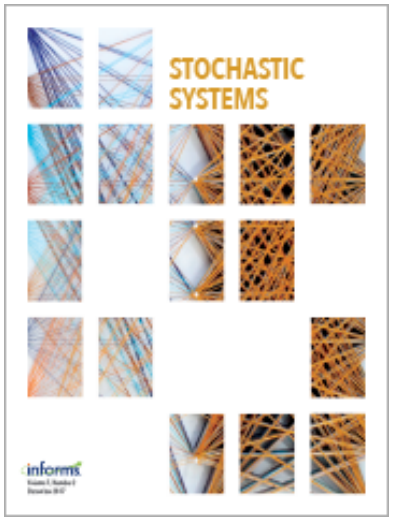
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WIENER-HOPF FACTORIZATIONS FOR A MULTIDIMENSIONAL MARKOV ADDITIVE PROCESS AND THEIR APPLICATIONS TO REFLECTED PROCESSES

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We extend the framework of Neuts' matrix analytic approach to a reflected process generated by a discrete time multidimensional Markov additive process. This Markov additive process has a general background state space and a real vector valued additive component, and generates a multidimensional reflected process. Our major interest is to derive a closed form formula for the stationary distribution of this reflected process. To this end, we introduce a real valued level, and derive new versions of the Wiener-Hopf factorization for the Markov additive process with the multidimensional additive component. In particular, it is represented by moment generating functions, and we consider the domain for it to be valid.

Our framework is general enough to include multi-server queues and/or queueing networks as well as non-linear time series which are currently popular in financial and actuarial mathematics. Our results yield structural results for such models. As an illustration, we apply our results to extend existing results on the tail behavior of reflected processes.

A major theme of this work is to connect recent work on matrix analytic methods to classical probabilistic studies on Markov additive processes. Indeed, using purely probabilistic methods such as censoring, duality, level crossing and time-reversal (which are known in the matrix analytic methods community but date back to Arjas & Speed [2] and Pitman [29]), we extend and unify existing results in both areas.

1. Introduction. The complexity of applied probability models has significantly increased over the last decades. Financial and actuarial mathematicians use time series which can be non-linear, and models in stochastic networks are typically required to consider multidimensional distributions. Thus, there is a need for general mathematical structures that allow for

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enough model flexibility and at the same time possesses enough structure to allow for a tractable performance analysis.

One success story in applied probability is the use of phase-type distributions and, more generally, matrix analytic methods. These objects enable a tractable algorithmic analysis of a significant class of stochastic models. The idea is to define a two-dimensional Markov process. The evolution in one dimension (called the level) is typically additive (up to some boundary reflection conditions), and other behavior is modeled by the other component, called the background process, or phase. From a probabilistic point of view, this is a special case of a reflected Markov additive process, but this connection has never been exploited in a systematic manner. On the other hand, the literature on Markov additive processes has not been driven by applications in stochastic networks, and the present state of the art calls for another look at such processes.

This brings us to the main aim of this paper: Our primary interest is in a reflected process driven by a discrete time Markov additive process. The latter process is referred to as MAP in short, which is also called a Markov random walk in the literature. It is assumed that the additive component of MAP is $\mathbb{R}^d$-valued, where d is a positive integer, and its background states take values in a locally compact, separable metric space. We define a mapping, which assigns a real value to the additive component, which is called a level. Reflection takes place when the level hits value 0. There are many applications of such reflected processes, particularly, in stochastic networks and queueing theory. For example, they can describe multi-server queues, parallel queues, fluid flow models and general queueing networks. Ideally, one would like to derive an expression for the stationary distribution which enables performance analysis of such systems.

We aim to derive a certain closed form formula for the stationary distribution of a reflected MAP, using the boundary transitions and hitting probabilities of the additive component. This allows us (i) to extend Neuts' matrix analytic framework and the existing theory on Markov additive processes in a unified way, and (ii) to study problems like obtaining the asymptotic tail behavior of the stationary distribution. Throughout the paper, we assume the existence of the stationary distribution of the reflected MAP. Of course, this existence is also an important issue, but it can usually be separately verified in specific applications. We give some sufficient conditions for existence in this paper.

The key ideas to derive the closed form formula are to use certain factorizations, called Wiener-Hopf factorizations for the MAP due to Arjas and Speed [2], and the representation of stationary distribution by occupation

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measures due to Pitman [29]. We combine these ideas with the exponential change of measure with respect to the additive component, which enables us to consider the asymptotic tail behavior through the extended factorizations. For this change of measure, we have to deal with operator moment generating functions, which are extension of matrix moment generating functions, and to consider their convergence parameters.

These observations allow us to extend Neuts' matrix analytic approach to a multidimensional additive components and to a general background state space. For this, we work with nonnegative kernels, which are operators on function spaces, rather than matrices. There have been few studies in this direction (e.g., see [32]). Another new feature is the multi-dimensionality of the additive component, never studied under the framework of the Neuts' matrix analytic approach. One may think that such a multi-dimensionality can be reduced to the one dimensional case by incorporating auxiliary information into the background state. This is possible, but it complicates the background process, which may make analysis intractable (see, e.g., [24]). We keep the multidimensional additive component as long as possible. Of course, this may cause another difficulty given the lack of total ordering of higher-dimensional state spaces. We deal with this by the level, which is different from the additive component in the multidimensional case while the additive component can be a level in the one dimensional case. Thus, the level is different from that of Neuts' approach. However, the exponential change of measure still works well.

In this paper, we only consider tail asymptotics of the stationary distributions of the reflecting MAP's for application. For the one-dimensional case, we extend existing results of Miyazawa and Zhao [24] from the case of countably many background states to real-valued background states. This allows us to consider precise asymptotics of a system with autoregressive input. For the multi-dimensional case, we give some preliminary results which, together with the other results in this paper, are further developed in [16]. We have not considered other applications, but one may think of them. For example, the resulting expressions for the steady-state distribution may be useful to derive (corrected)-diffusion approximations, or to derive error bounds associated to truncation of the background state space. A full development of such results will be undertaken elsewhere.

This paper is organized as follows. In Section 2, we first consider the censored process of a discrete time Markov process generated by a stopping time, and derive a decomposition identity for the occupation measure. In Section 3, we specialize this decomposition for a Markov additive process with a multidimensional additive component, and derive a version of the

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Wiener-Hopf factorization. As a corollary, we also derive the conventional Wiener-Hopf factorization. In Section 4, we introduce the reflected MAP, and derive some closed form expressions for its stationary distribution. These results are applied to study asymptotic tail behavior of the stationary distribution in Section 5. Concluding remarks are given in Section 6.

2. Censoring a discrete time Markov process. The purpose of this section is to derive a number of structural results, which will be specialized later on to Markov additive processes. Let $\{Z_n\} = \{Z_n; n = 0, 1, \ldots\}$ be a discrete-time Markov process with respect to filtration $\{\mathcal{F}_n\}$. Denote its state space by S. We assume that S is a complete separable metric space with the Borel σ-field $\mathcal{B}(S)$. A subset of S is said to be bounded if it is contained in some ball with a finite radius. Since S is separable, $\mathcal{B}(S)$ can be generated by a countable set of open balls with bounded radii, and a measure on $(S, \mathcal{B}(S))$ is uniquely determined by its restriction to the set of those balls or of all bounded sets in $\mathcal{B}(S)$ (e.g., see [4]). In our applications, S is typically a finite dimensional Euclidean space, so the reader may consider S as such a space. However, this restriction does not gain any profit in this section, so we just work with the general S.

We mainly work with probability transition kernels on $(S, \mathcal{B}(S))$, and consider random variables and stochastic processes on a probability space. However, many of our results are easily extended to nonnegative kernels on $(S, \mathcal{B}(S))$ and measurable functions. Thus, we first introduce them following the notions of [28].

Let $M_+(S)$, $M_{0+}(S)$, $M_b(S)$ and $M_b^0(S)$ be the set of all measurable functions from $(S, \mathcal{B}(S))$ to $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ which are positive, nonnegative, bounded, and bounded with bounded support, respectively. Let U be a function from $S \times \mathcal{B}(S)$ to $\overline{\mathbb{R}}_+ \equiv [0, +\infty]$ such that $(S, \mathcal{B}(S))$ if $U(\cdot, A) \in M_{0+}(S)$ for each $A \in \mathcal{B}(S)$ and $U(x, \cdot)$ is a measure on $(S, \mathcal{B}(S))$ for each $x \in S$. Then, U is said to be a nonnegative kernel on $S \times \mathcal{B}(S)$. For this nonnegative kernel U,

$$Uf(x) \equiv \int_S U(x, dx')f(x'), \qquad f \in M_{0+}(S), x \in S$$

defines a linear nonnegative operator on $M_+(S)$, where $Uf(x)$ may be infinite. U is said to be σ-finite if there is a $f \in M_+(S)$ such that $Uf(x) < \infty$ for all $x \in S$, finite if $U(x, S) < \infty$ for all $x \in S$, and bounded if $\sup_{x \in S} U(x, S) < \infty$. In particular, a bounded nonnegative kernel U is said to be sub-stochastic if $\sup_{x \in S} U(x, S) \leq 1$ and stochastic if $\sup_{x \in S} U(x, S) = 1$. For two nonnegative kernels U and V, their product UV is defined as

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$(UV)f(x) = U(Vf)(x)$, i.e.

$$(UV)f(x) = \int_{S\times S} U(x, dx')V(x', dx'')f(x''), \qquad f \in M_{0+}(S).$$

Similarly to Uf, we define an operation of U to a measure ν on $(S, \mathcal{B}(S))$ as

$$\nu U(B) = \int_S \nu(dx)U(x, B), \qquad B \in \mathcal{B}(S).$$

Then, U is an operator on the set of all nonnegative measures on $(S, \mathcal{B}(S))$.

We now consider the discrete time Markov chain Z_n with state space $(S.\mathcal{B}(S))$. We assume that it is time homogeneous and non-defective, and let

$$Q(x, A) = P(Z_{n+1} \in A | Z_n = x), \qquad x \in S, A \in \mathcal{B}(S).$$

Then, $Q \equiv \{Q(x, A); x \in S, A \in \mathcal{B}(S)\}$ is a stochastic kernel.

Let τ be a stopping time, that is, τ is a nonnegative integer-valued random variable such that $\{\tau \le n\} \in \mathcal{F}_n$ for all $n \ge 0$, where τ can be infinite. We consider censoring the Markov process $\{Z_n\}$ by a sequence of stopping times constructed from this stopping time. To this end, we introduce the following two sets of nonnegative kernels.

DEFINITION 2.1. Define nonnegative kernels $\underline{G}(s)$ and $\underline{H}(s)$ on $(S, \mathcal{B}(S))$ for each nonnegative number $s \in (0, 1)$ as

$$\underline{G}(s)f(x) = E_x(s^\tau f(Z_\tau); \tau < \infty), \qquad x \in S, f \in M_b(S),$$
$$\underline{H}(s)f(x) = E_x\left(\sum_{n=0}^{\tau-1} s^n f(Z_n); \tau < \infty\right), \qquad x \in S, f \in M_b(S),$$

and denote the corresponding kernels on $(S, \mathcal{B}(S))$ for $s = 1$ by $G = \underline{G}(1)$ and $H = \underline{H}(1)$, where E_x represents the conditional expectation given $Z_0 = x$.

Throughout the paper, we shall use underline "_" to indicate generating functions with respect to time index. For a stopping time τ, let τ_n be the n-th stopping time when the chain restarts at time τ_{n-1} with state $Z_{\tau_{n-1}}$, where $\tau_n - \tau_{n-1}$ is stochastically identical to τ under the conditional distribution given $Z_{\tau_{n-1}} = Z_0$.

The resulting embedded Markov chain Z_{τ_n} has transition probability kernel G, which may be defective. Motivated by the fact that τ is often a hitting time, we call this embedded Markov chain a censored process. On the other

hand, $H1_A$ represents the mean number of visiting $A \in \mathcal{B}(S)$ between embedded points (i.e. during the censoring). This censoring process is more general than the conventional one in which τ is a hitting time at a given set $C \in \mathcal{B}(S)$. In our applications, this level of generality is necessary.

We now give the following identity due to Pitman [29] (see Proposition 3.2 there). Since the expression is different from that in [29], we give its proof in Appendix A.

LEMMA 2.1. *For $s \in [0,1)$, we have*

$$\underline{H}(s) + \underline{G}(s) = I + s\underline{H}(s)Q, \tag{2.1}$$

or, equivalently,

$$\underline{H}(s)(I - sQ) = I - \underline{G}(s). \tag{2.2}$$

where I is an identity operator. Furthermore, (2.1) is valid also for $s = 1$, but both sides may be infinite.

We note that this lemma is useful to construct a stationary measure of Q.

COROLLARY 2.1. *Suppose the transition kernel G has a stationary measure ν_0, i.e.,*

$$\nu_0(A) = \nu_0 G 1_A, \qquad A \in \mathcal{B}(S), \tag{2.3}$$

and define a measure ν on $\mathcal{B}(S)$ as

$$\nu(A) = \nu_0 H 1_A, \qquad A \in \mathcal{B}(S).$$

Then, ν is a stationary measure for Q if $\nu(A) < \infty$ for all bounded $A \in \mathcal{B}(S)$.

PROOF. Suppose that $A \in \mathcal{B}(S)$ is bounded. Note that $G = \underline{G}(1)$ and $H = \underline{H}(1)$. Then, from $\nu(A) < \infty$ and (2.3), (2.2) implies $\nu_0 HQ1_A < \infty$. Hence, (2.2) with $s = 1$ can be evaluated with 1_A. Thus, it follows from (2.2) and (2.3) that

$$\nu_0 H(1 - Q)1_A = \nu_0 1_A - \nu_0 G 1_A = 0.$$

This immediately implies

$$\nu(A) = \nu_0 H 1_A = \nu_0 HQ 1_A = \nu Q(A),$$

which completes the proof. □

Corollary 2.1 may be interpreted as a regenerative construction of the stationary measure, where τ is the regeneration cycle. In applications of Corollary 2.1, we need to identify G and H. These are generally hard problems, and we need more structure on Q to go further.

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3. Markov additive process and the Wiener-Hopf factorization. Motivated by applications in stochastic networks, we are particularly interested in stochastic processes taking values in a nonnegative quadrant. These processes often have simple additive structure within the quadrant and are reflected at the boundary of the quadrant, where the reflection may complicate state transitions. In view of Corollary 2.1, if Q is the transition kernel of such a reflected process, we can study the stationary distribution ν of Q through H in which τ is chosen as a hitting time of the boundary. However, this direct approach is generally hard to be evaluated. So, before going to the reflected processes, we focus on a tractable additive structure inside the boundary.

3.1. *Definitions and basic properties.* Let $\{X_n\}$ be a discrete time Markov process with state space S_X, which is a complete and separable metric space. Let S_Y be either the d-dimensional Euclidean vector space $\mathbb{R}^d$ or the d-dimensional discrete space $\mathbb{Z}^d$, where d is a positive integer, and $\mathbb{Z}$ is the set of all integers. We use the standard Euclidean norm $\|\boldsymbol{u}\| \equiv \sqrt{u_1^2 + \cdots + u_d^2}$, and the inner product $\langle \boldsymbol{u}, \boldsymbol{v} \rangle \equiv \sum_{i=1}^d u_i v_i$. Let $\mathcal{B}(S_Y)$ be the Borel σ-field on S_Y. Note that we use the discrete topology for $\mathbb{Z}^d$. Many of our results can be extended to S_Y being a complete separable metric space with an additive operation which is closed in S_Y. However, $S_Y = \mathbb{R}^d$ or $\mathbb{Z}^d$ is sufficient for most applications. In what follows, we assume that $S_Y = \mathbb{R}^d$ unless stated otherwise for simplicity.

Let $\{Y_n\}$ be an S_Y-valued additive process such that the increment $\Delta Y_n \equiv Y_{n+1} - Y_n$ only depends on X_n and X_{n+1} with respect to the filtration $\sigma((X_{n'}, Y_{n'}); n' \le n)$. Namely, we assume that

$$\begin{aligned} &P(X_{n+1} \in A, \Delta Y_n \in B | X_{n'-1}, Y_{n'}, n' \le n, X_n = x) \\ &\quad = P(X_{n+1} \in A, \Delta Y_n \in B | X_n = x), \quad A \in \mathcal{B}(S_X), B \in \mathcal{B}(S_Y), x \in S_X. \end{aligned}$$

These transition probabilities are also assumed to be independent of n. Denote the right-hand side by $K(x, A \times B)$. Thus, we are concerned with the joint process $\{(X_n, Y_n)\}$, which has state space $S \equiv S_X \times S_Y$. This process is called a discrete-time Markov additive process, while $K \equiv \{K(x, A \times B)\}$ is called a Markov additive kernel. We simply refer to $\{(X_n, Y_n)\}$ as a Markov additive process while X_n and Y_n are referred to as a background state and an additive component, respectively. See [2] for more details.

Since $\{(X_n, Y_n)\}$ is also viewed as a Markov process with state space S, we can consider its transition kernel. Denote this kernel by Q, which is defined

as an operator on $M_b(S)$:

$$Qg(x,\boldsymbol{u}) = \int_S K(x, dy \times d\boldsymbol{v}) g(y, \boldsymbol{u}+\boldsymbol{v}), \qquad g \in M_b(S), (x,\boldsymbol{u}) \in S.$$

We shall use a time reversed version of this Markov process. However, we need to be careful since Q may not have a stationary measure. In what follows, we analytically construct such a process.

Assume that $K(x, S_X \times S_Y) = 1$, i.e. for every x, $\{(X_n, Y_n)\}$ is nonterminating. We always assume the following irreducibility.

(3a) The kernel $K_X \equiv \{K(x, A \times S_Y)\}$ of the background process is irreducible, that is, there exists a σ-finite measure ψ on $(S_X, \mathcal{B}(S_X))$ such that if $\psi(A) > 0$, then there is a positive integer n_0 for each $x \in S_X$ such that $K_X^{n_0}(x, A) > 0$.

REMARK 3.1. This irreducibility is referred to as ψ-irreducibility in the literature. Since such irreducibility implies the existence of a maximal irreducible measure, we can assume without loss of generality that ψ is maximal (see Section 2 of [28]).

We always assume (3a), but this is not a mandatory request. Some of our results may not need this assumption. By Theorem 2.1 of [28], (3a) implies that there exist a positive integer n_0, a nonnegative function $g_s(x)$ on S_X and a measure λ on $(S_X, \mathcal{B}(S_X))$ such that $\int g_s(x)\psi(dx) > 0$ for the irreducible measure ψ and

$$g_s(x)\lambda(A) \le K_X^{n_0}(x, A), \qquad x \in S, A \in \mathcal{B}(S_X). \tag{3.1}$$

g_s is called a small function, and K_X is said to satisfy the minorization condition. In the literature, the following stronger condition is often used (see, e.g. [26, 27]), but we only need it for some special cases in Section 5.

(3b) For the Markov additive kernel K satisfying (3a), there are a positive integer n_0, a family of measures $\{h(x,\cdot); x \in S_X\}$ on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ and a measure λ on $(S_X, \mathcal{B}(S_X))$ such that

$$h(x,B)\lambda(A) \le K^{n_0}(x, A \times B), \qquad x \in S, A \in \mathcal{B}(S_X), B \in \mathcal{B}(\mathbb{R}^d). \tag{3.2}$$

In this case, K is said to satisfy the minorization condition.

REMARK 3.2. If S_X is countable, then (3b) is automatically satisfied under assumption (3a). For the general case, useful sufficient conditions are given in Proposition 3.1 of [26].

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Under the irreducibility assumption (3a), K_X has a sub-invariant measure, which is denoted by π (e.g., see Section 5.2 of [28]). Namely, there exists a measure π on S_X such that

$$\int_{S_X} \pi(dx)K(x, A \times S_Y) \le \pi(A), \qquad A \in \mathcal{B}(S_X). \tag{3.3}$$

Since $\sum_{n=0}^{\infty} 2^{-n-1}\pi K^n$ can be a maximal irreducible measure (see (2.1) on page 10 and Proposition 2.4 of [28]), we choose π itself as a maximal irreducible measure. Thus, (3.3) is equivalent to (3a). In Section 3.3, we use stronger condition (3b), but our main results do not need this condition.

It follows from (3.3) that $K_X(x, \cdot) \ll \pi$ for *a.s.* x with respect to π, where $\mu \ll \nu$ stands for μ to be absolutely continuous with respect to ν for measures μ, ν on $(S_X, \mathcal{B}(S_X))$, that is, $\mu(A) > 0$ implies $\nu(A) > 0$ for all $A \in \mathcal{B}(S_X)$. Hence, by the Radon-Nikodym theorem, there exists a Markov additive kernel $\tilde{K}$ such that

$$\int_A \pi(dx)K(x, A' \times B) = \int_{A'} \pi(dx')\tilde{K}(x', A \times B), \qquad A, A' \in \mathcal{B}(S_X), B \in \mathcal{B}(S_Y). \tag{3.4}$$

Since π is also the sub-invariant measure for $\tilde{K}(\cdot, \cdot \times S_Y)$, $\tilde{K}$ is a substochastic kernel, i.e., $\tilde{K}(x, S_X \times S_Y) \le 1$. This is formally verified by substituting $A = S_X$ and $B = S_Y$ in the above formula, which yields

$$\int_{A'} \pi(dx')\tilde{K}(x', S_X \times S_Y) = \int_{S_X} \pi(dx)K(x, A' \times S_Y) \le \pi(A'), \qquad A' \in \mathcal{B}(S_X).$$

In particular, if π is an invariant measure, then $\tilde{K}$ is also stochastic. Similarly, we can see that π is the invariant measure of $\tilde{K}_X \equiv \{\tilde{K}(x, A \times S_Y)\}$. The additive kernel $\tilde{K}$ may depend on the choice of π. From these observations, $\widetilde{(\tilde{K})}$ is identical to K if we choose the same π for $\tilde{K}_X$. Based on these facts, we introduce the following terminology.

Definition 3.1. Let $\{(\tilde{X}_n, \tilde{Y}_n)\}$ be the Markov additive process generated by $\tilde{K}$ with respect to the sub-invariant measure π of K_X. Then $\{(\tilde{X}_n, \tilde{Y}_n)\}$ is said to be a dual Markov additive process of $\{(X_n, Y_n)\}$ with respect to π, dual MAP for short.

Remark 3.3. The minimal sub-invariant measure is known to be proportional to an occupation measure, i.e., the expected sojourn time before

returning to an appropriately chosen base set, so the sub-invariant measure π can not be finite if K is transient, while any sub-invariant measure must be invariant if K is recurrent (see [28] for details).

Define the transition kernel $\tilde{Q}$ for the Markov additive kernel $\tilde{K}$ as

$$\tilde{Q}g(x,\boldsymbol{u}) = \int_S \tilde{K}(x, dy \times d\boldsymbol{v}) g(y, \boldsymbol{u}+\boldsymbol{v}), \qquad g \in M_b(S), (x,\boldsymbol{u}) \in S.$$

Then, $\tilde{Q}$ is the transition kernel of the dual MAP $\{(\tilde{X}_n, \tilde{Y}_n)\}$.

We now give a sample path construction of the dual MAP with $\tilde{Q}$ from the original MAP $\{(X_n, Y_n)\}$. The following fact is obvious, but a key for this construction. For convenience of the reader, its proof is given in Appendix B.

LEMMA 3.1. *Let m be the Lebesgue measure on S_Y $(= \mathbb{R}^d)$, and π be the sub-invariant measure of K_X. Define the product measure of π and m as*

$$\pi \otimes m(A \times B) = \pi(A)m(B), \qquad A \in \mathcal{B}(S_X), B \in \mathcal{B}(S_Y)$$

Then $\pi \otimes m$ is the sub-invariant measure of Q, which is invariant if and only if π is invariant.

REMARK 3.4. If the additive component Y_n only takes values on a lattice, i.e., the set of all $(i_1, \ldots, i_d)a$ for all integers i_j and some number $a > 0$, then this lemma is valid for the uniform measure on the lattice instead of m. Throughout the paper, we use m in this sense.

The following fact shows how a sample path of the dual MAP is obtained from that of the original MAP under measure $\pi \otimes m$.

LEMMA 3.2. *For $A, A' \in \mathcal{B}(S_X)$ and $B, B' \in \mathcal{B}(S_Y)$, we have*

$$\begin{aligned} (3.5) \qquad & \int_{A\times B} \pi(dx)m(d\boldsymbol{u})Q((x,\boldsymbol{u}), A' \times B') \\ & \qquad = \int_{A' \times (-B')} \pi(dx)m(d\boldsymbol{u})\tilde{Q}((x,\boldsymbol{u}), A \times (-B)). \end{aligned}$$

Hence, we have, for $h \in M_+(S^{n+1})$,

$$\begin{aligned} (3.6) \qquad & E_{\pi\otimes m}\left(h(X_0, Y_0, X_1, Y_1, \ldots, X_n, Y_n)\right) \\ & \quad = E_{\pi\otimes m}\left(h(\tilde{X}_n, -\tilde{Y}_n, \tilde{X}_{n-1}, -\tilde{Y}_{n-1}, \ldots, \tilde{X}_0, -\tilde{Y}_0)\right), \qquad n \geq 1, \end{aligned}$$

where E_μ represents the expectation under measure μ.

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REMARK 3.5. From (3.5), $Q((x,\boldsymbol{u}),\cdot) \ll \pi \otimes m$ and $\tilde{Q}((x,\boldsymbol{u}),\cdot) \ll \pi \otimes m$ for *a.s.* $(x,\boldsymbol{u})$ with respect to $\pi \otimes m$.

The proof of Lemma 3.2 is deferred to Appendix C because it is just mechanical. From (3.6), we may set $\tilde{Y}_n = -Y_{-n}$ for all n. This is the sample path definition of the dual MAP. From this, one can see that $\tilde{Y}_n$ increases in n only if Y_n increases. So, the drift is unchanged under the dual operation. In view of $\tilde{K}$ and $\tilde{Q}$, it will be convenient to introduce the following transpositions of kernels.

DEFINITION 3.2. Let U_X be a nonnegative kernel on $(S_X, \mathcal{B}(S_X))$. Let ζ_X be a measure on $(S_X, \mathcal{B}(S_X))$. If there exists a nonnegative kernel $U_X^{\mathrm{T}(\zeta_X)}$ such that

$$\int_{S_X} f_1(x)(U_X f_2)(x)\zeta_X(dx) = \int_{S_X} f_2(x)(U_X^{\mathrm{T}(\zeta_X)} f_1)(x)\zeta_X(dx), \qquad f_1, f_2, \in M_b^0(S_X),$$

then $U_X^{\mathrm{T}(\zeta_X)}$ is said to be a transposition of U_X with respect to ζ_X. Similarly with,

$$g_i^-(x,\boldsymbol{u}) = g_i(x,-\boldsymbol{u}), \qquad i = 1,2. \tag{3.7}$$

the transposition $U^{\dagger(\zeta)}$ of the nonnegative kernel U on $(S, \mathcal{B}(S))$ is defined for a measure ζ on $(S, \mathcal{B}(S))$ by

$$\int_S g_1(x,\boldsymbol{u})(Ug_2)(x,\boldsymbol{u})\zeta(dx \times d\boldsymbol{u}) = \int_S g_2^-(x,\boldsymbol{u})(U^{\dagger(\zeta)} g_1^-)(x,\boldsymbol{u})\zeta(dx \times d\boldsymbol{u}), \qquad g_1, g_2, \in M_b^0(S).$$

We also define these transpositions for measures. For example, let ν be a σ-finite measure on $(S, \mathcal{B}(S))$, if there exists a measurable function $\nu^{\dagger(\zeta)}$ on S satisfying

$$\int_S g(x,\boldsymbol{u})\nu(dx \times d\boldsymbol{u}) = \int_S g(x,-\boldsymbol{u})\nu^{\dagger(\zeta)}(x,\boldsymbol{u})\zeta(dx \times d\boldsymbol{u}), \qquad g \in M_b^0(S).$$

$\nu^\dagger$ is said to be a transpose of ν. Obviously, $\nu \ll \zeta$, and $\nu^{\dagger(\zeta)}$ is a Radon-Nikodym derivative of ν with respect to ζ.

It is easy to see that $(U_X V_X)^{\mathrm{T}(\zeta_X)} = V_X^{\mathrm{T}(\zeta_X)} U_X^{\mathrm{T}(\zeta_X)}$ and $(UV)^{\dagger(\zeta)} = V^{\dagger(\zeta)} U^{\dagger(\zeta)}$ for kernels T_X, U_X and T, U on $(S_X, \mathcal{B}(S_X))$ and $(S, \mathcal{B}(S))$, respectively. It should be noted that transpose $\dagger$ involves sign changes concerning the additive component through g_i^- while transpose T does not include the additive component.

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3.2. *Decomposition formula.* We are interested in developing fluctuation identities for the MAP, as the additive component of the MAP reaches a certain level. This concept is clear in one dimension, but in general we need to specify what we mean by the level. Therefore, we formally introduce the concept of a level for the MAP.

DEFINITION 3.3. Let ℓ be a real-valued, measurable function on $S_Y(\equiv \mathbb{R}^d)$ such that $\ell(\mathbf{0}) = 0$. Then, ℓ is called a level function, and $J_n \equiv \ell(Y_n)$ is said to be a level for Z_n. The level process $\{J_n\}$ is called additive if ℓ is additive, that is,

$$\ell(\boldsymbol{v} + \boldsymbol{u}) = \ell(\boldsymbol{u}) + \ell(\boldsymbol{v}), \qquad \boldsymbol{u}, \boldsymbol{v} \in S_Y.$$

Let $\boldsymbol{c} = (c_1, \ldots, c_d)$ be the vector in $\mathbb{R}^d$ with $\|\boldsymbol{c}\| = 1$, and refer it as a direction vector. Then, $\{J_n\}$ is called partially additive in direction $\boldsymbol{c}$ if

$$\ell(\boldsymbol{u} + t\boldsymbol{c}) = \ell(\boldsymbol{u}) + t\ell(\boldsymbol{c}), \qquad \boldsymbol{u} \in S_Y, t \in \mathbb{R}.$$

For convenience, we always put $J_n = Y_n$ when $d = 1$. That is, the level is identical to the additive component for $d = 1$.

For our applications in Section 4, it is convenient to think of the level set, $\{(x, \boldsymbol{u}) \in S; \ell(\boldsymbol{u}) = t\}$, for each $t \in \mathbb{R}$. We here introduce $S_t^+ \equiv \{(x, \boldsymbol{u}) \in S; \ell(\boldsymbol{u}) > t\}$.

EXAMPLE 3.1. Let $\ell(\boldsymbol{u}) = u_1$ for $\boldsymbol{u} = (u_1, \ldots, u_d)$. Then J_n is an additive level. Another additive level is given by $\ell(\boldsymbol{u}) = \sum_{i=1}^d c_i u_i$ for a direction vector $\boldsymbol{c}$. On the other hand, let

$$\ell(\boldsymbol{u}) = \min\left\{\frac{u_i}{c_i}; c_i \neq 0, i = 1, \ldots, d\right\} \tag{3.8}$$

for a direction vector $\boldsymbol{c}$. Then, the level is partially additive in direction $\boldsymbol{c}$ with $\ell(\boldsymbol{c}) = 1$. If ℓ_1 is partially additive and if ℓ_2 is additive, then $\ell_1 + \ell_2$ is partially additive. So, the partial additivity does not imply (3.8). In our applications, (3.8) with $\boldsymbol{c} > \mathbf{0}$ is important since the set $\{\boldsymbol{u} \in S_Y; \ell(\boldsymbol{u}) > 0\}$ is the positive orthant as we shall see in Section 4.

If the level is not additive, then J_n does depend on the initial additive component Y_0. This does not fit the traditional framework for the Wiener-Hopf factorization of a Markov additive process (e.g., see Theorem 6.3 of [2]), so we need to extend that framework. To this end, let $\tau_y^{0-} = \inf\{n \geq 1; J_n -$

$J_0 \le y\}$. For $s \in (-1,1)$, $g \in M_b(S)$, and $(x,\boldsymbol{u}) \in S$, define nonnegative kernels by

$$\underline{H}^+(s)g(x,\boldsymbol{u}) = E_{(x,\boldsymbol{u})}\left(\sum_{n=0}^{\tau_0^{0-}-1} s^n g(X_n, Y_n)\right),$$
$$\underline{G}^{0-}(s)g(x,\boldsymbol{u}) = E_{(x,\boldsymbol{u})}\left(s^{\tau_0^{0-}} g(X_{\tau_0^{0-}}, Y_{\tau_0^{0-}})\right).$$

Clearly, τ_y^{0-} is a stopping time, but generally not a hitting time, as the set that needs to be reached depends on the starting position. For convenience, we write $\underline{H}^+(s)g(x,\boldsymbol{u})$ for $g(x,\boldsymbol{u}) = 1((x,\boldsymbol{u}) \in A \times B)$ as $\underline{H}^+(s)((x,\boldsymbol{u}), A \times B)$. Similarly, $\underline{G}^{0-}(s)((x,\boldsymbol{u}), \in A \times B)$ is defined. We also represent $\underline{G}^{0-}(s)$ by probabilities:

$$G_\bullet^{0-}((x,\boldsymbol{u}), A \times B; n) = P_{(x,\boldsymbol{u})}(X_n \in A, Y_n \in B, \tau_0^{0-} = n), \quad n \ge 1, (x,\boldsymbol{u}) \in S.$$

Similar to this expression, we define $R_\bullet^+$ and $\underline{R}^+(s)$ as

$$R_\bullet^+((x,\boldsymbol{u}), A \times B; n) = P_{(x,\boldsymbol{u})}(X_n \in A, Y_n \in B, J_0 < J_n \le \min(J_1, \ldots, J_{n-1})),$$
$$\underline{R}^+(s)g(x,\boldsymbol{u}) = \sum_{n=1}^{\infty} s^n \int_S R_\bullet^+((x,\boldsymbol{u}), dy \times d\boldsymbol{v}; n) g(y,\boldsymbol{v}).$$

We here use subscript $\bullet$ to indicate detailed probabilities for each time index n. Similarly to the underline, we shall use this notation throughout the paper. By R^+, we denote $\underline{R}^+(1)$, that is, the marginal of $R_\bullet^+$ summing over the time index. Similarly, G^{0-} denotes $\underline{G}^{0-}(1)$.

It should be noted that the kernels $\underline{H}^+(s)g(x,\boldsymbol{u}), \underline{G}^{0-}(s)g(x,\boldsymbol{u})$ and $\underline{R}^+(s)g(x,\boldsymbol{u})$ may depend on the initial value $\boldsymbol{u}$ of the additive component in addition to the background state x unless the level process $\{J_n\}$ is additive. We first note the following fact, which is not used in this section, but gives an interpretation of $\underline{R}^+(s)$.

LEMMA 3.3. *For $s \in [0,1)$,*

$$\underline{R}^+(s)g(x,\boldsymbol{u}) = sQ|_{S^+_{\ell(\boldsymbol{u})}}(I - \underline{G}^{0-}(s))^{-1}|_{S^+_{\ell(\boldsymbol{u})}} g(x,\boldsymbol{u}), \quad g \in M_b(S), \tag{3.9}$$

where $T|_C g(x,\boldsymbol{u}) = \int_C T((x,\boldsymbol{u}), dy \times d\boldsymbol{v}) g(y,\boldsymbol{v})$ for the kernel T on $(S, \mathcal{B}(S))$ and $C \in \mathcal{B}(S)$.

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Since the proof of this lemma is rather mechanical, we defer it to Appendix D. From (3.9), $\underline{R}^+(s)$ is interpreted as the mean discounted numbers of visiting states above the initial level before going below this level. To give another interpretation of $\underline{R}^+(s)$, we use the dual MAP $\{(\tilde{X}_n, \tilde{Y}_n)\}$. Define the level function $\tilde{\ell}$ for this dual MAP by

$$\tilde{\ell}(\boldsymbol{u}) = -\ell(-\boldsymbol{u}),$$

and let $\tilde{J}_n = \tilde{\ell}(\tilde{Y}_n)$. Similarly to the case of the original MAP, we also introduce the stopping time $\tilde{\tau}_0^+ = \inf\{n \geq 1; \tilde{J}_n - \tilde{J}_0 > 0\}$, and define

$$\underline{\tilde{G}}^+(s)g(x,\boldsymbol{u}) = E_{(x,\boldsymbol{u})}\left(s^{\tilde{\tau}_0^+} g\left(\tilde{X}_{\tilde{\tau}_0^+}, \tilde{Y}_{\tilde{\tau}_0^+}\right)\right), \qquad (x,\boldsymbol{u}) \in S, g \in M_b(S).$$

Similarly, $\underline{\tilde{R}}^-(s)$ is defined as the corresponding kernel with $\underline{R}^+(s)$ in the opposite direction for the dual MAP. The following result will be used to verify our formulas for $s = 1$.

LEMMA 3.4. *Let π be the sub-invariant measure of K_X, and let $\underline{\tilde{G}}^+(s)$ and $\underline{\tilde{R}}^-(s)$ denote kernels corresponding with $\underline{G}^+(s)$ and $\underline{R}^-(s)$, respectively, for the dual MAP with respect to π. Then, under the assumptions (3a), for $s \in [0,1]$,*

$$\underline{R}^+(s) = (\underline{\tilde{G}}^+(s))^{\dagger(\pi\otimes m)}, \tag{3.10}$$

$$\underline{G}^-(s) = (\underline{\tilde{R}}^-(s))^{\dagger(\pi\otimes m)}. \tag{3.11}$$

Hence, $\underline{R}^+(s)((x,\boldsymbol{u}),\cdot)$, $\underline{\tilde{G}}^+(s)((x,\boldsymbol{u}),\cdot)$, $\underline{G}^-(s)^\dagger((x,\boldsymbol{u}),\cdot)$ and $\underline{\tilde{R}}^-(s))^\dagger((x,\boldsymbol{u}),\cdot)$ are absolutely continuous with respect to $\pi\otimes m$ for almost all $(x,\boldsymbol{u})$.

PROOF. From (3.6) of Lemma 3.2, we have, for $g_1, g_2 \in M_b^0(S)$,

$$\begin{aligned}
&E_{\pi\otimes m}(g_1(X_0,Y_0)g_2(X_n,Y_n); J_0 < J_n \leq J_{n'}, n' = 1,\ldots,n-1)\\
&\quad = E_{\pi\otimes m}(g_1(\tilde{X}_n,-\tilde{Y}_n)g_2(\tilde{X}_0,-\tilde{Y}_0); \tilde{J}_0 < \tilde{J}_n, \tilde{J}_{n-n'} \leq \tilde{J}_0, n' = 1,\ldots,n-1)\\
&\quad = E_{\pi\otimes m}(g_2^-(\tilde{X}_0,\tilde{Y}_0)g_1^-(\tilde{X}_n,\tilde{Y}_n); \tilde{\tau}_0^+ = n),
\end{aligned}$$

where g_i^- are defined by (3.7). Hence, applying the definition of the transposition $\dagger(\pi\otimes m)$, we have

$$\begin{aligned}
\int_S g_1(x,\boldsymbol{y})\underline{R}^+(s)g_2(x,\boldsymbol{y})\pi(dx)m(\boldsymbol{y}) &= \int_S g_2^-(x,\boldsymbol{y})\underline{\tilde{G}}^+(s)g_1^-(x,\boldsymbol{y})\pi(dx)m(\boldsymbol{y})\\
&= \int_S g_1(x,\boldsymbol{y})(\underline{\tilde{G}}^+(s))^{\dagger(\pi\otimes m)}g_2(x,\boldsymbol{y})\pi(dx)m(\boldsymbol{y}).
\end{aligned}$$

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Thus, we get (3.10) for $s \in [0,1)$ since $\pi(dx)$ is positive almost everywhere due to the irreducibility of K_X. Since $\tilde{G}^{0-} \equiv \underline{\tilde{G}}^{0-}(1)$ is sub-stochastic, (3.11) is also valid for $s = 1$. Similarly, (3.11) is obtained for $s < 1$. □

We now present key identities.

THEOREM 3.1. *We have the following identities for kernels on* $(S, \mathcal{B}(S))$. *For* $s \in [0,1)$,

$$\underline{H}^{+}(s) = (I - \underline{R}^{+}(s))^{-1}, \tag{3.12}$$

and, for $s \in [0,1]$,

$$I - sQ = (I - (\underline{\tilde{G}}^{+}(s))^{\dagger(\pi\otimes m)})(I - \underline{G}^{0-}(s)). \tag{3.13}$$

REMARK 3.6. $I - \underline{G}^{0-}(s)$ can be decomposed further. For this, let $\tau_y^0 = \inf\{n \geq 1; J_n - J_0 = y\}$ and $\tau_y^- = \inf\{n \geq 1; J_n - J_0 < y\}$, and define

$$\underline{G}^{0}(s)g(x,\boldsymbol{u}) = E_{(x,\boldsymbol{u})}\left(s^{\tau_0^0} g(X_{\tau_0^0}, Y_{\tau_0^0})\right),$$
$$\underline{G}^{-}(s)g(x,\boldsymbol{u}) = E_{(x,\boldsymbol{u})}\left(s^{\tau_0^-} g(X_{\tau_0^-}, Y_{\tau_0^-})\right).$$

Since $\underline{G}^{-}(s) = (I - \underline{G}^{0}(s))^{-1}(\underline{G}^{0-}(s) - \underline{G}^{0}(s))$, we have

$$I - \underline{G}^{0-}(s) = (I - \underline{G}^{0}(s))(I - \underline{G}^{-}(s)). \tag{3.14}$$

Substituting this into (3.13) yields so called a symmetric decomposition.

REMARK 3.7. In our applications, the case $s = 1$ is important. $H^{+}f$ may diverge even for $f \in M_b^0(S)$, so we can not verify (3.12) for $s = 1$ in general while (3.13) holds.

REMARK 3.8. Equation (3.13) looks like the Wiener-Hopf factorization of Arjas & Speed [2], but this is not a precise interpretation because the additive components in $\underline{R}^{+}(s)$, $(\underline{G}^{+}(s))^{\dagger(\pi\otimes m)}$ and $\underline{G}^{0-}(s)$ may depend on the initial value of the additive component. Thus, (3.13) is more close to the factorization of Dinges [8]. However, we derive it using the same spirit as in [2], so we still call it the Wiener-Hopf factorization.

PROOF. We first derive (3.12). Decomposing the probability $P_{(x,\boldsymbol{u})}(\tau_0^{0-} \geq n+1, (X_n, Y_n) \in A \times B)$ by the last time when the minimum of the level is attained during the time interval from 1 to n, we have, for $n \geq 1$ and $(x,\boldsymbol{u}) \in S$,

$$
\begin{aligned}
&P_{(x,\boldsymbol{u})}(\tau_0^{0-} \geq n+1, (X_n, Y_n) \in A \times B) \\
&\quad = P_{(x,\boldsymbol{u})}\left(J_0 < \min_{1\leq n'\leq n} J_{n'}, X_n \in A, Y_n \in B\right) \\
&\quad = \sum_{k=1}^{n} P_{(x,\boldsymbol{u})}\left(J_0 < J_k \leq \min_{1\leq n'\leq k-1} J_{n'}, J_k < \min_{k+1\leq n'\leq n} J_{n'}, X_n \in A, Y_n \in B\right) \\
&\quad = \sum_{k=1}^{n} E_{(x,\boldsymbol{u})}\Bigg(1(J_0 < J_k \leq \min_{1\leq n'\leq k-1} J_{n'}) \\
&\qquad\qquad \times P\Big(J_k < \min_{k+1\leq n'\leq n} J_{n'}, X_n \in A, Y_n \in B \Big| \mathcal{F}_k\Big)\Bigg) \\
&\quad = \underline{R}^{+}_{\bullet}((x,\boldsymbol{u}), A \times B; n) \\
&\qquad + \sum_{k=1}^{n-1} \int_S \underline{R}^{+}_{\bullet}((x,\boldsymbol{u}), dx' \times d\boldsymbol{u}'; k) \\
&\qquad\qquad \times P_{(x',\boldsymbol{u}')}(\tau_0^{0-} \geq n-k+1, (X_{n-k}, Y_{n-k}) \in dy \times d\boldsymbol{v}).
\end{aligned}
$$

Multiplying both sides of the above equation with s^n, summing over $n \geq 1$, and rearranging terms, we have

$$(I - \underline{R}^{+}(s))\underline{H}^{+}(s) = I. \tag{3.15}$$

Similarly, decomposing the same probability by the last time when the level is less than J_n, we have

$$\underline{H}^{+}(s)(I - \underline{R}^{+}(s)) = I.$$

This concludes (3.12) since $\underline{H}^{+}(s)$ and $\underline{R}^{+}(s)$ are nonnegative and $\underline{H}^{+}(s)g$ is finite for $s \in [0,1)$ and $g \in M_b(S)$. We next prove (3.13). For this, we apply Lemma 2.1 for $\tau = \tau_0^{0-}$ and $Z_n = (X_n, Y_n)$. Then, (2.2) is

$$\underline{H}^{+}(s)(I - sQ) = I - \underline{G}^{0-}(s). \tag{3.16}$$

This and (3.15) together with Lemma 3.4 yield (3.13). □

3.3. *Operator moment generating functions.* We are interested in tail asymptotics of some distributions connected to $R^{+} \equiv \underline{R}^{+}(1)$ in our applications. This is the problem of large deviations, and it is well known that exponential change of measure is useful for them (see, e.g., [9, 31]). In this section, we introduce operators under exponential change of measure, which will be useful in applications. For this, we rewrite the Wiener-Hopf factorization (3.13) in terms of operator moment generating functions.

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Recall that the inner product of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is denoted by $\langle \boldsymbol{a}, \boldsymbol{b}\rangle$. For $\boldsymbol{\theta} \in \mathbb{R}^d_+$, define $K_*(\boldsymbol{\theta})$ and $Q_*(\boldsymbol{\theta})$ as

$$K_*(\boldsymbol{\theta})f(x) = E\Big(e^{\langle \boldsymbol{\theta},(Y_1-Y_0)\rangle} f(X_1)\Big| X_0 = x\Big), \quad x \in S_X, f \in M_b(S_X),$$
$$Q_*(\boldsymbol{\theta})g(x,\boldsymbol{u}) = E_{(x,\boldsymbol{u})}\Big(e^{\langle \boldsymbol{\theta},(Y_1-Y_0)\rangle} g(X_1,Y_1)\Big), \quad (x,\boldsymbol{u}) \in S, g \in M_b(S),$$

as long as the right-hand side exists. Similarly, $\underline{G}^+_*(s,\boldsymbol{\theta})$, $\underline{G}^{0-}_*(s,\boldsymbol{\theta})$, $\underline{H}^+_*(s,\boldsymbol{\theta})$ and $\underline{R}^+_*(s,\boldsymbol{\theta})$ are defined. For example, for $(x,\boldsymbol{u}) \in S, g \in M_b(S)$,

$$\underline{G}^+_*(s,\boldsymbol{\theta})g(x,\boldsymbol{u}) = E_{(x,\boldsymbol{u})}\Big(s^{\tau_0^+} e^{\langle \boldsymbol{\theta},(Y_{\tau_0^+}-Y_0)\rangle} g(X_{\tau_0^+}, Y_{\tau_0^+})\Big),$$
$$\underline{H}^+_*(s,\boldsymbol{\theta})g(x,\boldsymbol{u}) = E_{(x,\boldsymbol{u})}\left(\sum_{n=0}^{\tau_0^{0-}-1} s^n e^{\langle \boldsymbol{\theta},(Y_n-Y_0)\rangle} g(X_n,Y_n)\right).$$

Then, (3.10), (3.13) and (3.16) can be written as, for $s \in [0,1)$ and $\Re\boldsymbol{\theta} = \mathbf{0}$,

$$\underline{R}^+_*(s,\boldsymbol{\theta}) = (\tilde{\underline{G}}^+_*(s,\boldsymbol{\theta}))^{\dagger(\pi\otimes m)}, \tag{3.17}$$
$$I - sQ_*(\boldsymbol{\theta}) = (I - \underline{R}^+_*(s,\boldsymbol{\theta}))(I - \underline{G}^{0-}_*(s,\boldsymbol{\theta})), \tag{3.18}$$
$$\underline{H}^+_*(s,\boldsymbol{\theta})(I - sQ_*(\boldsymbol{\theta})) = I - \underline{G}^{0-}_*(s,\boldsymbol{\theta}). \tag{3.19}$$

Similarly to R^+ and some others, we remove the underline in the above notation for $s = 1$ in such a way that $R^+_*(\boldsymbol{\theta}) = \underline{R}^+_*(1,\boldsymbol{\theta})$.

To consider those formulas under an exponential change of measure, we need to find their domains concerning s and $\boldsymbol{\theta}$ on which they are σ-finite nonnegative kernels on $(S,\mathcal{B}(S))$. For this, we will use a sub-invariant measure and superharmonic function for $sK_*(\boldsymbol{\theta})$. Following Nummelin [28], we introduce some basic notions.

We define the irreducibility of a nonnegative kernel U on $(S_X,\mathcal{B}(S_X))$ similar to (3a) for K_X. Assume that U is irreducible. Then, similar to (3.1) for K_X, there is a small function g_s, that is, there are a positive integer m, a nonnegative function g_s on S_X, a nonzero measure ν on $(S_X,\mathcal{B}(S_X))$ such that $\int g_s(x)\psi(dx) > 0$ for the irreducible measure ψ of U and

$$g_s(x)\nu(A) \le U^m(x,A), \qquad x \in S_X, A \in \mathcal{B}(S_X).$$

Let $M_s(S_X)$ be the set of all small functions on S_X. Since the irreducibility of U implies $M_s(S_X) \neq \emptyset$ by Theorem 2.1 of [28], we can define $c_p(U)$ as

$$c_p(U) = \sup\left\{z \ge 0; \sum_{n=0}^{\infty} z^n U^n g_s(x) < \infty, \forall g_s \in M_s(S_X), \exists x \in S_X\right\},$$

which is referred to as a convergence parameter of U (see Theorems 3.2 and 3.3 of [28]).

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The following lemma characterizes the convergence parameter by sub-invariant measures or super-harmonic functions.

LEMMA 3.5 (Propositions 5.2 and 5.7 of [28]). *For a nonnegative irreducible kernel U on $(S_X, \mathcal{B}(S_X))$, $c_p(U) \geq s$ is equivalent to either one of the following conditions.*

(i) There is a measure μ such that $s\mu U(A) \leq \mu(A)$ for all $A \in \mathcal{B}(S_X)$.
(ii) There is a function $h \in M_+(S_X)$ such that $sUh(x) \leq h(x)$ for all $x \in S_X$.

REMARK 3.9. If (i) (or (ii)) holds, μ (or h) is called a sub-invariant measure (or a super-harmonic function) of kernel sU.

We now consider the convergence parameter for $K_*(\boldsymbol{\theta})$. Since it is a nonnegative kernel and the irreducibility of K_X implies that of $K_*(\boldsymbol{\theta})$, $c_p(K(\boldsymbol{\theta}))$ is well defined. Thus, we define

$$\mathcal{C}_0(K) = \{\boldsymbol{\theta} \in \mathbb{R}^d; c_p(K_*(\boldsymbol{\theta})) > 0\},$$
$$\underline{\mathcal{C}}_0(K) = \{(t, \boldsymbol{\theta}) \in \mathbb{R} \times S_X; tK_*(\boldsymbol{\theta})h \leq h, \text{ for some } h \in M_+(S_X)\}.$$

Similarly, $\mathcal{C}_0(\tilde{K})$ and $\underline{\mathcal{C}}_0(\tilde{K})$ are defined for K and $\tilde{K}$. Note that $\mathcal{C}_0(K)$ may be smaller than the convergence domain of $K(\boldsymbol{\theta})$, that is, the set of all $\boldsymbol{\theta} \in \mathbb{R}^d$ such that $K(\boldsymbol{\theta})$ is a σ-finite nonnegative operator. From (3.4), it follows that

$$\int_A \pi(dx)(K_*(\boldsymbol{\theta})h)(x) = \int \pi(dx')h(x')\tilde{K}_*(\boldsymbol{\theta})(x', A), \quad A \in \mathcal{B}(S_X).$$

This implies that, for each fixed $t > 0$, h is super-harmonic for $tK_*(\boldsymbol{\theta})$ if and only if $\nu(A) \equiv \int_A \pi(dx)h(x)$ is sub-invariant for $t\tilde{K}_*(\boldsymbol{\theta})$. Hence, Lemma 3.5 leads to the following lemma.

LEMMA 3.6. *$\boldsymbol{\theta} \in \mathcal{C}_0(K)$ if and only if $(t, \boldsymbol{\theta}) \in \underline{\mathcal{C}}_0(K)$ for some $t > 0$. Furthermore,*

$$c_p(\tilde{K}_*(\boldsymbol{\theta})) = c_p(K_*(\boldsymbol{\theta})) = \sup\{t; (t, \boldsymbol{\theta}) \in \underline{\mathcal{C}}_0(K)\}. \tag{3.20}$$

We further note the following facts.

LEMMA 3.7. *$\Lambda(\boldsymbol{\theta}) \equiv -\log c_p(K_*(\boldsymbol{\theta}))$ is a convex function on $\mathcal{C}_0(K)$. If (3b) holds, then $\Lambda(\boldsymbol{\theta})$ is lower semi-continuous. Furthermore,*

$$c_p(Q_*(\boldsymbol{\theta})) \geq \sup_{\boldsymbol{\theta}' \in \mathcal{C}_0(K)} c_p(K_*(\boldsymbol{\theta}')), \qquad \boldsymbol{\theta} \in \mathcal{C}_0(K), \tag{3.21}$$

and therefore $\mathcal{C}_0(K) \subset \mathcal{C}_0(Q)$.

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REMARK 3.10. The convexity and lower semi-continuity of $\Lambda(\boldsymbol{\theta})$ are essentially known. For example, see Corollary 3.3 and Lemma 3.4 of [26], in which finer results are obtained under extra conditions including (3b). On the other hand, (3.21) may imply $c_p(Q^{(t,\boldsymbol{\theta})}) > 1$ for $(t,\boldsymbol{\theta}) \in \underline{\mathcal{C}}_0(K)$. This is not surprising since the Markov additive process is generally transient.

PROOF. We prove the convexity without condition (3b). Let $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2$ be arbitrarily chosen elements of $\mathcal{C}_0(K)$. It follows from Lemma 3.6 and the definition of $\Lambda(\boldsymbol{\theta})$ that there exist positive functions h_1 and h_2 such that

$$K_*(\boldsymbol{\theta}_i)h_i \le e^{\Lambda(\boldsymbol{\theta}_i)}h_i, \qquad i = 1, 2,$$

where the variable with parenthesis (x) is omitted for simplicity. This convention will be used below as long as there will be no confusion. For $\lambda \in (0,1)$, these inequalities imply

$$(K_*(\boldsymbol{\theta}_1)h_1)^{\lambda}(K_*(\boldsymbol{\theta}_2)h_2)^{1-\lambda} \le e^{\lambda\Lambda(\boldsymbol{\theta}_1)+(1-\lambda)\Lambda(\boldsymbol{\theta}_2)}h_1^{\lambda}h_2^{1-\lambda} \tag{3.22}$$

By Hölder's inequality with $p = \frac{1}{\lambda}$ and $q = \frac{1}{1-\lambda}$, we have

$$\begin{aligned}
&\int_{S_X} K_*(\lambda\boldsymbol{\theta}_1 + (1-\lambda)\boldsymbol{\theta}_2)(x, dy)h_1^{\lambda}(y)h_2^{1-\lambda}(y) \\
&\quad = \int_S K(x, dy \times d\boldsymbol{u})(e^{\langle\boldsymbol{\theta}_1,\boldsymbol{u}\rangle}h_1(y))^{\lambda}(e^{\langle\boldsymbol{\theta}_2,\boldsymbol{u}\rangle}h_2(y))^{1-\lambda} \\
&\quad \le \left(\int_S K(x, dy \times d\boldsymbol{u})e^{\langle\boldsymbol{\theta}_1,\boldsymbol{u}\rangle}h_1(y)\right)^{\lambda} \\
&\qquad\qquad \times \left(\int_S K(x, dy \times d\boldsymbol{u})e^{\langle\boldsymbol{\theta}_2,\boldsymbol{u}\rangle}h_2(y)\right)^{1-\lambda} \\
&\quad = (K_*(\boldsymbol{\theta}_1)h_1(x))^{\lambda}(K_*(\boldsymbol{\theta}_2)h_2(x))^{1-\lambda}..
\end{aligned} \tag{3.23}$$

Hence, letting $h(x) = h_1^{\lambda}(x)h_2^{1-\lambda}(x)$, inequalities (3.22) and (3.23) yield

$$K_*(\lambda\boldsymbol{\theta}_1 + (1-\lambda)\boldsymbol{\theta}_2)h \le e^{\lambda\Lambda(\boldsymbol{\theta}_1)+(1-\lambda)\Lambda(\boldsymbol{\theta}_2)}h.$$

This and Lemma 3.5 imply that $\lambda\boldsymbol{\theta}_1 + (1-\lambda)\boldsymbol{\theta}_2 \in \mathcal{C}_0(K)$, and therefore Lemma 3.6 yields

$$e^{\lambda\Lambda(\boldsymbol{\theta}_1)+(1-\lambda)\Lambda(\boldsymbol{\theta}_2)} \le c_p(K_*(\lambda\boldsymbol{\theta}_1 + (1-\lambda)\boldsymbol{\theta}_2)) = e^{-\Lambda(\lambda\boldsymbol{\theta}_1+(1-\lambda)\boldsymbol{\theta}_2)}$$

Hence, we have the convexity of Λ:

$$\Lambda(\lambda\boldsymbol{\theta}_1 + (1-\lambda)\boldsymbol{\theta}_2) \le \lambda\Lambda(\boldsymbol{\theta}_1) + (1-\lambda)\Lambda(\boldsymbol{\theta}_2).$$

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We next assume (3b). In this case, Λ can be defined by (2.5) of [27], and it can be shown that Λ is lower semi-continuous.

We next prove (3.21). For $\boldsymbol{\theta}' \in \mathcal{C}_0(K)$, let $t = c_p(K_*(\boldsymbol{\theta}'))$, then $tK_*(\boldsymbol{\theta}')$ has a super-harmonic function h, and therefore

$$\begin{aligned} tQ_*(\boldsymbol{\theta})(h \otimes e^{(\boldsymbol{\theta}'-\boldsymbol{\theta})})(x,\boldsymbol{u}) &= t\int_S Q((x,\boldsymbol{u}), dx' \times d\boldsymbol{u}')e^{\langle \boldsymbol{\theta},(\boldsymbol{u}'-\boldsymbol{u})\rangle}h(x')e^{\langle \boldsymbol{\theta}'-\boldsymbol{\theta},\boldsymbol{u}'\rangle} \\ &= t\int_S Q((x,\boldsymbol{u}), dx' \times d\boldsymbol{u}')e^{\langle \boldsymbol{\theta}',\boldsymbol{u}'-\boldsymbol{u}\rangle}h(x')e^{\langle \boldsymbol{\theta}'-\boldsymbol{\theta},\boldsymbol{u}\rangle} \\ &= tK_*(\boldsymbol{\theta}')h(x)e^{\langle \boldsymbol{\theta}'-\boldsymbol{\theta},\boldsymbol{u}\rangle} \le h \otimes e^{(\boldsymbol{\theta}'-\boldsymbol{\theta})}(x,u), \end{aligned}$$

where $e^{(\boldsymbol{\theta})}(\boldsymbol{u}) = e^{\langle \boldsymbol{\theta},\boldsymbol{u}\rangle}$. Hence, $c_p(Q_*(\boldsymbol{\theta})) \ge c_p(K_*(\boldsymbol{\theta}'))$. This proves (3.21). □

To derive the Wiener-Hopf factorization under change of measure, we will use the following notation.

DEFINITION 3.4. For $(t,\boldsymbol{\theta}) \in \underline{\mathcal{C}}_0(K)$, let $h^{(t,\boldsymbol{\theta})}$ be the super-harmonic function of $tK_*(\boldsymbol{\theta})$ and define kernels $K^{(t,\boldsymbol{\theta})}$ and $Q^{(t,\boldsymbol{\theta})}$ on S as, for $g \in M_b(S)$,

$$\begin{aligned} &K^{(t,\boldsymbol{\theta})}g(x) = (h^{(t,\boldsymbol{\theta})}(x))^{-1}\int_S tK(x, dx' \times d\boldsymbol{u}')e^{\langle \boldsymbol{\theta},\boldsymbol{u}'\rangle}h^{(t,\boldsymbol{\theta})}(x')g(x',\boldsymbol{u}'), \\ &Q^{(t,\boldsymbol{\theta})}g(x,\boldsymbol{u}) \\ &\quad = (h^{(t,\boldsymbol{\theta})}(x))^{-1}\int_S tQ_*(\boldsymbol{\theta})((x,\boldsymbol{u}), dx' \times d\boldsymbol{u}')h^{(t,\boldsymbol{\theta})}(x')g(x',\boldsymbol{u}'), \\ &\underline{R}^{(t,\boldsymbol{\theta})+}(t)g(x,\boldsymbol{u}) \\ &\quad = (h^{(t,\boldsymbol{\theta})}(x))^{-1}\int_S \underline{R}^{+}_*(st,\boldsymbol{\theta})((x,\boldsymbol{u}), dx' \times d\boldsymbol{u}')h^{(t,\boldsymbol{\theta})}(x')g(x',\boldsymbol{u}'), \\ &\underline{G}^{(t,\boldsymbol{\theta})0-}(t)g(x,\boldsymbol{u}) \\ &\quad = (h^{(t,\boldsymbol{\theta})}(x))^{-1}\int_S \underline{G}^{0-}_*(st,\boldsymbol{\theta})((x,\boldsymbol{u}), dx' \times d\boldsymbol{u}')h^{(t,\boldsymbol{\theta})}(x')g(x',\boldsymbol{u}'). \end{aligned}$$

These kernels are said to be twisted with parameter $(t,\boldsymbol{\theta})$. Similarly, $\tilde{K}^{(t,\boldsymbol{\theta})}$ and $\tilde{Q}^{(t,\boldsymbol{\theta})}$ are defined for the dual kernels $\tilde{K}$ and $\tilde{Q}$.

Note that $Q^{(t,\boldsymbol{\theta})}$ is the transition probability kernel of the Markov additive process generated by $K^{(t,\boldsymbol{\theta})}$, and $R^+(s)$ and $G^{0-}(s)$ are transferred to $R^{(t,\boldsymbol{\theta})+}(s)$ and $G^{(t,\boldsymbol{\theta})0-}(s)$ under this change of measure. Hence, the following theorem is immediate from Theorem 3.1 and Lemmas 3.6 and 3.7.

THEOREM 3.2. *For $(t,\boldsymbol{\theta}) \in \underline{\mathcal{C}}_0(K)$, we have, for $s \in [0,1]$,*

$$\underline{R}^{(t,\boldsymbol{\theta})+}(s) = (\tilde{G}^{(t,\boldsymbol{\theta})+}(s))^{\dagger(\pi^{(t,\boldsymbol{\theta})}\otimes m)}, \tag{3.24}$$

$$I - sQ^{(t,\boldsymbol{\theta})} = (I - \underline{R}^{(t,\boldsymbol{\theta})+}(s))(I - \underline{G}^{(t,\boldsymbol{\theta})0-}(s)), \tag{3.25}$$

where $\pi^{(t,\boldsymbol{\theta})}$ is the sub-invariant measure of $K_X^{(t,\boldsymbol{\theta})}$. Furthermore, (3.17) and (3.18) hold for $(s,\boldsymbol{\theta}) \in \underline{\mathcal{C}}_0(K)$.

REMARK 3.11. In view of this theorem, we can see that Theorem 3.1 can be extended to any nonnegative and irreducible kernel $\{K(x, A \times B)\}$. In this case, the range of s in (3.13) is changed to $s \in [0, c_p(K))$ and the conjugate operation $\dagger(\pi \otimes m)$ must be appropriately modified.

We next specialize the decomposition (3.18) for additive and partially additive cases.

3.4. *Additive and partially additive levels.* Assume that the level J_n is additive, that is, ℓ is an additive function. In this case, $\underline{H}_*^+(s,\boldsymbol{\theta})|_{S_X} g(x,\boldsymbol{u})$ does not depend on $\boldsymbol{u}$, so we denote it by $\underline{H}_{*\triangle}^+(s,\boldsymbol{\theta})g(x)$. Similarly, $\underline{\tilde{G}}_{*\triangle}^+(s,\boldsymbol{\theta})$ and $\underline{\tilde{R}}_{*\triangle}^+(s,\boldsymbol{\theta})$ and some others are defined. For example,

$$\underline{\tilde{G}}_{*\triangle}^+(s,\boldsymbol{\theta})g(x) = E_x\left(s^{\tilde{\tau}_0^+}\exp(\langle\boldsymbol{\theta}, \tilde{Y}_{\tilde{\tau}_0^+} - \tilde{Y}_0\rangle)g(\tilde{X}_{\tilde{\tau}_0^+})\right), \qquad g \in M_b(S_X),$$

where $\tilde{\tau}_0^+ = \inf\{n \geq 1; \tilde{J}_n > \tilde{J}_0\}$. Let $\underline{H}_\triangle^+(s)$, $\underline{\tilde{G}}_\triangle^+(s)$ and $\underline{R}_\triangle^+(s)$ be the kernels on $(S, \mathcal{B}(S))$ corresponding to $[\underline{H}_{*\triangle}^+(s,\boldsymbol{\theta})]_{S_X}$, $[\underline{\tilde{G}}_{*\triangle}^+(s,\boldsymbol{\theta})]_{S_X}$ and $[\underline{R}_{*\triangle}^+(s,\boldsymbol{\theta})]_{S_X}$, respectively. In this notation system, K and $\tilde{K}$ can be written as $Q_\triangle$ and $\tilde{Q}_\triangle$, respectively. For these kernels, we can naturally define convolutions. For example, for $g \in M_b(S_X)$,

$$\begin{aligned}&(s\tilde{Q}_\triangle * \underline{\tilde{G}}_\triangle^+(s))g(x)\\ &\quad = \int_S s\tilde{Q}_\triangle(x, dx' \times d\boldsymbol{u}') \int_S \underline{\tilde{G}}_\triangle^+(s)(x', dx'' \times d\boldsymbol{u}'')g(x'', \boldsymbol{u}' + \boldsymbol{u}'').\end{aligned}$$

Since $\tilde{\ell}(\boldsymbol{u}) = \ell(\boldsymbol{u})$, the following formula is immediate from Lemma 3.4.

$$\underline{R}_{*\triangle}^+(s,\boldsymbol{\theta}) = (\underline{\tilde{G}}_{*\triangle}^+)^{\mathrm{T}(\pi)}(s,\boldsymbol{\theta}), \qquad (s,\boldsymbol{\theta}) \in \underline{\mathcal{C}}_0(K). \tag{3.26}$$

Since $Q_{*\triangle}(\boldsymbol{\theta}) = K_*(\boldsymbol{\theta})$, Theorem 3.1 yields:

COROLLARY 3.1. *If the level is additive, then, for $(s,\boldsymbol{\theta}) \in \underline{\mathcal{C}}_0(K)$,*

$$I - sK_*(\boldsymbol{\theta}) = (I - (\underline{\tilde{G}}_{*\triangle}^+)^{\mathrm{T}(\pi)}(s,\boldsymbol{\theta}))(I - \underline{G}_{*\triangle}^{0-}(s,\boldsymbol{\theta})). \tag{3.27}$$

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REMARK 3.12. For $\Re\boldsymbol{\theta} = \mathbf{0}$ and $s \in [0,1)$, (3.27) is well known as the Wiener-Hopf factorization for a Markov additive process (see Theorem 6.3 and Remark 6.6 of [2]). We here define the dual process and transpose operation in a slightly different way, which avoids the sign change that is needed in [2]. Another different feature is expanding the set of possible $(s,\boldsymbol{\theta})$ within nonnegative vector values, which is convenient for change of measure.

REMARK 3.13. (3.27) is often derived for $d=1$, $s=1$ and $\Re\theta = 0$ assuming that S is a finite set. See Asmussen [3] for more about this type of Wiener-Hopf factorization. The formulas replacing $\underline{\tilde{G}}^{+}_{*\triangle}$ in (3.27) by $\underline{R}^{+}_{*\triangle}(s,\boldsymbol{\theta})$ is also known as R-G decomposition when Y_n is integer valued and S_X is a finite or countable set (e.g., see [13, 33]).

We next consider the case that the level is partially additive. Let $\boldsymbol{c}$ be the direction of the partially additive level. Let $Z_n = Y_n - \ell(Y_n)\boldsymbol{c}$. Since any difference $J_{n'+1} - J_{n'}$ only depends on $Y_{n'}$ through $Z_{n'} \equiv Y_{n'} - \ell(Y_n)\boldsymbol{c} \in S_Y$, we take (X_n, Z_n) as the background state instead of X_n and J_n as the additive component. Let $K_{\boldsymbol{c}}((x,\boldsymbol{z}), A \times B)$, $(x,\boldsymbol{z}) \in S, A \in \mathcal{B}(S), B \in \mathcal{B}(R)$ be the Markov additive kernel of the Markov additive process $((X_n, Z_n), J_n)$. Namely, $K_{\boldsymbol{c}}$ is defined by

$$\int_{S\times B} K_{\boldsymbol{c}}((x,\boldsymbol{z}), dx' \times d\boldsymbol{z}' \times du) g(x',\boldsymbol{z}')$$
$$= \int_S K(x, dx' \times d\boldsymbol{u}) 1(\ell(\boldsymbol{u}) \in B) g(x', \boldsymbol{u} + \boldsymbol{z} - \ell(\boldsymbol{u}+\boldsymbol{z})\boldsymbol{c}),$$
$$(x,\boldsymbol{z}) \in S, g \in M_b(S), B \in \mathcal{B}(S_Y).$$

Similarly to Lemma 3.1, we can see that $\pi \otimes m$ is a sub-invariant measure of $K_{\boldsymbol{c}}$. Since the additive component J_n is itself a level, the MAP $((X_n, Z_n))$ has an additive level process (J_n).

Hence, Corollary 3.1 yields

COROLLARY 3.2. *If the level is partially additive in the direction* $\boldsymbol{c}$*, we have*

$$I - sK_{\boldsymbol{c}*}(\theta) = (I - (\underline{\tilde{G}}^{+}_{*\boldsymbol{c}})^{\mathrm{T}(\pi\otimes m)}(s,\theta))(I - \underline{G}^{0-}_{*\boldsymbol{c}}(s,\theta)), \quad (s,\boldsymbol{\theta}) \in \underline{\mathcal{C}}_0(K), \tag{3.28}$$

where $K_{*\boldsymbol{c}}(\theta)$ *is the moment generating function operator of* $K_{\boldsymbol{c}}$*, that is,*

$$K_{\boldsymbol{c}*}(\theta) g(x,\boldsymbol{u}) = E_{(x,\boldsymbol{u})}\left(e^{\theta(J_1 - J_0)} g(X_1, Y_1 - J_1\boldsymbol{c})\right).$$

The operator generating functions $\underline{\tilde{G}}^{+}_{*\boldsymbol{c}}(s,\theta)$ *and* $\underline{G}^{0-}_{*\boldsymbol{c}}(s,\theta)$ *are defined similarly.*

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4. Reflected Markov additive process. We now modify the Markov additive process $\{(X_n, Y_n)\}$ in such a way that the level J_n is kept nonnegative. For this, we change the level function ℓ to a level function $\hat{\ell}$ which satisfies that, for each $\boldsymbol{u} \in S_Y$,

$$\hat{\ell}(\boldsymbol{u}) > 0 \quad \text{if and only if} \quad \ell(\boldsymbol{u}) > 0. \tag{4.1}$$

As long as this condition satisfied, $\hat{\ell}(\boldsymbol{u})$ can be different from $\ell(\boldsymbol{u})$. Denote this modified process by $\{(\hat{X}_n, \hat{Y}_n)\}$. To describe the state space for this process, let

$$\hat{S}_Y^0 = \{\boldsymbol{u} \in S_Y; \hat{\ell}(\boldsymbol{u}) = 0\}, \qquad \hat{S}_Y^+ = \{\boldsymbol{u} \in S_Y; \hat{\ell}(\boldsymbol{u}) > 0\},$$

where $S_Y = \mathbb{R}^d$. For an arbitrarily given measurable space $(\hat{S}_X^0, \mathcal{B}(\hat{S}_X^0))$, we put

$$\hat{S}^0 = \hat{S}_X^0 \times \hat{S}_Y^0, \qquad \hat{S}_Y = \hat{S}_Y^0 \cup \hat{S}_Y^+, \qquad \hat{S}^+ = S_X \times \hat{S}_Y.$$

The modified process has state space $\hat{S} \equiv \hat{S}^0 \cup \hat{S}^+$. We refer to $\hat{S}^0$ and $\hat{S}^+$ as boundary and interior, respectively. Note that the background state space at level 0 may be different from those at positive levels.

To describe the evolution of our process at the boundary, we assume the following transition probabilities:

$$\begin{aligned} &P(\hat{X}_{n+1} \in A, \hat{Y}_{n+1} \in B | \hat{X}_n = x, \hat{Y}_n = \boldsymbol{u}) \\ &\qquad = \begin{cases} K(x, A \times (B - \boldsymbol{u})), & (x, \boldsymbol{u}) \in \hat{S}^+, A \times B \in \mathcal{B}(\hat{S}^+), \\ L^0((x, \boldsymbol{u}), A \times B), & (x, \boldsymbol{u}) \in \hat{S}^0, A \times B \in \mathcal{B}(\hat{S}), \\ L^{+0}((x, \boldsymbol{u}), A \times B), & (x, \boldsymbol{u}) \in \hat{S}^+, A \times B \in \mathcal{B}(\hat{S}^0), \end{cases} \end{aligned}$$

where L^0 and L^{+0} are transition kernels such that

$$\begin{aligned} &L^0((x, \boldsymbol{u}), \hat{S}) = 1, && (x, \boldsymbol{u}) \in \hat{S}^0, \\ &L^{+0}((x, \boldsymbol{u}), \hat{S}^0) + K(x, \hat{S}_Y^+ - \boldsymbol{u}) = 1, && (x, \boldsymbol{u}) \in \hat{S}^+. \end{aligned}$$

The kernels L^0 and L^{+0} can be arbitrarily given as long as the above conditions are satisfied.

Often, the kernels L^0 and L^{+0} can be expressed in terms of K, but this is not necessary, which is common in the matrix analytic framework (see, e.g., [25]). This modified process $\{\hat{Z}_n, n \geq 0\} \equiv \{(\hat{X}_n, \hat{Y}_n)\}$ is a discrete time Markov process, and referred to as a Markov additive process with reflection or a reflected Markov additive process, respectively; MAP with reflection or reflected MAP for short. We will discuss about examples later.

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We now consider the stationary distribution of this MAP with reflection, which is the main subject of this section. We construct this stationary distribution using Corollary 2.1 with $\tau = \hat{\tau}_0^{0-}$, where $\hat{\tau}_y^{0-} = \inf\{n \geq 1; \hat{J}_n \leq y\}$. We also consider the time elapsed after leaving level 0. Let μ_0 be a distribution on $\hat{S}^0$ (to be chosen later), and define, for $n \geq 1$,

$$\begin{aligned}
&\nu_{0\bullet}(A \times B; n) \\
&\quad = \int_{\hat{S}^0} \mu_0(dx \times d\boldsymbol{u}) \hat{P}_{(x,\boldsymbol{u})}((\hat{X}_n, \hat{Y}_n) \in A \times B, \hat{\tau}_0^{0-} = n, \hat{J}_0 = 0), \\
&\nu_{+\bullet}(A \times B; n) \\
&\quad = \int_{\hat{S}^0} \mu_0(dx \times d\boldsymbol{u}) \hat{P}_{(x,\boldsymbol{u})}((\hat{X}_n, \hat{Y}_n) \in A \times B, \hat{\tau}_0^{0-} \geq n+1, \hat{J}_0 = 0), \\
&\nu_{\bullet}(A \times B; n) = \nu_{0\bullet}(A \times B; n) + \nu_{+\bullet}(A \times B; n).
\end{aligned}$$

We also define the measure $\underline{\nu}_0(s)$ as

$$\underline{\nu}_0(s)g = \sum_{n=1}^{\infty} s^n \int_{\hat{S}^0} \nu_{0\bullet}(dx \times d\boldsymbol{u}; n) g(x, \boldsymbol{u}), \qquad g \in M_b(\hat{S}).$$

In the same way, $\underline{\nu}_+(s)$ is defined for $\nu_{+\bullet}$, and we set $\underline{\nu}(s) = \underline{\nu}_0(s) + \underline{\nu}_+(s)$. As usual, $\underline{\nu}_0(1)$, $\underline{\nu}_+(1)$ and $\underline{\nu}(1)$ are simply denoted by ν_0, ν_+ and ν, respectively.

By Corollary 2.1, ν is the stationary measure of the reflected process $\{\hat{Z}_n\}$ if $\nu \equiv \underline{\nu}(1)$ is a measure and if $\mu_0 = \nu_0$, that is, for all $A \times B \in \mathcal{B}(\hat{S}^0)$,

$$\begin{aligned}
&\mu_0(A \times B) \\
&\quad = \int_{\hat{S}^0} \mu_0(dx \times d\boldsymbol{u}) \hat{P}_{(x,\boldsymbol{u})}(\hat{X}_{\hat{\tau}_0^{0-}} \in A, \hat{Y}_{\hat{\tau}_0^{0-}} \in B, \hat{\tau}_0^{0-} < \infty, \hat{J}_0 = 0).
\end{aligned} \tag{4.2}$$

Clearly, it is necessary for the existence of such μ_0 that $\hat{\tau}_0^{0-} < \infty$ with probability 1, but even the finiteness of $E_{\mu_0}(\hat{\tau}_0^{0-})$ may not be sufficient. In general, we have to work on each specific model individually (see e.g., [10, 11]). In queueing applications, it is often the case that the system becomes empty with positive probability, which makes it relatively easy to see the existence of μ_0 (see e.g., [17]).

In what follows, we focus on obtaining expressions for ν, $\underline{\nu}_0(s)$ and $\underline{\nu}_+(s)$ for μ_0 satisfying (4.2). To consider them, we introduce some notation. Define

$$\underline{\nu}_{+*}(s, \boldsymbol{\theta})g = \sum_{n=1}^{\infty} s^n \int_{\hat{S}^0} \nu_{+\bullet}(dx \times d\boldsymbol{u}; n) g(x, \boldsymbol{u}) e^{\langle \boldsymbol{\theta}, \boldsymbol{u} \rangle}, \qquad g \in M_b(S).$$

In the same way, $\underline{\nu}_{0*}(s, \boldsymbol{\theta})$ is defined for ν_0. Similarly to $\underline{R}^+(s)$, we define

$$\underline{R}^{0+}(s)g(x, \boldsymbol{u}) = \sum_{n=1}^{\infty} s^n \hat{E}_{(x,\boldsymbol{u})} \left(g(\hat{X}_n, \hat{Y}_n) 1(n \leq \hat{\tau}_{\hat{J}_n}^-, \hat{J}_n > 0, \hat{J}_0 = 0) \right).$$

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We also need the following notation.

$$L^{00} = L^0|_{\hat{S}^0}, \qquad L^{0+} = L^0|_{\hat{S}^+}.$$

Recall that $T|_C f(x,\boldsymbol{u}) = \int_C T((x,\boldsymbol{u}), dy \times d\boldsymbol{v}) f(y,\boldsymbol{v})$ for an operator T and $C \in \mathcal{B}(\hat{S})$. The following fact can be proved in the exactly same way as Lemma 3.3, so its proof is omitted.

LEMMA 4.1. *For $s \in [0,1)$,*

$$\underline{R}^{0+}(s) = sL^{0+}(I - \underline{G}^{0-}(s))^{-1}|_{\hat{S}^+}, \tag{4.3}$$

where $(I - \underline{G}^{0-}(s))^{-1}|_{\hat{S}^+}$ means $\sum_{n=0}^{\infty} \underline{G}^{0-}(s)^n|_{\hat{S}^+}$.

Similarly to $R_*^+(\boldsymbol{\theta})$, we denote the operator moment generating function of L^{00} and L^{0+}, respectively, by $L_*^{00}(\boldsymbol{\theta})$ and $L_*^{0+}(\boldsymbol{\theta})$. Then, we have the following representations.

THEOREM 4.1. *For any measure μ_0 on $(\hat{S}^0, \mathcal{B}(\hat{S}^0))$, we have, for $s \in [0,1)$,*

$$\underline{\nu}_0(s) = s\mu_0 L^{00} + s^2 \mu_0 \underline{R}^{0+}(s) L^{+0}, \tag{4.4}$$

$$\underline{\nu}_+(s) = \mu_0 \underline{R}^{0+}(s) + \underline{\nu}_+(s) \tilde{\underline{G}}^+(s)^{\dagger(\pi\otimes m)}. \tag{4.5}$$

Hence,

$$\underline{\nu}_+(s) = \mu_0 \underline{R}^{0+}(s)(I - \tilde{\underline{G}}^+(s)^{\dagger(\pi\otimes m)})^{-1}, \tag{4.6}$$

equivalently,

$$\underline{\nu}_+(s) = \mu_0 \underline{R}^{0+}(s)(I - \underline{G}^{0-}(s))(I - sQ)^{-1}. \tag{4.7}$$

In particular, if μ_0 satisfies

$$\mu_0 = \mu_0(L^{00} + R^{0+}L^{+0}), \tag{4.8}$$

and if $\mu_0 R^{0+} H^+$ is finite, where $R^{0+} = \underline{R}^{0+}(1)$, then (4.4) and (4.5) are valid for $s = 1$, and $\nu = \nu_0 + \nu_+$ is the stationary measure of the reflected MAP.

REMARK 4.1. From (4.3) and (4.6), $\mu_0 L^{0+} \ll \pi \otimes m$ implies $\underline{\nu}_+(s) \ll \pi \otimes m$.

PROOF. The idea is to apply censoring at level 0. It follows from Corollary 2.1 and the arguments in the proof of Theorem 3.1 that, for $n \geq 1$,

$$\begin{aligned}\nu_{+\bullet}(A\times B;n)&=\int_{S_X}\int_{\hat{S}_Y^0}\mu_0(dx,d\boldsymbol{u})\hat{P}_{(x,\boldsymbol{u})}((\hat{X}_n,\hat{Y}_n)\in A\times B,n\le\hat{\tau}^-_{\hat{J}_n},\hat{J}_n>0,\hat{J}_0=0)\\&\quad+\sum_{n'=1}^{n-1}\int_{S_X}\int_{\hat{S}_Y^+}\nu_{+\bullet}(dx\times d\boldsymbol{u},n-n')R^+((x,\boldsymbol{u}),A\times B;n').\end{aligned}$$

Multiplying both sides with s^n and summing over n, $n\ge 1$ yields the following recursive equation

$$\underline{\nu}_+(s)=\mu_0\underline{R}^{0+}(s)+\underline{\nu}_+(s)\underline{R}^+(s).$$

This and Lemma 3.4 imply (4.5). Similarly, we have (4.4). If H^+ is finite, then $R^+g(x,\boldsymbol{u})$ is finite for all $(x,\boldsymbol{u})\in S$ and $g\in M_b(S)$ by Theorem 3.1. So, $(I-G^{0-})^{-1}|_{\hat{S}^+}$ must be finite as well. This implies that (4.6) is valid for $s=1$. As we already noted, if (4.4), equivalently, (4.8), holds, ν is the stationary measure of the reflected MAP. □

Computing $\underline{R}^{0+}(s)$ is generally complicated because $L^{0+}(I-\underline{G}^{0-}(s))^{-1}|_{\hat{S}^+}$ can not be simplified as well as involving with the boundary transition L^{0+}. There is a simpler case, which can be used for some queueing and risk models.

COROLLARY 4.1. *If* $\hat{S}_X^0=S_X$ *and if*

$$L^{0+}((x,\boldsymbol{u}),A\times B)=K(x,A\times(B+\boldsymbol{u})),\qquad \boldsymbol{u}\in\hat{S}_Y^0. \tag{4.9}$$

then $\underline{R}^{0+}(s)=\underline{R}^+(s)$. *Hence, (4.6) implies*

$$\underline{\nu}_{+*}(s)=\mu_0\underline{R}^+(s)(I-\underline{R}^+(s))^{-1}. \tag{4.10}$$

Furthermore, $\mu_0\ll\pi\otimes m$, *then (4.10) implies that* $\underline{\nu}_{+*}(s)\ll\pi\otimes m$.

PROOF. If $\ell(\boldsymbol{u})=0$, then (4.9) implies

$$\begin{aligned}R^{0+}((x,\boldsymbol{u}),A\times B)&=\hat{P}_{(x,\boldsymbol{u})}((\hat{X}_n,\hat{Y}_n)\in A\times B,n\le\hat{\tau}^-_{\hat{J}_n},\hat{J}_n>0)\\&=P_{(x,\boldsymbol{u})}(X_n\in A,Y_n\in B,n\le\tau^{0-}_{J_n},J_n>J_0)\\&=R^+((x,\boldsymbol{u}),A\times B).\end{aligned}$$

This proves the corollary. □

The following corollaries are immediate from Theorems 4.1 and 3.2 and Corollary 3.1.

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COROLLARY 4.2. *For* $(s,\boldsymbol{\theta}) \in \underline{\mathcal{C}}_0(K)$, *we have*

$$\underline{\nu}_{0*}(s,\boldsymbol{\theta}) = s\mu_{0*}(\boldsymbol{\theta})L_*^{00}(\boldsymbol{\theta}) + s^2\mu_0\underline{R}^{0+}(s,\boldsymbol{\theta})L_*^{+0}(\boldsymbol{\theta}), \tag{4.11}$$

$$\underline{\nu}_{+*}(s,\boldsymbol{\theta}) = \mu_{0*}(\boldsymbol{\theta})\underline{R}_*^{0+}(s,\boldsymbol{\theta})\left(I - [\tilde{\underline{G}}_*^{+}(s,\boldsymbol{\theta})]^{\dagger(\pi\otimes m)}\right)^{-1}, \tag{4.12}$$

and, if $\underline{\nu}_{+*}(s,\boldsymbol{\theta})|_{S_X}$ *is finite, then*

$$\underline{\nu}_{+*}(s,\boldsymbol{\theta})|_{S_X} = \mu_{0*}(\boldsymbol{\theta})\underline{R}_*^{0+}(s,\boldsymbol{\theta})(I - \underline{G}_*^{0-}(s,\boldsymbol{\theta}))|_{S_X}(I - sK_*(\boldsymbol{\theta}))^{-1}. \tag{4.13}$$

COROLLARY 4.3. *If* ℓ *is additive and if* $\underline{\nu}_{+*}(s,\boldsymbol{\theta})|_{S_X}$ *is finite, then we have, for* $(s,\boldsymbol{\theta}) \in \underline{\mathcal{C}}_0(K)$,

$$\begin{aligned}\underline{\nu}_{0*}(s,\boldsymbol{\theta})|_{S_X} &\\ = s\mu_{0*}(\boldsymbol{\theta})|_{S_X}[L_*^{00}(\boldsymbol{\theta})]_{S_X} &+ s^2\mu_0[\underline{R}_*^{0+}(s,\boldsymbol{\theta})]_{S_X}[L_*^{+0}(\boldsymbol{\theta})]_{S_X},\end{aligned} \tag{4.14}$$

$$\begin{aligned}\underline{\nu}_{+*}(s,\boldsymbol{\theta})|_{S_X} &\\ = \mu_{0*}(\boldsymbol{\theta})|_{S_X}[\underline{R}_*^{0+}(s,\boldsymbol{\theta})]_{S_X}&(I - [\underline{G}_*^{0-}(s,\boldsymbol{\theta})]|_{S_X})(I - sK_*(\boldsymbol{\theta}))^{-1}.\end{aligned} \tag{4.15}$$

REMARK 4.2. (4.13) and (4.15) can be considered as extensions of the Pollaczek-Khinchine formula for the $M/G/1$ queue.

In computing the stationary distribution, it is often convenient to use the recursive equation (4.5). It may be considered as a sort of Neuts' matrix analytic approach. In Neuts' approach, a great effort is dedicated to compute R^+, equivalently, $\tilde{G}^+$, under the assumption that $s = 1$ and the level is one dimensional, i.e., additive. It also requires the further assumption that the level is skip free or has exponentially distributed jumps at least in one direction when the level is integer valued or continuous, respectively. In our setting, these problems can be reduced to solve the Wiener-Hopf factorization.

We illustrate a typical example of reflected MAP below, which is a generalization of the 0-partially homogeneous chain studied in [5] (see also [22]).

EXAMPLE 4.2 (RMAP with homogeneous boundary transitions). Under the general setting with $\hat{S}_Y = \mathbb{R}_+^d$ in Section 4, we take as a level function for reflection

$$\hat{\ell}(\boldsymbol{u}) = \min\{u_i; i = 1,2,\ldots,d\}, \qquad \text{for } \boldsymbol{u} = (u_1,\ldots.u_d). \tag{4.16}$$

For each subset F of $D \equiv \{1,2,\ldots,d\}$, let

$$\hat{S}_Y^F = \{\boldsymbol{u} \in \hat{S}_Y; u_i > 0 \text{ for all } i \in F, u_j = 0 \text{ for all } j \in F^c\}.$$

Then, $\hat{S}^F \equiv \hat{S}_X^0 \times \hat{S}_Y^F$ stands for the boundary face for $F \neq D$. We refer to it as the F-face. Obviously, $\hat{S}^0 = \cup_{F\neq D}\hat{S}^F$ is the reflection boundary, and

$\hat{S}^D$ is the interior of the boundaries. Thus, we have a reflected MAP on the nonnegative orthant with boundary faces $\hat{S}^F$ with $D \neq F$.

To make this model more specific, we assume the boundary transitions $L^0((x,\boldsymbol{u}), A \times (B-\boldsymbol{u}))$ and $L^{+0}((x,\boldsymbol{u}), A \times (B-\boldsymbol{u}))$ are independent of the additive component $\boldsymbol{u}$ within each boundary face $\hat{S}^F$. This reflected MAP is said to have homogenous boundary transitions at each boundary face. In particular, if there is no background state, then this process is termed as a 0-partially homogenous chain in [5].

EXAMPLE 4.3 ($M/G/1$-type model). Assume that $S_Y = \mathbb{Z}^d$ and Q is skip free towards the boundary, which implies that $L^{+0}((x,\boldsymbol{y}), A \times B) = Q((x,\boldsymbol{y}), A \times B)$ for $x \in S_X$, $\hat{\ell}(\boldsymbol{y}) = 1$, $A \in S_X$ and $B \in \hat{S}^0$, where $\hat{\ell}$ is defined by (4.16). This model is referred to as $M/GI/1$-type since it is the multi-dimensional version of the $M/G/1$-type queue (see, e.g., [20]). For this model, the stationary equations can be written as

$$\nu_0 = \nu_0 L^{00} + \nu_+ L^{+0}, \qquad \nu_+ = \nu_0 L^{0+} + \nu_+ Q|_{\hat{S}^+}.$$

Hence, summing up these equations, we have $\nu_+(I-Q) = \nu_0(L^0 - I)$, and therefore

$$\nu_{+*}(\boldsymbol{\theta}) \quad = \quad \nu_{0*}(\boldsymbol{\theta})(L^0_*(\boldsymbol{\theta}) - I)(I - Q_*(\boldsymbol{\theta}))^{-1}, \tag{4.17}$$

as long as the right-hand side exists. This form looks to be simpler than the corresponding formula (4.13), but they must be identical. When there is no background state, (4.17) is nothing but the stationary equation, which is stated in [22].

For the case of a continuous state space, it may be convenient to consider densities rather than measures and distributions. In the rest of this section, we consider those densities. Assume that $\mu_0 L^{0+} \ll \pi \otimes m$, then there exist densities

$$\underline{k}_+(s) \equiv \frac{d\underline{\nu}_+(s)}{d\pi \otimes m} \geq 0, \qquad \underline{k}_0(s) = \frac{d\mu_0 \underline{R}^{0+}(s)}{d\pi \otimes m},$$

by Remark 4.1. Hence, from (4.5), we have, using the notation $\underline{k}^-_+(x,\boldsymbol{u}) \equiv \underline{k}_+(x,-\boldsymbol{u})$,

$$\begin{aligned}
&\int_S \underline{k}_+(s)(x,\boldsymbol{u}) g(x,\boldsymbol{u}) \pi(dx) m(d\boldsymbol{u}) \\
&\quad = \int_S \underline{k}_0(s)(x,\boldsymbol{u}) g(x,\boldsymbol{u}) \pi(dx) m(d\boldsymbol{u}) \\
&\qquad\qquad + \int_S \int_S \underline{k}_+(s)(x,\boldsymbol{u}) \underline{\tilde{G}}^+(s)^{\dagger(\pi\otimes m)} g(x,\boldsymbol{u}) \pi(dx) m(d\boldsymbol{u})
\end{aligned}$$

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$$= \int_S \underline{k}_0(s)(x,\boldsymbol{u})g(x,\boldsymbol{u})\pi(dx)m(d\boldsymbol{u}) + \int_S\int_S g(x,-\boldsymbol{u})\underline{\tilde{G}}^+(s)^{\dagger(\pi\otimes m)}\underline{k}_+^-(s)(x,\boldsymbol{u})\pi(dx)m(d\boldsymbol{u})$$

$$= \int_S \underline{k}_0(s)(x,\boldsymbol{u})g(x,\boldsymbol{u})\pi(dx)m(d\boldsymbol{u}) + \int_S\int_S g(x,\boldsymbol{u})\underline{\tilde{G}}^+(s)\underline{k}_+^-(s)(x,-\boldsymbol{u})\pi(dx)m(d\boldsymbol{u}).$$

This concludes that, for $(x,\boldsymbol{u}) \in \hat{S}$,

$$\underline{k}_+(s)(x,\boldsymbol{u}) = \underline{k}_0(s)(x,\boldsymbol{u}) + \underline{\tilde{G}}^+(s)\underline{k}_+^-(s)(x,-\boldsymbol{u}), \qquad s \in [0,1] \tag{4.18}$$

holds almost surely concerning $\pi\otimes m$. This is a version of the Markov renewal equation with kernel $\underline{\tilde{G}}^+(s)$, cf. Alsmeyer [1]

We next specialize (4.18) to a one dimensional additive component, so the level is additive and $s = 1$. In this case, letting $k_+ = \underline{k}_+(1)$ and $u = \boldsymbol{u}$, we have

$$\begin{aligned}\underline{\tilde{G}}^+(1)\underline{k}_+^-(1)(x,-u) &= \int_S \underline{\tilde{G}}^+(1)((x,-u),dy\times dv)k_+(x,-v)\\ &= \int_S \tilde{G}_\triangle^+(x,dy\times dv)k_+(y,u-v)\\ &= (\tilde{G}_\triangle^+ * k_+)(x,u).\end{aligned}$$

Hence, (4.18) is written as

$$k_+(x,u) = k_0(x,u) + (\tilde{G}_\triangle^+ * k_+)(x,u), \qquad u > 0, \tag{4.19}$$

where $k_0 = \underline{k}_0(1)$. Thus, we have Markov renewal equations for the densities k_+.

In the above arguments, we have considered the joint density $k_+(x,u)$. However, we can also consider the tail probability for the additive component. Namely, let

$$\overline{k}_+(x,t) = \frac{\mu_0 R^+(dx,[t,\infty))}{\pi(dx)},$$

which denotes the Radon-Nikodym derivative of measure $\mu_0 R^+(A,[t,\infty))$ for $A \in \mathcal{B}(S_X)$ with respect to π. Similarly, we define Radon-Nikodym derivative $\overline{k}_0(x,t)$ for R^{0+}, provided $L^{0+} \ll K$. Then, from (4.18) and Corollary 3.1, we have

$$\overline{k}_+(x,t) = \overline{k}_0(x,t) + (\tilde{G}_\triangle^+ * \overline{k}_+)(x,t), \qquad t > 0. \tag{4.20}$$

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5. Light tail asymptotics for the stationary distribution. We now apply our results in the previous sections to study tail asymptotics of the stationary distribution of a reflected MAP, provided it exists. We here produce this reflected MAP by the level function $\hat{\ell}(\boldsymbol{u}) = \min(u_1, \ldots, u_d)$. An example of this reflected process is considered in Example 4.2. Thus, the boundary and interior are given by

$$\hat{S}^0 = \{(x, \boldsymbol{u}) \in \hat{S}; \min(u_1, \ldots, u_d) = 0\}, \quad \hat{S}^+ = \{(x, \boldsymbol{u}) \in \hat{S}; \boldsymbol{u} > \boldsymbol{0}\}.$$

As we discussed in the first part of Section 4, we can consider another level function $\hat{\ell}$ as long as it satisfies the compatibility condition (4.1). We define the level process by the level function of (3.8) for each direction vector $\boldsymbol{c}$, which is denoted by $\ell_{\boldsymbol{c}}$. This enables us to consider the stationary tail asymptotics in the direction $\boldsymbol{c}$.

We are here only concerned with the light tail case. For this, we will assume a condition on $K_*(\boldsymbol{\theta})$. In our formulation of the multidimensional reflected process, there are two approaches to study the tail behavior of its stationary distribution. The first approach is to reformulate the Markov additive process to have one dimensional additive component by putting all necessary information into the background state. In this case, a major difficulty is that the Markov additive kernel K may be complicated.

The second approach is to directly consider the multidimensional additive component. In this case, we can work with the original Markov additive kernel. This simplifies descriptions, but we have to directly work on the multidimensional reflected process. This creates another difficulty. In this case, we only consider rough asymptotics for each finite set of background states.

5.1. *One dimensional formulation.* We consider a simple situation such that the reflected MAP is already formulated so as to have a one dimensional additive component. So, we let $\hat{\ell}(u) = \ell(u) = u$, that is, the level is identical with the additive component. Let $\{(X_n, Y_n)\}$ be a real-valued MAP with a general background state space S and Markov additive kernel K, and let $\{(\hat{X}_n, \hat{Y}_n)\}$ be its reflected process with boundary transition operators L^0 and L^{+0}. Thus, K, L^0 and L^{+0} are primitive data to uniquely determine the reflected MAP.

We first consider a general situation, assuming that the reflected MAP has the stationary distribution ν, which is decomposed as ν_0 and ν_+. Then, Corollary 4.3 yields the following result.

PROPOSITION 5.1. *Assume that the stationary distribution exists, and denote a random vector subject to the stationary distribution of the reflected*

MAP with one dimensional additive component by $(\hat{X}, \hat{Y})$. *Then,*

$$\limsup_{t\to\infty} \frac{1}{t} \log P(\hat{Y} > t) \geq -\sup\{\theta > 0; c_p(K_*(\theta)) \geq 1\}. \tag{5.1}$$

PROOF. Since $\underline{G}_*^{0-}(1,\theta)|_{S_X}(A) < 1$ for $\theta > 0$ and any $A \in \mathcal{B}(S_X)$, it follows from (4.15) that, if $\underline{\nu}_{+*}(1,\boldsymbol{\theta})|_{S_X} 1 < \infty$, then

$$(I - K_*(\boldsymbol{\theta}))^{-1} 1 < \infty.$$

This further implies that $c_p(K_*(\boldsymbol{\theta})) \geq 1$. This yields (5.1) since its left-hand side equals $-\sup\{\theta \geq 0; \nu_{+*}(\theta)|_{S_X} 1 < \infty\}$ by the continuous parameter version of the well known Cauchy-Hadamard theorem (see, e.g., Theorem 16.1 of [19]). □

This result can be sharpened in some cases.

PROPOSITION 5.2. *In addition to the assumptions of Proposition 5.1, assume that the minorization condition (3b) is satisfied,* $S_Y = \mathbb{Z}$ *and* $\{Y_n\}$ *is skip free in downward direction, that is,* Y_n *is decreased at most by* 1, *then*

$$\liminf_{n\to\infty} \frac{1}{n} \log P(\hat{Y} > n) \geq -\sup\{\theta > 0; c_p(K_*(\theta)) \geq 1\}. \tag{5.2}$$

The proof of this proposition is quite technical and the assumption on S_X can be relaxed. However, full discussions for them are complicated, so we defer them in Appendix F.

We next consider to refine the above results to be so called exact asymptotics. For this, we use the following condition.

(5a) There exist a positive constant α, a positive measure η on $(S_X, \mathcal{B}(S_X))$ and positive and measurable function h from $(S_X, \mathcal{B}(S_X))$ to $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that

$$\eta K_*(\alpha) = \eta, \tag{5.3}$$

$$K_*(\alpha) h = h. \tag{5.4}$$

This condition is stronger than $c_p(K_*(\alpha)) = 1$, but it is still too weak for us. We assume further conditions which enable us to extend the approach of [24] that considers an integer-valued Markov additive process with a countable background state space.

(5b) The additive process $\{Y_n\}$ is non-lattice.
(5c) $K_*(\alpha + \epsilon) < \infty$ for some $\epsilon > 0$.
(5d) $\eta h \equiv \int_{S_X} \eta(dx) h(x) < \infty$.

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(5e) The restriction of ν on level 0, that is, ν_0, satisfies

$$\nu_0 R_*^{0+}(\alpha)h < \infty. \tag{5.5}$$

REMARK 5.1. Since the one dimensional additive component is identical to the level (see Remark 3.13), $R_*^{0+}(\alpha)$ is the same as $R_{*\triangle}^{0+}(\alpha)$. So, the subscript "$_\triangle$" is omitted in this subsection.

Under the above conditions, we can define the operator $K^{(\alpha)}$ as

$$K^{(\alpha)}(x, dy \times du) = e^{\alpha u} K(x, dy \times du) \frac{h(y)}{h(x)}.$$

That is, $K^{(\alpha)} = K^{(1,\alpha)}$ in the notation of Section 3.3. The conditions (5b) and (5d) are equivalent to $K^{(\alpha)}$ being positive recurrent.

Note that $K^{(\alpha)}$ has stationary measure $\eta \circ h(dx) \equiv \eta(dx)h(x)$, which is finite by (5d). Namely,

$$\int_{S_X} \eta(dx)h(x)K^{(\alpha)}(x, A \times \mathbb{R}) = \int_A \eta(dx)h(x), \qquad A \in \mathcal{B}(\mathbb{R}).$$

Denote the Markov additive process generated by $K^{(\alpha)}$ by $\{(X_n^{(\alpha)}, Y_n^{(\alpha)})\}$. The operators corresponding to R^+, G^{0-}, G^+ are denoted by $R^{(\alpha)+}, G^{(\alpha)0-}$, $G^{(\alpha)+}$, respectively.

We also define a dual Markov additive process $\{(\tilde{X}_n^{(\alpha)}, \tilde{Y}_n^{(\alpha)})\}$ for $\{(X_n^{(\alpha)}, Y_n^{(\alpha)})\}$ using the dual kernel:

$$\begin{aligned} \tilde{K}^{(\alpha)}(x, dy \times du) &\equiv \frac{K^{(\alpha)}(y, dx \times du)}{\eta(dx)h(x)} h(y)\eta(dy) \\ &= e^{\alpha u} \frac{K(y, dx \times du)}{\eta(dx)} \eta(dy), \end{aligned}$$

where $\frac{K(y,dx\times du)}{\eta(dx)}$ is the Radon-Nikodym derivative of measure $\{K(y, A \times B); A \in \mathcal{B}(S_X)\}$ with respect to η for each fixed $y \in S_X$ and $B \in \mathcal{B}(S_Y)$. In what follows, this convention for a Radon-Nikodym derivative will be used, but we also use the conventional notation $\frac{d\eta_1}{d\eta_2}(x)$ for measures η_1 and η_2 such that $\eta_2 \ll \eta_1$, that is, η_2 is absolutely continuous with respect to η_1. The following lemma is a key to consider the asymptotic behavior of the stationary distribution ν.

LEMMA 5.1. *Under the conditions (5a) and (5b),*

$$\tilde{G}^{(\alpha)+}(x, dy \times du) = e^{\alpha u} \frac{R^+(y, dx \times du)}{\eta(dx)} \eta(dy). \tag{5.6}$$

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Furthermore, $\tilde{G}^{(\alpha)+}(x, dy \times (0,\infty))$ *is a stochastic kernel and has the finite invariant measure* ξ *given by*

$$\xi = (\eta \circ h)(I - \tilde{R}_*^{(\alpha)0-}(0)), \tag{5.7}$$

so it is positive recurrent.

We defer the proof of this lemma to Appendix E since it is similar to Lemma 4.2 of [24]. We are now ready to present a main result of this subsection.

THEOREM 5.1. *Assume that the reflected MAP* $\{(\hat{X}_n, \hat{Y}_n)\}$ *satisfies conditions (5a)–(5e). Let*

$$b(\alpha) = \alpha \int_0^\infty \eta(dx) K'_*(\alpha) h(x).$$

Then $b(\alpha)$ *is finite, and we have, for a.s.* η,

$$\begin{aligned} \lim_{t\to\infty} e^{\alpha t} \frac{\nu(dx \times [t,\infty))}{\eta(dx)} \\ = \frac{1}{b(\alpha)} \int_0^\infty e^{\alpha u} \int_{S_X} \frac{\nu_0 R^{0+}(dx \times [u,\infty))}{\eta(dx)} \xi(dx)du. \end{aligned} \tag{5.8}$$

Furthermore, if $L^{0+} \ll K$, *then, for a.s.* $\eta \otimes m$,

$$\lim_{t\to\infty} e^{\alpha t} \frac{d\nu}{d\eta \otimes m}(x,t) = \frac{1}{b(\alpha)} \int_0^\infty e^{\alpha u} \int_{S_X} \frac{d\nu_0 R^{0+}}{d\eta \otimes m}(x,u)\xi(dx)du, \tag{5.9}$$

PROOF. From (3.26) and (4.15) with $s = 1$, we have

$$\begin{aligned} &\nu_+(dx \times du) \\ &\quad = \mu_0 R^{0+}(dx \times du) + \int_{v=0}^u \int_{S_X} \nu_+(dy \times dv) R^+_\triangle(y, dx \times (du - v)) \\ &\quad = \mu_0 R^{0+}(dx \times du) + \int_{v=0}^u \int_{S_X} \nu_+(dy \times du - v) R^+_\triangle(y, dx \times dv). \end{aligned}$$

Integrating both sides concerning u over $[t,\infty)$, we have

$$\begin{aligned} &\nu_+(dx \times [t,\infty)) \\ &\quad = \mu_0 R^{0+}(dx \times [t,\infty)) + \int_{v=0}^t \int_{S_X} \nu_+(dy \times [t-v,\infty)) R^+_\triangle(y, dx \times dv). \end{aligned}$$

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Let

$$\psi(x,t) = e^{\alpha t}\frac{\nu_+(dx\times[t,\infty))}{\eta(dx)}, \qquad \psi_0(x,t) = e^{\alpha t}\frac{\mu_0 R^{0+}(dx\times[t,\infty))}{\eta(dx)}.$$

Then, applying Lemma 5.1 to the above formula yields

$$\begin{aligned}\psi(x,t) &= \psi_0(x,t) + \int_{u=0}^{t}\int_{S_X} \tilde{G}_{\triangle}^{(\alpha)+}(x, dy\times du)\psi(y,t-u)\\ &= \psi_0(x,t) + (\tilde{G}_{\triangle}^{(\alpha)+}) * \psi(x,t).\end{aligned}$$

Hence, by the Markov renewal theorem, we have

$$\lim_{t\to\infty}\psi(x,t) = \frac{1}{b(\alpha)}\int_{u=0}^{\infty}\int_{S_X}\psi_0(y,u)\xi(dy)du.$$

This yields (5.8) since $\nu = \nu_+$ on $S_X\times(0,\infty)$. Since the assumption $L^{0+}\ll K$ implies $\nu \ll \eta\otimes m$, similar arguments can be applied to the density $\frac{d\nu}{d\eta\otimes m}(x,t)$, and (5.8) is obtained. It remains to prove the finiteness of $b(\alpha)$. This can be done similarly to the proof of Theorem 4.1 of [24]. □

EXAMPLE 5.4. Let (X_n) be an AR(1) process, i.e., for some constant $a\in[0,1)$, let $X_{n+1} = aX_n + B_{n+1}$, with (B_n) an i.i.d. sequence with negative mean and moment generating function $\beta(s) = E[\exp\{sB\}]$. Let the reflected process $(\hat{Y}_n, X_n)$ be given by

$$\hat{Y}_{n+1} = (\hat{Y}_n + X_{n+1})^+. \tag{5.10}$$

Assume there exists a strictly positive solution to the equation $E[\exp\{sB\}] = 1$ and $E[\exp\{(s+\epsilon)B\}] < \infty$ for some $\epsilon > 0$. For convenience, we also assume that B has a continuous distribution. The associated free Markov additive process (Y_n, X_n) is defined by $Y_n = X_1 + \cdots + X_n$. Then, the n-th increment of the additive component is identical with the n-th background state X_n, and the Markov additive kernel is given by

$$K_*(\theta)f(x) = e^{\theta a x}E(e^{\theta B}f(ax+B)), \qquad \theta\ge 0, f\in M_+(\mathbb{R}), \tag{5.11}$$

as long as it exists.

We now show that all five conditions of the above theorem hold with $\alpha > 0$ satisfying $\beta(\alpha/(1-a)) = 1$. We first note that $L^{0+}\ll K$ is automatically satisfied since the background process is unchanged at the boundary. Since $L^{0+}((x,0),[0,u]\times[0,u]) = P(ax+B\le u)$, we can apply Corollary 4.1. The conditions (5b) and (5c) follow from our assumption on the distribution of

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B. Set $h(y) = \exp\{y\alpha a/(1-a)\}$, and let η be the distribution of a random variable V which has moment generating function

$$v(s) = E[e^{sV}] = \prod_{n=0}^{\infty} \frac{\beta(\frac{1-a^{n+1}}{1-a}\alpha + a^n s)}{\beta(\frac{1-a^{n+1}}{1-a}\alpha)}, \qquad s \in [0, s_0), \tag{5.12}$$

where $s_0 = \epsilon + \frac{a}{1-a}\alpha$. Obviously, all individual components in this infinite product are finite. To see its convergence, set $c_n = \frac{1-a^{n+1}}{1-a}\alpha$. Since β is log-convex and $\beta(s)$ is increasing around $s = \alpha/(1-a)$, for sufficiently large n,

$$0 \le \log \beta(c_n + a^n s) - \log \beta(c_n) \le a^n s \frac{\beta'(c_n + a^n s)}{\beta(c_n + a^n s)}$$

This is converging to 0 exponentially fast as $n \to \infty$, guaranteeing the finiteness of the infinite product.

We show that these particular choices of α, η and h yield condition (5a). Namely, we need to verify

$$\int_{-\infty}^{+\infty} \eta(dx) E(e^{(s+\alpha)(ax+B)}) = \int_{-\infty}^{+\infty} \eta(dx) e^{sx}.$$

This is equivalent to

$$v((s+\alpha)a)\beta(s+\alpha) = v(s).$$

It can be verified by substitution or iteration that the solution of this equation is given by (5.12). Condition (5d) reduces to showing $v(\alpha a/(1-a)) < \infty$, which follows from the expression for $v(\cdot)$ and the fact that

$$\beta\left(\frac{1-a^{n+1}}{1-a}\alpha + a^n \frac{\alpha a}{(1-a)}\right) = \beta\left(\frac{\alpha}{1-a}\right) = 1 < \infty.$$

It remains to prove (5e). Recall that Corollary 4.1 can be applied, and (in the stationary setting),

$$\nu_0(A) = P(\hat{Y}_n = 0, X_n \in A) \le \pi(A),$$

where π is the stationary distribution of the background process $\{X_n\}$. Hence, it is sufficient to prove (5e) that $\pi h < \infty$, which indeed holds since $\pi_*(s) = \prod_{n=0}^{\infty} \beta(a^n s)$, and

$$\int_{-\infty}^{+\infty} \pi(dx) h(x) = \pi_*\left(\frac{a\alpha}{1-a}\right).$$

Here, we should note that the infinite product representation of π_* is well defined by similar reasons as before.

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Thus, Theorem 5.1 implies that the stationary distribution of the level process is exponential with rate α, up to a function that converges to a constant. This is an improvement of the logarithmic asymptotics which follows from the standard framework of Glynn & Whitt [12]. A related model is considered by Kim & Sohraby [15], who consider that a is a 0-1 random variable, allowing the usage of more basic methods.

5.2. *Multidimensional formulation.* We next consider the case that the additive component is multidimensional. One may expect that Proposition 5.1 can be extended. However, things are not so simple as it may look since $\underline{G}_*^{0-}(1,\boldsymbol{\theta})|_{S_X}(A) < 1$ may not be true for $\boldsymbol{\theta} \geq \mathbf{0}$. This is because some of the additive components may be positive even if the level is below zero.

In this subsection, S_Y is either $\mathbb{R}^d$ or $\mathbb{Z}^d$, and S_X is arbitrary. However, we mainly consider rough asymptotics of the additive component for each fixed finite set of background states. This means that our results are useful only for the background states which are atomic. This is particularly the case when S_X is finite or countable.

Definition 5.1. For a random element $(X,Y) \in S_X \times S_Y$ and a direction vector $\boldsymbol{c} \geq \mathbf{0}$ such that $\|\boldsymbol{c}\| = 1$, define $\overline{\alpha}_Y(A,\boldsymbol{c})$ and $\underline{\alpha}_Y(A,\boldsymbol{c})$ for $A \in \mathcal{B}(S_X)$ as

$$\limsup_{t\to\infty} \frac{1}{t} \log P(X \in A, Y > t\boldsymbol{c}) = -\overline{\alpha}_Y(A,\boldsymbol{c}),$$
$$\liminf_{t\to\infty} \frac{1}{t} \log P(X \in A, Y > t\boldsymbol{c}) = -\underline{\alpha}_Y(A,\boldsymbol{c}),$$

which are referred to as upper and lower decay rates, respectively, of Y in the direction $\boldsymbol{c}$ given $X = x$.

From these definitions, we have, for sufficiently large t,

$$e^{-(\underline{\alpha}_Y(A,\boldsymbol{c})+\epsilon)t} \leq P(X \in A, Y > t\boldsymbol{c}) \leq e^{-(\overline{\alpha}_Y(A,\boldsymbol{c})-\epsilon)t}. \tag{5.13}$$

In the theory of large deviations, it typically occurs that $\overline{\alpha}_Y(A,\boldsymbol{c}) = \underline{\alpha}_Y(A,\boldsymbol{c})$. However, there are few results for computing the decay rates from primitive modeling parameters. Moreover, most of them are limited to the two dimensional case without background state although exact asymptotics are also obtained (e.g., see [5, 21, 22]). We are interested to find the rough asymptotics for the multidimensional reflected MAP which may have background states, where reflection at the boundary may be arbitrarily given. In light of the above discussions, it would be very hard even to compute the decay rates. So, we consider to bound the decay rates.

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Let $\{(\hat{X}_n, \hat{Y}_n)\}$ be the multidimensional reflected MAP with the level function $\hat{\ell}$ of (4.16), and assume that it has the stationary distribution ν. Let $(\hat{X}, \hat{Y})$ be a random vector subject to this distribution ν. Denote its domain by

$$\mathcal{D}_{\hat{Y}} = \{\boldsymbol{\theta} \in \mathbb{R}^d; \nu_{+*}(\boldsymbol{\theta})|_{S_X} g < \infty, \exists g \in M_+(S_X)\}.$$

Since $t > 0$ and $\hat{Y} > t\boldsymbol{c}$ imply

$$t \inf_{\boldsymbol{u} > \boldsymbol{c}} \langle \boldsymbol{u}, \boldsymbol{\theta} \rangle \le \langle \boldsymbol{\theta}, \hat{Y} \rangle,$$

we have

$$\inf_{\boldsymbol{u} > \boldsymbol{c}} e^{t\langle \boldsymbol{u}, \boldsymbol{\theta} \rangle} E(g(\hat{X}) 1(\hat{Y} > t\boldsymbol{c})) \le E(g(\hat{X}) e^{\langle \boldsymbol{\theta}, \hat{Y} \rangle}) = \nu_*(1, \boldsymbol{\theta})|_{S_X} g.$$

Hence, if A is a finite set, then we have

$$\min_{x \in A} g(x) e^{t \inf_{\boldsymbol{u} > \boldsymbol{c}} \langle \boldsymbol{u}, \boldsymbol{\theta} \rangle} P(\hat{X} \in A, \hat{Y} > t\boldsymbol{c}) \le \nu_*(1, \boldsymbol{\theta})|_{S_X} g$$

because $e^{t\langle \boldsymbol{u}, \boldsymbol{\theta} \rangle}$ is continuous in $\boldsymbol{u} \in \mathbb{R}^d$. Since the right-hand side is finite for $\boldsymbol{\theta} \in \mathcal{D}_{\hat{Y}}$ and $h > 0$, we have, for finite set $A \in \mathcal{B}(S_X)$,

$$-\overline{\alpha}_Y(A, \boldsymbol{c}) \le -\sup_{\boldsymbol{\theta} \in \mathcal{D}_{\hat{Y}}} \inf_{\boldsymbol{u} > \boldsymbol{c}} \langle \boldsymbol{u}, \boldsymbol{\theta} \rangle. \tag{5.14}$$

Thus, the domain $\mathcal{D}_{\hat{Y}}$ is important to bound the decay rates. To consider this domain, we introduce the following sets. Here, S_X can be general.

$$\mathcal{C}_1(K) = \{\boldsymbol{\theta} \in \mathbb{R}^d; c_p(K_*(\boldsymbol{\theta})) > 1\},$$
$$\mathcal{D}_0 = \{\boldsymbol{\theta} \in \mathbb{R}^d; \mu_{0*}(\boldsymbol{\theta}) L_*^{0+}(\boldsymbol{\theta})|_{S_X} h < \infty, \exists h \in \mathcal{H}(\boldsymbol{\theta})\},$$

where for each $\boldsymbol{\theta} \in \mathbb{R}^d$,

$$\mathcal{H}(\boldsymbol{\theta}) = \{h \in M_+(S_X); tK_*(\boldsymbol{\theta})h \le h \text{ for some } t > 0\}.$$

We will use the following fact, which is proved in Appendix G.

LEMMA 5.2. *$\mathcal{D}_0$ is a convex set.*

Note that $\mathcal{C}_1(K)$ is a convex open set by Lemma 3.7. Hence, by Lemma 5.2, $\mathcal{C}_1(K) \cap \mathcal{D}_0$ is a convex set.

THEOREM 5.2. *For the reflected MAP satisfying the irreducibility assumption (3a), if the stationary distribution exists, then, for any direction vector $\boldsymbol{c} \ge \boldsymbol{0}$ and any finite set $A \in \mathcal{B}(S_X)$,*

$$\limsup_{t\to\infty}\frac{1}{t}\log P(X\in A, Y>t\boldsymbol{c}) \le -\sup_{\boldsymbol{\theta}\in\mathcal{C}_1(K)\cap\mathcal{D}_0}\inf_{\boldsymbol{u}>\boldsymbol{c}}\langle \boldsymbol{u},\boldsymbol{\theta}\rangle. \tag{5.15}$$

In particular, if $\mathcal{C}_1(K)$ is bounded, then

$$\limsup_{t\to\infty}\frac{1}{t}\log P(X\in A, Y>t\boldsymbol{c}) \le -\inf_{\boldsymbol{u}>\boldsymbol{c}}\sup_{\boldsymbol{\theta}\in\mathcal{C}_1(K)\cap\mathcal{D}_0}\langle \boldsymbol{u},\boldsymbol{\theta}\rangle. \tag{5.16}$$

REMARK 5.2. If $\boldsymbol{\theta}$ in (5.15) is limited in $\mathbb{R}^d_+$, then (5.15) becomes

$$\limsup_{t\to\infty}\frac{1}{t}\log P(X\in A, Y>t\boldsymbol{c}) \le -\sup_{\boldsymbol{\theta}\in\mathcal{C}_1(K)\cap\mathcal{D}_0\cap\mathbb{R}^d_+}\langle \boldsymbol{c},\boldsymbol{\theta}\rangle. \tag{5.17}$$

This is simpler, but less sharp than (5.15) as well as (5.16) in general.

REMARK 5.3. If Q is irreducible, then $K_*(t\boldsymbol{u})(x)$ diverges as $t\to\infty$ for each non-zero $\boldsymbol{u}\in\mathbb{R}^d$ and almost all $x\in S_X$ with respect to the subinvariant measure π of K_X, and therefore $c_p(K_*(t\boldsymbol{u}))$ converges to zero as $t\to\infty$. Hence, $\mathcal{C}_1(K)$ is bounded, and we have (5.16). The irreducibility of Q means that the Markov modulated random walk $\{Y_n\}$ eventually hits any bounded open ball in $\mathbb{R}^d$ with positive probability, so it is relatively easily checked in application.

REMARK 5.4. In the large deviations context, $\mathcal{C}_1(K)$ comes from the Markov additive process without reflection, while $\mathcal{D}_0$ reflects the boundary effect. They also correspond with (5a) and (5c) of the one dimensional case. So, they are quite natural, and we conjecture that the interiors of $\mathcal{D}_{\hat{Y}}$ and $\mathcal{C}_1(K)\cap\mathcal{D}_0$ are identical. We may also conjecture this upper bound is tight, that is, identical with the decay rate.

PROOF. Obviously $\mathbf{0}\in\mathcal{C}_1(K)\cap\mathcal{D}_0$. If there is no other vector in $\mathcal{C}_1(K)\cap\mathcal{D}_0$, (5.15) clearly holds with 0 in the right-hand side. So, we consider the case that there is a nonzero vector in $\mathcal{C}_1(K)\cap\mathcal{D}_0$. By Lemma 4.1, it is easy to see that

$$\underline{\nu}_+(s)\underline{R}^{0+}(s)f \le s\underline{\nu}_+(s)L^{0+}(I-\underline{G}^{0-}(s))^{-1}f, \qquad f\in M_+(S),$$

and therefore (4.7) of Theorem 4.1 yields

$$\begin{aligned}\underline{\nu}_+(s)f &\le s\mu_0 L^{0+}(I-\underline{G}^{0-}(s))^{-1}(I-\underline{G}^{0-}(s))(I-sQ)^{-1}f\\ &= s\mu_0 L^{0+}(I-sQ)^{-1}f,\end{aligned}$$

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where we have used the fact that

$$(I - \underline{G}^{0-}(s))(I - sQ)^{-1} = (I - \tilde{\underline{G}}^{+}(s)^{\dagger(\pi\otimes m)})^{-1}$$

is nonnegative. Thus, we have

$$\underline{\nu}_{+*}(s,\boldsymbol{\theta})|_{S_X} g \le s\mu_{0*}(\boldsymbol{\theta})L_*^{0+}(\boldsymbol{\theta})|_{S_X}(I - sK_*(\boldsymbol{\theta}))^{-1}g, \quad g \in M_+(S_X), \tag{5.18}$$

as long as the right-hand side is finite. For any $\boldsymbol{\theta} \in \mathcal{C}_1(K) \cap \mathcal{D}_0$, there exists $\delta > 0$ such that $c_p(K_*(\boldsymbol{\theta})) \ge 1 + \delta$. For this $\boldsymbol{\theta}$, there is a superharmonic function $h \in \mathcal{H}(\boldsymbol{\theta}')$ such that

$$\mu_{0*}(\boldsymbol{\theta})L^{0+}(\boldsymbol{\theta})|_{S_X} h < \infty,$$

and $c_p(K_*(\boldsymbol{\theta})) \ge 1 + \delta$ implies

$$(1 + \delta)K_*(\boldsymbol{\theta})h \le h.$$

Hence, (5.18) with $g = h$ and $s = 1$ yields

$$\underline{\nu}_{+*}(1,\boldsymbol{\theta})|_{S_X} h \le \frac{(1+\delta)}{\delta}\mu_{0*}(\boldsymbol{\theta})L_*^{0+}(\boldsymbol{\theta})|_{S_X} h < \infty,$$

which implies that $-\overline{\alpha}_{\hat{Y}}(A,\boldsymbol{c}) \le -\langle \boldsymbol{c},\boldsymbol{\theta}\rangle$ since $h(x) > 0$. Thus, (5.15) is obtained if $\mathcal{C}_1(K) \cap \mathcal{D}_0 \ne \emptyset$. Otherwise, we have the trivial bound 0.

If $\mathcal{C}_1(K)$ is bounded, then $\mathcal{C}_1(K) \cap \mathcal{D}_0$ is a bounded convex set. Then, by Corollary 37.3.2 of [30], we have

$$\sup_{\boldsymbol{\theta}\in\mathcal{C}_1(K)\cap\mathcal{D}_0} \inf_{\boldsymbol{u}>\boldsymbol{c}} \langle \boldsymbol{u},\boldsymbol{\theta}\rangle = \inf_{\boldsymbol{u}>\boldsymbol{c}} \sup_{\boldsymbol{\theta}\in\mathcal{C}_1(K)\cap\mathcal{D}_0} \langle \boldsymbol{u},\boldsymbol{\theta}\rangle.$$

Thus, we get (5.16) from (5.15). □

In light of Proposition 5.2 and Remark 5.4, we conjecture:

CONJECTURE 5.1. *For the reflected MAP with general S_X satisfying (3a) and (3b), if the stationary distribution exists, then, for any direction vector $\boldsymbol{c} \ge \mathbf{0}$ and $A \in \mathcal{B}(S_X)$,*

$$\liminf_{t\to\infty} \frac{1}{t}\log P(\hat{X} \in A, \hat{Y} > t\boldsymbol{c}) \ge -\sup\{\langle \boldsymbol{c},\boldsymbol{\theta}\rangle; c_p(K_*(\boldsymbol{\theta})) \ge 1\}. \tag{5.19}$$

If there is no background state, this conjecture can be verified for the two dimensional 0 partially chain of [5] and for multi-dimensional reflected process with simple boundaries of [18]. However, it seems hard to prove it for the case with background states.

6. Concluding remarks. There are a number of possibilities for extensions of the present results.

1. The model of Example 5.4 may be extended to higher order autoregressive processes and/or more general background Markov chains.
2. The upper bound (5.15) may not be easy to use since it involves the unknown factor μ_0. There have been some studies to find bounds for $\mu_{0*}(\boldsymbol{\theta})$ for the two dimensional reflecting process without background state (e.g., see [7, 21, 22, 23]).
3. We have only considered discrete time Markov additive process. Using an embedded process, it is possible to convert a continuous time process to a discrete time process. This is particularly the case when the additive component is monotone between embedded epochs. However, this is not always the case. So, it is still interesting to consider a continuous time Markov additive process, and to find decomposition formulas similar to the Wiener Hopf factorization and their extended versions. There are some results on this problem, but they need strong assumptions such that the additive component is one dimensional and the background states are finitely many (e.g., see [14]). So far, it is also very challenging to find such decomposition formulas for more general continuous time processes.

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APPENDIX A: PROOF OF LEMMA 2.1

Similarly to the proof of Proposition 3.2 of [29], we use the following identity for $f \in M_b(S)$.

$$\text{(A.1)}\quad \sum_{n=0}^{\infty} s^n f(Z_n)1(\tau > n) + s^\tau f(Z_\tau)1(\tau < \infty) = f(Z_0) + \sum_{n=0}^{\infty} s^{n+1} f(Z_{n+1})1(\tau > n)$$

Since τ is a stopping time,

$$\begin{aligned} E_x(f(Z_{n+1})1(\tau > n)) &= E_x(E(f(Z_{n+1})|\mathcal{F}_n)1(\tau > n)) \\ &= E_x(E(f(Z_{n+1})|Z_n)1(\tau > n)) \\ &= E_x(1(\tau > n)Qf(Z_n)), \end{aligned}$$

Taking the expectation of (A.1) with respect to E_x, we have (2.1).

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APPENDIX B: PROOF OF LEMMA 3.1

The subinvariance is equivalent to

$$\int_S \pi(dx)m(d\boldsymbol{u})Q((x,\boldsymbol{u}),dx'\times d\boldsymbol{u}')g(x',\boldsymbol{u}') \le \int_S \pi(dx)m(d\boldsymbol{u})g(x,\boldsymbol{u}), \quad g\in M_b^0(S). \tag{B.1}$$

Substituting the definition of Q, we have

$$\begin{aligned}
&\text{The left-hand side of (B.1)}\\
&\quad=\int_S \pi(dx)m(d\boldsymbol{u})\int_S K(x,dx'\times d\boldsymbol{u}')g(x',\boldsymbol{u}+\boldsymbol{u}')\\
&\quad=\int_S \pi(dx)m(d\boldsymbol{u})\int_S K(x,dx'\times (d\boldsymbol{u}'-\boldsymbol{u}))g(x',\boldsymbol{u})\\
&\quad=\int_S \pi(dx)m(d\boldsymbol{u})\int_{S_X} K(x,dx'\times S_Y)g(x',\boldsymbol{u})\\
&\quad\le\int_{S_Y} m(d\boldsymbol{u})\int_{S_X}\pi(dx')g(x',\boldsymbol{u}),
\end{aligned}$$

where the second equality is obtained since the Lebesgue measure is shift invariant, and the last inequality is obtained from the subinvariance of π. Thus we get (B.1).

APPENDIX C: PROOF OF LEMMA 3.2

We first prove (3.5). From the definition of Q and $\tilde{K}$, we have

(C.1) The left-hand side of (3.5)

$$\begin{aligned}
&=\int_{A\times B}\pi(dx)m(d\boldsymbol{u})K(x,A'\times(-\boldsymbol{u}+B'))\\
&=\int_{A'}\pi(dx)\int_{S_Y} m(d\boldsymbol{u})\tilde{K}(x,A\times(-\boldsymbol{u}+B'))1(\boldsymbol{u}\in B),
\end{aligned}$$

where $\boldsymbol{u}+B=\{\boldsymbol{u}+\boldsymbol{y};\boldsymbol{y}\in B\}$. Since the Lebesgue measure m is shift invariant, the integration with respect to m in the last term becomes

$$\begin{aligned}
&\int_{S_Y}\int_A m(d\boldsymbol{u})\tilde{K}(x,dy\times(-\boldsymbol{u}+B'))1(\boldsymbol{u}\in B)\\
&\qquad=\int_{S_Y}\int_{S_Y}\int_A m(d\boldsymbol{u})\tilde{K}(x,dy\times d\boldsymbol{v})1(\boldsymbol{u}\in B,\boldsymbol{v}\in-\boldsymbol{u}+B')
\end{aligned}$$

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$$
\begin{aligned}
&= \int_{S_Y}\int_{S_Y}\int_A m(d\boldsymbol{u}-\boldsymbol{v})\tilde{K}(x,dy\times d\boldsymbol{v})1(\boldsymbol{u}-\boldsymbol{v}\in B,\boldsymbol{u}\in B')\\
&= \int_{S_Y} m(d\boldsymbol{u})\int_{S_Y}\int_A \tilde{K}(x,dy\times d\boldsymbol{v})1(\boldsymbol{v}\in \boldsymbol{u}-B,\boldsymbol{u}\in B')\\
&= \int_{S_Y} m(-d\boldsymbol{u})\int_{S_Y}\int_A \tilde{K}(x,dy\times d\boldsymbol{v})1(\boldsymbol{v}\in -\boldsymbol{u}-B,\boldsymbol{u}\in -B')\\
&= \int_{-B'} m(d\boldsymbol{u})\tilde{K}(x,A\times(-\boldsymbol{u}-B)).
\end{aligned}
$$

Substituting this into (C.1), we have (3.5) since

$$
\begin{aligned}
\tilde{Q}1_{A\times(-B)}(x,\boldsymbol{u}) &= \int_S \tilde{K}(x,dy\times d\boldsymbol{v})1(y\in A,\boldsymbol{u}+\boldsymbol{v}\in -B)\\
&= \tilde{K}(x,A\times(-\boldsymbol{u}-B)).
\end{aligned}
$$

We can write (3.5) symbolically as

$$
\pi(dx)m(d\boldsymbol{u})Q((x,\boldsymbol{u}),dy\times d\boldsymbol{v}) = \pi(dy)m(d\boldsymbol{v})\tilde{Q}((y,\boldsymbol{v}),dx\times d\boldsymbol{u}).
$$

Repeatedly using this expression, we obtain

$$
\begin{aligned}
&\pi(dx)m(d\boldsymbol{u})Q((x,\boldsymbol{u}),dy\times d\boldsymbol{v})Q((y,\boldsymbol{v}),dz\times d\boldsymbol{w})\\
&\qquad = \pi(dz)m(d\boldsymbol{w})\tilde{Q}((z,\boldsymbol{w}),dy\times d\boldsymbol{v})\tilde{Q}((y,\boldsymbol{v}),dx\times d\boldsymbol{u}).
\end{aligned}
$$

We further repeat this computation post-multiplying Q for $n-2$ times and integrating h over it, to arrive at (3.6).

APPENDIX D: PROOF OF LEMMA 3.3

We first note that, for $n\geq 1$,

$$
\begin{aligned}
&R^+((x,\boldsymbol{u}),A\times B;n)\\
&\quad = P_{(x,\boldsymbol{u})}(X_n\in A,Y_n\in B,J_0<J_n\leq \min(J_1,\ldots,J_{n-1}))\\
&\quad = E_{(x,\boldsymbol{u})}(P_{(X_1,Y_1)}(X_n\in A,Y_n\in B,\ell(\boldsymbol{u})<J_n\leq \min(J_1,\ldots,J_{n-1})))\\
&\quad = \int_{S_X}\int_{\hat{S}_Y^{\ell(\boldsymbol{u})+}} Q((x,\boldsymbol{u}),dy\times d\boldsymbol{v})\\
&\qquad \times P_{(y,\boldsymbol{v})}(X_{n-1}\in A,Y_{n-1}\in B\cap \hat{S}_Y^{\ell(\boldsymbol{u})+},J_{n-1}\leq \min(J_0,\ldots,J_{n-2})).
\end{aligned}
\tag{D.1}
$$

To compute the last probability term, we tentative use the following notation for $n\geq 0$.

$$
U((x,\boldsymbol{u}),A\times B;n) = P_{(x,\boldsymbol{u})}((X_n,Y_n)\in A\times B,J_n\leq \min(J_0,\ldots,J_{n-1})).
$$

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Then, from the Markov property,

$$\begin{aligned} &U((x,\boldsymbol{u}),A\times B;n)\\ &\quad=\sum_{n'=1}^{n} P_{(x,\boldsymbol{u})}((X_n,Y_n)\in A\times B,\tau_0^{0-}=n',J_{n-1}\\ &\qquad\qquad\qquad\qquad\qquad\qquad\le \min(J_{n'},J_{n'+1},\dots,J_{n-2}))\\ &\quad=\sum_{n'=1}^{n}\int_S G^{0-}((x,\boldsymbol{u}),dy\times d\boldsymbol{v};n')U((y,\boldsymbol{v}),A\times B;n-n'),\qquad n\ge 1. \end{aligned}$$

Denote the moment generation function operator corresponding with U by $U(s)$. This yields $U(s)=G^{0-}(s)\,(I+U(s))$, which concludes

$$U(s)=(I-G^{0-}(s))^{-1}.$$

Thus, summing (D.1) multiplied with s^n over $n\ge 1$, we have (3.9) for $s\in[0,1)$.

APPENDIX E: PROOF OF LEMMA 5.1

From (3.26), it follows that

$$\begin{aligned} \tilde{G}^{(\alpha)+}(x,dy\times du) &= R^{(\alpha)+}(y,dx\times du)\frac{\eta(dy)h(y)}{\eta(dx)h(x)}\\ &= e^{\alpha u}R^{+}(y,dx\times du)\frac{\eta(dy)}{\eta(dx)}. \end{aligned}\tag{E.1}$$

Hence, we have (5.6). The Wiener-Hopf factorization (3.25) for $d=1$ and $s=t=1$ can be written as

$$I-K_*^{(\alpha)}(\theta)=(I-R_*^{(\alpha)+}(\theta))(I-G_*^{(\alpha)0-}(\theta)),\qquad \theta\le 0.\tag{E.2}$$

Note that $\eta\circ h$ is the invariant measure of $K_*^{(\alpha)}(0)$, and $G_*^{(\alpha)0-}(0)$ with $\alpha>0$ is strictly sub-stochastic. These imply that $\eta\circ h$ is also the invariant measure of $R_*^{(\alpha)+}(0)$. Hence, from (E.1), we can see that $\tilde{G}^{(\alpha)+}(x,S)=1$, that is, $\tilde{G}_*^{(\alpha)+}(0)$ is stochastic.

Taking the transpose of both sides of (E.2), we have

$$I-\tilde{K}_*^{(\alpha)}(\theta)=(I-\tilde{R}_*^{(\alpha)0-}(\theta))(I-\tilde{G}_*^{(\alpha)+}(\theta)),\qquad \theta\le 0.\tag{E.3}$$

This yields

$$\xi=\xi\tilde{G}_*^{(\alpha)+}(0),$$

for ξ of (5.7) since η is also the invariant measure of $\tilde{K}_*^{(\alpha)}(0)$. Since the total variation of ξ is finite and $\tilde{G}_*^{(\alpha)+}(0)$ is stochastic, ξ must be the finite invariant measure of $\tilde{G}_*^{(\alpha)+}(0)$ (see, e.g., the arguments in the proof of Lemma 4.2 of [24]). This proves Lemma 5.1.

APPENDIX F: PROOF OF PROPOSITION 5.1

We prove Proposition 5.2 under the multidimensional additive component setting because preliminary results are obtained under such situation. For this, we refer to the following results on large deviations of a Markov additive process. They are given for the dual MAP $\{(\tilde{X}_n, \tilde{Y}_n)\}$ for our convenience.

(L1) [**Theorem 1 of [27]**] If $\{(\tilde{X}_n, \tilde{Y}_n)\}$ satisfies (3a) and there exist some $n_0 \geq 1$, a family of measure $\{\tilde{h}(x, \cdot)\}$ and a measure $\tilde{\lambda}$ on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ such that

$$\tilde{h}(x)\tilde{\lambda}(A \times B) \leq \tilde{K}^{n_0}(x, A \times B), \quad x \in S, A \in \mathcal{S}_X, B \in \mathcal{B}(\mathbb{R}^d), \tag{F.1}$$

then large deviations lower bound for $\{P_x(\tilde{X}_n \in A, \tilde{Y}_n \in nB)\}$ with open set B and $\pi(A) > 0$ is given by

$$\liminf_{n\to\infty} \frac{1}{n} \log P_{(\tilde{X}_0, \tilde{Y}_0)}(\tilde{X}_n \in A, \tilde{Y}_n \in nB) \geq - \inf_{\boldsymbol{u} \in G} \tilde{\Lambda}^*(\boldsymbol{u}), \tag{F.2}$$

where $\tilde{\Lambda}^*(\boldsymbol{u})$ is the convex conjugate of $\tilde{\Lambda}(\boldsymbol{\theta}) \equiv -\log c_p(\tilde{K}_*(\boldsymbol{\theta}))$, that is,

$$\tilde{\Lambda}^*(\boldsymbol{u}) = \sup_{\boldsymbol{\theta} \in \mathbb{R}^d} \{\langle \boldsymbol{\theta}, \boldsymbol{u} \rangle - \tilde{\Lambda}(\boldsymbol{\theta})\}.$$

(L2) [**Lower bound part of Theorem 2.1 of [6]**] Let $\tilde{\sigma}^+(n) = \inf\{m \geq \tilde{\sigma}^+(n-1); \ell_{\boldsymbol{c}}(\tilde{Y}_m) - \ell_{\boldsymbol{c}}(\tilde{Y}_{\tilde{\sigma}^+(n-1)}) > 0\}$ for $n = 1, 2, \ldots$, where $\tilde{\sigma}(0) = 0$. That is, $\tilde{\sigma}^+(n)$ is the n-th ascending ladder epoch of the dual MAP $\{(\tilde{X}_n, \tilde{Y}_n)\}$ with respect to level function $\ell_{\boldsymbol{c}}$. Then, apply the lower bound part of Theorem 2.1 of [6] to the process $\{\tilde{Y}_n 1(\tilde{X}_n \in A)\}$, which satisfies the large deviations lower bound (F.2) with rate function $\tilde{\Lambda}^*$ for each A satisfying $\pi(A) > 0$. Hence, for nonnegative vector $\boldsymbol{c} \in \mathbb{R}^d$ and sufficiently large n_0,

$$\liminf_{n\to\infty} \frac{1}{n} \log P_{(\tilde{X}_0, \tilde{Y}_0)}(\tilde{X}_{\tilde{\sigma}^+(n')} \in A, \tilde{Y}_{\tilde{\sigma}^+(n')} > n\boldsymbol{c}, \text{ some } n' \geq n_0) \geq - \inf_{\boldsymbol{u} > \boldsymbol{c}} \tilde{I}(\boldsymbol{u}), \tag{F.3}$$

where $\tilde{I}(\boldsymbol{u}) = \sup_{\boldsymbol{\theta} \in \mathbb{R}^d}\{\langle \boldsymbol{\theta}, \boldsymbol{u} \rangle; \tilde{\Lambda}^{**}(\boldsymbol{\theta}) \leq 0\}$ for

$$\tilde{\Lambda}^{**}(\boldsymbol{\theta}) \equiv \sup_{\boldsymbol{u} \in \mathbb{R}^d} \{\langle \boldsymbol{\theta}, \boldsymbol{u} \rangle - \tilde{\Lambda}^*(\boldsymbol{u})\}.$$

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REMARK F.1. It should be noted that level function $\ell_{\boldsymbol{c}}$ is used rather than $\tilde{\ell}_{\boldsymbol{c}}$ in (L2), where $\tilde{\ell}_{\boldsymbol{c}}(\boldsymbol{u}) = -\ell_{\boldsymbol{c}}(-\boldsymbol{u})$.

We now derive the following result from (L1) and (L2).

LEMMA F.1. *If (3b) is satisfied and if $\pi(A) > 0$, then, for $\boldsymbol{c} \geq \boldsymbol{0}$,*

$$\liminf_{t\to\infty} \frac{1}{t} \log P_{(\tilde{X}_0,\tilde{Y}_0)}(\cup_{n=1}^{\infty}\{\tilde{X}_{\tilde{\sigma}^+(n)} \in A, \tilde{Y}_{\tilde{\sigma}^+(n)} > t\boldsymbol{c}\}) \geq -\sup_{\boldsymbol{\theta}\in\mathbb{R}^d}\{\langle \boldsymbol{\theta}, \boldsymbol{c}\rangle; c_p(K_*(\boldsymbol{\theta})) \geq 1\}. \tag{F.4}$$

PROOF. Since $\langle \boldsymbol{\theta}, \boldsymbol{c}\rangle$ can be nonnegative for $\tilde{\Lambda}(\boldsymbol{\theta}) \leq 0$, we have

$$\inf_{\boldsymbol{u}>\boldsymbol{c}} \sup_{\boldsymbol{\theta}\in\mathbb{R}^d}\{\langle \boldsymbol{\theta}, \boldsymbol{u}\rangle; \tilde{\Lambda}(\boldsymbol{\theta}) \leq 0\} = \sup_{\boldsymbol{\theta}\in\mathbb{R}^d}\{\langle \boldsymbol{\theta}, \boldsymbol{c}\rangle; \tilde{\Lambda}(\boldsymbol{\theta}) \leq 0\},$$

On the other hand, by Lemma 3.6, $c_p(\tilde{K}_*(\boldsymbol{\theta})) = c_p(K_*(\boldsymbol{\theta}))$, so $\tilde{\Lambda}(\boldsymbol{\theta}) \leq 0$ is equivalent to $c_p(K_*(\boldsymbol{\theta})) \geq 1$. By Lemma 3.7, $\Lambda(\boldsymbol{\theta})$ is convex and lower semicontinuous, so $\Lambda^{**}(\boldsymbol{\theta}) = \Lambda(\boldsymbol{\theta})$. Furthermore, (3.b) implies (F.1). Hence, from (L1) and (L2), we have (F.4). □

Unfortunately, we can not use Lemma F.1 except for the one dimensional case since the level functions ℓ and $\tilde{\ell}$ are different. In the one dimensional case, they are identical, and we have the following results, which includes Proposition 5.2 as a special case.

LEMMA F.2. *For the reflected MAP with one dimensional additive component, assume that it has the stationary distribution and (F.1) holds. If there is a constant $C_A > 0$ for each $A \in \mathcal{B}(S_X)$ such that, for all $u \in \{\tilde{Y}_{\tilde{\tau}_n^+} - \tilde{Y}_{\tilde{\tau}_{n-1}^+}; n = 1, 2, \ldots\}$,*

$$\frac{d\mu_0 R^{0+}}{d\pi \otimes m}(x, u) \geq C_A, \qquad (x, u) \in A \times \hat{S}_Y^+ \ a.s.\ \pi \otimes m, \tag{F.5}$$

then we have

$$\liminf_{n\to\infty} \frac{1}{t} \log P(\hat{X} \in A, \hat{Y} > t) \geq -\sup\{\theta; c_p(K_*(\theta)) \geq 1\}. \tag{F.6}$$

PROOF. From Lemma F.1, we get (F.6) if, for some constant $b > 0$ for each A and some $(x, v) \in S$,

$$P(\hat{X} \in A, \hat{Y} > t) \quad \geq \quad bP_{(x,0)}(\cup_{n=1}^{\infty}\{\tilde{X}_{\tilde{\sigma}^+(n)} \in A, \tilde{Y}_{\tilde{\sigma}^+(n)} > t\}). \tag{F.7}$$

Let $\tilde{\tau}^+(n) = \inf\{n' \geq \tilde{\tau}^+(n-1); \tilde{\ell}_1(\tilde{Y}_{n'}) - \tilde{\ell}_1(\tilde{Y}_{\tilde{\tau}^+(n-1)}) > 0\}$ for $n \geq 1$, where $\tilde{\tau}(0) = 0$. Since $\ell_1(u) = \tilde{\ell}_1(u)$, we have

$$\begin{aligned} &P_{(x,0)}(\cup_{n=1}^{\infty}\{\tilde{X}_{\tilde{\sigma}^+(n)} \in A, \tilde{Y}_{\tilde{\sigma}^+(n)} > t\}) \\ &\quad = P_{(x,-t)}(\cup_{n=1}^{\infty}\{\tilde{X}_{\tilde{\tau}^+(n)} \in A, \tilde{Y}_{\tilde{\tau}^+(n)} > 0\}) \\ &\quad \leq \frac{1}{C_A}\int_A \int_{u'=0}^{\infty} \sum_{n=0}^{\infty} (\tilde{G}^+)^n((x,-t), dx' \times du') \frac{d\mu_0 R^{0+}}{d\pi \otimes m}(x', u'), \end{aligned}$$

where the 2nd inequality is obtained by (F.5). Taking the transposition of (4.5) of Theorem 4.1, we have

$$\nu_+^{\dagger(\pi\otimes m)}(x,t) = \sum_{n=0}^{\infty} (\tilde{G}^+)^n (\mu_0 (R^{0+})^{\dagger(\pi\otimes m)}(x,t).$$

This can be written as

$$\begin{aligned} &\int_S g(x,u)\nu_+(dx \times du) \\ &\qquad = \int_S g(x,-u) \sum_{n=0}^{\infty} (\tilde{G}^+)^n (\mu_0 (R^{0+})^{\dagger(\pi\otimes m)}(x,u)\pi(dx)m(du). \end{aligned}$$

Thus, letting $g(x,u) = 1(x \in A, u > t)$, we have (F.7) with $C_A = b$. □

APPENDIX G: PROOF OF LEMMA 5.2

Assume that $\boldsymbol{\theta}_i \in \mathcal{D}_0$ for $i = 1, 2$. Let $\eta_*(\boldsymbol{\theta}) = \mu_{0*}(\boldsymbol{\theta}) L_*^{0+}(\boldsymbol{\theta})|_{S_X}$. Then, for $i = 1, 2$, we can find $h_i \in M_+(S_X)$ such that

$$\eta_*(\boldsymbol{\theta}_i)h_i < \infty, \qquad K_*(\boldsymbol{\theta}_i)h_i \leq e^{\Lambda(\boldsymbol{\theta}_i)} h_i.$$

Let $h(x) = h_1^{\lambda}(x) h_2^{1-\lambda}(x)$ for each fixed $\lambda \in (0,1)$. Then,

$$\begin{aligned} &\left([K_*(\lambda\boldsymbol{\theta}_1)h_1](x)\right)^{\lambda} \left([K_*((1-\lambda)\boldsymbol{\theta}_2)h_2](x)\right)^{1-\lambda} \\ &\qquad \leq e^{\lambda\Lambda(\boldsymbol{\theta}_1) + (1-\lambda)\Lambda(\boldsymbol{\theta}_2)} h_1(x)^{\lambda} h_2(x)^{1-\lambda}. \end{aligned}$$

By (3.23), this yields

$$[\eta_*(\lambda\boldsymbol{\theta}_1 + (1-\lambda)\boldsymbol{\theta}_2)h](x) \leq e^{\lambda\Lambda(\boldsymbol{\theta}_1) + (1-\lambda)\Lambda(\boldsymbol{\theta}_2)} h(x).$$

Hence, $h \in \mathcal{H}(\lambda\boldsymbol{\theta}_1 + (1-\lambda)\boldsymbol{\theta}_2)$, and therefore $\lambda\boldsymbol{\theta}_1 + (1-\lambda)\boldsymbol{\theta}_2 \in \mathcal{D}_0$.

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