x e^{\lambda_2(y-a)}h'(y)dy}{e^{\lambda_2(x-a)}} \\
&\qquad = \lim_{x\uparrow\infty} h'(x),
\end{aligned} \tag{5.81}$$

where the last equlity follows from (2.10). Equalities (5.46) and (5.81) yield

$$\lim_{A\uparrow\overline{A}}\big(A - \lambda_1\int_a^{x_2(A,B)} e^{-\lambda_1(y-a)}h'(y)dy\big)e^{\lambda_1(x_2(A,B)-a)} = \lim_{x\uparrow\infty} h'(x)$$

Therefore, using the expression in (5.21) for g, we have

$$\begin{aligned}
&\lim_{A\uparrow\overline{A}} g_{A,B}(x_2(A,B)) \\
&\quad = \frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2}\big[(\frac{1}{\lambda_1}+\frac{1}{\lambda_2})\lim_{x\uparrow\infty} h'(x)\big] \\
&\quad = \frac{1}{\beta}\lim_{x\uparrow\infty} h'(x) \\
&\quad > \ell,
\end{aligned}$$

where the second equality uses $\lambda_1\lambda_2 = \frac{2\beta}{\sigma^2}$ and the inequality is due to the first part of (2.5).

It remains to prove (5.80). We consider two cases: $B \in (0, h'(a+)\wedge \ell\beta)$ and $B \in (-\infty, 0]$. If $B \in (0, h'(a+)\wedge\ell\beta)$, $\lim_{A\downarrow B} x_2(A,B) = a$ in (5.62) implies

$$\begin{aligned}
\lim_{A\downarrow B} g_{A_2,B_2}(x_2(A,B)) &= \frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2}\big[(\frac{1}{\lambda_1}+\frac{1}{\lambda_2})(B-h'(a+))\big]+\frac{1}{\beta}h'(a+) \\
&= \frac{B}{\beta}.
\end{aligned}$$

Because $B < \ell\beta$, we have $\lim_{A\downarrow B} g_{A,B}(x_2(A,B)) < \ell$.

On the other hand, if $B \in (-\infty, 0]$, (5.46) implies

$$\begin{aligned}
&\lim_{A\downarrow 0}\big(B+\lambda_2\int_a^{x_2(A,B)} e^{\lambda_2(y-a)}h'(y)dy\big)e^{-\lambda_2(x_2(A,B)-a)} \\
&\quad = \lim_{A\downarrow 0}\big(A-\lambda_1\int_a^{x_2(A,B)} e^{-\lambda_1(y-a)}h'(y)dy\big)e^{\lambda_1(x_2(A,B)-a)} \\
&\quad \le 0.
\end{aligned}$$

Using the expression in (5.21) for g, we have

$$\lim_{A\downarrow 0} g_{A,B}(x_2(A,B)) \le 0 < \ell.$$

(b) For a fixed A that satisfies (5.73). We can prove similarly that for $B \in (\underline{B}, A\wedge 0)$,

$$\frac{\partial g_{A,B}(x_1(A,B))}{\partial B} = \frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2}\frac{1}{\lambda_2}e^{-\lambda_2(x_1(A,B)-a)} > 0 \tag{5.82}$$

and

$$\lim_{B\downarrow\underline{B}} g_{A,B}(x_1(A,B)) = \frac{1}{\beta}\lim_{x\downarrow-\infty} h'(x) < -k, \tag{5.83}$$

where the inequality is due to the second part of (2.5).

If $A \in (h'(a-) \vee (-k\beta), 0)$,

$$\lim_{B \uparrow A} g_{A,B}(x_1(A,B)) = \frac{A}{\beta}.$$

Because $A > -k\beta$, we have $\lim_{B\uparrow A} g_{A,B}(x_1(A,B)) > -k$. Then (5.82) and (5.83) imply that there exists a unique $\overline{B}(A) \in (\underline{B}, A)$ such that

$$g_{A,\overline{B}(A)}(x_1(A, \overline{B}(A))) = -k$$

and for $B \in (\underline{B}, \overline{B}(A))$,

$$g_{A,B}(x_1(A,B)) < -k.$$

If $A \in [0, \infty)$,

$$\lim_{B \uparrow 0} g_{A,B}(x_1(A,B)) \geq 0 > -k.$$

Then (5.82) implies that there exists a unique $\overline{B}(A) \in (\underline{B}, 0)$ such that

$$g_{A,\overline{B}(A)}(x_1(A, \overline{B}(A))) = -k$$

and for $B \in (\underline{B}, \overline{B}(A))$,

$$g_{A,B}(x_1(A,B)) < -k.$$

Applying the Implicit Function Theorem to (5.74), we also have (5.75).

(c) First consider the curve $\{(\underline{A}(B), B) : B \in (-\infty, h'(a+) \wedge \ell\beta)\}$ that is determined by equation $g_{\underline{A}(B),B}(x_2(\underline{A}(B), B)) = \ell$. Consider two points

$$(\underline{A}(0), 0) \quad \text{and} \quad (\underline{A}(\underline{B}), \underline{B})$$

on the curve $\{(\underline{A}(B), B) : B \in (-\infty, h'(a+) \wedge \ell\beta)\}$ (see Figure 4). By part (a) of this lemma, we have

$$0 < \underline{A}(0) < \underline{A}(\underline{B}) < \overline{A}. \tag{5.84}$$

Similarly, consider two points

$$(0, \overline{B}(0)) \quad \text{and} \quad (\overline{A}, \overline{B}(\overline{A}))$$

on the curve determined by $g_{A,\overline{B}(A)}(x_1(A, \overline{B}(A))) = -k$. Similar to (5.84), by part (b) of this lemma, we have

$$\underline{B} < \overline{B}(\overline{A}) < \overline{B}(0) < 0.$$

Therefore, point $(\overline{A}, \overline{B}(\overline{A}))$ is on the right side of the curve $g_{\underline{A}(B),B}(x_2(\underline{A}(B), B) = \ell$ and point $(0, \overline{B}(0))$ is on the left side of the curve. The continuity and monotonicity of the two curves imply that there is a unique point

$$(A^{\rm int}, B^{\rm int})$$

at which the two curves intersect. See Figure 4 for an illustration. It is clear from Figure 4 that (5.77) and (5.78) hold. ☐

Let

$$G = \{(A, B) : \underline{A}(B) < A < \overline{A}, \quad \underline{B} < B < \overline{B}(A)\} \tag{5.85}$$

be the shaded region in Figure 4. Region G has four corners. They are $(A^{\rm int}, B^{\rm int})$, $(\overline{A}, \overline{B}(\overline{A}))$, $(\overline{A}, \underline{B})$ and $(\underline{A}(\underline{B}), \underline{B})$. Its boundary has four pieces: the top, the right, the bottom and the left.

For $(A, B) \in G$, we have

$$g_{A,B}(x_1(A, B)) < -k, \quad g_{A,B}(x_2(A, B)) > \ell. \tag{5.86}$$

It follows from part (b) of Lemma 5.2 and (5.86) that there exist unique $d(A, B)$, $D(A, B)$, $U(A, B)$ and $u(A, B)$ such that

$$d(A, B) < x_1(A, B) < D(A, B) < U(A, B) < x_2(A, B) < u(A, B), \tag{5.87}$$

$$g_{A,B}(d(A, B)) = g_{A,B}(D(A, B)) = -k, \tag{5.88}$$

$$g_{A,B}(U(A, B)) = g_{A,B}(u(A, B)) = \ell, \tag{5.89}$$

$$g'_{A,B}(d(A, B)) < 0, \quad g'_{A,B}(D(A, B)) > 0, \tag{5.90}$$

$$g'_{A,B}(U(A, B)) > 0, \quad g'_{A,B}(u(A, B)) < 0. \tag{5.91}$$

For each $(A, B) \in G$, define

$$\Lambda_1(A, B) = \int_{d(A,B)}^{D(A,B)} \big[g_{A,B}(x) + k\big] dx, \;\; \Lambda_2(A, B) = \int_{U(A,B)}^{u(A,B)} \big[g_{A,B}(x) - \ell\big] dx.$$

Although $(A, \overline{B}(A))$ is not in G for $A \in (A^{\rm int}, \overline{A})$, these points are on the upper boundary of G, and

$$\Lambda_2(A, \overline{B}(A))$$

is also well defined for $A \in (A^{\rm int}, \overline{A})$.

LEMMA 5.5. *There exists a unique* $A = \overline{A}_1 \in (A^{\rm int}, \overline{A})$ *such that*

$$\Lambda_2(\overline{A}_1, \overline{B}(\overline{A}_1)) = L, \tag{5.92}$$

and for $A \in (\overline{A}_1, \overline{A})$,

$$\Lambda_2(A, \overline{B}(A)) > L. \tag{5.93}$$

PROOF. When A goes to A^{int}, $(A, \overline{B}(A))$ goes to $(A^{\text{int}}, B^{\text{int}})$. Then the definition of $(A^{\text{int}}, B^{\text{int}})$ in Lemma 5.4 implies

$$\lim_{A \downarrow A^{\text{int}}} U(A, \overline{B}(A)) = \lim_{A \downarrow A^{\text{int}}} u(A, \overline{B}(A)) = x_2(A^{\text{int}}, B^{\text{int}}).$$

Therefore,

$$\lim_{A \downarrow A^{\text{int}}} \Lambda_2(A, \overline{B}(A)) = 0. \tag{5.94}$$

Fixing $A \in (A^{\text{int}}, \overline{A})$ gives

$$\begin{aligned}
\frac{\partial \Lambda_2(A, \overline{B}(A))}{\partial A} &= \int_{U(A,\overline{B}(A))}^{u(A,\overline{B}(A))} \frac{\partial g_{A,\overline{B}(A)}(x)}{\partial A} dx \\
&\quad + \frac{\partial u(A, \overline{B}(A))}{\partial A} \big[g_{A,\overline{B}(A)}(u(A, \overline{B}(A))) - \ell \big] \\
&\quad - \frac{\partial U(A, \overline{B}(A))}{\partial A} \big[g_{A,\overline{B}(A)}(U(A, \overline{B}(A))) - \ell \big] \\
&= \int_{U(A,\overline{B}(A))}^{u(A,\overline{B}(A))} \frac{\partial g_{A,\overline{B}(A)}(x)}{\partial A} dx \\
&= \frac{2}{\sigma^2} \frac{1}{\lambda_1 + \lambda_2} \int_{U(A,\overline{B}(A))}^{u(A,\overline{B}(A))} \Big[\frac{1}{\lambda_1} e^{\lambda_1 (x-a)} + \frac{1}{\lambda_2} \frac{d\overline{B}(A)}{dA} e^{-\lambda_2 (x-a)} \Big] dx \\
&= \frac{2}{\sigma^2} \frac{1}{\lambda_1 + \lambda_2} \frac{1}{\lambda_1} \\
&\quad \times \int_{U(A,\overline{B}(A))}^{u(A,\overline{B}(A))} \Big[e^{\lambda_1 (x-a)} - e^{(\lambda_1+\lambda_2)(x_1(A,\overline{B}(A)) - a)} e^{-\lambda_2 (x-a)} \Big] dx \\
&> 0,
\end{aligned} \tag{5.95}$$

where the second equality is due to $g_{A,\overline{B}(A)}(U(A, \overline{B}(A))) = g_{A,\overline{B}(A)}(u(A, \overline{B}(A))) = \ell$, the forth equality is from (5.75), and the inequality is due to $u(A, \overline{B}(A)) > U(A, \overline{B}(A)) > x_1(A, \overline{B}(A))$. Therefore $\Lambda_2(A, \overline{B}(A))$ is increasing in $A \in (A^{\text{int}}, \overline{A})$.

Next, we show that

$$\lim_{A \uparrow \overline{A}} \Lambda_2(A, \overline{B}(A)) = \infty. \tag{5.96}$$

It follows from (5.94), (5.96) and the monotonicity of $\Lambda_2(A, \overline{B}(A))$ that there exists unique $\overline{A}_1 \in (A^{\text{int}}, \overline{A})$ such that (5.92) and (5.93) hold.

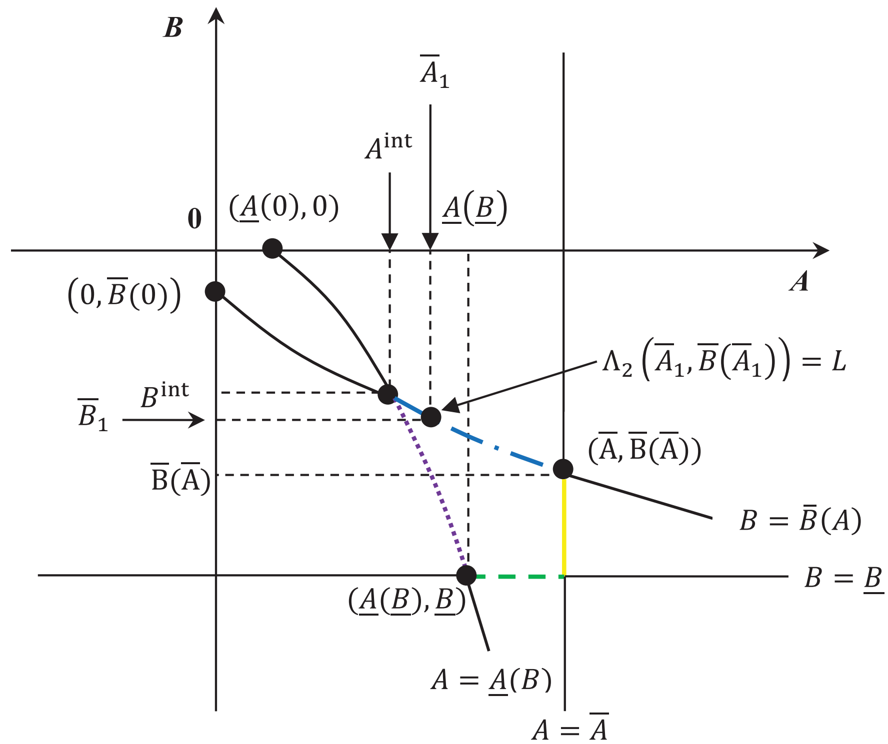

FIG 5. *Point* $(\overline{A}_1, \overline{B}_1)$ *is the unique point on the top boundary of* G *in (5.85) such that* $\Lambda_2(\overline{A}_1, \overline{B}_1) = L$. *For any point* (A, B) *on the top boundary and to the right of* $(\overline{A}_1, \overline{B}_1)$, $\Lambda_2(A, B) > L$.

To prove (5.96), note that (5.75) implies

$$
\begin{aligned}
\frac{\partial g_{A,\overline{B}(A)}(x_2(A, \overline{B}(A)))}{\partial A} &= g'_{A,\overline{B}(A)}(x_2(A, \overline{B}(A)))\frac{\partial x_2(A, \overline{B}(A))}{\partial A} \\
&\quad + \frac{2}{\sigma^2}\frac{1}{\lambda_1 + \lambda_2}\left[\frac{1}{\lambda_1}e^{\lambda_1(x_2(A,\overline{B}(A))-a)} + \frac{1}{\lambda_2}\frac{d\overline{B}(A)}{dA}e^{-\lambda_2(x_2(A,\overline{B}(A))-a)}\right] \\
&= \frac{2}{\sigma^2}\frac{1}{\lambda_1 + \lambda_2}\left[\frac{1}{\lambda_1}e^{\lambda_1(x_2(A,\overline{B}(A))-a)}\right. \\
&\quad \left. - \frac{1}{\lambda_1}e^{(\lambda_1+\lambda_2)(x_1(A,\overline{B}(A))-a)}e^{-\lambda_2(x_2(A,\overline{B}(A))-a)}\right] \\
&> 0,
\end{aligned}
\tag{5.97}
$$

where the second equality is due to $g'_{A,\overline{B}(A)}(x_2(A, \overline{B}(A))) = 0$, and the inequality is due to $x_2(A, \overline{B}(A)) > x_1(A, \overline{B}(A))$.

For $A \in (A^{\text{int}}, \overline{A})$, $(A, \overline{B}(A))$ on the right side of the curve $g_{A,B}(x_2(A,B)) = \ell$ and therefore

$$g_{A,\overline{B}(A)}(x_2(A, \overline{B}(A))) > \ell. \tag{5.98}$$

Fix an $A' \in (A^{\text{int}}, \overline{A})$ and let

$$M_1 = \Big(g_{A',\overline{B}(A')}(x_2(A', \overline{B}(A'))) - \ell\Big)/2.$$

It follows from (5.98) that $M_1 > 0$. Then (5.97) implies that for each $A \in [A', \overline{A})$,

$$g_{A,\overline{B}(A)}(x_2(A, \overline{B}(A))) \geq g_{A',\overline{B}(A')}(x_2(A', \overline{B}(A'))) = \ell + 2M_1 > \ell + M_1.$$

Therefore, for each $A \in [A', \overline{A})$, there exist unique $U_1(A, \overline{B}(A))$ and $u_1(A, \overline{B}(A))$ such that

$$\begin{aligned}
&U_1(A, \overline{B}(A)) < x_2(A, \overline{B}(A)) < u_1(A, \overline{B}(A)) \\
&g_{A,\overline{B}(A)}(U_1(A, \overline{B}(A))) = g_{A,\overline{B}(A)}(u_1(A, \overline{B}(A))) = \ell + M_1, \\
&g'_{A,\overline{B}(A)}(U_1(A, \overline{B}(A))) > 0,\ g'_{A,\overline{B}(A)}(u_1(A, \overline{B}(A))) < 0.
\end{aligned}$$

The properties of $g_{A,B}$ in Lemma 5.2 imply that for $A \in [A', \overline{A})$,

$$U(A, \overline{B}(A)) < U_1(A, \overline{B}(A)) < x_2(A, \overline{B}(A)) < u_1(A, \overline{B}(A)) < u(A, \overline{B}(A)).$$

This implies

$$\begin{aligned}
\lim_{A \uparrow \overline{A}} u_1(A, \overline{B}(A)) &\geq \lim_{A \uparrow \overline{A}} x_2(A, \overline{B}(A)) \\
&\geq \lim_{A \uparrow \overline{A}} x_2(A, B^{\text{int}}) \\
&= \infty,
\end{aligned} \tag{5.99}$$

where the second inequality holds because (5.67) and the equality is due to (5.63). Therefore, for $A \in [A', \overline{A})$,

$$\begin{aligned}
\Lambda_2(A, \overline{B}(A)) &= \int_{U(A,\overline{B}(A))}^{u(A,\overline{B}(A))} \big[g_{A,\overline{B}(A)}(x) - \ell\big]\,dx \\
&\geq \int_{U_1(A,\overline{B}(A))}^{u_1(A,\overline{B}(A))} \big[g_{A,\overline{B}(A)}(x) - \ell\big]\,dx \\
&\geq M_1(u_1(A, \overline{B}(A)) - U_1(A, \overline{B}(A))).
\end{aligned}$$

Applying the Implicit Function Theorem to $g_{A,\overline{B}(A)}(U_1(A,\overline{B}(A))) = \ell + M_1$, we have

$$
\begin{aligned}
&\frac{\partial U_1(A,\overline{B}(A))}{\partial A}\\
&\quad= -\frac{\frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2}\left[\frac{1}{\lambda_1}e^{\lambda_1(U_1(A,\overline{B}(A))-a)} + \frac{1}{\lambda_2}\frac{d\overline{B}(A)}{dA}e^{-\lambda_2(U_1(A,\overline{B}(A))-a)}\right]}{g'_{A,\overline{B}(A)}(U_1(A,\overline{B}(A)))}\\
&\quad= -\frac{1}{g'_{A,\overline{B}(A)}(U_1(A,\overline{B}(A)))}\frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2}\\
&\qquad\times\left[\frac{1}{\lambda_1}e^{\lambda_1(U_1(A,\overline{B}(A))-a)} - \frac{1}{\lambda_1}e^{(\lambda_1+\lambda_2)(x_1(A,\overline{B}(A))-a)}e^{-\lambda_2(U_1(A,\overline{B}(A))-a)}\right]\\
&\quad< 0,
\end{aligned}
$$

where the second equality is due to (5.75), and the inequality is due to $U_1(A,\overline{B}(A)) > x_1(A,\overline{B}(A))$ and $g'_{A,\overline{B}(A)}(U_1(A,\overline{B}(A))) > 0$. Thus, for any $A \in [A', \overline{A})$,

$$U_1(A,\overline{B}(A)) \le U_1(A',\overline{B}(A')).$$

Therefore, for any $A \in [A', \overline{A})$,

$$\Lambda_2(A,\overline{B}(A)) \ge M_1(u_1(A,\overline{B}(A)) - U_1(A',\overline{B}(A'))),$$

which, together with (5.99), implies (5.96). □

Define

$$\overline{B}_1 = \overline{B}(\overline{A}_1). \tag{5.100}$$

From (5.75) and (5.78), it follows that

$$\underline{B} < \overline{B}(\overline{A}) < \overline{B}_1 < B^{\text{int}} < 0.$$

See Figure 5 for point $(\overline{A}_1, \overline{B}_1)$.

LEMMA 5.6. *(a) For $B \in (\underline{B}, \overline{B}_1]$, there exists unique $A^*(B) \in [\overline{A}_1, \overline{A})$ such that*

$$\Lambda_2(A^*(B), B) = L. \tag{5.101}$$

(b) For $B \in (\underline{B}, \overline{B}_1]$,

$$\frac{dA^*(B)}{dB} = \frac{\lambda_1^2(e^{-\lambda_2(u(A^*(B),B)-a)} - e^{-\lambda_2(U(A^*(B),B)-a)})}{\lambda_2^2(e^{\lambda_1(u(A^*(B),B)-a)} - e^{\lambda_1(U(A^*(B),B)-a)})} < 0. \tag{5.102}$$

PROOF. (a) For $B = \overline{B}_1$, recall that Lemma 5.5 showed

$$A^*(\overline{B}_1) = \overline{A}_1. \tag{5.103}$$

For $B \in (\underline{B}, \overline{B}_1)$ and $(A, B) \in G$, we first have

$$\begin{aligned}\frac{\partial \Lambda_2(A,B)}{\partial A} &= \int_{U(A,B)}^{u(A,B)} \frac{\partial g_{A,B}(x)}{\partial A} dx \\ &= \int_{U(A,B)}^{u(A,B)} \frac{2}{\sigma^2} \frac{1}{\lambda_1 + \lambda_2} \frac{1}{\lambda_1} e^{\lambda_1(x-a)} dx \\ &> 0.\end{aligned} \tag{5.104}$$

From the definition of $\underline{A}(B)$ in (5.71), we have

$$\lim_{A \downarrow \underline{A}(B)} U(A,B) = \lim_{A \downarrow \underline{A}(B)} u(A,B) = \lim_{A \downarrow \underline{A}(B)} x_2(A,B) = x_2(\underline{A}(B), B).$$

Therefore, for a fixed $B \in (\underline{B}, \overline{B}_1)$,

$$\lim_{A \downarrow \underline{A}(B)} \Lambda_2(A,B) = 0 < L. \tag{5.105}$$

Next for $B \in (\underline{B}, \overline{B}_1)$, we consider two cases depending on whether $B \in (\underline{B}, \overline{B}(\overline{A})]$ or $B \in (\overline{B}(\overline{A}), \overline{B}_1)$. See Figure 6 for an illustration.

We first assume that $B \in (\overline{B}(\overline{A}), \overline{B}_1)$. For a fixed $B \in (\overline{B}(\overline{A}), \overline{B}_1)$, by the monotonicity of $\overline{B}(\cdot)$ in (5.75), there exists an $A(B) \in (\overline{A}_1, \overline{A})$ such that $(A(B), B)$ is on the upper boundary of G. From (5.95) and the definition of $\overline{A}_1$ in Lemma 5.5, it follows that

$$\Lambda_2(A(B), B) = \Lambda_2(A(B), \overline{B}(A(B))) > \Lambda_2(\overline{A}_1, \overline{B}(\overline{A}_1)) = L,$$

which, together with (5.104) and (5.105), implies that there exists a unique

$$A^*(B) \in (\underline{A}(B), A(B))$$

such that (5.101) holds.

Now assume that $B \in (\underline{B}, \overline{B}(\overline{A})]$. Following the proof for (5.96), similarly we prove

$$\lim_{A \uparrow \overline{A}} \Lambda_2(A,B) = \infty,$$

which, together with (5.104) and (5.105), implies that there exists a unique $A^*(B) \in (\underline{A}(B), \overline{A})$ such that (5.101) holds. By (5.72) and (5.103), we have for $B \in (\underline{B}, \overline{B}_1)$,

$$A^*(B) > \overline{A}_1.$$

(b) Applying the Implicit Function Theorem to $\Lambda_2(A^*(B), B) = L$, we have (5.102). $\square$

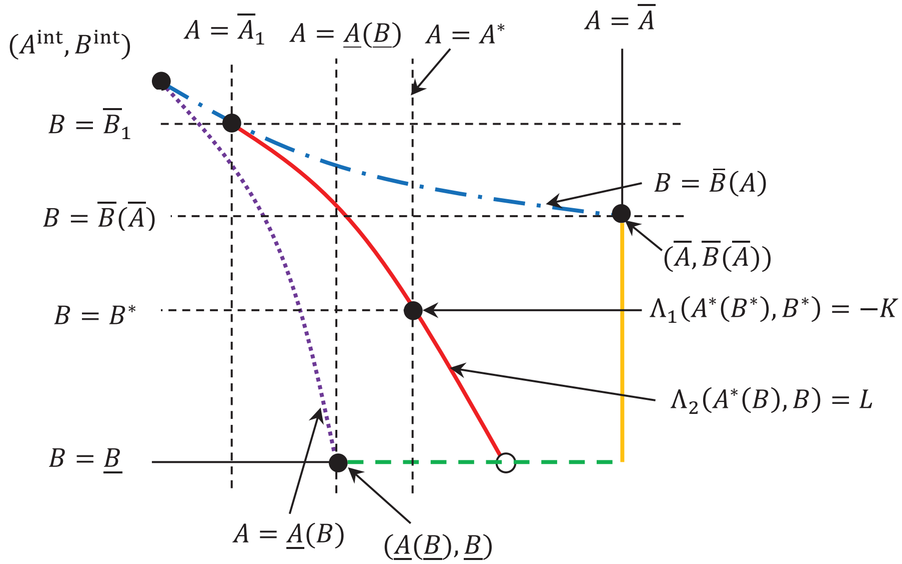

FIG 6. *For $B \in (\underline{B}, \overline{B}_1]$, there exists a unique $A^*(B) \in [\overline{A}_1, \overline{A})$ such that $\Lambda_2(A^*(B), B) = L$. Thus, on the solid red curve $A = A^*(B)$, $\Lambda_2(A, B) = L$. There is a unique $B^* \in (\underline{B}, \overline{B}_1)$ that satisfies $\Lambda_1(A^*(B^*), B^*) = -K$.*

For each $B \in (\underline{B}, \overline{B}_1)$, Lemma 5.6 shows that $(A^*(B), B) \in G$. Thus,

$$g_{A^*(B),B}(x_1(A^*(B), B)) < -k$$

and

$$\Lambda_1(A^*(B), B) = \int_{d(A^*(B),B)}^{D(A^*(B),B)} [g_{A^*(B),B}(x) + k]dx$$

is well defined.

LEMMA 5.7. *There exists a unique B^* with $B^* \in (\underline{B}, \overline{B}_1)$ such that $\Lambda_1(A^*(B^*), B^*) = -K$.*

PROOF. We only need to show that $\Lambda_1(A^*(B), B)$ can take any value in $(-\infty, 0)$ for $B \in (\underline{B}, \overline{B}_1)$ and is strictly increasing in B.

Recall that Lemma 5.5 showed that $A^*(\overline{B}_1) = \overline{A}_1$ and $(\overline{A}_1, \overline{B}_1)$ is on the upper boundary of G (the dashed blue curve in Figure 6). Therefore

$$g_{\overline{A}_1,\overline{B}_1}(x_1(\overline{A}_1, \overline{B}_1)) = -k$$

and

$$\lim_{B\uparrow\overline{B}_1} g_{A^*(B),B}(x_1(A^*(B),B)) = g_{A^*(\overline{B}_1),\overline{B}_1}(x_1(A^*(\overline{B}_1),\overline{B}_1)) = g_{\overline{A}_1,\overline{B}_1}(x_1(\overline{A}_1,\overline{B}_1)) = -k. \tag{5.106}$$

It follows that

$$\lim_{B\uparrow\overline{B}_1} \Lambda_1(A^*(B),B) = 0. \tag{5.107}$$

We now prove

$$\lim_{B\downarrow\underline{B}} \Lambda_1(A^*(B),B) = -\infty. \tag{5.108}$$

First, we prove

$$\frac{\partial g_{A^*(B),B}(x_1(A^*(B),B))}{\partial B} > 0. \tag{5.109}$$

To see this, for $B \in (\underline{B}, \overline{B}_1)$,

$$\begin{aligned}
&\frac{\partial g_{A^*(B),B}(x_1(A^*(B),B))}{\partial B} \\
&\quad = \frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2}\Big[\frac{1}{\lambda_1}\frac{dA^*(B)}{dB}e^{\lambda_1(x_1(A^*(B),B)-a)} + \frac{1}{\lambda_2}e^{-\lambda_2(x_1(A^*(B),B)-a)}\Big] \\
&\quad = \frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2}\Big[\frac{1}{\lambda_1}\frac{\lambda_1^2(e^{-\lambda_2(u(A^*(B),B)-a)} - e^{-\lambda_2(U(A^*(B),B)-a)})}{\lambda_2^2(e^{\lambda_1(u(A^*(B),B)-a)} - e^{\lambda_1(U(A^*(B),B)-a)})} \\
&\qquad \times e^{\lambda_1(x_1(A^*(B),B)-a)} + \frac{1}{\lambda_2}e^{-\lambda_2(x_1(A^*(B),B)-a)}\Big],
\end{aligned}$$

where the second equality follows from (5.102). Using the *Lagrange Mean Value Theorem*, there exist $y_1 \in \big(U(A^*(B),B), u(A^*(B),B)\big)$ and $y_2 \in \big(U(A^*(B),B), u(A^*(B),B)\big)$ such that

$$\begin{aligned}
&e^{-\lambda_2(u(A^*(B),B)-a)} - e^{-\lambda_2(U(A^*(B),B)-a)} \\
&\qquad = -\lambda_2 e^{-\lambda_2(y_1-a)}\big(u(A^*(B),B) - U(A^*(B),B)\big),
\end{aligned} \tag{5.110}$$

$$\begin{aligned}
&e^{\lambda_1(u(A^*(B),B)-a)} - e^{\lambda_1(U(A^*(B),B)-a)} \\
&\qquad = \lambda_1 e^{\lambda_1(y_2-a)}\big(u(A^*(B),B) - U(A^*(B),B)\big).
\end{aligned} \tag{5.111}$$

Therefore, for $B \in (\underline{B}, \overline{B}_1)$,

$$\begin{aligned}
&\frac{\partial g_{A^*(B),B}(x_1(A^*(B),B))}{\partial B} \\
&\quad = \frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2}\Big[-\frac{1}{\lambda_2}\frac{e^{-\lambda_2(y_1-a)}}{e^{\lambda_1(y_2-a)}}e^{\lambda_1(x_1(A^*(B),B)-a)}+\frac{1}{\lambda_2}e^{-\lambda_2(x_1(A^*(B),B)-a)}\Big] \\
&\quad = \frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2}\frac{1}{\lambda_2}e^{-\lambda_2(y_1-a)}\Big[-e^{\lambda_1(x_1(A^*(B),B)-y_2)}+e^{-\lambda_2(x_1(A^*(B),B)-y_1)}\Big] \\
&\quad > 0,
\end{aligned}$$

where the inequality holds because $x_1(A^*(B),B) < D(A^*(B),B) < U(A^*(B),B) < y_1$ and $x_1(A^*(B),B) < D(A^*(B),B) < U(A^*(B),B) < y_2$. Thus, we have proved (5.109).

Fix an $\overline{B}_2 \in (\underline{B}, \overline{B}_1)$. Define

$$M_2 = -\frac{g_{A^*(\overline{B}_2),\overline{B}_2}(x_1(A^*(\overline{B}_2),\overline{B}_2)) + k}{2}.$$

From (5.106) and (5.109), it follows that $g_{A^*(\overline{B}_2),\overline{B}_2}(x_1(A^*(\overline{B}_2),\overline{B}_2)) < -k$ and thus $M_2 > 0$. Inequality (5.109) implies that for $B \in (\underline{B}, \overline{B}_2)$,

$$\begin{aligned}
g_{A^*(B),B}(x_1(A^*(B),B)) &< g_{A^*(\overline{B}_2),\overline{B}_2}(x_1(A^*(\overline{B}_2),\overline{B}_2)) \\
&= -k - 2M_2 \\
&< -k - M_2.
\end{aligned}$$

Therefore, for $B \in (\underline{B}, \overline{B}_2)$, there exist unique $d_1(A^*(B),B)$ and $D_1(A^*(B),B)$ such that

$$\begin{aligned}
&d_1(A^*(B),B) < x_1(A^*(B),B) < D_1(A^*(B),B), \\
&g_{A^*(B),B}(d_1(A^*(B),B)) = g_{A^*(B),B}(D_1(A^*(B),B)) = -k - M_2, \\
&g'_{A^*(B),B}(d_1(A^*(B),B)) < 0, \quad g'_{A^*(B),B}(D_1(A^*(B),B)) > 0.
\end{aligned}$$

The properties of $g_{A^*(B),B}$ in Lemma 5.2 imply that for $B \in (\underline{B}, \overline{B}_2)$,

$$\begin{aligned}
d(A^*(B),B) &< d_1(A^*(B),B) < x_1(A^*(B),B) < D_1(A^*(B),B) \\
&< D(A^*(B),B).
\end{aligned}$$

Therefore, for $B \in (\underline{B}, \overline{B}_2)$,

$$\begin{aligned}
\Lambda_1(A^*(B),B) &= \int_{d(A^*(B),B)}^{D(A^*(B),B)} \big[g_{A^*(B),B}(x) + k\big]dx \\
&\le \int_{d_1(A^*(B),B)}^{D_1(A^*(B),B)} \big[g_{A^*(B),B}(x) + k\big]dx \\
&\le -M_2(D_1(A^*(B),B) - d_1(A^*(B),B)).
\end{aligned}$$

By (5.60), (5.64) and $A^*(B) \geq A^{\text{int}}$, we have

$$\lim_{B \downarrow \underline{B}} x_1(A^*(B), B) \leq \lim_{B \downarrow \underline{B}} x_1(A^{\text{int}}, B) = -\infty. \tag{5.112}$$

Because $d_1(A^*(B), B) < x_1(A^*(B), B)$, (5.112) implies

$$\lim_{B \downarrow \underline{B}} d_1(A^*(B), B) = -\infty. \tag{5.113}$$

Now we prove

$$\lim_{B \downarrow \underline{B}} D_1(A^*(B), B) > -\infty, \tag{5.114}$$

which, together with (5.113), implies

$$\begin{aligned}
&\lim_{B \downarrow \underline{B}} \Lambda_1(A^*(B), B) \\
&\qquad \leq \lim_{B \downarrow \underline{B}} -M_2(D_1(A^*(B), B) - d_1(A^*(B), B)) \\
&\qquad = -\infty,
\end{aligned}$$

and proves (5.108).

To prove (5.114), noting the definitions of y_1 and y_2, for $B \in (\underline{B}, \overline{B}_1)$, we have

$$\begin{aligned}
\frac{\partial D_1(A^*(B), B)}{\partial B} &= -\frac{2}{\sigma^2} \frac{1}{\lambda_1 + \lambda_2} \frac{1}{g'_{A^*(B),B}(D_1(A^*(B), B))} \\
&\quad \times \left[\frac{1}{\lambda_1} \frac{dA^*(B)}{dB} e^{\lambda_1 (D_1(A^*(B),B)-a)} + \frac{1}{\lambda_2} e^{-\lambda_2 (D_1(A^*(B),B)-a)} \right] \\
&= -\frac{2}{\sigma^2} \frac{1}{\lambda_1 + \lambda_2} \frac{1}{g'_{A^*(B),B}(D_1(A^*(B), B))} \left[\frac{1}{\lambda_2} e^{-\lambda_2 (D_1(A^*(B),B)-a)} \right. \\
&\quad \left. + \frac{\lambda_1 (e^{-\lambda_2 (u(A^*(B),B)-a)} - e^{-\lambda_2 (U(A^*(B),B)-a)})}{\lambda_2^2 (e^{\lambda_1 (u(A^*(B),B)-a)} - e^{\lambda_1 (U(A^*(B),B)-a)})} e^{\lambda_1 (D_1(A^*(B),B)-a)} \right] \\
&= \frac{2}{\sigma^2} \frac{1}{\lambda_1 + \lambda_2} \frac{1}{\lambda_2} \frac{1}{g'_{A^*(B),B}(D_1(A^*(B), B))} \\
&\quad \times \left[\frac{e^{-\lambda_2 (y_1 - a)}}{e^{\lambda_1 (y_2 - a)}} e^{\lambda_1 (D_1(A^*(B),B)-a)} - e^{-\lambda_2 (D_1(A^*(B),B)-a)} \right] \\
&= \frac{2}{\sigma^2} \frac{1}{\lambda_1 + \lambda_2} \frac{1}{\lambda_2} \frac{1}{g'_{A^*(B),B}(D_1(A^*(B), B))} e^{-\lambda_2 (y_1 - a)} \\
&\quad \times \left[e^{\lambda_1 (D_1(A^*(B),B)-y_2)} - e^{-\lambda_2 (D_1(A^*(B),B)-y_1)} \right] \\
&< 0,
\end{aligned} \tag{5.115}$$

where the inequality is due to $D_1(A^*(B),B) < U(A^*(B),B) < y_1$, $D_1(A^*(B),B) < U(A^*(B),B) < y_2$ and $g'_{A^*(B),B}(D_1(A^*(B),B)) > 0$. Therefore, we have proved (5.114).

Finally we show that $\frac{\partial \Lambda_1(A^*(B),B)}{\partial B} > 0$. From (5.102), it follows that

$$\begin{aligned}
\frac{\partial \Lambda_1(A^*(B),B)}{\partial B} &= \int_{d(A^*(B),B)}^{D(A^*(B),B)} \frac{\partial g_{A^*(B),B}(x)}{\partial B} dx \\
&\quad + \frac{\partial D(A^*(B),B)}{\partial B}\big[g_{A^*(B),B}(D(A^*(B),B)) + k\big] \\
&\quad - \frac{\partial d(A^*(B),B)}{\partial B}\big[g_{A^*(B),B}(d(A^*(B),B)) + k\big] \\
&= \frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2}\int_{d(A^*(B),B)}^{D(A^*(B),B)} \Big[\frac{1}{\lambda_1}\frac{dA^*(B)}{dB}e^{\lambda_1(x-a)} + \frac{1}{\lambda_2}e^{-\lambda_2(x-a)}\Big]dx \\
&= \frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2}\int_{d(A^*(B),B)}^{D(A^*(B),B)} \Big[\frac{1}{\lambda_2}e^{-\lambda_2(x-a)} \\
&\quad + \frac{\lambda_1(e^{-\lambda_2(u(A^*(B),B)-a)} - e^{-\lambda_2(U(A^*(B),B)-a)})}{\lambda_2^2(e^{\lambda_1(u(A^*(B),B)-a)} - e^{\lambda_1(U(A^*(B),B)-a)})}e^{\lambda_1(x-a)}\Big]dx \\
&= \frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2}\frac{1}{\lambda_2^2}\frac{1}{e^{\lambda_1(u(A^*(B),B)-a)} - e^{\lambda_1(U(A^*(B),B)-a)}}\Big[\\
&\quad \big(e^{-\lambda_2(u(A^*(B),B)-a)} - e^{-\lambda_2(U(A^*(B),B)-a)}\big) \\
&\quad \times\big(e^{\lambda_1(D(A^*(B),B)-a)} - e^{\lambda_1(d(A^*(B),B)-a)}\big) \\
&\quad - \big(e^{-\lambda_2(D(A^*(B),B)-a)} - e^{-\lambda_2(d(A^*(B),B)-a)}\big) \\
&\quad \times\big(e^{\lambda_1(u(A^*(B),B)-a)} - e^{\lambda_1(U(A^*(B),B)-a)}\big)\Big].
\end{aligned} \tag{5.116}$$

If the expression inside the bracket is positive, then $\frac{\partial \Lambda_1(A^*(B),B)}{\partial B} > 0$. Note that $d(A^*(B),B) < D(A^*(B),B) < U(A^*(B),B) < u(A^*(B),B)$. Thus, the positivity of the expression is equivalent to

$$\begin{aligned}
&\frac{e^{-\lambda_2(u(A^*(B),B)-a)} - e^{-\lambda_2(U(A^*(B),B)-a)}}{e^{\lambda_1(u(A^*(B),B)-a)} - e^{\lambda_1(U(A^*(B),B)-a)}} \\
&\qquad > \frac{e^{-\lambda_2(D(A^*(B),B)-a)} - e^{-\lambda_2(d(A^*(B),B)-a)}}{e^{\lambda_1(D(A^*(B),B)-a)} - e^{\lambda_1(d(A^*(B),B)-a)}}.
\end{aligned} \tag{5.117}$$

Using the *Lagrange Mean Value Theorem*, there exist $z_1 \in (d(A^*(B),B), D(A^*(B),B))$ and $z_2 \in (d(A^*(B),B), D(A^*(B),B))$ such that

$$\begin{aligned}
&e^{-\lambda_2(D(A^*(B),B)-a)} - e^{-\lambda_2(d(A^*(B),B)-a)} \\
&\qquad = -\lambda_2 e^{-\lambda_2(z_1-a)}(D(A^*(B),B) - d(A^*(B),B)),
\end{aligned}$$

$$e^{\lambda_1(D(A^*(B),B)-a)} - e^{\lambda_1(d(A^*(B),B)-a)}$$
$$= \lambda_1 e^{\lambda_1(z_2-a)}(D(A^*(B),B) - d(A^*(B),B)).$$

Using (5.110) and (5.111), inequality (5.117) is equivalent to

$$\frac{e^{-\lambda_2(y_1-a)}}{e^{\lambda_1(y_2-a)}} < \frac{e^{-\lambda_2(z_1-a)}}{e^{\lambda_1(z_2-a)}},$$

which is further equivalent to

$$e^{-\lambda_2(y_1-z_1)} < e^{\lambda_1(y_2-z_2)}. \tag{5.118}$$

Inequality (5.118) holds because $y_1 > U(A^*(B),B) > D(A^*(B),B) > z_1$ and $y_2 > U(A^*(B),B) > D(A^*(B),B) > z_2$ imply

$$y_1 - z_1 > 0, \quad y_2 - z_2 > 0.$$

Therefore, we have proved

$$\frac{\partial \Lambda_1(A^*(B),B)}{\partial B} > 0, \tag{5.119}$$

which completes the proof of the lemma. □

Finally, we give the following lemma that completes the proof of (5.29) in Theorem 5.2 and will be used in the proof of Theorem 6.2.

LEMMA 5.8. *Under Assumption 1, for any $(A,B) \in G$, we have*

$$h'(x_1(A,B)) \le -\beta k \quad \text{and} \quad h'(x_2(A,B)) \ge \beta \ell. \tag{5.120}$$

PROOF. From (5.41), it follows that

$$\begin{aligned}
&\left(B - h'(a-) + \int_{x_1(A,B)}^{a} e^{\lambda_2(y-a)} h''(y)dy\right) e^{-\lambda_2(x_1(A,B)-a)} \\
&\qquad = \left(A - h'(a-) + \int_{x_1(A,B)}^{a} e^{-\lambda_1(y-a)} h''(y)dy\right) e^{\lambda_1(x_1(A,B)-a)} \\
&\qquad \ge 0,
\end{aligned} \tag{5.121}$$

where the inequality follows from (5.43). Therefore, the definition of g in (5.35) and inequality (5.121) imply

$$\frac{1}{\beta} h'(x_1(A,B)) \le g_{A,B}(x_1(A,B)). \tag{5.122}$$

Since $(A,B) \in G$, we have $g_{A,B}(x_1(A,B)) \le -k$. This inequality and (5.122) imply the first part of (5.120). The proof for the second part of (5.120) is similar and omitted. □

6. Singular controls. We assume that $K = 0$ and $L = 0$. Our feasible policies (Y_1, Y_2) in (2.1) are all adaptive, nondecreasing processes that include singular controls, also known as instantaneous controls. Under a singular control, (Y_1, Y_2) has infinitely many increases in each finite interval $[0, t]$. An example of a singular control policy is a two-parameter control band policy, which is defined by two parameters d, u with $d < u$. No control is exercised until the inventory level $Z(t)$ reaches the lower boundary d or the upper boundary u. When $Z(t)$ reaches a boundary, there is no advantage in using impulse control because there is no fixed cost.

6.1. *Control band policies.* Let us fix a two-parameter control band policy $\varphi = \{d, u\}$. See Section 6.1 in [6] for a mathematical description of the control process (Y_1, Y_2) under policy φ. To find the expected total discounted cost under policy $\varphi = \{d, u\}$, we use the following theorem.

THEOREM 6.1. *Fix a control band policy $\varphi = \{d, u\}$. If there exists a twice continuously differentiable function $V : [d, u] \to \mathbb{R}$ that satisfies*

$$\Gamma V(x) - \beta V(x) + h(x) = 0, \quad d \le x \le u, \tag{6.1}$$

with boundary conditions

$$V'(d) = -k, \tag{6.2}$$

$$V'(u) = \ell. \tag{6.3}$$

Then for each starting point $x \in \mathbb{R}$, the expected total discounted cost $DC(x, \varphi)$ is given by

$$DC(x, \varphi) = \begin{cases} V(d) + k(d - x) & \text{for } x \in (-\infty, d), \\ V(x) & \text{for } x \in [d, u], \\ V(u) + \ell(x - u) & \text{for } x \in (u, \infty), \end{cases}$$

where V is in (6.1).

PROOF. Consider control band policy $\{d, u\}$. Let V be a twice continuously differentiable function on $[d, u]$ that satisfies (6.1)-(6.3). Because $d \le Z(t) \le u$, Lemma 3.1 gives

$$\begin{aligned} e^{-\beta t} V(Z(t)) &= V(Z(0)) + \int_0^t e^{-\beta s} \big(\Gamma V(Z(s)) - \beta V(Z(s))\big) ds \\ &\quad + \sigma \int_0^t e^{-\beta s} V'(Z(s)) dW(s) + \int_0^t e^{-\beta s} V'(Z(s-)) dY_1(s) \\ &\quad - \int_0^t e^{-\beta s} V'(Z(s-)) dY_2(s) \end{aligned}$$

$$
\begin{aligned}
&= V(Z(0)) + \int_0^t e^{-\beta s}\big(\Gamma V(Z(s)) - \beta V(Z(s))\big)ds \\
&\quad + \sigma \int_0^t e^{-\beta s} V'(Z(s))dW(s) + V'(d)\int_0^t e^{-\beta s} dY_1(s) \\
&\quad - V'(u)\int_0^t e^{-\beta s} dY_2(s) \\
&= V(Z(0)) - \int_0^t e^{-\beta s} h(Z(s))ds + \sigma \int_0^t e^{-\beta s} V'(Z(s))dW(s) \\
&\quad - k\int_0^t e^{-\beta s} dY_1(s) - \ell \int_0^t e^{-\beta s} dY_2(s).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\mathbb{E}_x\big[e^{-\beta t} V(Z(t))\big] &= \mathbb{E}_x[V(Z(0))] - \mathbb{E}_x\Big[\int_0^t e^{-\beta s} h(Z(s))ds \\
&\quad + k\int_0^t e^{-\beta s} dY_1(s) + \ell \int_0^t e^{-\beta s} dY_2(s)\Big].
\end{aligned}
$$

Taking the limit as $t \to \infty$, we have

$$
\mathrm{DC}(x, \varphi) = \mathbb{E}_x[V(Z(0))] + \mathbb{E}_x[kY_1(0) + \ell Y_2(0)] \tag{6.4}
$$

because

$$
\lim_{t\to\infty} \mathbb{E}_x\big[e^{-\beta t} V(Z(t))\big] = 0.
$$

When $Z(0-) = x \in [d, u]$, we have $Z(0) = Z(0-) = x$ and $Y_1(0) = Y_2(0) = 0$, then

$$
\mathrm{DC}(x, \varphi) = V(x).
$$

When $Z(0-) < d$, we assume Z immediately jumps up to d at time 0. Therefore, $Z(0) = d$, $Y_1(0) = d - x$ and $Y_2(0) = 0$, and then

$$
\mathbb{E}_x[V(Z(0))] = V(d), \quad \mathbb{E}_x[kY_1(0) + \ell Y_2(0)] = k(d - x),
$$

which, together with (6.4), implies

$$
\mathrm{DC}(x, \varphi) = V(d) + k(d - x).
$$

The analysis for case $x > u$ is analogous and is omitted. □

PROPOSITION 2. *Let $\varphi = \{d, u\}$ be a control band policy with $d < u$. Define*

$$V(x) = A_1 e^{\lambda_1 x} + B_1 e^{-\lambda_2 x} + V_0(x), \tag{6.5}$$

as in (5.7), where $V_0(x)$ is defined in (5.8),

$$A_1 = \frac{d_2(V_0'(d) + k) - d_1(V_0'(u) - \ell)}{\lambda_1(c_2 d_1 - c_1 d_2)}, \tag{6.6}$$

$$B_1 = \frac{c_2(V_0'(d) + k) - c_1(V_0'(u) - \ell)}{\lambda_2(c_1 d_2 - c_2 d_1)}. \tag{6.7}$$

Then V is a solution to (6.1)-(6.3)*. In* (6.6) *and* (6.7)*, we set*

$$c_1 = e^{\lambda_1 d}, \quad c_2 = e^{\lambda_1 u}, \quad d_1 = -e^{-\lambda_2 d}, \quad d_2 = -e^{-\lambda_2 u}.$$

PROOF. From the proof of Proposition 1, we have that a general solution $V(x)$ to (6.1) is given by (6.5) with $V_0(x)$ being defined in (5.8). Boundary conditions (6.2) and (6.3) become

$$\begin{aligned} A_1\lambda_1 e^{\lambda_1 d} - B_1\lambda_2 e^{-\lambda_2 d} + V_0'(d) &= -k, \\ A_1\lambda_1 e^{\lambda_1 u} - B_1\lambda_2 e^{-\lambda_2 u} + V_0'(u) &= \ell. \end{aligned}$$

from which we have a unique solution for A_1 and B_1 given in (6.6) and (6.7). $\square$

6.2. *Optimal policy and optimal parameters.* Theorem 4.1 suggests the following strategy to obtain an optimal policy. We hope that a control band policy is optimal. Therefore, the first task is to find an optimal policy among all control band policies. We denote this optimal control band policy by $\{d^*, u^*\}$ with expected total discounted cost

$$\bar{V}(x) = \begin{cases} V(d^*) + k(d^* - x) & \text{for } x \in (-\infty, d^*), \\ V(x) & \text{for } x \in [d^*, u^*], \\ V(u^*) + \ell(x - u^*) & \text{for } x \in (u^*, \infty). \end{cases} \tag{6.8}$$

We hope that $\bar{V}$ can be used as the function f in Theorem 4.1. To find the corresponding f that satisfies all the conditions of Theorem 4.1, we provide the following conditions that should be imposed on the optimal parameters d^* and u^*:

$$V''(d^*+) = 0, \quad V''(u^*-) = 0, \tag{6.9}$$

which is identical to condition (6.18) in [6]. See Section 6.2 of [6] for an intuitive explanation of these conditions.

First we need to prove the existence of parameters d^* and u^* such that the value function V, defined on $[d^*, u^*]$, corresponding to control band policy $\varphi = \{d^*, u^*\}$ satisfies (6.1)-(6.3) and (6.9). Since part of the solution is to find the boundary points d^* and u^*, (6.1)-(6.3) and (6.9) define a free boundary problem. We then prove in Theorem 6.3 that function $\bar{V}$ in (6.8) with parameters d^* and u^* satisfies all conditions in Theorem 4.1.

Recall that (5.21) defines function $g(x) = g_{A,B}(x)$.

THEOREM 6.2. *Assume that holding cost function h satisfies Assumption 1. There exist unique A^*, B^*, d^* and u^* with*

$$d^* < u^* \tag{6.10}$$

such that $g(x) = g_{A^*,B^*}(x)$ in (5.21) satisfies

$$g(d^*) = -k, \tag{6.11}$$

$$g(u^*) = \ell, \tag{6.12}$$

$$g'(d^*) = 0, \tag{6.13}$$

$$g'(u^*) = 0. \tag{6.14}$$

Furthermore,

$$h'(d^*) \le -k\beta \quad \text{and} \quad h'(u^*) \ge \beta\ell. \tag{6.15}$$

Function $g(x)$ decreases in $(-\infty, d^*)$, increases in (d^*, u^*), and decreases again in (u^*, ∞).

PROOF. Recall the definitions A^{int} and B^{int} in (5.76). By Lemma 5.4 (c), the point $(A^{\text{int}}, B^{\text{int}})$ is the unique point satisfying

$$g_{A^{\text{int}},B^{\text{int}}}(x_1(A^{\text{int}}, B^{\text{int}})) = -k, \quad \text{and} \quad g_{A^{\text{int}},B^{\text{int}}}(x_2(A^{\text{int}}, B^{\text{int}})) = \ell.$$

See Figure 4 for an illustration. Then $A^* = A^{\text{int}}$, $B^* = A^{\text{int}}$, $g(x) = g_{A^*,B^*}(x)$, $d^* = x_1(A^*, B^*)$ and $u^* = x_2(A^*, B^*)$ satisfy (6.10)-(6.15). □

Now we show that control band policy $\varphi^* = \{d^*, u^*\}$ is optimal among all feasible policies.

THEOREM 6.3. *Assume that h satisfies Assumption 1. Let d^* and u^*, along with constants A^* and B^*, be the unique solution in Theorem 6.2. Therefore, control band policy $\varphi^* = \{d^*, u^*\}$ is optimal among all feasible policies.*

PROOF. Let $g(x)$, $x \in \mathbb{R}$, be the solution in (5.21) with $A = A^*$, $B = B^*$. Let

$$\bar{g}(x) = \begin{cases} -k, & x < d^*, \\ g(x), & d^* \le x \le u^*, \\ \ell, & x > u^*, \end{cases}$$

and

$$\bar{V}(x) = \begin{cases} V(d^*) + k(d^* - x), & x < d^*, \\ V(x), & d^* \le x \le u^*, \\ V(u^*) + \ell(x - u^*), & x > u^*, \end{cases} \tag{6.16}$$

with

$$V(x) = A_1^* e^{\lambda_1 x} + B_1^* e^{-\lambda_2 x} + V_0(x),$$

where $\frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2}\frac{1}{\lambda_1}e^{-\lambda_1 a}A^* = \lambda_1 A_1^*$, $\frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2}\frac{1}{\lambda_2}e^{\lambda_2 a}B^* = -\lambda_2 B_1^*$ and $V_0(x)$ is given by (5.8). Therefore,

$$\bar{V}'(x) = \bar{g}_{A^*,B^*}(x), \quad \text{for } x \in \mathbb{R}.$$

We now show that $\bar{V}$ satisfies all of the conditions in Theorem 4.1. Thus, Theorem 4.1 shows that the expected total discounted cost under any feasible policy is at least $\bar{V}$. Since $\bar{V}(x)$ is the expected total discounted cost under control band policy $\varphi^* = \{d^*, u^*\}$ with starting point x, $\bar{V}(x)$ is the optimal cost and control band policy φ^* is optimal among all feasible policies.

First, $\bar{V}(x)$ is in $C^2((d^*, u^*))$. Theorem 6.2 and the definition of $\bar{V}$ in (6.16) imply

$$\lim_{x\uparrow d^*} \bar{V}''(x) = 0 = \lim_{x\downarrow d^*} \bar{V}''(x), \quad \text{and} \quad \lim_{x\uparrow u^*} \bar{V}''(x) = 0 = \lim_{x\downarrow u^*} \bar{V}''(x).$$

Then $\bar{V}''(x)$ is continuous at d^* and u^*. Note that $\bar{V}''(x) = 0$ in $(-\infty, d^*)$ and (u^*, ∞). Therefore, $V(x)$ is in $C^2(\mathbb{R})$. Let

$$M = \sup_{x \in [d^*, u^*]} |\bar{g}(x)|,$$

we have $|\bar{V}'(x)| \le M$ for all $x \in \mathbb{R}$.

To check (4.1), first, we find that

$$\Gamma\bar{V}(x) - \beta\bar{V}(x) + h(x) = \Gamma V(x) - \beta V(x) + h(x) = 0 \quad \text{for } x \in [d^*, u^*].$$

In particular

$$\Gamma\bar{V}(d^*) - \beta\bar{V}(d^*) + h(d^*) = 0 \tag{6.17}$$

and

$$\Gamma\bar{V}(u^*) - \beta\bar{V}(u^*) + h(u^*) = 0.$$

For $x < d^*$,

$$\begin{aligned}
&\Gamma\bar{V}(x) - \beta\bar{V}(x) + h(x)\\
&\quad = \frac{\sigma^2}{2}\bar{V}''(x) + \mu\bar{V}'(x) - \beta\bar{V}(x) + h(x)\\
&\quad = \frac{\sigma^2}{2}\bar{V}''(d^*) + \mu\bar{V}'(d^*) - \beta(\bar{V}(d^*) + k(d^* - x)) + h(x)\\
&\quad = \frac{\sigma^2}{2}\bar{V}''(d^*) + \mu\bar{V}'(d^*) - \beta\bar{V}(d^*) + h(d^*) + h(x) - h(d^*) - \beta k(d^* - x)\\
&\quad = h(x) - h(d^*) - \beta k(d^* - x)\\
&\quad \geq 0,
\end{aligned}$$

where the last equality follows from (6.17), and the inequality follows from (6.15) and the convexity of h. Similarly, we can check that

$$\Gamma\bar{V}(x) - \beta\bar{V}(x) + h(x) \geq 0$$

for $x > u^*$.

Finally, (4.2) and (4.3) hold because $\bar{V}'(x) = \bar{g}(x)$ is strictly increasing in x, $x \in [d^*, u^*]$, and $\bar{V}'(x) = -k$ for $x \in (-\infty, d^*)$, $\bar{V}'(x) = \ell$ for $x \in (u^*, \infty)$. Thus, Theorem 4.1 implies the optimality of control band policy $\varphi^* = \{d^*, u^*\}$. □

7. No inventory backlog. Prohibiting inventory backlog, we add constraint $Z(t) \geq 0$ for all $t \geq 0$. The holding cost function $h(\cdot)$ is defined on $[0, \infty)$, and $a \in [0, \infty)$ is its minimum point. We focus on the impulse control case when $K > 0$ and $L > 0$. Thus, this section parallels Section 5. In particular, the results and proofs in this section are analogous to those in Section 5. Below, we highlight the differences and note that cases when $K = 0$ and $L = 0$, when $K > 0$ and $L = 0$, and when $K = 0$ and $L > 0$, are analogous to this case. For example, we can obtain the optimal policy for the case when $K = 0$ and $L = 0$ with constraint $Z(t) \geq 0$ by adapting the arguments presented in Section 6.

For control band policy $\{d, D, U, u\}$ with $0 \leq d < D < U < u$, we continue to use Theorem 5.1 to evaluate its performance and to obtain its expected total discounted cost function. However, we need to modify Theorem 4.1, the lower bound theorem, slightly as follows.

THEOREM 7.1. *Suppose that $f \in C^1([0,\infty))$ and f' is absolutely continuous with f'' being locally in L^1. Suppose there exists a constant $M > 0$ such that $|f'(x)| \le M$ for all $x \in [0,\infty)$. Assume further*

$$\Gamma f(x) - \beta f(x) + h(x) \geq 0 \quad \text{for almost all } x \in [0,\infty), \tag{7.1}$$

$$f(y) - f(x) \leq K + k(x-y) \quad \text{for } 0 \leq y < x, \tag{7.2}$$

$$f(y) - f(x) \leq L + \ell(y-x) \quad \text{for } 0 \leq x < y. \tag{7.3}$$

Then $DC(x,\varphi) \geq f(x)$ for each feasible policy φ and each initial state $Z(0-) = x \in [0,\infty)$.

7.1. *Optimal policy parameters.* Recall that for a given control band policy $\{d, D, U, u\}$ with $0 \le d < D < U < u$, the corresponding value function satisfies (5.1)-(5.3). To search for the optimal parameters (d^*, D^*, U^*, u^*), we impose the following conditions on $\{d, D, U, u\}$ and V

$$V'(U) = l, \tag{7.4}$$

$$V'(u) = l, \tag{7.5}$$

$$V'(D) = -k, \tag{7.6}$$

$$V'(d) = -k - \alpha, \tag{7.7}$$

$$0 \leq d < D < U < u, \tag{7.8}$$

$$\alpha d = 0, \quad \text{and} \tag{7.9}$$

$$\alpha \geq 0. \tag{7.10}$$

This section is analogous to Section 5.2 and Section 5.3. We highlight the differences and omit some details to avoid repetition.

Recall that a is the minimum point of holding cost function $h(x)$ on $[0,\infty)$. It is possible $a = 0$ or $a > 0$. In the following, whenever we invoke Assumption 1 for h, we ignore any condition on $h(x)$ with $x < 0$. By convention, we set $h'(a-) = h'(a+)$ when $a = 0$. The following theorem solves the free boundary problem when inventory backlog is not allowed. For a graphical illustration of function g in the theorem, see Figure 3 in [6].

THEOREM 7.2. *Assume that holding cost function h satisfies Assumption 1. There exist unique A^*, B^*, x_1^*, x_2^*, d^*, D^*, U^*, u^* and α^* with*

$$0 \leq d^* \leq x_1^* < D^* \quad \text{and} \quad U^* < x_2^* < u^* \tag{7.11}$$

such that $g(x) = g_{A^*,B^*}(x)$ *in* (5.21) *satisfies*

$$\int_{d^*}^{D^*} [g(x) + k]dx = -K, \tag{7.12}$$

$$\int_{U^*}^{d^*} [g(x) - \ell]dx = L, \tag{7.13}$$

$$g(d^*) = -k - \alpha^*, \quad g(D^*) = -k, \tag{7.14}$$

$$g(U^*) = g(u^*) = \ell, \tag{7.15}$$

$$\alpha^* d^* = 0, \quad \textit{and} \tag{7.16}$$

$$\alpha^* \geq 0. \tag{7.17}$$

Furthermore,

$$h'(x_1^*) \leq -\beta k \quad \textit{if } x_1^* > 0 \quad \textit{and} \quad h'(x_2^*) \geq \beta\ell. \tag{7.18}$$

Function $g(x)$ *has a local minimum in* $[0, a]$ *at* $x_1^* \in [0, a]$ *and* g *has the maximum at* $x_2^* \in (a, \infty)$. *Function* g *is decreasing on* $(0, x_1^*)$, *increasing on* (x_1^*, x_2^*) *and decreasing again on* (x_2^*, ∞).

We leave the proof of Theorem 7.2 to the end of this section.

THEOREM 7.3. *Assume that holding cost function* h *satisfies Assumption 1. Let* $0 \leq d^* < D^* < U^* < u^*$, *along with constants* A^*, B^* *and* α^*, *be the unique solution in Theorem 7.2. Then control band policy* $\varphi^* = \{d^*, D^*, U^*, u^*\}$ *is optimal among all feasible policies to minimize the expected total discounted cost when inventory backlog is not allowed.*

PROOF. The proof is identical to that of Theorem 5.3. □

Next, we explain the proof for Theorem 7.2, which is similar to that for Theorem 5.2. We provide an outline of the proof here, highlighting the differences between the two proofs. In the following, we assume $a > 0$; when $a = 0$, the proof is simpler and is omitted.

Similar to (5.36), we define

$$\underline{B}_1 = h'(a-) - \int_0^a e^{\lambda_2(y-a)} h''(y)dy. \tag{7.19}$$

First, we have the following lemma.

LEMMA 7.1. *For $A > h'(a-)$, there exists a unique $\hat{B}(A) \in (\underline{B}_1, A]$ such that*

$$\Big(A - h'(a-) + \int_0^a e^{-\lambda_1(y-a)} h''(y)dy\Big)e^{-\lambda_1 a} = \Big(\hat{B}(A) - h'(a-) + \int_0^a e^{\lambda_2(y-a)} h''(y)dy\Big)e^{\lambda_2 a}. \tag{7.20}$$

PROOF. Fix $A > h'(a-)$. Then, $\big(A-h'(a-)+\int_0^a e^{-\lambda_1(y-a)} h''(y)dy\big)e^{-\lambda_1 a} > 0$ is a constant. Consider function $H(B) = \big(B - h'(a-) + \int_0^a e^{\lambda_2(y-a)} \times h''(y)dy\big)e^{\lambda_2 a}$. The function is linear in B and is strictly increasing. By the definition of $\underline{B}_1$, we have $H(\underline{B}_1) = 0$. At $B = A$,

$$\begin{aligned} H(A) &= \big(A - h'(a-)\big)e^{\lambda_2 a} + \int_0^a e^{\lambda_2 y} h''(y)dy \\ &\geq \big(A - h'(a-)\big)e^{-\lambda_1 a} + \int_0^a e^{-\lambda_1 y} h''(y)dy \\ &= \big(A - h'(a-) + \int_0^a e^{-\lambda_1(y-a)} h''(y)dy\big)e^{-\lambda_1 a}. \end{aligned}$$

Thus, there exists a unique $\hat{B}(A) \in (\underline{B}_1, A]$ such that

$$H(\hat{B}(A)) = \big(A - h'(a-) + \int_0^a e^{-\lambda_1(y-a)} h''(y)dy\big)e^{-\lambda_1 a},$$

and the lemma is proved. $\square$

Recall the expression of $g'(x)$ in (5.47). Definition (7.20) implies

$$g'_{A,\hat{B}(A)}(0) = 0. \tag{7.21}$$

From (7.20), it follows that

$$\begin{aligned} &\frac{d\hat{B}(A)}{dA} = e^{-(\lambda_1+\lambda_2)a} > 0, \\ &\lim_{A\downarrow h'(a-)} \hat{B}(A) = B_{\min}, \\ &\lim_{A\uparrow +\infty} \hat{B}(A) = \infty, \end{aligned}$$

where

$$B_{\min} = h'(a-) + \int_0^a e^{-\lambda_1 y} h''(y)dy \cdot e^{-\lambda_2 a} - \int_0^a e^{\lambda_2(y-a)} h''(y)dy > \underline{B}_1.$$

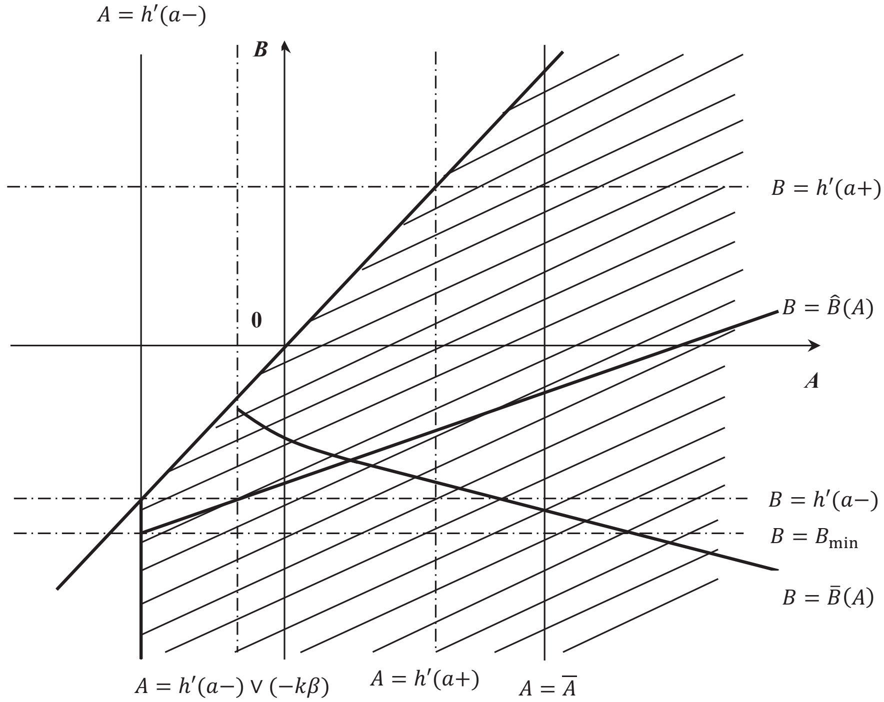

FIG 7. *(a) The shaded region is the set of (A, B) that satisfies (7.22)-(7.23). The unique minimum $x_1 = x_1(A, B) \in [0, a)$ is well defined for all (A, B) in this region.*
(b) For each $A \in \big(h'(a-) \vee (-k\beta), +\infty\big)$, there exists a unique $\overline{B}(A) \in \big(-\infty, A \wedge 0\big)$ such that $g_{A,\overline{B}(A)}(x_1(A, \overline{B}(A))) = -k$. Thus, on curve $B = \overline{B}(A)$, $g_{A,B}(x_1(A, B)) = -k$. Curve $B = \overline{B}(A)$ is decreasing.

Therefore, $\hat{B}(A)$ is linear in A and is strictly increasing, mapping $(h'(a-), \infty)$ onto $(B_{\min}, \infty)$. See Figure 7 for the line $\{(A, \hat{B}(A)) : A \in (h'(a-), \infty)\}$.

The following lemma is analogous to Lemma 5.2. The only difference is that the expression for $x_1 = x_1(A, B)$ has two forms in Lemma 7.2.

LEMMA 7.2. *(a) For each A satisfying*

$$A > h'(a-) \tag{7.22}$$

and each B satisfying

$$B < A \tag{7.23}$$

$g_{A,B}(x)$ attains a unique minimum in $[0, a]$ at $x_1 = x_1(A, B) \in [0, a]$.

For each (A, B) *satisfying*

$$B < A < \overline{A}, \tag{7.24}$$

$g_{A,B}(x)$ *attains a unique maximum in* (a, ∞) *at* $x_2 = x_2(A, B) \in (a, \infty)$.
(b) For each fixed A *satisfying* $A > h'(a-)$ *and* B *satisfying*

$$\hat{B}(A) < B < A$$

the local minimizer $x_1 = x_1(A, B) > 0$ *is a unique solution to* (5.41)*, and the local minimizer*

$$x_1 = x_1(A, B) = 0 \text{ for each } (A, B) \text{ with } A > h'(a-) \text{ and } B \le \hat{B}(A). \tag{7.25}$$

For each (A, B) *satisfying* (7.24)*, the local maximizer* $x_2 = x_2(A, B)$ *is the unique solution in* (a, ∞) *to* (5.44)*.*
(c) Furthermore, for each A *satisfying* $h'(a-) < A < \overline{A}$ *and each* B *satisfying* $B < A$*, we have* $g'_{A,B}(x) < 0$ *for* $x \in (0, x_1(A, B))$*,* $g'_{A,B}(x) > 0$ *for* $x \in (x_1(A, B), x_2(A, B))$*, and* $g'_{A,B}(x) < 0$ *for* $x \in (x_2(A, B), \infty)$*.*

Remark. The set of (A, B) that satisfies (7.22)-(7.23) is the shaded region in Figure 7.

Proof. The proof for the existence of $x_2 = x_2(A, B)$ and for its properties is the same to that in Lemma 5.2. We now prove the existence and uniqueness of $x_1 = x_1(A, B)$.

For $A > h'(a-)$ and $\hat{B}(A) < B < A$, the proof is similar to that in Lemma 5.2 and the local minimizer $x_1 = x_1(A, B) > 0$ is the unique solution to (5.41). We next prove the case when $A > h'(a-)$ and $B \le \hat{B}(A)$, which we subdivide into two cases: $A > h'(a-)$ and $B \le \underline{B}_1$ or $A > h'(a-)$ and $\underline{B}_1 < B \le \hat{B}(A)$.

Case 1: $A > h'(a-)$ and $B \le \underline{B}_1$. In this case, recall the expression for g' in (5.47) and $\underline{B}_1$ in (7.19). Condition $A > h'(a-)$ implies $g'_{A,B}(x) > 0$ for $0 < x < a$. Thus, the local minimizer $x_1 = x_1(A, B) = 0$.

Case 2: $A > h'(a-)$ and $\underline{B}_1 < B \le \hat{B}(A)$. From the expression of g' in (5.47), it follows that

$$g'_{A,B}(x) \ge g'_{A,\hat{B}(A)}(x) \quad \text{for any } \underline{B}_1 < B \le \hat{B}(A) \text{ and } x \in (0, a).$$

Next, we prove that for $A > h'(a-)$,

$$g'_{A,\hat{B}(A)}(x) > 0 \quad \text{for } x \in (0, a); \tag{7.26}$$

therefore, the local minimizer of $g_{A,B}(x)$ is $x_1 = x_1(A,B) = 0$. Because $\hat{B}(A) \in (\underline{B}_1, A]$, the proof for (7.26) is divided into either $\hat{B}(A) \in (\underline{B}_1, h'(a-)]$ or $\hat{B}(A) \in (h'(a-), A]$.

If $\hat{B}(A) \in (\underline{B}_1, h'(a-)]$, there exists $x'' \in [0,a)$ such that

$$\hat{B}(A) = h'(a-) - \int_{x''}^{a} e^{\lambda_2(y-a)} h''(y)dy$$

and

$$\hat{B}(A) - h'(a-) + \int_{x}^{a} e^{\lambda_2(y-a)} h''(y)dy \quad \geq \quad 0 \quad \text{for } x \in [0, x''), \tag{7.27}$$

$$\hat{B}(A) - h'(a-) + \int_{x}^{a} e^{\lambda_2(y-a)} h''(y)dy \quad \leq \quad 0 \quad \text{for } x \in [x'', a). \tag{7.28}$$

Recall the expression for g'' in (5.48). Inequality (7.27) implies $g''_{A,\hat{B}(A)}(x) > 0$ for $x \in [0, x'')$, which, together with (7.21), implies

$$g'_{A,\hat{B}(A)}(x) > 0 \quad \text{for } x \in (0, x'').$$

Also, (5.47) and (7.28) imply

$$g'_{A,\hat{B}(A)}(x) > 0 \quad \text{for } x \in [x'', a).$$

Therefore, we have proved (7.26) when $\hat{B}(A) \in (\underline{B}_1, h'(a-)]$.

If $\hat{B}(A) \in (h'(a-), A]$, from the expression for g'' in (5.48), we have $g''_{A,\hat{B}(A)}(x) > 0$ for $x \in [0,a)$, which, together with (7.21), implies

$$g'_{A,\hat{B}(A)}(x) > 0 \quad \text{for } x \in (0, a).$$

Therefore, we have proved (7.26) when $\hat{B}(A) \in (h'(a-), A]$. □

The following lemma is analogous to Lemma 5.3.

LEMMA 7.3. *Suppose (A,B) satisfies (7.22)-(7.23). For each B, local minimizer $x_1(A,B)$ is continuous and nonincreasing in A. For each A, local minimizer $x_1(A,B)$ is continuous and nondecreasing in B. Suppose (A,B) satisfies (7.24). For each B, local maximizer $x_2(A,B)$ is continuous and strictly increasing in A. For each A, local maximizer $x_2(A,B)$ is continuous and strictly decreasing in B. Furthermore, (5.61)-(5.63) hold.*

PROOF. Fix $A \in (h'(a-), \infty)$. We have $x_1(A, B) = 0$ for $B \in (-\infty, \hat{B}(A))$. Thus $x_1(A, B)$ is continuous in B for $B \in (-\infty, \hat{B}(A))$. It follows the proof of Lemma 5.3 that $x_1(A, B)$ is continuous in $B \in (\hat{B}(A), A)$. It is easy to check that $x_1(A, B)$ is also continuous at $B = \hat{B}(A)$ and that $x_1(A, B)$ has the desired monotonicity property for $B \in (-\infty, A)$. The proof for properties of $x_1(A, B)$ in A is similar and is omitted.

The proof for the properties of $x_2(A, B)$ and (5.61)-(5.63) is identical to that in Lemma 5.3. □

The following lemma is analogous to Lemma 5.4. The only difference is that we modify the range of $\overline{B}(A)$ in part (b).

LEMMA 7.4. *(a) For each B satisfying (5.70), there exists a unique*

$$\underline{A}(B) \in \big(B \vee 0, \overline{A}\big)$$

such that

$$g_{\underline{A}(B),B}(x_2(\underline{A}(B), B)) = \ell. \tag{7.29}$$

Furthermore, for $B \in (-\infty, h'(a+) \wedge \ell\beta)$,

$$\frac{d\underline{A}(B)}{dB} \quad = \quad -\frac{\lambda_1}{\lambda_2} e^{-(\lambda_1+\lambda_2)(x_2(\underline{A}(B),B)-a)} < 0. \tag{7.30}$$

Therefore, function $A = \underline{A}(B)$ is strictly decreasing in $B \in (-\infty, h'(a+) \wedge \ell\beta)$; see Figure 3 for an illustration. For $A \in (\underline{A}(B), \overline{A})$,

$$g_{A,B}(x_2(A, B)) > \ell. \tag{7.31}$$

(b) For each A satisfying (5.73), there exists a unique

$$\overline{B}(A) \in \big(-\infty, A \wedge 0\big)$$

such that

$$g_{A,\overline{B}(A)}(x_1(A, \overline{B}(A))) = -k. \tag{7.32}$$

Furthermore, for $A \in (h'(a-) \vee (-k\beta), \infty)$,

$$\frac{d\overline{B}(A)}{dA} \quad = \quad -\frac{\lambda_2}{\lambda_1} e^{(\lambda_1+\lambda_2)(x_1(A,\overline{B}(A))-a)} < 0. \tag{7.33}$$

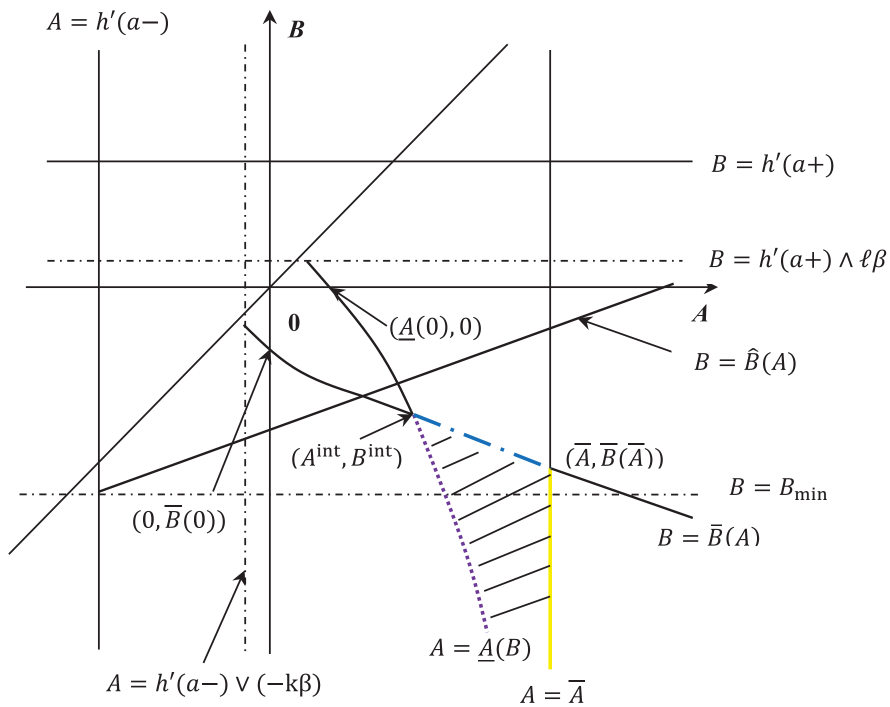

FIG 8. *(a) On the curve $B = \overline{B}(A)$, $g_{A,B}(x_1(A,B)) = -k$ and on the curve $A = \underline{A}(B)$, $g_{A,B}(x_2(A,B)) = \ell$.*
(b) The two curves $\{(\underline{A}(B), B) : B \in (-\infty, 0]\}$ and $\{(A, \overline{B}(A)) : A \in [0, \overline{A}]\}$ have a unique intersection point $(A^{\text{int}}, B^{\text{int}})$ that satisfies $0 < \underline{A}(0) < A^{\text{int}} < \overline{A}$ and $0 > \overline{B}(0) > B^{\text{int}} > \overline{B}(\overline{A})$. For any (A, B) in the shaded region G_1 definied in (7.40), $g_{A,B}(x_1(A,B)) < -k$ and $g_{A,B}(x_2(A,B)) > \ell$.

Therefore, function $B = \overline{B}(A)$ is strictly decreasing in $A \in (h'(a-) \vee (-k\beta), \infty)$; see Figure 7 for an illustration. For $B \in (-\infty, \overline{B}(A))$,

$$g_{A,B}(x_1(A,B)) < -k. \tag{7.34}$$

(c) The two curves $\{(\underline{A}(B), B) : B \in (-\infty, 0]\}$ and $\{(A, \overline{B}(A)) : A \in [0, \infty)\}$ have a unique intersection point $(A^{\text{int}}, B^{\text{int}})$ that satisfies

$$\overline{B}(A^{\text{int}}) = B^{\text{int}} \quad \textit{and} \quad \underline{A}(B^{\text{int}}) = A^{\text{int}} \tag{7.35}$$

with

$$0 < \underline{A}(0) < A^{\text{int}} < \underline{A}(\underline{B}) < \overline{A}, \tag{7.36}$$

$$0 > \overline{B}(0) > B^{\text{int}} > \overline{B}(\overline{A}) > \underline{B}. \tag{7.37}$$

See Figure 8 for an illustration.

PROOF. The proofs for (a) and (c) are identical to those in Lemma 5.4. We now prove (b). Fix an A that satisfies (5.73). We first prove that for $B \in (-\infty, A \wedge 0)$,

$$\frac{\partial g_{A,B}(x_1(A,B))}{\partial B} = \frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2}\frac{1}{\lambda_2}e^{-\lambda_2(x_1(A,B)-a)} > 0. \tag{7.38}$$

For (7.38), we first assume $B \in (\hat{B}(A), A \wedge 0)$. In this case, $x_1(A,B) > 0$ and the expression for $\partial g_{A,B}(x_1(A,B))/\partial B$ is given by (5.82), which is identical to (7.38).

When $B \in (-\infty, \hat{B}(A)]$, from (7.25), it follows that $x_1(A,B) = 0$. Using the expression of $g_{A,B}(0)$ in (5.21), we have

$$\frac{\partial g_{A,B}(x_1(A,B))}{\partial B} = \frac{\partial g_{A,B}(0)}{\partial B} = \frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2}\frac{1}{\lambda_2}e^{\lambda_2 a} > 0.$$

Thus, (7.38) continues to hold in this case.

From (7.25), it follows that $x_1(A,B) = 0$ when $B \le \hat{B}(A)$. Therefore,

$$\lim_{B\downarrow -\infty} g_{A,B}(x_1(A,B)) = \lim_{B\downarrow -\infty} g_{A,B}(0) = -\infty, \tag{7.39}$$

where the latter limit follows from the expression of $g_{A,B}(0)$ in (5.21).

If $A \in (h'(a-) \vee (-k\beta), 0)$, (5.21) and (5.61) imply

$$\lim_{B\uparrow A} g_{A,B}(x_1(A,B)) = \frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2}\left(\frac{1}{\lambda_1}+\frac{1}{\lambda_2}\right)A = \frac{A}{\beta}.$$

Because $A > -k\beta$,

$$\lim_{B\uparrow A} g_{A,B}(x_1(A,B)) > -k.$$

which, together with (7.38)-(7.39), implies that there exists unique $\overline{B}(A) \in (-\infty, A)$ such that

$$g_{A,\overline{B}(A)}(x_1(A,\overline{B}(A))) = -k$$

and for $B \in (-\infty, \overline{B}(A))$

$$g_{A,B}(x_1(A,B)) < -k.$$

If $A \in [0,\infty)$, (5.21) implies

$$\begin{aligned}&\lim_{B\uparrow 0} g_{A,B}(x_1(A,B))\\ &\quad = \lim_{B\uparrow 0}\frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2}\Big[\frac{1}{\lambda_1}\big(A+\lambda_1\int_{x_1(A,B)}^{a} e^{-\lambda_1(y-a)}h'(y)dy\big)e^{\lambda_1(x_1(A,B)-a)}\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\lambda_2}\big(B-\lambda_2\int_{x_1(A,B)}^{a} e^{\lambda_2(y-a)}h'(y)dy\big)e^{-\lambda_2(x_1(A,B)-a)}\Big]\\
&=\lim_{B\uparrow 0}\frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2}\Big[\frac{1}{\lambda_1}Ae^{\lambda_1(x_1(A,B)-a)}+\frac{1}{\lambda_2}Be^{-\lambda_2(x_1(A,B)-a)}\\
&\quad+\int_{x_1(A,B)}^{a}\big(e^{\lambda_1(x_1(A,B)-y)}-e^{-\lambda_2(x_1(A,B)-y)}\big)h'(y)dy\Big]\\
&\geq 0\\
&>-k,
\end{aligned}
$$

where the first inequality has used $h'(x)\leq 0$ for $x<a$. This plus (7.38)-(7.39) imply that there exists a unique $\overline{B}(A)\in(-\infty,0)$ such that

$$g_{A,\overline{B}(A)}(x_1(A,\overline{B}(A)))=-k$$

and for $B\in(-\infty,\overline{B}(A))$

$$g_{A,B}(x_1(A,B))<-k.$$

Applying the Implicit Function Theorem to (7.32), we also have (7.33). ☐

Let

$$G_1=\{(A,B):\underline{A}(B)<A<\overline{A},\ -\infty<B<\overline{B}(A)\} \tag{7.40}$$

be the shaded region in Figure 8. Region G_1 has two corners. They are $(A^{\text{int}},B^{\text{int}})$, $(\overline{A},\overline{B}(\overline{A}))$. Its boundary has three pieces: the top, the right and the left. The bottom has no boundary, extending the region all the way to $-\infty$ in the B axis. For any $(A,B)\in G_1$, there exist $U(A,B)$ and $u(A,B)$ that satisfy

$$U(A,B)<x_2(A,B)<u(A,B),$$

(5.89) and (5.91).

With Lemma 7.4 replacing Lemma 5.4, Lemma 5.5 holds without any modification. Recall the definition of $\overline{B}_1$ in (5.100) and see Figure 9 for point $(\overline{A}_1,\overline{B}_1)$. The following lemma is analogous to Lemma 5.6. The only difference is that we modify the range of B to be $(-\infty,\overline{B}_1]$.

LEMMA 7.5. *(a) For $B\in(-\infty,\overline{B}_1]$, there exists unique $A^*(B)\in[\overline{A}_1,\overline{A})$ such that*

$$\Lambda_2(A^*(B),B)=L. \tag{7.41}$$

See Figure 10 for an illustration.

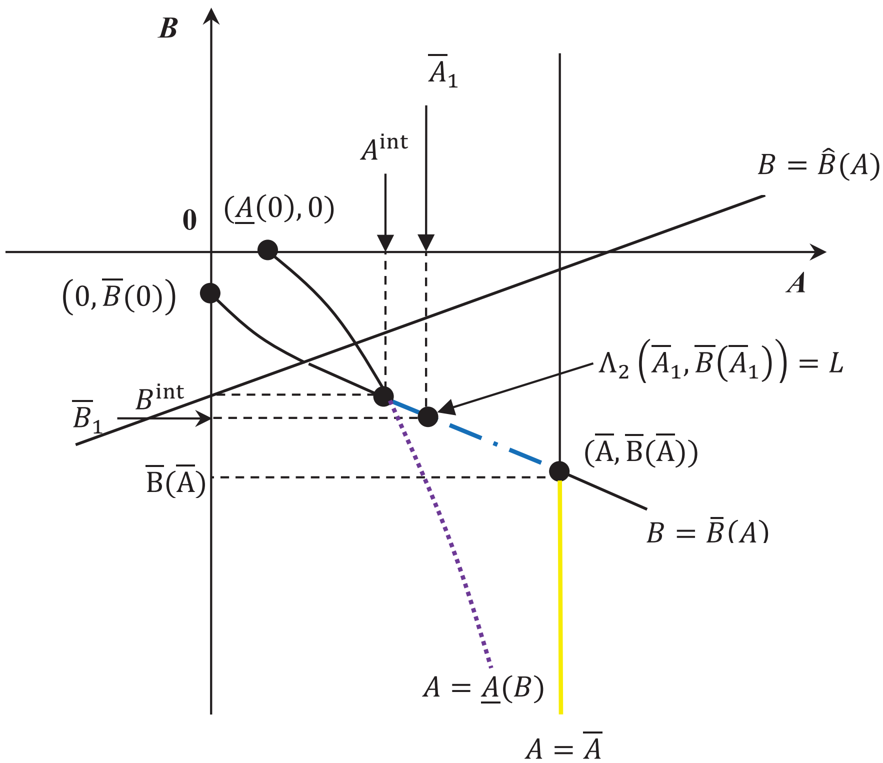

FIG 9. *Point $(\overline{A}_1, \overline{B}_1)$ is the unique point on the top boundary of G_1 such that $\Lambda_2(\overline{A}_1, \overline{B}_1) = L$. For any point (A, B) on the top boundary and to the right of $(\overline{A}_1, \overline{B}_1)$, $\Lambda_2(A, B) > L$.*

(b) For $B \in (-\infty, \overline{B}_1]$,

$$\frac{dA^*(B)}{dB} = \frac{\lambda_1^2(e^{-\lambda_2(u(A^*(B),B)-a)} - e^{-\lambda_2(U(A^*(B),B)-a)})}{\lambda_2^2(e^{\lambda_1(u(A^*(B),B)-a)} - e^{\lambda_1(U(A^*(B),B)-a)})} < 0. \tag{7.42}$$

PROOF. The proof is identical to that of Lemma 5.6. □

The following lemma gives a proof of Theorem 7.2 except for (7.18). The latter will be proved in Lemma 7.7 below.

LEMMA 7.6. *There exists a unique B^* with $B^* \in (-\infty, \overline{B}_1)$, d^*, D^*, x_1^* and α^* with $0 \le d^* \le x_1^* < D^*$ such that*

$$g_{A^*(B^*),B^*}(D^*) = -k,$$
$$g_{A^*(B^*),B^*}(d^*) = -k - \alpha^*,$$

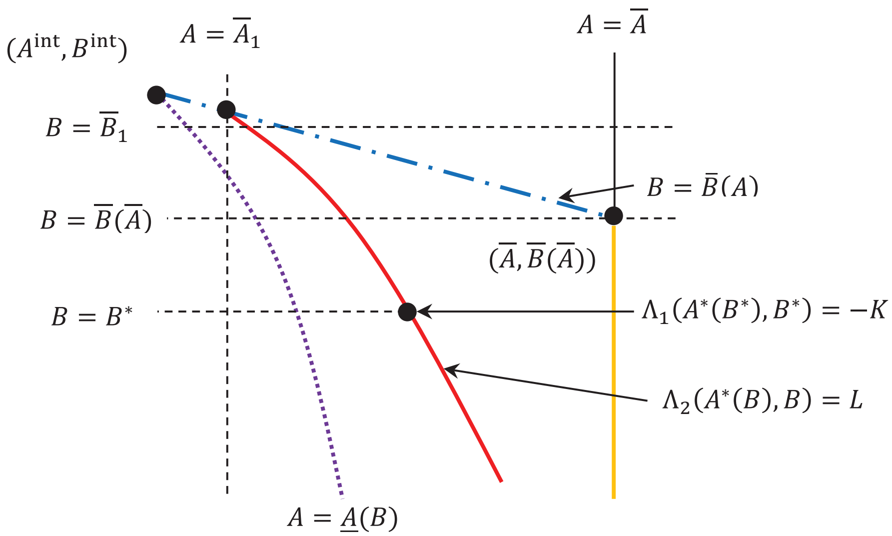

FIG 10. *For $B \in (-\infty, \overline{B}_1]$, there exists unique $A^*(B) \in [\overline{A}_1, \overline{A})$ such that $\Lambda_2(A^*(B), B) = L$. There exists a unique B^* with $B^* \in (-\infty, \overline{B}_1)$ that satisfies $\Lambda_1(A^*(B^*), B^*) = -K$.*

$$\int_{d^*}^{D^*} \left[g_{A^*(B^*),B^*}(x) + k\right] dx = -K,$$
$$\alpha^* d^* = 0, \quad \textit{and}$$
$$d^* \geq 0.$$

See Figure 10 for point $(A^*(B^*), B^*)$.

PROOF. For any $B \in (-\infty, \overline{B}_1)$, $(A^*(B), B) \in G_1$ and then

$$g_{A^*(B),B}(x_1(A^*(B), B)) < -k.$$

Therefore, there exists a unique $D(B) > x_1(A^*(B), B)$ such that

$$D(B) > 0, \quad g_{A^*(B),B}(D(B)) = -k, \quad g'_{A^*(B),B}(D(B)) > 0. \tag{7.43}$$

If

$$g_{A^*(B),B}(0) > -k, \tag{7.44}$$

there is a unique $d(B) < x_1(A^*(B), B)$ such that

$$d(B) > 0, \quad g_{A^*(B),B}(d(B)) = -k, \quad g'_{A^*(B),B}(d(B)) < 0. \tag{7.45}$$

To investigate when (7.44) holds, we study $\partial g_{A^*(B),B}(0)/\partial B$. For $B \in (-\infty, \overline{B}_1)$, from the expression of $g_{A,B}(x)$ in (5.21), we have

$$
\begin{aligned}
\frac{\partial g_{A^*(B),B}(0)}{\partial B} &= \frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2}\Big[\frac{1}{\lambda_1}\frac{dA^*(B)}{dB}e^{-\lambda_1 a} + \frac{1}{\lambda_2}e^{\lambda_2 a}\Big] \\
&= \frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2}\frac{1}{\lambda_2^2}\frac{1}{e^{\lambda_1(u(A^*(B),B)-a)} - e^{\lambda_1(U(A^*(B),B)-a)}} \\
&\quad \times\Big[\lambda_1\big(e^{-\lambda_2(u(A^*(B),B)-a)} - e^{-\lambda_2(U(A^*(B),B)-a)}\big)e^{-\lambda_1 a} \\
&\quad + \lambda_2\big(e^{\lambda_1(u(A^*(B),B)-a)} - e^{\lambda_1(U(A^*(B),B)-a)}\big)e^{\lambda_2 a}\Big] \\
&= \frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2}\frac{\lambda_1}{\lambda_2}\frac{1}{e^{\lambda_1(u(A^*(B),B)-a)} - e^{\lambda_1(U(A^*(B),B)-a)}} \\
&\quad \times\Big[e^{\lambda_1(y_2-a)}e^{\lambda_2 a} - e^{-\lambda_2(y_1-a)}e^{-\lambda_1 a}\Big] \\
&\quad \times\big(u(A^*(B),B) - U(A^*(B),B)\big) \\
&> 0,
\end{aligned}
$$

where the second equality follows from (7.42), the third equality follows from (5.110)-(5.111), and the inequality follows from the fact that $a < U(A^*(B),B) < y_1 < u(A^*(B),B)$ and $a < U(A^*(B),B) < y_2 < u(A^*(B),B)$. Therefore, $g_{A^*(B),B}(0)$ is strictly decreasing in $B \in (-\infty, \overline{B}_1)$. Let $(\overline{B}_2, \overline{B}_1)$ be the interval over which $g_{A^*(B),B}(0) > -k$. If there is no B that satisfies $g_{A^*(B),B}(0) > -k$, then set $\overline{B}_2 = \overline{B}_1$. Thus, for $B \in (\overline{B}_2, \overline{B}_1)$, (7.44) holds and $d(B) > 0$ satisfying (7.45) is well defined. For $B \in (-\infty, \overline{B}_2]$, (7.44) fails to hold and thus we set $d(B) = 0$. In this case, $g_{A^*(B),B}(d(B)) \le -k$ and

$$
\frac{\partial d(B)}{\partial B} = 0. \tag{7.46}
$$

The remainder of the proof mimics the proof of Lemma 5.7. Define

$$
\Lambda_1(A^*(B),B) = \int_{d(B)}^{D(B)} \big[g_{A^*(B),B}(x) + k\big]dx.
$$

First, we need to prove that $\Lambda_1(A^*(B),B)$ is continuous and strictly increasing in $B \in (-\infty, \overline{B}_1)$ and

$$
\lim_{B\downarrow -\infty} \Lambda_1(A^*(B),B) = -\infty \quad \text{and} \quad \lim_{B\uparrow \overline{B}_1} \Lambda_1(A^*(B),B) = 0.
$$

Therefore, there exists a unique $B^* \in (-\infty, \overline{B}_1)$ such that

$$
\Lambda_1(A^*(B^*),B^*) = -K,
$$

from which we prove the lemma by choosing $A^* = A^*(B^*)$, $d^* = d(B^*)$, $D^* = D(B^*)$, $x_1^* = x_1(A^*(B^*), B^*)$ and $\alpha^* = (k + g_{A^*(B^*),B^*}(0))^-$.

We need to show that $\Lambda_1(A^*(B), B)$ is continuous and strictly increasing in $B \in (-\infty, \overline{B}_1)$. It suffices to prove that (5.116) continues to hold for $B \in (-\infty, \overline{B}_1)$. Examining the proof of (5.116), we conclude that all of its equalities continue to hold where the second equality follows from

$$\frac{\partial d(A^*(B), B)}{\partial B}\big[g_{A^*(B),B}(d(A^*(B), B)) + k\big] = 0,$$

which holds because either (7.45) or (7.46) is true.

It is also easy to see that the limit (5.107) continues to hold as well. It remains to prove

$$\lim_{B\downarrow -\infty} \Lambda_1(A^*(B), B) = -\infty. \tag{7.47}$$

First, we need to prove that

$$\lim_{B\downarrow -\infty} \frac{\partial \Lambda_1(A^*(B), B)}{\partial B} > 0, \tag{7.48}$$

from which (7.47) immediately follows.

To study the limit (7.48), we only need to consider $\Lambda_1(A^*(B), B)$ for $B \in (-\infty, \overline{B}_2)$. When $B \in (-\infty, \overline{B}_2)$, $d(B) = 0$ and hence

$$\Lambda_1(A^*(B), B) = \int_0^{D(B)} \big[g_{A^*(B),B}(x) + k\big]\,dx.$$

For $B \in (-\infty, \overline{B}_2)$,

$$\begin{aligned}
&\frac{\partial \Lambda_1(A^*(B), B)}{\partial B} \\
&\quad= \int_0^{D(B)} \frac{\partial g_{A^*(B),B}(x)}{\partial B}\,dx + \frac{dD(B)}{dB}\big[g_{A^*(B),B}(D(B)) + k\big] \\
&\quad= \frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2}\int_0^{D(B)} \Big[\frac{1}{\lambda_1}\frac{dA^*(B)}{dB}e^{\lambda_1(x-a)} + \frac{1}{\lambda_2}e^{-\lambda_2(x-a)}\Big]\,dx \\
&\quad= \frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2}\int_0^{D(B)} \Big[\frac{1}{\lambda_2}e^{-\lambda_2(x-a)} \\
&\qquad + \frac{\lambda_1(e^{-\lambda_2(u(A^*(B),B)-a)} - e^{-\lambda_2(U(A^*(B),B)-a)})}{\lambda_2^2(e^{\lambda_1(u(A^*(B),B)-a)} - e^{\lambda_1(U(A^*(B),B)-a)})}e^{\lambda_1(x-a)}\Big]\,dx \\
&\quad= \frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2}
\end{aligned} \tag{7.49}$$

$$
\begin{aligned}
&\times \int_0^{D(B)} \Big[\frac{1}{\lambda_2} e^{-\lambda_2(x-a)} - \frac{1}{\lambda_2} e^{-\lambda_2(y_1-a)-\lambda_1(y_2-a)} e^{\lambda_1(x-a)}\Big] dx \\
&\ge \frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2}\int_0^{D(B)} \Big[\frac{1}{\lambda_2} e^{-\lambda_2(x-a)} - \frac{1}{\lambda_2} e^{-\lambda_2(y_1-a)}\Big] dx \\
&\ge \frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2}\frac{1}{\lambda_2}\int_0^{D(B)} \Big[e^{-\lambda_2(x-a)} - e^{-\lambda_2(D(B)-a)}\Big] dx,
\end{aligned}
$$

where the third equality follows from (7.42), the fourth equality is due to (5.110)-(5.111), the first inequality is due to $D(B) < U(A^*(B), B) < y_2$, and the second inequality is due to $D(B) < U(A^*(B), B) < y_1$.

Similar to proving (5.115), for $B \in (-\infty, \overline{B}_1)$, we can prove

$$\frac{dD(B)}{dB} < 0,$$

which implies that for $B \in (-\infty, \overline{B}_1)$,

$$
\begin{aligned}
&\frac{d}{dB}\int_0^{D(B)} \Big[e^{-\lambda_2(x-a)} - e^{-\lambda_2(D(B)-a)}\Big] dx \\
&\quad = \lambda_2 \int_0^{D(B)} e^{-\lambda_2(D(B)-a)} \frac{dD(B)}{dB} dx \\
&\quad < 0.
\end{aligned} \tag{7.50}
$$

Therefore, (7.49) implies

$$
\begin{aligned}
&\lim_{B\downarrow-\infty} \frac{\partial \Lambda_1(A^*(B), B)}{\partial B} \\
&\quad \ge \lim_{B\downarrow-\infty} \frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2}\frac{1}{\lambda_2}\int_0^{D(B)} \Big[e^{-\lambda_2(x-a)} - e^{-\lambda_2(D(B)-a)}\Big] dx \\
&\quad \ge \frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2}\frac{1}{\lambda_2}\int_0^{D(\overline{B}_3)} \Big[e^{-\lambda_2(x-a)} - e^{-\lambda_2(D(\overline{B}_3)-a)}\Big] dx \\
&\quad > 0,
\end{aligned}
$$

where $\overline{B}_3$ is a fixed number satisfying $\overline{B}_3 \in (-\infty, \overline{B}_2)$ and the second inequality is due to (7.50).

Therefore, we have proved $\lim_{B\downarrow-\infty} \partial\Lambda_1(A^*(B), B)/\partial B > 0$, which completes the proof of the lemma. □

The following lemma proves (7.18).

Lemma 7.7. *For $(A, B) \in G_1$, we have*

$$h'(x_1(A,B)) \le -\beta k \quad \textit{if } x_1(A,B) > 0 \quad \textit{and} \quad h'(x_2(A,B)) \ge \beta \ell.$$

PROOF. The proof is identical to the proof of Lemma 5.8. □

PROOF OF THEOREM 7.2. We have defined B^*, d^*, D^*, x_1^* and α^* in Lemma 7.6. As mentioned, we have defined $U(A,B)$ and $u(A,B)$ in this section as same as those in (5.87), (5.89) and (5.91). Recall the definition of $x_2(A,B)$ in Lemma 7.2. We set $U^* = U(A^*,B^*)$, $u^* = u(A^*,B^*)$ and $x_2^* = x_2(A^*,B^*)$. Next, we check that $g_{A^*,B^*}(x)$, together with d^*, D^*, U^*, u^*, x_1^*, x_2^* and α^*, satisfies (7.11)-(7.18) and the monotonicity properties of g in Theorem 7.2.

First, (5.87) implies the second part of (7.11), and (5.89) implies (7.15). Second, Lemma 7.5 implies (7.13). Furthermore, Lemma 7.6 implies the first part of (7.11), (7.12), (7.14) and (7.16)-(7.17). Finally, Lemma 7.2 implies the monotonicity properties of $g_{A^*,B^*}(x)$ and Lemma 7.7 implies (7.18). □

Acknowledgments. The authors thank the associate editor and two referees for improving the exposition of the paper. The authors also thank Hanqin Zhang of Chinese Academy of Sciences and National University of Singapore for stimulating discussions. Part of work was done when the second author visited School of Industrial and Systems Engineering, Georgia Institute of Technology. He thanks the school for its hospitality.

REFERENCES

[1] BACCARIN, S. (2002). Optimal impulse control for cash management with quadratic holding-penalty costs. *Decisions in Economics and Finance* **25** 19–32. MR1961219

[2] BENSOUSSAN, A. and LIONS, J.-L. (1974/75). Nouvelles méthodes en contrôle impulsionnel. *Applied Mathematics and Optimization* **1** 289–312. MR0390886

[3] BENSOUSSAN, A. and LIONS, J.-L. (1984). *Impulse control and quasivariational inequalities*. Gauthier-Villars, Montrouge. Translated from the French by J. M. Cole. MR0756234

[4] BERTOLA, G. and CABALLERO, R. J. (1990). *Kinked Adjustment Costs and Aggregate Dynamics*. MIT Press, 237–296.

[5] CONSTANTINIDES, G. M. and RICHARD, S. (1978). Existence of optimal simple policies for discounted-cost inventory and cash management in continous time. *Operations Research* **26** 620–636. MR0496558

[6] DAI, J. G. and YAO, D. (2013). Brownian inventory models with convex holding cost, part 1: average-optimal controls. *Stochastic Systems* **3**(2) 442–499.

[7] FENG, H. and MUTHURAMAN, K. (2010). A computational method for stochastic impulse control problems. *Mathematics of Operations Research* **35** 830–850. MR2777517

[8] HARRISON, J. M., SELLKE, T. M., and TAKSAR, M. I. (1983). Impulse control of Brownian motion. *Mathematics of Operations Research* **8** 454–466. MR0716124

[9] HARRISON, J. M. and TAKSAR, M. I. (1983). Instantaneous control of Brownian motion. *Mathematics of Operations Research* **8** 439–453. MR0716123

[10] PLEHN-DUJOWICH, J. M. (2005). The optimality of a control band policy. *Review of Economic Dyanmics* **8** 877–901.

[11] Protter, P. E. (2005). *Stochastic Integration and Differential Equations*. Stochastic Modelling and Applied Probability, Vol. **21**. Springer-Verlag, Berlin. Second edition. Version 2.1, Corrected third printing. MR2273672

[12] Richard, S. F. (1977). Optimal impulse control of a diffusion process with both fixed and proportional costs of control. *SIAM Journal on Control and Optimization* **15** 79–91. MR0490377

[13] Taksar, M. (1997). Infinite dimensional linear programming approach to singular stochastic control problems. *SIAM Journal on Control and Optimization* **35** 604–625. MR1436641

School of Operations Research
and Information Engineering
Cornell University
136 Roy Road
Ithaca, New York, 14853, U.S.A.
E-mail: jim.dai@cornell.edu

Institute of Applied Mathematics
Academy of Mathematics and Systems Science
Chinese Academy of Sciences
No. 55, Zhongguancun East Road
Haidian District, Beijing, 100190, China
E-mail: dachengyao@amss.ac.cn