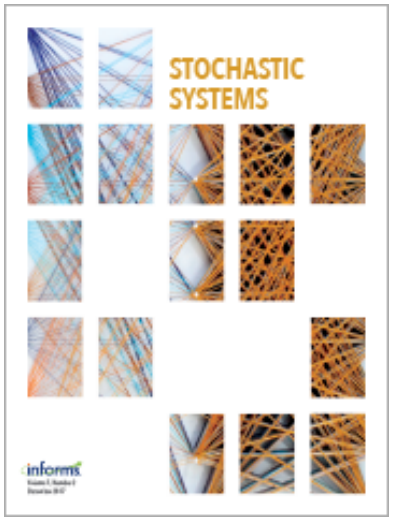
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# POWER IDENTITIES FOR LÉVY RISK MODELS UNDER TAXATION AND CAPITAL INJECTIONS*

By Hansjörg Albrecher and Jevgenijs Ivanovs

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In this paper we study a spectrally negative Lévy process which is refracted at its running maximum and at the same time reflected from below at a certain level. Such a process can for instance be used to model an insurance surplus process subject to tax payments according to a loss-carry-forward scheme together with the flow of minimal capital injections required to keep the surplus process non-negative. We characterize the first passage time over an arbitrary level and the cumulative amount of injected capital up to this time by their joint Laplace transform, and show that it satisfies a simple power relation to the case without refraction, generalizing results by Albrecher and Hipp (2007) and Albrecher, Renaud and Zhou (2008). It turns out that this identity can also be extended to a certain type of refraction from below. The net present value of tax collected before the cumulative injected capital exceeds a certain amount is determined, and a numerical illustration is provided.

**1. Introduction.** The aim of this paper is to study certain power relations of level crossing quantities for spectrally negative Lévy processes, which are motivated by insurance applications. Concretely, assume that the surplus process of an insurance portfolio is modeled by a spectrally negative Lévy process, and tax payments on profits according to a loss-carry-forward scheme are implemented by paying a certain proportion $\gamma$ of the premium income, whenever the surplus process is at its running maximum. For a constant value of $\gamma$, it was shown by Albrecher and Hipp (2007) and Albrecher, Renaud and Zhou (2008) that the probability of the resulting process to stay positive is intimately connected to the one without tax payments by a simple power relation (see also Albrecher et al. (2009); Kyprianou and Ott (2012); Kyprianou and Zhou (2009) for extensions). The implemented tax rule can alternatively be seen as a general profit participation scheme for shareholders, which for the special case of $\gamma = 1$ reduces to a horizontal

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dividend barrier strategy. Whereas in classical models business is stopped as soon as the surplus is negative, it is natural to consider the amount of capital needed to bring the surplus back to zero whenever it turns negative and henceforth continue the business operations. Under horizontal dividend payments and a compound Poisson model for insurance claims, this question was considered by Dickson and Waters (2004), and Kulenko and Schmidli (2008) showed that it can be optimal for shareholders to "save" the insurance business in this way (for another injection scheme see Nie, Dickson and Li (2011)).

In this paper we consider capital injections below zero for the general case $\gamma \leq 1$. This amounts to study level crossing events for a spectrally negative Lévy process refracted at its running maximum and at the same time reflected at zero. We characterize the first passage time over an arbitrary level and the cumulative amount of injected capital up to this time by their joint Laplace transform, and establish a simple power relation to the case without refraction. From the proof it becomes clear that such a power identity can not hold, if reflection from below is generalized to refraction at the running minimum. However, if refraction always starts at the same fixed level, a power identity still holds.

In Section 2, we discuss simultaneous refraction and reflection. Section 3 then states the main results, which are proved in Section 4. In Section 5 we consider an application of the obtained formula to determine the net present value of tax collected before the cumulative injected capital exceeds an exponential amount, and give a concrete numerical example for a compound Poisson risk model. Finally, in Section 6 we illustrate with yet another example that power identities hold in wide generality. Concretely, we use our proof technique to extend the power tax identity for first passage times (without capital injections) to a relaxed concept of ruin which was considered recently in the literature.

**2. Refraction and reflection.** For a càdlàg sample path $X_t$ of any stochastic process, consider reflection of $X_t$ at a level $b$ (from above) defined by $Y_t = X_t - U_t \leq b$, where $U_t$ is a non-decreasing càdlàg function with $U_0 = 0 \vee (X_0 - b)$, whose points of increase are contained in the set $\{t \geq 0 : Y_t = b\}$. This identifies $U_t$ in a unique way, and implies that $U_t = 0 \vee (\overline{X}_t - b)$, where $\overline{X}_t = \sup\{X_s : 0 \leq s \leq t\}$, see e.g. Kella (2006). Essentially, $U_t$ evolves as the supremum process.

For an arbitrary $\gamma \in \mathbb{R}$ we call the process $X_t - \gamma U_t$ a refraction from above, which has some interpretations in insurance risk theory. For $\gamma = 1$ we retrieve the reflected process, which can model an insurance surplus pro-

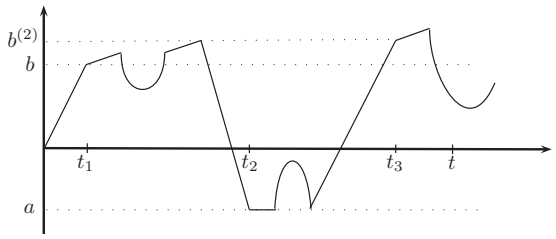



FIG 1. *A sample path refracted from above and reflected from below.*

cess with dividends paid out according to a barrier strategy with barrier $b$, whereas $\gamma \in (0,1)$ refers to an insurance surplus process taxed according to a loss-carried forward scheme (see e.g. Albrecher and Hipp (2007); Albrecher, Renaud and Zhou (2008)). A value $\gamma < 0$ could refer to a model with stimulation proportional to the increase of the maximum. Finally, the case of $\gamma > 1$ can be interpreted as inhibition, which will not be considered further in the sequel. In general, $\gamma$ could be allowed to depend on the current value of $U_t$ (or on the running maximum of the refraction itself), which leads to a more general process of the form $X_t - \int_0^{U_t} \gamma(x)\mathrm{d}x$. For simplicity, we will however assume throughout this work that $\gamma$ is a constant, and only give some comments in Remark 4.1.

This paper focuses on processes refracted from above with rate $\gamma \leq 1$ and reflected from below. Such a process can be defined by using one-sided refraction from above and one-sided reflection from below locally, and then gluing segments of paths together, see also (Asmussen, 2003, Sec. XIV.3) where a similar procedure is used to define a two-sided reflection. More precisely, we do the following for a given interval $[a,b]$, where $a$ is the level for reflection from below, and $b$ is the initial level for refraction from above. First, we consider a free process $X_t$ until it exits $[a,b]$, at which moment we start either reflection from below (it exits through $a$) or refraction from above (it exits through $b$). Assuming (w.l.o.g.) the latter, we consider the time at which the corresponding refraction goes below $a$, and then start reflection from below. When this reflection goes above the running maximum, the refraction from above starts, and so on, see Figure 1 for an illustration of such a process.

The above procedure is described rigorously in the form of an algorithm in the Appendix, where we also allow for two-sided refraction. For the present model it results in a representation

$$Y_t = X_t + L_t - \gamma U_t, \tag{1}$$

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where it is assumed that $X_0 \in [a, b]$, and $\gamma \leq 1$ to avoid the case of inhibition. Moreover, $L_t$ and $U_t$ are non-decreasing càdlàg functions, and the points of increase of $L_t$ and $U_t$ are contained in the sets $\{t \geq 0 : Y_t = a\}$ and $\{t \geq 0 : Y_t = \overline{Y_t} \vee b\}$ respectively. Finally, note that $L_t$ and $U_t$ are interrelated and both depend on the parameter $\gamma$.

**3. A power identity.** Throughout this work we assume that $X_t$ is a spectrally negative Lévy process with Laplace exponent $\psi(\alpha)$ so that $\mathbb{E}e^{\alpha X_t} = e^{\psi(\alpha)t}$ for $\alpha \geq 0$. Define the first passage times

$$\tau_y^{\pm} = \inf\{t \geq 0 : \pm X_t > y\}$$

and recall that for all $q \geq 0$ there exists a unique continuous function $W^q : [0, \infty) \to \mathbb{R}_+$, such that $W^q(y) > 0$ for $y > 0$,

$$\mathbb{E}_x[e^{-q\tau_y^+}; \tau_y^+ < \tau_0^-] = W^q(x)/W^q(y) \text{ for } y \geq x \geq 0, y > 0, \tag{2}$$

and $\int_0^\infty e^{-\alpha y} W^q(y) \mathrm{d}y = 1/(\psi(\alpha) - q)$ for $\alpha$ larger than the rightmost zero of $\psi(\alpha) - q$. This $W^q$ is called a scale function.

For a Lévy risk model with tax, it was shown by Albrecher, Renaud and Zhou (2008) that certain probabilities and transforms can be related to their analogues under no taxation by power identities. We will now generalize such power identities to the setting of a refraction from above and reflection from below. Consider a process $Y_t$ given by (1), where $X_0 = x > 0$, the reflection barrier is placed at the level $a = 0$, and the refraction from above at rate $\gamma \leq 1$ is applied immediately, i.e. $b = x$ (it is straightforward to extend our result to $b > x$ using identities for reflected Lévy processes). Let also

$$T_y = \inf\{t \geq 0 : Y_t > y\}$$

be the first passage time of the refraction above the level $y$.

THEOREM 3.1. *For $\gamma < 1$ and $q, \theta \geq 0$ it holds that*

$$\mathbb{E}_x^\gamma e^{-qT_y - \theta L_{T_y}} = \left(\mathbb{E}_x^0 e^{-qT_y - \theta L_{T_y}}\right)^{\frac{1}{1-\gamma}}, \tag{3}$$

*where $y \geq x > 0$ and $\mathbb{E}_x^\gamma$ denotes the expectation operator for the model defined by (1) with $a = 0$ and $b = x$.*

It should be noted that the right hand side of (3) can be identified using results on reflected Lévy processes. In particular, Ivanovs (2011) shows that

$$\mathbb{E}_x^0 e^{-qT_y - \theta L_{T_y}} = Z^{q,\theta}(x)/Z^{q,\theta}(y), \tag{4}$$

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where $Z^{q,\theta}(x)$ is a so-called second scale function given by

$$Z^{q,\theta}(x) = e^{\theta x}[1 - (\psi(\theta) - q)\int_0^x e^{-\theta y} W^q(y)\mathrm{d}y],$$

see also Pistorius (2004) for the case when $\theta = 0$. Observe that

$$\lim_{\theta\to\infty} \mathbb{E}_x^0 e^{-qT_y - \theta L_{T_y}} = \mathbb{E}_x^0[e^{-qT_y}; L_{T_y} = 0] = \mathbb{E}_x[e^{-q\tau_y^+}; \tau_y^+ < \tau_0^-] = \frac{W^q(x)}{W^q(y)}.$$

Similarly, for $\theta \to \infty$ the left-hand side of (3) becomes the transform of the first passage time $T_y$ on the event that it precedes ruin, hence we recover the tax identity (3.1) of Albrecher, Renaud and Zhou (2008) as a special case.

In the case $\gamma = 1$ (corresponding to payments of dividends according to a barrier strategy at the level $x$) we have $T_y = \infty$ for all $y \geq x$. Instead we look at

$$\rho_y = \inf\{t \geq 0 : U_t > y\}, \tag{5}$$

which is the first time that the amount of accumulated dividends (or taxes) exceeds a level $y$.

THEOREM 3.2. *For $q, \theta \geq 0$ and $x > 0, y \geq 0$ it holds that*

$$\mathbb{E}_x^1 e^{-q\rho_y - \theta L_{\rho_y}} = e^{-\lambda^{q,\theta}(x)y}, \tag{6}$$

*where*

$$\lambda^{q,\theta}(x) = Z^{q,\theta'}(x)/Z^{q,\theta}(x) = \theta - \frac{(\psi(\theta) - q)W^q(x)}{Z^{q,\theta}(x)}.$$

In a somewhat different form this formula appears also in Ivanovs (2011). We note that for $\theta = \infty$ one has to take $\lambda^q(x) = W^{q\prime}_+(x)/W^q(x)$, which is intimately related to the excursion measure, see e.g. (Kyprianou, 2006, Lem. 8.2).

REMARK 3.1. The power identity (3) fails to hold for a two-sided refraction (defined in Appendix) with $\gamma_L < 1$. The case of reflection $\gamma_L = 1$ is special because in this case we know the distance to the (lower) reflection barrier at the first passage time $T_y$ (in other words, $a^{(n)}$ in the algorithm defining the two-sided refraction is constant, see Appendix).

Nevertheless, from the proof in Section 4 it becomes clear that if one modifies the model and considers either refraction from below always starting at a fixed level $a$ or always starting at a fixed distance from the running maximum (rather than starting at the current running minimum), then the power identity (3) is preserved also in the case $\gamma_L < 1$.

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**4. Proofs.** In this section we prove Theorem 3.1 and Theorem 3.2. We construct an auxiliary process by a certain modification of paths of the simultaneously refracted and reflected process. This modification preserves excursions from the maximum, but leads to the same 'behavior at the maximum' as the one of the free process. Furthermore, the auxiliary process corresponding to $\gamma = 1$ exhibits a lack of memory property at its first passage times, because the lower reflection barrier is always placed at a constant distance from the maximum. This gives rise to a certain exponent $\lambda(x)$, and allows to relate this process to the processes corresponding to different $\gamma$, see Lemma 4.1. Subsequently the strong Markov property is applied to establish a differential equation for the quantity of interest, which then yields the results.

It is convenient to shift our process, so that $X_0 = 0$ and reflection from below is applied at the level $-x < 0$. Recall also that refraction from above is applied immediately. Note that $\mathbb{E}^\gamma e^{-qT_y}$ can be written as $\mathbb{P}^\gamma(T_y < \infty)$ for an exponentially killed process, i.e. when $X_t$ is sent to an additional absorbing state at an independent exponentially distributed time $e_q$ with rate $q \geq 0$. The double transform $\mathbb{E}^\gamma e^{-qT_y - \theta L_{T_y}}$ is obtained by additional killing at the time when $L_t$ surpasses an independent exponentially distributed $e_\theta$. Hence it suffices to analyze $\mathbb{P}^\gamma(T_y < \infty)$ for a doubly killed process.

Let us fix some terminology and notation concerning the paths of $Y_t$. Segments of a path of the process $Y_t - \overline{Y}_t$ in the intervals when this difference is strictly negative are called excursions of $Y_t$ (from the maximum). The starting level of an excursion is the corresponding value of $\overline{Y}_t$. Next, consider a triplet $(Y_t, L_t, U_t)$ of paths (where each component depends on the choice of $\gamma$) and define

$$\tilde{Y}_t = X_t + L_t = Y_t + \gamma U_t.$$

From the construction of $Y_t$ one can see that $\overline{Y_t} = (1-\gamma)U_t$, which immediately yields $\overline{\tilde{Y}_t} = U_t$. Letting

$$\tilde{T}_y = \inf\{t \geq 0 : \tilde{Y}_t > y\}$$

we see that $\tilde{T}_y = \rho_y$ and for $\gamma < 1$ also

$$\tilde{T}_y = T_{(1-\gamma)y}. \tag{7}$$

It is noted that we could have avoided constructing the auxiliary process, since it is possible to use the stopping time $\rho_y$ instead of $\tilde{T}_y$. But then the following arguments would become less visually appealing.

When $\gamma = 1$ the reflecting barrier is always placed at a constant distance $x$ from the maximum, which together with the strong Markov property of

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$X_t$ implies that

$$\mathbb{P}^1(\tilde{T}_{y+z} < \infty | \tilde{T}_y < \infty) = \mathbb{P}^1(\tilde{T}_z < \infty) \tag{8}$$

for all $y, z > 0$ (note that the memoryless property of the killing times $e_q$ and $e_\theta$ is essential here). From (8) it follows that there exists a $\lambda(x) \geq 0$ such that

$$\mathbb{P}^1(\tilde{T}_y < \infty) = e^{-\lambda(x)y}, \tag{9}$$

where $x$ denotes the distance between the reflecting barriers. This provides the proof of Theorem 3.2 up to the identification of $\lambda(x)$.

LEMMA 4.1. *It holds for all $\gamma \leq 1$ that*

$$\mathbb{P}^\gamma(\tilde{T}_h < \infty) = \mathbb{P}^1(\tilde{T}_h < \infty) + o(h) \text{ as } h \downarrow 0.$$

PROOF. In the following we will need to compare the sample paths of $\tilde{Y}_t$ processes for different $\gamma$, hence throughout this proof we write $\tilde{Y}_t^\gamma$ and $\tilde{T}_y^\gamma$ to make their dependence on $\gamma$ explicit. For the ease of exposition, consider first the case $\gamma = 0$, where $\tilde{Y}_t^0$ is a process $X_t$ reflected at the level $-x$. Let $\delta \geq 0$ be the starting level of the first excursion of $X_t$ from the maximum exceeding height $x$; this is also the starting level of the first excursion of $\tilde{Y}_t^1$ leading to reflection (i.e. an increase of $L_t^1$). Note that on the event $\{\delta > h\}$ the times $\tilde{T}_h^0$ and $\tilde{T}_h^1$ coincide. In the following we exclusively work on the complementary event $\{\delta \leq h\}$.

The lack of memory of $\tilde{Y}_t^1$ at its first passage times implies that the number of excursions of $\tilde{Y}_t^1$ starting in $[0, h]$ and leading to reflection defines a (killed) Lévy process indexed by $h$. Hence on the event $\{\tilde{T}_h^1 < \infty\}$ this number is Poisson distributed. Using the lack of memory of $\tilde{Y}_t^1$ at $\tilde{T}_h^1$ we see that

$$\begin{aligned}&\mathbb{P}(\delta \leq h, \tilde{T}_h^1 < \infty, \tilde{T}_{2h}^1 = \infty) = \mathbb{P}(\delta \leq h, \tilde{T}_h^1 < \infty)\mathbb{P}(\tilde{T}_h^1 = \infty)\\ &= O(h)(\lambda(x)h + o(h)) = o(h).\end{aligned}$$

Hence considering $\{\delta \leq h, \tilde{T}_h^1 < \infty\}$ we can assume that $\tilde{T}_{2h}^1 < \infty$ and also there is only one excursion of $\tilde{Y}_t^1$ starting in $[0, 2h]$ and leading to reflection. Comparison of the sample paths of $\tilde{Y}_t^1$ and $\tilde{Y}_t^0$, see Figure 2, reveals that $\tilde{T}_h^0 < \infty$, because the difference between them is bounded by $h$. For an arbitrary $\gamma \leq 1$ it is bounded by $(1-\gamma)h$, hence one can take $h + (1-\gamma)h$ instead of $2h$ to finish this part of the proof.

Let us now consider $\{\tilde{T}_h^0 < \infty\}$. Note that $\tilde{T}_h^1 = \infty$ can only happen as a consequence of killing according to $e_\theta$. Hence it is only required to show

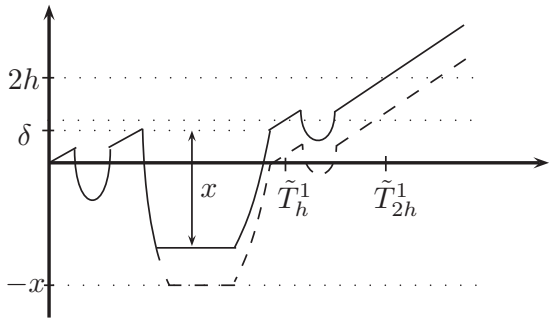



FIG 2. *A schematic sample path of* $\tilde{Y}^1_t$ *and* $\tilde{Y}^0_t$ *(with a dashed line).*

that this happens with probability $o(h)$. In fact, it is enough to show that for a non-killed process $\tilde{Y}^1_t$ it holds that

$$\mathbb{P}^1(\delta \le h, e_\theta \in (L_{\tilde{T}_\delta} - h, L_{\tilde{T}_\delta})) = o(h),$$

which follows from the independence of $e_\theta$. Again, for general $\gamma$, $h$ in the above display is replaced by $(1-\gamma)h$. □

Combining Lemma 4.1, (9), and (7) we get for $\gamma < 1$

$$\mathbb{P}^\gamma(T_h < \infty) = \mathbb{P}^1(\tilde{T}_{h/(1-\gamma)} < \infty) + o(h) = 1 - \frac{\lambda(x)}{1-\gamma}h + o(h) \text{ as } h \downarrow 0. \tag{10}$$

Let us now return to the original set-up, where $X_0 = x$ and the reflecting barrier is placed at the level 0; we use $\mathbb{P}_x$ to denote the corresponding law.

PROOF OF THEOREM 3.1. Assume that $\gamma < 1$ and write using the strong Markov property

$$\mathbb{P}^\gamma_x(T_y < \infty) = \mathbb{P}^\gamma_x(T_{x+h} < \infty)\mathbb{P}^\gamma_{x+h}(T_y < \infty).$$

According to (10) we have $\mathbb{P}^\gamma_x(T_{x+h} < \infty) = 1 - \frac{\lambda(x)}{1-\gamma}h + o(h)$ as $h \downarrow 0$. Hence $\mathbb{P}^\gamma_{x+h}(T_y < \infty) \to \mathbb{P}^\gamma_x(T_y < \infty)$, and moreover

$$\frac{\partial}{\partial x}\mathbb{P}^\gamma_x(T_y < \infty) = \frac{\lambda(x)}{1-\gamma}\mathbb{P}^\gamma_x(T_y < \infty). \tag{11}$$

Formally, this computation gives only the right derivative.

Let us identify $\lambda(x)$ using the existing theory. In particular (4) states that $\mathbb{P}^0_x(T_y < \infty) = Z(x)/Z(y)$. Hence we obtain $Z'(x)/Z(y) = \lambda(x)Z(x)/Z(y)$ yielding

$$\lambda(x) = Z'(x)/Z(x) \text{ for } x > 0, \tag{12}$$

which also shows that $\lambda(x)$ is continuous on $(0, \infty)$.

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It is not hard to see that for any $\gamma < 1$ and fixed $y > 0$ the function $\mathbb{P}_x^\gamma(T_y < \infty), x \in (0, y]$ is continuous and non-zero. Hence for all $x \in (0, y)$ we have the following right derivative:

$$\frac{\partial}{\partial x} \ln \mathbb{P}_x^\gamma(T_y < \infty) = \frac{\lambda(x)}{1-\gamma},$$

which together with $\mathbb{P}_y^\gamma(T_y < \infty) = 1$ yields

$$\ln \mathbb{P}_x^\gamma(T_y < \infty) = -\frac{1}{1-\gamma} \int_x^y \lambda(u) \mathrm{d}u.$$

Uniqueness of the solution is based on the fact that a continuous function with right derivative 0 at every point of an interval is constant on this interval. So we have

$$\mathbb{P}_x^\gamma(T_y < \infty) = e^{-\frac{1}{1-\gamma} \int_x^y \lambda(u) \mathrm{d}u}, \tag{13}$$

which immediately yields the power relation of Theorem 3.1. □

Finally, Theorem 3.2 is a direct consequence of (9) and (12).

REMARK 4.1. When the refraction rate $\gamma(x)$ depends on the level, assuming some regularity conditions (e.g. $\gamma(x)$ is continuous and bounded away from 1), one can still apply Lemma 4.1 to derive the differential equation (11). In this case the solution takes the form

$$\mathbb{P}_x^\gamma(T_y < \infty) = e^{-\int_x^y \lambda(u)/(1-\gamma(u)) \mathrm{d}u}.$$

**5. An application: Profit participation and capital injection.** As an application of Theorem 3.1, interpret $Y_t$ in (1) as an insurance surplus process at time $t$, where $\gamma U_t$ is a profit participation scheme for an investor (a proportion $\gamma$ of the profits is paid out to the investor) and, in turn, if needed the investor injects a minimal flow of capital into the company to prevent its bankruptcy, i.e. to keep the surplus non-negative, with $L_t$ being the total amount injected up to time $t$. Alternatively, one can think of $\gamma U_t$ as tax payments for profits up to time $t$ according to a loss-carry forward scheme with constant tax rate $0 < \gamma < 1$ (cf. Albrecher and Hipp (2007)) and $L_t$ would then be the necessary amount of capital up to time $t$ to bail out the insurance company to prevent bankruptcy. Consider an upper limit $e_\theta$ for the cumulative amount that the investor is willing to inject, which is assumed to be an independent exponential random variable with rate parameter $\theta \geq 0$ (it can be interpreted as impatience of the investor).

Whenever this limit is exceeded the company is not bailed out anymore and has to go out of business. Put differently, for each infinitesimal required injection $h$, the investor will stop payments with probability $\theta h$ (independently of everything else). This concept extends the notion of classical ruin (which is retrieved for $\theta = \infty$), and leads to an interesting trade-off between collected profits (or tax) and injected capital.

The expected discounted profit (tax) payments for this model can be written as

$$V(\gamma) = \frac{\gamma}{1-\gamma}\mathbb{E}_x^\gamma \int_0^\infty e^{-qt} 1_{\{L_t < e_\theta\}} \mathrm{d}\overline{Y}_t, \quad \gamma < 1,$$

where $q > 0$ is the discount rate. Note that each $\mathrm{d}\overline{Y}_t = \mathrm{d}y$ corresponds to $\gamma/(1-\gamma)\mathrm{d}y$ tax payment. Recalling that $\overline{Y}_t$ is continuous, and using a standard change of variable argument with $\overline{Y}_t = y$ and $t = T_y$ we obtain

$$\begin{aligned} V(\gamma) &= \frac{\gamma}{1-\gamma}\mathbb{E}_x^\gamma \int_x^\infty e^{-qT_y} 1_{\{L_{T_y} < e_\theta\}} \mathrm{d}y = \frac{\gamma}{1-\gamma}\int_x^\infty \mathbb{E}_x^\gamma[e^{-qT_y - \theta L_{T_y}}]\mathrm{d}y \\ &= \frac{\gamma}{1-\gamma}\int_x^\infty \left(\frac{Z^{q,\theta}(x)}{Z^{q,\theta}(y)}\right)^{\frac{1}{1-\gamma}} \mathrm{d}y, \end{aligned} \tag{14}$$

where in the second step we use Fubini's theorem and the independence of $e_\theta$, and in the last step we invoke Theorem 3.1. This formula is an extension of Equation (3.2) of Albrecher, Renaud and Zhou (2008), which is retained for $\theta \to \infty$ (the case without capital injections).

If we choose $\gamma = 1$ (in which case the profit participation reduces to a horizontal dividend barrier strategy), then we get in a similar way by using Theorem 3.2 that the expected discounted dividends $V(1)$ are given by

$$\begin{aligned} V(1) &= \mathbb{E}_x^1 \int_0^\infty e^{-qt} 1_{\{L_t < e_\theta\}} \mathrm{d}U_t = \int_0^\infty \mathbb{E}_x^1[e^{-q\rho_y - \theta L_{\rho_y}}]\mathrm{d}y \\ &= \frac{1}{\lambda^{q,\theta}(x)} = Z^{q,\theta}(x)/Z^{q,\theta'}(x). \end{aligned} \tag{15}$$

As above, for $\theta \to \infty$ we get back to the classical formula without capital injections, where $Z$ is replaced by $W$ (see e.g. Equation (3) in Renaud and Zhou (2007)).

The quantity $V(\gamma)$ can consequently be computed explicitly whenever the function $Z$ has an explicit representation. This is for instance the case for a Poisson stream of phase-type claims (for a detailed discussion of explicit cases cf. Hubalek and Kyprianou (2011)).

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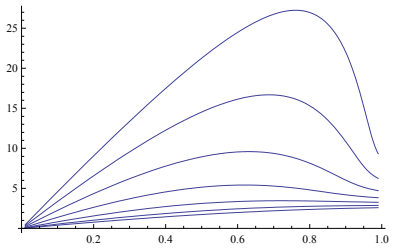
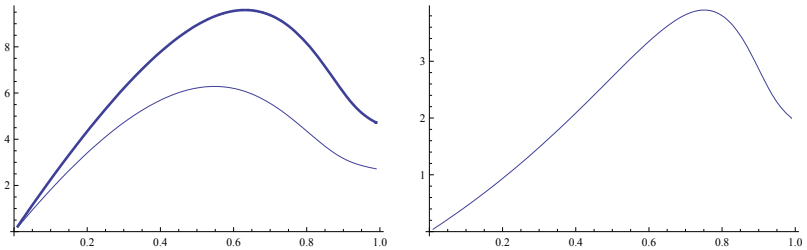



FIG 3. $V(\gamma)$ *for drift* $= (0.5, 0.4, 0.3, 0.2, 0.1, 0, -0.1)$*; from top to bottom.*




(a) $V^1(\gamma)$ (thick) and $V^\infty(\gamma)$ (b) $V^1(\gamma) - V^\infty(\gamma)$

FIG 4. $V(\gamma)$ *for* $\theta = 1$ *and* $\theta = \infty$.

5.1. *A numerical example.* Let us consider a concrete simple example, for which the scale function $W(x)$ has an explicit form, and hence the expected discounted profit (tax) payments $V(\gamma)$ as identified in (14) can be easily evaluated. We assume that the driving process is a Cramèr-Lundberg risk process $X_t = x + ct - \sum_{n=1}^{N(t)} M_i$, where $N(t)$ is a homogeneous Poisson process with rate 1, the insurance claims $M_i$ are independent and identically distributed exponential random variables with mean $m$ and the constant premium intensity is chosen as $c = 1$, so that the drift of $X$ is then given by $\mathbb{E}X(1) = 1 - m$. Choose further the initial capital $x = 1$, the discount factor $q = 0.01$, and the investor impatience parameter $\theta = 1$.

Figure 3 depicts $V$ as a function of $\gamma$ for different values of the drift. Essentially, the shape of these functions is the same as in the case of classical ruin ($\theta = \infty$), but higher in absolute value due to the longer life-time of the process. This shape reflects that overly large values of $\gamma$ may lead to an early ruin resulting in a smaller profit.

In Figure 4(a), this is visualized by comparing $V(\gamma)$ for $\theta = 1$ and $\theta = \infty$ for a fixed drift of $\mathbb{E}X(1) = 0.3$, and Figure 4(b) depicts the increase of $V(\gamma)$ as compared to the case of classical ruin. This expected increase of profit comes at the cost of the capital injections, whose expected value does not

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exceed $\mathbb{E}e_\theta = 1$. The latter is in fact a crude upper bound, because of two reasons: no discounting, and the fact that cumulative injections may never reach the threshold $e_\theta$. These results show that on average it can be quite advantageous for an investor to perform these capital injections, in particular for those $\gamma$ for which the difference $V^1(\gamma) - V^\infty(\gamma)$ is larger than 1. If one would compare this difference to the actual expected discounted investments, the effect would be even more pronounced. The analysis of the net present value of injections is, however, considerably more involved, and could be an interesting direction for future work.

**6. Power identities under a relaxed ruin concept.** It turns out that power relations similar to (3) hold in quite wide generality. Essentially, it is only required that killing and modification (such as reflection) of excursions of the (non-taxed) process is done in a memoryless way (in other words, what happens after the first passage time $T_y^0$ is independent from the past and has the same law as the original process started in $y$). Of course, one still has to handle model-specific technical details similar to those contained in Lemma 4.1.

For illustration, let us consider an example from Albrecher, Gerber and Shiu (2011) and Albrecher and Lautscham (2013), where bankruptcy is declared at some rate $\theta > 0$ when the risk process is below zero (there is no reflection from below). In other words, the killing occurs when the cumulative time $X_t$ spent below zero surpasses an independent exponential random variable $e_\theta$ (one can also introduce dependence of $\theta$ on the level, but for clarity we refrain from doing so, and only note that generalizations of power identities to arbitrary measurable, locally bounded functions $\theta(x)$ do not cause additional problems). As before we assume that $X_t$ is a spectrally negative Lévy process (no reflection from below). The concept of occupation times plays an important role in this setting. Let

$$M(A,t) = \int_0^t 1_{\{X_s \in A\}} \mathrm{d}s$$

be the time $X$ spends in a Borel set $A$ up to time $t$.

THEOREM 6.1. *Consider the model* (1) *without reflection from below* ($a = -\infty, b = x \geq 0$)*, and let* $\nu_\theta$ *be the time of bankruptcy:*

$$\nu_\theta = \inf\{t \geq 0 : M((-\infty, 0), t) > e_\theta\}.$$

*Then for all* $\gamma < 1$ *and* $q \geq 0$ *it holds that*

$$\mathbb{E}_x^\gamma[e^{-qT_y}; T_y < \nu_\theta] = \left(\mathbb{E}_x^0[e^{-qT_y}; T_y < \nu_\theta]\right)^{\frac{1}{1-\gamma}}.$$

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PROOF. Without real loss of generality one can assume that $q = 0$. One can repeat the arguments from the previous section. In fact, many things simplify since there is no process $L_t$. In particular, paths of the processes $\tilde{Y}_t^\gamma$ (and $X_t$) are the same, but the intervals of times when the processes are in danger of bankruptcy are different for different $\gamma$, and so the killing points are different. In order to (re-)establish Lemma 4.1, we have to show that the differences between 'in danger' sets up to the time $\tau_h^+$ are small in certain sense. It is enough to show that

$$\mathbb{P}(M([-x+\gamma h, -x+h), \tau_h^+) > e_\theta) = o(h) \tag{16}$$

as $h \downarrow 0$. Moreover, to establish the differential equation (11) we have to show (for the reason of continuity) that

$$M(\{x\}, t) = 0 \text{ a.s. for any } t, x. \tag{17}$$

The latter fact is well-known, see (Bertoin, 1996, Prop. I.15). So it is only left to show that (16) holds.

The probability in (16) can be bounded from above by

$$\mathbb{P}(\tau_{x-h}^- < \tau_h^+)\mathbb{P}(M([-(1-\gamma)h, (1-\gamma)h], \tau_{x+(1-\gamma)h}^+) > e_\theta).$$

In short, the process must go below the upper boundary of the interval, then we start it at the lower boundary and make the strip twice as large, so that it starts in the middle. The first probability is given by $1 - W(x-h)/W(x) = W'_-(x)/W(x)h + o(h)$, and the second decreases to 0 as $h \downarrow 0$, because $M([-h, h], \tau_y^+) \to 0$ for any $y > 0$ a.s. (use (17) and the fact that either $\overline{X}_t \to \infty$ a.s. or $X_t \to -\infty$ a.s.). This concludes the proof. □

COROLLARY 6.1. *For the model of Theorem 6.1 and $q \geq 0$ it holds that*

$$\mathbb{E}_x^\gamma[e^{-qT_y}; T_y < \nu_\theta] = \left(\frac{Z^{q,\Phi}(x)}{Z^{q,\Phi}(y)}\right)^{\frac{1}{1-\gamma}}, \qquad \gamma < 1,$$

$$\mathbb{E}_x^\gamma[e^{-q\rho_y}; \rho_y < \nu_\theta] = \exp\left(-\frac{Z^{q,\Phi'}(x)}{Z^{q,\Phi}(x)} y\right), \qquad \gamma = 1,$$

*where $\Phi$ is the unique positive solution of $\phi(\Phi) = q + \theta$.*

PROOF. It holds that

$$\mathbb{E}_x[e^{-q\tau_y^+}; \tau_y^+ < \nu_\theta] = Z^{q,\Phi}(x)/Z^{q,\Phi}(y),$$

which can be easily deduced from the results by Loeffen, Renaud and Zhou (2014) or Albrecher and Ivanovs (2013). The rest follows from Theorem 6.1 and its proof which employs the ideas of Section 4. □

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## APPENDIX

In the following we present an algorithm defining a two-sided refraction of a càdlàg sample path $X_t$ corresponding to the interval $[a, b]$. It is assumed that $X_0 \in [a, b]$, and $\gamma_L, \gamma_U \leq 1$ to avoid the case of inhibition. The triplet of processes $(Y_t, L_t, U_t)$ is defined iteratively as follows (cf. Figure 1 depicting refraction from above at $b$ and reflection from below at $a$).

**Algorithm**:

**Initialization** $(n = 0)$: $Y_t^{(0)} = X_t, U_t^{(0)} = 0, L_t^{(0)} = 0, t_0 = 0$ and $a^{(1)} = a, b^{(1)} = b$,

$$t_1 = \inf\{t \geq 0 : X_t \notin [a, b]\}.$$

**Step** $(n = n + 1)$: $X_t^{(n)} = Y_{t_n}^{(n-1)} + X_{t_n+t} - X_{t_n}$ for $t \geq 0$.

**If** $X_0^{(n)} \geq b^{(n)}$: $L_t^{(n)} = 0$ and $Y_t^{(n)} = X_t^{(n)} - \gamma_U U_t^{(n)}$ is the refraction of $X_t^{(n)}, t \geq 0$ from above at the level $b^{(n)}$. Put

$$\Delta_n = \inf\{t \geq 0 : Y_t^{(n)} < a^{(n)}\}$$

and $t_{n+1} = t_n + \Delta_n, a^{(n+1)} = a^{(n)}, b^{(n+1)} = \overline{Y_{\Delta_n}^{(n)}}$.

**If** $X_0^{(n)} \leq a^{(n)}$: $U_t^{(n)} = 0$ and $Y_t^{(n)} = X_t^{(n)} + \gamma_L L_t^{(n)}$ is the refraction of $X_t^{(n)}, t \geq 0$ from below at the level $a^{(n)}$. Put

$$\Delta_n = \inf\{t \geq 0 : Y_t^{(n)} > b^{(n)}\}$$

and $t_{n+1} = t_n + \Delta_n, a^{(n+1)} = \underline{Y_{\Delta_n}^{(n)}}, b^{(n+1)} = b^{(n)}$.

Finally, we set

$$Y_t = Y_{t-t_n}^{(n)}, L_t = \sum_{i=0}^{n-1} L_{\Delta_i}^{(i)} + L_{t-t_n}^{(n)}, U_t = \sum_{i=0}^{n-1} U_{\Delta_i}^{(i)} + U_{t-t_n}^{(n)} \text{ for } t \in [t_n, t_{n+1}).$$

Observe that the above procedure defines the process $Y_t$ for all $t \geq 0$, i.e. $t_n \to \infty$ as $n \to \infty$, because a càdlàg function can not cross the interval $[a, b]$ infinitely many times in finite time; here we use the fact that the intervals $[a^{(n)}, b^{(n)}]$ are increasing. Careful examination of the above algorithm (together with known properties of a one-sided refraction) shows that

$$Y_t = X_t + \gamma_L L_t - \gamma_U U_t,$$

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where $L_t$ and $U_t$ are non-decreasing càdlàg functions. Moreover, the points of increase of $L_t$ and $U_t$ are contained in the sets $\{t \geq 0 : Y_t = \underline{Y_t} \wedge a\}$ and $\{t \geq 0 : Y_t = \overline{Y_t} \vee b\}$ respectively. It may be interesting to find an explicit representation of the two-sided refraction similar to those given by Andersen and Mandjes (2009) and Kruk et al. (2007) for the two-sided reflection.

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