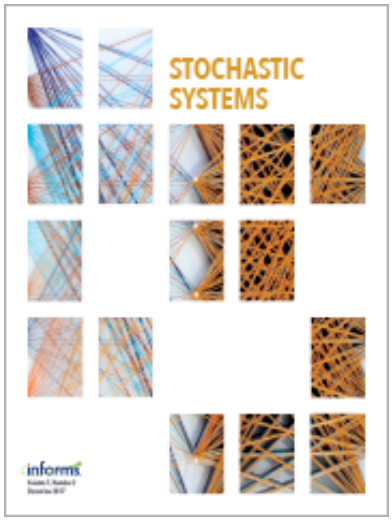
This article was downloaded by: [216.73.217.172] On: 15 November 2026, At: 06:26
Publisher: Institute for Operations Research and the Management Sciences (INFORMS)
INFORMS is located in Maryland, USA



Stochastic Systems

Publication details, including instructions for authors and subscription information:
http://pubsonline.informs.org

Convergence Properties of Weighted Particle Islands with Application to the Double Bootstrap Algorithm

Pierre Del Moral, Eric Moulines, Jimmy Olsson, Christelle Vergé

To cite this article:
Pierre Del Moral, Eric Moulines, Jimmy Olsson, Christelle Vergé (2016) Convergence Properties of Weighted Particle Islands with Application to the Double Bootstrap Algorithm. Stochastic Systems 6(2):367-419. https://doi.org/10.1287/15-SSY190

This work is licensed under a Creative Commons Attribution 4.0 International License. You are free to copy, distribute, transmit and adapt this work, but you must attribute this work as "*Stochastic Systems*. Copyright 2016 The Author(s). https://doi.org/10.1287/15-SSY190, used under a Creative Commons Attribution License: http://creativecommons.org/licenses/by/4.0/."

Copyright © 2016, The author(s)

Please scroll down for article—it is on subsequent pages

With 12,500 members from nearly 90 countries, INFORMS is the largest international association of operations research (O.R.) and analytics professionals and students. INFORMS provides unique networking and learning opportunities for individual professionals, and organizations of all types and sizes, to better understand and use O.R. and analytics tools and methods to transform strategic visions and achieve better outcomes.
For more information on INFORMS, its publications, membership, or meetings visit http://www.informs.org

Stochastic Systems
2016, Vol. 6, No. 2, 367–419
DOI: 10.1214/15-SSY190

Downloaded from informs.org by [216.73.217.172] on 15 November 2026, at 06:26 . For personal use only, all rights reserved.

CONVERGENCE PROPERTIES OF WEIGHTED PARTICLE ISLANDS WITH APPLICATION TO THE DOUBLE BOOTSTRAP ALGORITHM

By Pierre Del Moral[*], Eric Moulines[†], Jimmy Olsson[‡,¶], Christelle Vergé[§,‖]

University of New South Wales[*], École Polytechnique, INRIA XPOP[†], KTH Royal Institute of Technology[‡], ONERA/CNES[§]

Particle island models [31] provide a means of parallelization of sequential Monte Carlo methods, and in this paper we present novel convergence results for algorithms of this sort. In particular we establish a central limit theorem—as the number of islands and the common size of the islands tend jointly to infinity—of the double bootstrap algorithm with possibly adaptive selection on the island level. For this purpose we introduce a notion of archipelagos of weighted islands and find conditions under which a set of convergence properties are preserved by different operations on such archipelagos. This theory allows arbitrary compositions of these operations to be straightforwardly analyzed, providing a very flexible framework covering the double bootstrap algorithm as a special case. Finally, we establish the long-term numerical stability of the double bootstrap algorithm by bounding its asymptotic variance under weak and easily checked assumptions satisfied typically for models with non-compact state space.

1. Introduction. This paper discusses approaches to parallelization of *sequential Monte Carlo* (SMC) methods (or *particle filters*) approximating normalized *Feynman-Kac distribution flows*. At present, SMC methods are used successfully for online sampling from sequences of complex distributions in a wide range of applications, including nonlinear filtering, signal processing, data assimilation [see, e.g., 19, 6, 27, 3, 9, and the references therein], and rare event analysis [11, 5]. These algorithms evolve, recursively and randomly in time, a sample of random draws, *particles*, with associated *importance weights*. The particle cloud is updated through *selection* and *mutation* operations, where the former duplicates or eliminates, through

Received June 2015.

¶J. Olsson is supported by the Swedish Research Council, Grant 2011-5577.

‖C. Vergé is supported by CNES (Centre National d'Études Spatiales) and ONERA, The French Aerospace Lab.

Keywords and phrases: Central limit theorem, exponential deviation, parallelization, particle island models, particle filter, sequential Monte Carlo methods.

resampling, particles with large or small importance weights, respectively, while the latter disseminates randomly the particles over the state space and updates accordingly the importance weights for further selection.

SMC methods are computationally intensive, which may be critical in online applications. In particular, since the particle interaction enforced by the selection operation is of "global" nature (as it draws, with replacement, each particle from the entire particle population rather than from a subset of the same), running SMC methods in parallel on multicore processors is not straightforward. A natural ideal, which is the basis also for the present paper, is to parallelize the algorithm by, instead of considering a single batch of N particles, simply dividing the particle population into N_1 batches of each N_2 particles (i.e., $N = N_1 N_2$), where each batch is referred to as a *particle island* (or simply an *island*).

Parallel implementation of SMC was first proposed in [2] in the form of an algorithm referred to as the *local exchange particle filter* (LEPF), in which groups of particles are spread across computational units. This algorithm was later improved in [1] (see also [20] where a detailed convergence analysis of the LEPF is carried out). As indicated by the almost 300 Google Scholar citations at the time of writing, the LEPF has triggered a substantial interest in parallelization of SMC. Most notably, variations of the LEPF are found in the contexts of multitarget tracking [30], optical tracking [29], and state estimation [28].

In the present paper we consider an algorithm suggested in [31], which may be viewed as a variant of the LEPF algorithm. In this framework, each island evolves according to the standard SMC scheme subjecting alternatingly the subpopulation to selection and mutation. Unfortunately, the division of the particle population introduces additional bias which may be of note for moderate island sizes N_2. Thus, in [31] it is proposed to reduce this bias by performing additional selection also on the *island level* by resampling multinomially, when needed, the islands according to probabilities proportional to the weight averages over the different subpopulations. Selection on the island level may be performed systematically, as in the *double bootstrap* (B^2) *algorithm* (in the present paper we have chosen to denote the algorithm "B^2" rather than "2B", as we consider it more correctly described as a "square bootstrap" rather than a "double bootstrap"; nevertheless, the algorithm must not be confused with the *SMC square* (SMC^2) *algorithm* proposed in [8], which is, if still of a related form, of a different nature) or may be activated adaptively by some criterion measuring the skewness of the island weights. The latter approach will be referred to by us as the *double bootstrap algorithm with adaptive selection on the island level* (B^2ASIL).

Downloaded from informs.org by [216.73.217.172] on 15 November 2026, at 06:26 . For personal use only, all rights reserved.

Downloaded from informs.org by [216.73.217.172] on 15 November 2026, at 06:26 . For personal use only, all rights reserved.

At the end of the day, a sequence of Monte Carlo estimators is obtained by weighing up, using the island weights, the self-normalized empirical measures associated with the different particle islands.

Needless to say, the theoretical analysis of B^2-type algorithms is challenging due to the intricate dependence structure imposed by the island selection operation and the "double asymptotics" introduced by the two sample sizes N_1 and N_2. The authors of [31], who base their theoretical analysis on a reformulation of the particle island model as an extended Feynman-Kac model on an augmented space of dimension N_2, detour the latter difficulty by letting first the number N_1 of islands and then the number N_2 of individuals of each island tend to infinity. By *separating* the asymptotics in this manner, the analysis can, not surprisingly, at least in the case of the B^2 algorithm, be handled using classical techniques from SMC analysis, and in this way the authors establish convergence of bias and variance when these quantities are scaled with the size N of the system. However, working with this somewhat synthetic mode of convergence (with separated limits), the authors fail to supplement their consistency results with a central limit theorem (CLT). Moreover, they do not provide any convergence results for the B^2ASIL algorithm.

Nevertheless, even though the islands are allowed to interact through selection, any two individuals of the system should become more and more statistically independent as the number of islands as well as the size of the islands grow (cf. the *propagation of chaos* property of standard SMC methods [10]). Thus, we may expect a law of large number as well as a CLT to hold when N_1 and N_2 tend *jointly* to infinity. Moreover, in analogy with similar results for standard, single batch SMC methods [see 12, 7, 22, 15], we may expect the rate of such a CLT to be $\sqrt{N}$.

The aim of the present paper is to improve the existing theoretical analysis of particle island models by establishing results of the previous type. For this purpose we will introduce a notion of *archipelagos of weighted islands* that generalizes the particle models studied in [31] and consider three kinds of convergence properties of such archipelagos, namely *consistency* (convergence in probability), *asymptotic normality* (convergence in distribution in terms of a CLT with rate $\sqrt{N}$), and *large deviation* (an exponential inequality of Hoeffing-type that holds uniformly over all islands). After this, we perform single-step analyses of three kinds of operations on archipelagos, namely *selection on the island level*, *selection on the individual level*, and *mutation*, and show how these operations preserve the convergence properties under consideration. As a consequence, we are able to establish that the convergence properties in question are preserved through an *arbitrary* com-

position of the mentioned operations, including the B^2 algorithm as a special case, and to provide explicit expressions of the associated asymptotic variance. Moreover, the flexibility of our results, which generalize those obtained in [15] for standard, single batch SMC methods, makes these well-suited for analyzing particle island algorithms with adaptive resampling strategies such as the B^2ASIL scheme, for which we provide a detailed analysis (including a CLT). In our proofs, which rely on limit theorems for triangular arrays obtained in [15], the working process is highly inductive. Since the intricate dependence structures of the particle model force us to define triangular arrays on the island level, we will often, when establishing the preservation of a certain convergence property of a certain operation, face a situation where the only way of obtaining some critical limit or bound is to add the same to the list of induction hypotheses. After this, one establishes that the operation in question preserves also this additional property (limit or bound), by possibly adding, if needed, further assumptions to the list, and so on. At the end of the day, we have obtained a more or less minimal set (a hexad in the case of asymptotic normality) of properties that need to be checked at each induction step. In this machinery, the large deviation property is a critical component, since it provides, as a consequence of the distribution-free character of Hoeffding-type inequalities, uniform control of the deviation of the empirical measures associated with the different islands from their common mean.

As a last contribution, we establish the numerical stability of the B^2 algorithm by bounding uniformly the asymptotic variance of its output. We carry through this analysis under a *strong mixing condition* as well a *local Doeblin condition* (see Section 5.10 for details), where the latter is considerably weaker than the former and easily verified for a large variety of models with possibly non-compact state space. When operating under the local Doeblin condition, we let the Feynman-Kac model be indexed by a strictly stationary sequence of random parameters (corresponding, e.g., to random observations in the case of optimal filtering in *hidden Markov models*) and show, using novel results in [17], that the sequence of asymptotic variances is stochastically bounded (tight) in this setting. On the other hand, imposing the strong mixing assumption, which is classical in the literature of SMC analysis [13, 10], allows an explicit, deterministic uniform variance bound to be obtained using standard methods.

To sum up, the contribution of the present paper is threefold, since we

- introduce a general theory of archipelagos of weighted particle islands and analyze thoroughly the convergence properties, as the number N_1 of islands and the common size N_2 of the islands tend jointly to

Downloaded from informs.org by [216.73.217.172] on 15 November 2026, at 06:26 . For personal use only, all rights reserved.

infinity, of such objects when subjected to certain operations. For this purpose, we develop a machinery that allows triangular arrays defined on the island level to be analyzed and which may be used for handling double asymptotics appearing in other kinds of island-type particle algorithms.

- apply the previous theoretical results to the B^2 and B^2ASIL algorithms, yielding laws of large numbers and CLTs for these schemes.
- establish the long-term stability of the B^2 and B^2ASIL algorithms under weak and easily checked assumptions.

The paper is organized as follows. In Section 2 we introduce, after some prefatory notation, the concept of archipelagos of weighted islands, and define the three different convergence properties of such archipelagos. Our main results are, along with the three different operations under consideration, presented in Section 3, and Section 4 discusses the application of these results to the B^2ASIL algorithm. In particular, in Corollary 4.3 we establish the asymptotic normality of this algorithm, which implies the asymptotic normality of the B^2 algorithm as a special case (see Corollary 4.4), and provide a formula for the asymptotic variance; moreover, in Section 4.3 establish the long-term stability of the algorithm by showing that the asymptotic variance of the B^2 algorithm may, under suitable assumptions, be bounded uniformly. The proofs are gathered in Section 5. Finally, Appendix A provides some technical results that are used frequently in Section 5.

2. Preliminaries.

2.1. *Some notation.* For $(m, n) \in \mathbb{Z}^2$ such that $m \leq n$ we denote $[\![m, n]\!] \triangleq \{m, m+1, \ldots, n\} \subset \mathbb{Z}$. Moreover, we denote by and $\mathbb{R}_+$ and $\mathbb{R}_+^*$ the sets of nonnegative and positive real numbers, respectively, and by $\mathbb{N}^*$ the set of positive integers. For any quantities $\{a_\ell\}_{\ell=m}^n$ we will use the vector notation $a_{m:n} = (a_m, \ldots, a_n)$ with the convention $a_{m:n} = \varnothing$ if $m > n$.

In the sequel we assume that all random variables are defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For some given measurable space $(\mathsf{X}, \mathcal{X})$ we denote by $\mathsf{M}(\mathcal{X})$ and $\mathsf{M}_1(\mathcal{X}) \subset \mathsf{M}(\mathcal{X})$ the set of measures and probability measures on $(\mathsf{X}, \mathcal{X})$, respectively. In addition, we denote by $\mathsf{F}(\mathcal{X})$ the set of real-valued measurable functions on $(\mathsf{X}, \mathcal{X})$ and by $\mathsf{F}_\mathrm{b}(\mathcal{X}) \subset \mathsf{F}(\mathcal{X})$ the set of bounded such functions. For $h \in \mathsf{F}_\mathrm{b}(\mathcal{X})$ we denote the sup norm $\|h\|_\infty \triangleq \sup_{x \in \mathsf{X}} |h(x)|$ and the oscillator norm $\mathrm{osc}(h) \triangleq \sup_{(x,x') \in \mathsf{X}^2} |h(x) - h(x')|$. For any $\nu \in \mathsf{M}(\mathcal{X})$ and $f \in \mathsf{F}(\mathcal{X})$ we denote by $\nu f \triangleq \int f(x)\, \nu(\mathrm{d}x)$ the Lebesgue integral of f under ν whenever this is well-defined. Now, given also some other $(\mathsf{Y}, \mathcal{Y})$ measurable space, an *unnormalized transition kernel* K from

Downloaded from informs.org by [216.73.217.172] on 15 November 2026, at 06:26 . For personal use only, all rights reserved.

$(\mathsf{X}, \mathcal{X})$ to $(\mathsf{Y}, \mathcal{Y})$ is a mapping from $\mathsf{X} \times \mathcal{Y}$ to $\mathbb{R}_+$ such that for all $\mathsf{A} \in \mathcal{Y}$, $x \mapsto K(x, \mathsf{A})$ is a nonnegative measurable function on X and for all $x \in \mathsf{X}$, $\mathsf{A} \mapsto K(x, \mathsf{A})$ is a measure on $(\mathsf{Y}, \mathcal{Y})$. If $K(x, \mathsf{Y}) = 1$ for all $x \in \mathsf{X}$, then K is called a *transition kernel*. The kernel K induces two integral operators, one acting on functions and the other on measures. More specifically, let $f \in \mathsf{F}(\mathcal{X})$ and $\nu \in \mathsf{M}(\mathcal{X})$ and define the measurable function

$$Kf : \mathsf{X} \ni x \mapsto \int f(y)\, K(x, \mathrm{d}y)$$

and the measure

$$\nu K : \mathcal{Y} \ni \mathsf{A} \mapsto \int K(x, \mathsf{A})\, \nu(\mathrm{d}x)$$

whenever these quantities are well-defined. Finally, let K be as above and let L be another unnormalized transition kernels from $(\mathsf{Y}, \mathcal{Y})$ to some third measurable space $(\mathsf{Z}, \mathcal{Z})$; then we define the *product* of K and L as the unnormalized transition kernel

$$KL : \mathsf{X} \times \mathcal{Z} \ni (x, \mathsf{A}) \mapsto \int K(x, \mathrm{d}y)\, L(y, \mathsf{A})$$

from $(\mathsf{X}, \mathcal{X})$ to $(\mathsf{Z}, \mathcal{Z})$ whenever this is well-defined.

2.2. *Weighted particle islands and archipelagos.* Let $\{N_1(N)\}_{N \in \mathbb{N}^*}$ and $\{N_2(N)\}_{N \in \mathbb{N}^*}$ be sequences of positive integers such that $N_1(N) N_2(N) = N$ for all $N \in \mathbb{N}^*$ and $N_1(N) \to \infty$ and $N_2(N) \to \infty$ as $N \to \infty$. For lucidity we will often omit the index N from the notation and write simply N_1 and N_2. In the following, let $\{(\xi_N(i,j), \omega_N(i,j)); (i,j) \in [\![1, N_1]\!] \times [\![1, N_2]\!]\}$ be an array of X-valued random variables (the ξ_N) with associated nonnegative (possibly unnormalized) weights (the ω_N). For each $i \in [\![1, N_1]\!]$, the subset $\{(\xi_N(i,j), \omega_N(i,j))\}_{j=1}^{N_2}$ of the array will be referred to as an *island*. With this terminology, a random variable $\xi_N(i,j)$ in the array will be referred to as an *individual* or a *particle*. Finally, we associate each island $\{(\xi_N(i,j), \omega_N(i,j))\}_{j=1}^{N_2}$ with a nonnegative (possibly unnormalized) weight $\Omega_N(i)$. In the following, the set $\{(\Omega_N(i), \{(\xi_N(i,j), \omega_N(i,j))\}_{j=1}^{N_2})\}_{i=1}^{N_1}$ of islands with associated weights will be referred to as an *archipelago* on $(\mathsf{X}, \mathcal{X})$. We will always require the island weights to be positive and the particle weights to be positive and uniformly bounded, i.e., there exists some constant $|\omega|_\infty$ such that $0 < |\omega_N(i,j)| \leq |\omega|_\infty$ for all $(i,j) \in [\![1, N_1(N)]\!] \times [\![1, N_2(N)]\!]$ and $N \in \mathbb{N}^*$.

Downloaded from informs.org by [216.73.217.172] on 15 November 2026, at 06:26 . For personal use only, all rights reserved.

Downloaded from informs.org by [216.73.217.172] on 15 November 2026, at 06:26 . For personal use only, all rights reserved.

2.3. *Convergence properties of archipelagos.* In the following, any limit ($\longrightarrow$), limit in probability ($\xrightarrow{\mathbb{P}}$), and limit in distribution ($\xrightarrow{\mathcal{D}}$) is supposed to hold as $N \to \infty$ if not specified differently.

DEFINITION 2.1 (consistency). *An archipelago* $\{(\Omega_N(i), \{(\xi_N(i,j), \omega_N(i,j))\}_{j=1}^{N_2})\}_{i=1}^{N_1}$ *on* $(\mathsf{X}, \mathcal{X})$ *is said to be* consistent for $\eta \in \mathsf{M}_1(\mathcal{X})$ *if for all* $h \in \mathsf{F}_{\mathrm{b}}(\mathcal{X})$,

$$\textbf{(C1)} \qquad \sum_{i=1}^{N_1} \frac{\Omega_N(i)}{\sum_{i'=1}^{N_1} \Omega_N(i')} \sum_{j=1}^{N_2} \frac{\omega_N(i,j)}{\sum_{j'=1}^{N_2} \omega_N(i,j')} h(\xi_N(i,j)) \xrightarrow{\mathbb{P}} \eta h,$$

$$\textbf{(C2)} \qquad \max_{i \in [\![1,N_1]\!]} \frac{\Omega_N(i)}{\sum_{i'=1}^{N_1} \Omega_N(i')} \xrightarrow{\mathbb{P}} 0.$$

Note that the estimator in (**C1**) assigns the weight $\Omega_N(i)/\sum_{i'=1}^{N_1} \Omega_N(i')$ to the self-normalized importance sampling estimator $\sum_{j=1}^{N_2} \omega_N(i,j) h(\xi_N(i,j))/\sum_{j'=1}^{N_2} \omega_N(i,j')$ associated with island $i \in [\![1, N_1]\!]$, and the smallness condition (**C2**) formalizes the fact that this weight, and thus the contribution of each island to the estimator associated with the archipelago as a whole, should vanish asymptotically as $N \to \infty$.

DEFINITION 2.2 (exponential deviation). *In the following, let* $\eta \in \mathsf{M}_1(\mathcal{X})$ *and* ϱ *and* $\{c_\ell\}_{\ell=1}^2$ *be positive constants. An archipelago* $\{(\Omega_N(i), \{(\xi_N(i,j), \omega_N(i,j))\}_{j=1}^{N_2})\}_{i=1}^{N_1}$ *on* $(\mathsf{X}, \mathcal{X})$ *is said to satisfy* exponential deviation for $(\eta, \varrho, \{c_\ell\}_{\ell=1}^2)$ *if for all* $h \in \mathsf{F}_{\mathrm{b}}(\mathcal{X})$, $N_1 \in \mathbb{N}^*$, $N_2 \in \mathbb{N}^*$, *and* $\varepsilon > 0$,

$$\textbf{(D)} \quad \mathbb{P}\left(\max_{i \in [\![1,N_1]\!]} \left| \frac{1}{N_2} \sum_{j=1}^{N_2} \omega_N(i,j) h(\xi_N(i,j)) - \varrho \times \eta h \right| \geq \varepsilon \right) \leq c_1 N_1 \exp\left(-c_2 N_2 \frac{\varepsilon^2}{\|h\|_\infty^2} \right).$$

The exponential deviation inequality in (**D**) provides uniform control on the deviations of the unnormalized importance sampling estimators $\sum_{j=1}^{N_2} \omega_N(i,j) h(\xi_N(i,j))/N_2$, $i \in [\![1, N_1]\!]$, associated with the different islands from their common mean level $\varrho \times \eta h$. The factor N_1 on the right hand side of the equality is required to compensate for the maximum with respect to the island index. Assumption (**D**) implies, by a straightforward extension of the generalized Hoeffding inequality derived in [14, Lemma 4], that also the deviations of the properly normalized importance sampling estimators associated with the different islands from the expectations targeted by the archipelago can be uniformly controlled as follows.

Downloaded from informs.org by [216.73.217.172] on 15 November 2026, at 06:26 . For personal use only, all rights reserved.

LEMMA 2.3. *Assume that* ***(D)*** *holds for* $(\eta, \varrho, \{c_\ell\}_{\ell=1}^2)$. *Then for all* $h \in \mathsf{F}_{\mathrm{b}}(\mathcal{X})$, $N_1 \in \mathbb{N}^*$, $N_2 \in \mathbb{N}^*$, *and* $\varepsilon > 0$,

$$\mathbb{P}\left(\max_{i\in[\![1,N_1]\!]}\left|\sum_{j=1}^{N_2}\frac{\omega_N(i,j)}{\sum_{j'=1}^{N_2}\omega_N(i,j')}\{h(\xi_N(i,j))-\eta h\}\right|\geq\varepsilon\right) \leq 2c_1N_1\exp\left(-c_2N_2\frac{\varepsilon^2\varrho^2}{4\|h\|_\infty^2}\right). \tag{2.1}$$

Finally, we introduce a third convergence property describing weak convergence in the sense of a CLT. Let N denote the Gaussian distribution.

DEFINITION 2.4 (asymptotic normality). *In the following, let*

- $\sigma^2 : \mathsf{F}_{\mathrm{b}}(\mathcal{X}) \to \mathbb{R}_+^*$ *and* $\nu^2 : \mathsf{F}_{\mathrm{b}}(\mathcal{X}) \to \mathbb{R}_+^*$ *be functionals.*
- $\eta \in \mathsf{M}_1(\mathcal{X})$ *and* $\{\mu_\ell\}_{\ell=1}^3 \subset \mathsf{M}(\mathcal{X})$ *be measures.*

An archipelago $\{(\Omega_N(i), \{(\xi_N(i,j), \omega_N(i,j))\}_{j=1}^{N_2})\}_{i=1}^{N_1}$ *on* $(\mathsf{X}, \mathcal{X})$ *is said to be* asymptotically normal for $(\eta, \sigma^2, \nu^2, \{\mu_\ell\}_{\ell=1}^3)$ *if for all* $h \in \mathsf{F}_{\mathrm{b}}(\mathcal{X})$,

(AN1) $$\sqrt{N}\sum_{i=1}^{N_1}\frac{\Omega_N(i)}{\sum_{i'=1}^{N_1}\Omega_N(i')}\sum_{j=1}^{N_2}\frac{\omega_N(i,j)}{\sum_{j'=1}^{N_2}\omega_N(i,j')}\{h(\xi_N(i,j))-\eta h\} \xrightarrow{\mathcal{D}} \mathsf{N}(0,\sigma^2(h))$$

and, in addition,

(AN2) $$N_2\sum_{i=1}^{N_1}\frac{\Omega_N(i)}{\sum_{i'=1}^{N_1}\Omega_N(i')}\left(\sum_{j=1}^{N_2}\frac{\omega_N(i,j)}{\sum_{j'=1}^{N_2}\omega_N(i,j')}\{h(\xi_N(i,j))-\eta h\}\right)^2 \xrightarrow{\mathbb{P}} \nu^2(h),$$

(AN3) $$N_1\sum_{i=1}^{N_1}\left(\frac{\Omega_N(i)}{\sum_{i'=1}^{N_1}\Omega_N(i')}\right)^2\sum_{j=1}^{N_2}\frac{\omega_N(i,j)}{\sum_{j'=1}^{N_2}\omega_N(i,j')}h(\xi_N(i,j)) \xrightarrow{\mathbb{P}} \mu_1 h,$$

(AN4) $$N\sum_{i=1}^{N_1}\left(\frac{\Omega_N(i)}{\sum_{i'=1}^{N_1}\Omega_N(i')}\right)^2\sum_{j=1}^{N_2}\left(\frac{\omega_N(i,j)}{\sum_{j'=1}^{N_2}\omega_N(i,j')}\right)^2 h(\xi_N(i,j)) \xrightarrow{\mathbb{P}} \mu_2 h,$$

(AN5) $$N_2\sum_{i=1}^{N_1}\frac{\Omega_N(i)}{\sum_{i'=1}^{N_1}\Omega_N(i')}\sum_{j=1}^{N_2}\left(\frac{\omega_N(i,j)}{\sum_{j'=1}^{N_2}\omega_N(i,j')}\right)^2 h(\xi_N(i,j)) \xrightarrow{\mathbb{P}} \mu_3 h,$$

(AN6) $$\lim_{\lambda\to\infty}\sup_{N\in\mathbb{N}^*}\mathbb{P}\left(\max_{i\in[\![1,N_1]\!]}N_1\frac{\Omega_N(i)}{\sum_{i'=1}^{N_1}\Omega_N(i')}\geq\lambda\right)=0.$$

Here (**AN1**) corresponds to a CLT and implies straightforwardly (**C1**). In addition, since (**AN6**) implies immediately (**C2**) we may conclude that asymptotic normality is stronger than consistency. Assumptions (**AN2**–**6**) guarantee the existence of asymptotic variance; see Remark 3.11 for further comments.

3. Main results.

3.1. *Operations on weighted archipelagos.* In the following we let $\mathsf{P}(\{a(i)\}_{i=1}^M)$ denote the categorical probability distribution induced by a set $\{a(i)\}_{i=1}^M$ of positive (possibly unnormalized) numbers; thus, writing $V\sim\mathsf{P}(\{a(i)\}_{i=1}^M)$ means that the random variable V takes the value $i\in[\![1,M]\!]$ with probability $a(i)/\sum_{i'=1}^M a(i')$.

3.1.1. *Selection on the island level.* The first operation, described in Algorithm 1, is referred to as *multinomial selection on the island level* (SIL). This operation consists in converting an archipelago $\{(\Omega_N(i),\{(\xi_N(i,j),\omega_N(i,j))\}_{j=1}^{N_2})\}_{i=1}^{N_1}$ targeting some distribution η into an archipelago $\{(1,\{(\tilde{\xi}_N(i,j),\tilde{\omega}_N(i,j))\}_{j=1}^{N_2})\}_{i=1}^{N_1}$ with uniform island weights targeting the same distribution η. This step allows islands with small/large weights to be eliminated/duplicated, respectively. More precisely, a new family of islands is generated from the existing ones by selecting, conditionally independently given the input archipelago, new islands according to probabilities proportional to the island weights $\{\Omega_N(i)\}_{i=1}^{N_1}$. After this, the weights and the particles of the selected islands are copied deterministically (which of course implies that the particle weights of the new archipelago are bounded by the same constant $|\omega|_\infty$ as the ancestor archipelago).

Data: $\{(\Omega_N(i),\{(\xi_N(i,j),\omega_N(i,j))\}_{j=1}^{N_2})\}_{i=1}^{N_1}$
Result: $\{(1,\{(\tilde{\xi}_N(i,j),\tilde{\omega}_N(i,j))\}_{j=1}^{N_2})\}_{i=1}^{N_1}$
for $i\leftarrow 1$ **to** N_1 **do**
  draw $I_N(i)\sim\mathsf{P}(\{\Omega_N(i')\}_{i'=1}^{N_1})$;
  for $j\leftarrow 1$ **to** N_2 **do**
    set $\tilde{\xi}_N(i,j)\leftarrow\xi_N(I_N(i),j)$;
    set $\tilde{\omega}_N(i,j)\leftarrow\omega_N(I_N(i),j)$;
  end
end

Algorithm 1: Multinomial selection on the island level (SIL)

Downloaded from informs.org by [216.73.217.172] on 15 November 2026, at 06:26 . For personal use only, all rights reserved.

Downloaded from informs.org by [216.73.217.172] on 15 November 2026, at 06:26 . For personal use only, all rights reserved.

In the following we will abbreviate Algorithm 1 by writing

$$\text{“}\{(1,\{(\tilde{\xi}_N(i,j),\tilde{\omega}_N(i,j))\}_{j=1}^{N_2})\}_{i=1}^{N_1} \leftarrow \mathsf{SIL}\left(\{(\Omega_N(i),\{(\xi_N(i,j),\omega_N(i,j))\}_{j=1}^{N_2})\}_{i=1}^{N_1}\right)\text{”}.$$

The following theorems state conditions under which SIL preserves consistency, exponential deviation, and asymptotic normality. The input and output in Algorithm 1 are respectively denoted by $\{(\Omega_N(i),\{(\xi_N(i,j),\omega_N(i,j))\}_{j=1}^{N_2})\}_{i=1}^{N_1}$ and $\{(1,\{(\tilde{\xi}_N(i,j),\tilde{\omega}_N(i,j))\}_{j=1}^{N_2})\}_{i=1}^{N_1}$ and all proofs are found in Section 5.

THEOREM 3.1. *Assume that* $\{(\Omega_N(i),\{(\xi_N(i,j),\omega_N(i,j))\}_{j=1}^{N_2})\}_{i=1}^{N_1}$ *is consistent for* η. *Then also* $\{(1,\{(\tilde{\xi}_N(i,j),\tilde{\omega}_N(i,j))\}_{j=1}^{N_2})\}_{i=1}^{N_1}$ *is consistent for* η.

THEOREM 3.2. *Assume that* $\{(\Omega_N(i),\{(\xi_N(i,j),\omega_N(i,j))\}_{j=1}^{N_2})\}_{i=1}^{N_1}$ *satisfies exponential deviation for* $(\eta,\varrho,\{c_\ell\}_{\ell=1}^2)$. *Then also* $\{(1,\{(\tilde{\xi}_N(i,j),\tilde{\omega}_N(i,j))\}_{j=1}^{N_2})\}_{i=1}^{N_1}$ *satisfies exponential deviation for* $(\eta,\varrho,\{c_\ell\}_{\ell=1}^2)$.

We impose the following assumption, guaranteeing that N_1 grows only subexponentially fast with respect to N_2.

(S) For all $\beta>0$, $N_1\exp(-\beta N_2)\to 0$ as $N\to\infty$.

THEOREM 3.3. *Assume (**S**) and that* $\{(\Omega_N(i),\{(\xi_N(i,j),\omega_N(i,j))\}_{j=1}^{N_2})\}_{i=1}^{N_1}$ *satisfies exponential deviation for* $(\eta,\varrho,\{c_\ell\}_{\ell=1}^2)$ *and is asymptotically normal for* $(\eta,\sigma^2,\nu^2,\{\mu_\ell\}_{\ell=1}^3)$. *Then also* $\{(1,\{(\tilde{\xi}_N(i,j),\tilde{\omega}_N(i,j))\}_{j=1}^{N_2})\}_{i=1}^{N_1}$ *is asymptotically normal for* $(\eta,\tilde{\sigma}^2,\nu^2,\{\eta,\mu_3,\mu_3\})$, *where for all* $h\in\mathsf{F}_{\mathrm{b}}(\mathcal{X})$,

$$\tilde{\sigma}^2(h)=\sigma^2(h)+\nu^2(h)$$

(i.e. the SIL operation modifies only σ^2, μ_1, *and* μ_2*).*

3.1.2. *Selection on the individual level.* A second operation, described in Algorithm 2, is referred to as *multinomial selection on the individual level* (SiL), and consists in converting a weighted archipelago $\{(\Omega_N(i),\{(\xi_N(i,j),\omega_N(i,j))\}_{j=1}^{N_2})\}_{i=1}^{N_1}$ targeting some distribution η into an archipelago $\{\Omega_N(i),\{(\tilde{\xi}_N(i,j),1)\}_{j=1}^{N_2}\}_{i=1}^{N_1}$ with uniform particle weights targeting the same distribution η. This step allows particles with large/small weights to be duplicated/eliminated, respectively. Note that the island weights remain unaffected.

Data: $\{(\Omega_N(i), \{(\xi_N(i,j), \omega_N(i,j))\}_{j=1}^{N_2})\}_{i=1}^{N_1}$
Result: $\{(\Omega_N(i), \{(\tilde{\xi}_N(i,j), 1)\}_{j=1}^{N_2})\}_{i=1}^{N_1}$
for $i \leftarrow 1$ **to** N_1 **do**
  for $j \leftarrow 1$ **to** N_2 **do**
    draw $J_N(i,j) \sim \mathsf{P}(\{\omega_N(i,j')\}_{j'=1}^{N_2})$;
    set $\tilde{\xi}_N(i,j) \leftarrow \xi_N(i, J_N(i,j))$;
  end
end

Algorithm 2: Multinomial selection on the individual level (SiL)

Trivially, the particle weights are bounded by $|\omega|_\infty = 1$ in this case. As for the SIL operation, we will express Algorithm 2 in a compact form by writing

$$\text{“}\{(\Omega_N(i), \{(\tilde{\xi}_N(i,j), 1)\}_{j=1}^{N_2})\}_{i=1}^{N_1} \leftarrow \mathsf{SiL}\left(\{(\Omega_N(i), \{(\xi_N(i,j), \omega_N(i,j))\}_{j=1}^{N_2})\}_{i=1}^{N_1}\right)\text{”}.$$

The following theorems state conditions under which SiL preserves consistency, exponential deviation inequality, and asymptotic normality. Here, $\{(\Omega_N(i), \{(\xi_N(i,j), \omega_N(i,j))\}_{j=1}^{N_2})\}_{i=1}^{N_1}$ and $\{(\Omega_N(i), \{(\tilde{\xi}_N(i,j), 1)\}_{j=1}^{N_2})\}_{i=1}^{N_1}$ denote the input and output, respectively, of Algorithm 2.

THEOREM 3.4. *Assume that* $\{(\Omega_N(i), \{(\xi_N(i,j), \omega_N(i,j))\}_{j=1}^{N_2})\}_{i=1}^{N_1}$ *is consistent for* η. *Then also* $\{(\Omega_N(i), \{(\tilde{\xi}_N(i,j), 1)\}_{j=1}^{N_2})\}_{i=1}^{N_1}$ *is consistent for* η.

THEOREM 3.5. *Assume that* $\{(\Omega_N(i), \{(\xi_N(i,j), \omega_N(i,j))\}_{j=1}^{N_2})\}_{i=1}^{N_1}$ *satisfies exponential deviation for* $(\eta, \varrho, \{c_\ell\}_{\ell=1}^2)$. *Then also* $\{(\Omega_N(i), \{(\tilde{\xi}_N(i,j), 1)\}_{j=1}^{N_2})\}_{i=1}^{N_1}$ *satisfies exponential deviation for* $(\eta, 1, \{\tilde{c}_\ell\}_{\ell=1}^2)$, *where* $\tilde{c}_1 = 4(1 \vee c_1)$ *and* $\tilde{c}_2 = (1 \wedge (c_2\varrho^2/2))/8$.

THEOREM 3.6. *Assume that* $\{(\Omega_N(i), \{(\xi_N(i,j), \omega_N(i,j))\}_{j=1}^{N_2})\}_{i=1}^{N_1}$ *satisfies exponential deviation for* $(\eta, \varrho, \{c_\ell\}_{\ell=1}^2)$ *and is asymptotically normal for* $(\eta, \sigma^2, \nu^2, \{\mu_\ell\}_{\ell=1}^3)$.

Then also $\{(\Omega_N(i), \{(\tilde{\xi}_N(i,j), 1)\}_{j=1}^{N_2})\}_{i=1}^{N_1}$ *is asymptotically normal for* $(\eta, \tilde{\sigma}^2, \tilde{\nu}^2, \{\mu_1, \mu_1, \eta\})$, *where for all* $h \in \mathsf{F}_\mathrm{b}(\mathcal{X})$,

$$\begin{cases} \tilde{\sigma}^2(h) = \sigma^2(h) + \mu_1\{(h - \eta h)^2\}, \\ \tilde{\nu}^2(h) = \nu^2(h) + \eta\{(h - \eta h)^2\}. \end{cases}$$

Again, proofs are found in Section 5.

Downloaded from informs.org by [216.73.217.172] on 15 November 2026, at 06:26 . For personal use only, all rights reserved.

Downloaded from informs.org by [216.73.217.172] on 15 November 2026, at 06:26 . For personal use only, all rights reserved.

3.1.3. *Mutation.* The last operation we consider is *Mutation*, described in Algorithm 3. This operation converts, using importance sampling on the individual level, an archipelago $\{(\Omega_N(i), \{(\xi_N(i,j), \omega_N(i,j))\}_{j=1}^{N_2})\}_{i=1}^{N_1}$ targeting $\eta \in \mathsf{M}_1(\mathcal{X})$ into another archipelago $\{\tilde{\Omega}_N(i), \{(\tilde{\xi}_N(i,j), \tilde{\omega}_N(i,j))\}_{j=1}^{N_2}\}_{i=1}^{N_1}$ targeting some *other* probability distribution $\tilde{\eta}$, defined on another state space $(\tilde{\mathsf{X}}, \tilde{\mathcal{X}})$. The distribution $\tilde{\eta}$ is related to η through the identity

$$\tilde{\eta}h = \frac{\eta Q h}{\eta Q \mathbb{1}_{\tilde{\mathsf{X}}}} \quad (h \in \mathsf{F}_{\mathrm{b}}(\tilde{\mathcal{X}})), \tag{3.1}$$

where $Q : \mathsf{X} \times \tilde{\mathcal{X}} \to \mathbb{R}_+$ is a possibly unnormalized transition kernel. In the algorithm that follows, let $R : \mathsf{X} \times \tilde{\mathcal{X}} \to \mathbb{R}_+$ be a (normalized) transition kernel such that $Q(x,\cdot) \ll R(x,\cdot)$ for all $x \in \mathsf{X}$, and denote the corresponding Radon-Nikodym derivatives by

$$w(x,\tilde{x}) \triangleq \frac{\mathrm{d}Q(x,\cdot)}{\mathrm{d}R(x,\cdot)}(\tilde{x}) \quad ((x,\tilde{x}) \in \mathsf{X} \times \tilde{\mathsf{X}}).$$

In the sequel, we will refer to the mapping w as the *importance weight function* and assume that $w \in \mathsf{F}_{\mathrm{b}}(\mathcal{X} \otimes \tilde{\mathcal{X}})$ and $Q\mathbb{1}_{\tilde{\mathsf{X}}} \in \mathsf{F}_{\mathrm{b}}(\mathcal{X})$.

Data: $\{(\Omega_N(i), \{(\xi_N(i,j), \omega_N(i,j))\}_{j=1}^{N_2})\}_{i=1}^{N_1}$, Q, R
Result: $\{(\tilde{\Omega}_N(i), \{(\tilde{\xi}_N(i,j), \tilde{\omega}_N(i,j))\}_{j=1}^{N_2})\}_{i=1}^{N_1}$
for $i \leftarrow 1$ **to** N_1 **do**
  for $j \leftarrow 1$ **to** N_2 **do**
    draw $\tilde{\xi}_N(i,j) \sim R(\xi_N(i,j), \cdot)$;
    set $\tilde{\omega}_N(i,j) \leftarrow w(\xi_N(i,j), \tilde{\xi}_N(i,j))\omega_N(i,j)$;
  end
  set $\tilde{\Omega}_N(i) \leftarrow \Omega_N(i) \dfrac{\sum_{j'=1}^{N_2} \tilde{\omega}_N(i,j')}{\sum_{j''=1}^{N_2} \omega_N(i,j'')}$;
end

Algorithm 3: Mutation

As before, we will abbreviate Algorithm 3 by writing

$$\text{“}\{(\tilde{\Omega}_N(i), \{(\tilde{\xi}_N(i,j), \tilde{\omega}_N(i,j))\}_{j=1}^{N_2})\}_{i=1}^{N_1} \leftarrow \mathsf{Mut}\langle Q\rangle \left(\{(\Omega_N(i), \{(\xi_N(i,j), \omega_N(i,j))\}_{j=1}^{N_2})\}_{i=1}^{N_1}, R\right)\text{”},$$

where the kernel Q is included in the notation for the sake of completeness. Note that the Mutation operation forms indeed a proper weighted

Downloaded from informs.org by [216.73.217.172] on 15 November 2026, at 06:26 . For personal use only, all rights reserved.

archipelago with $|\tilde{\omega}|_\infty = |\omega|_\infty \|w\|_\infty$. In conformity with the SIL and SiL operations, the Mutation operation preserves consistency, exponential deviation, and asymptotic normality. This is established below, where $\{(\Omega_N(i), \{(\xi_N(i,j), \omega_N(i,j))\}_{j=1}^{N_2})\}_{i=1}^{N_1}$ and $\{(\tilde{\Omega}_N(i), \{(\tilde{\xi}_N(i,j), \tilde{\omega}_N(i,j))\}_{j=1}^{N_2})\}_{i=1}^{N_1}$ denote consequently the input and output of Algorithm 3, respectively.

THEOREM 3.7. *Assume that* $\{(\Omega_N(i), \{(\xi_N(i,j), \omega_N(i,j))\}_{j=1}^{N_2})\}_{i=1}^{N_1}$ *is consistent for* η. *Then* $\{(\tilde{\Omega}_N(i), \{(\tilde{\xi}_N(i,j), \tilde{\omega}_N(i,j))\}_{j=1}^{N_2})\}_{i=1}^{N_1}$ *is consistent for* $\tilde{\eta}$ *defined in* (3.1).

THEOREM 3.8. *Assume that* $\{(\Omega_N(i), \{(\xi_N(i,j), \omega_N(i,j))\}_{j=1}^{N_2})\}_{i=1}^{N_1}$ *satisfies exponential deviation for* $(\eta, \varrho, \{c_\ell\}_{\ell=1}^2)$. *Then* $\{(\tilde{\Omega}_N(i), \{(\tilde{\xi}_N(i,j), \tilde{\omega}_N(i,j))\}_{j=1}^{N_2})\}_{i=1}^{N_1}$ *satisfies exponential deviation for* $(\tilde{\eta}, \tilde{\varrho}, \{\tilde{c}_\ell\}_{\ell=1}^2)$, *where* $\tilde{\varrho} = \varrho \times \eta Q \mathbb{1}_{\tilde{\mathsf{X}}}$, $\tilde{c}_1 = 2(2 \vee c_1)$, *and*

$$\tilde{c}_2 = \frac{1}{2}\left(\frac{1}{\delta^2} \wedge \frac{c_2}{2\|Q\mathbb{1}_{\tilde{\mathsf{X}}}\|_\infty^2}\right),$$

with $\delta \triangleq |\tilde{\omega}|_\infty + |\omega|_\infty \|Q\mathbb{1}_{\tilde{\mathsf{X}}}\|_\infty$.

THEOREM 3.9. *Assume that* $\{(\Omega_N(i), \{(\xi_N(i,j), \omega_N(i,j))\}_{j=1}^{N_2})\}_{i=1}^{N_1}$ *satisfies exponential deviation for* $(\eta, \varrho, \{c_\ell\}_{\ell=1}^2)$ *and is asymptotically normal for* $(\eta, \sigma^2, \nu^2, \{\mu_\ell\}_{\ell=1}^3)$. *Then the mutated archipelago* $\{(\tilde{\Omega}_N(i), \{(\tilde{\xi}_N(i,j), \tilde{\omega}_N(i,j))\}_{j=1}^{N_2})\}_{i=1}^{N_1}$ *is asymptotically normal for* $(\tilde{\eta}, \tilde{\sigma}^2, \tilde{\nu}^2, \{\tilde{\mu}_\ell\}_{\ell=1}^3)$, *where* $\tilde{\eta}$ *is defined in* (3.1) *and for all* $h \in \mathsf{F}_{\mathrm{b}}(\tilde{\mathcal{X}})$,

$$\begin{cases} \tilde{\sigma}^2(h) = \big(\sigma^2\{Q(h - \tilde{\eta}h)\} + \mu_2 R\{w^2(h - \tilde{\eta}h)^2\} \\ \qquad\qquad -\mu_2\{Q^2(h - \tilde{\eta}h)\}\big) / (\eta Q \mathbb{1}_{\tilde{\mathsf{X}}})^2 , \\ \tilde{\nu}^2(h) = \big(\nu^2\{Q(h - \tilde{\eta}h)\} + \mu_3 R\{w^2(h - \tilde{\eta}h)^2\} \\ \qquad\qquad -\mu_3\{Q^2(h - \tilde{\eta}h)\}\big) / (\eta Q \mathbb{1}_{\tilde{\mathsf{X}}})^2 , \\ \tilde{\mu}_1 h = \mu_1 Qh / \eta Q \mathbb{1}_{\tilde{\mathsf{X}}}, \\ \tilde{\mu}_2 h = \mu_2 R(w^2 h) / (\eta Q \mathbb{1}_{\tilde{\mathsf{X}}})^2, \\ \tilde{\mu}_3 h = \mu_3 R(w^2 h) / (\eta Q \mathbb{1}_{\tilde{\mathsf{X}}})^2, \end{cases}$$

(where $Q^2 h(x) \triangleq \{Qh(x)\}^2$ *and* $R(w^2 h)(x) \triangleq \int w^2(x, x') h(x')\, R(x, \mathrm{d}x')$ *for all* $x \in \mathsf{X}$ *and* $h \in \mathsf{F}_{\mathrm{b}}(\mathcal{X})$*).*

REMARK 3.10. *Note that Theorem 3.6 and Theorem 3.9 hold true regardless of the intermutual rates by which* N_1 *and* N_2 *tend to infinity with*

*N. In particular, these results do not, on the contrary to Theorem 3.3, require the condition (**S**). This is in line with what we expect, as the SiL and Mutation operations do not involve any island interaction.*

REMARK 3.11. *As clear from the previous, the SiL, SIL, and Mutation operations modify a given archipelago by means of either resampling of islands, local, island-wise resampling of individuals or random mutation of all the individuals of the archipelago. Assumptions (**AN2–4**) regulate the increase of asymptotic variance brought forth by subjecting the archipelago to each of these operations, respectively. Thus, when the archipelago is subjected to a given operation, only one of these conditions plays the active role for the propagation of the CLT in (**AN1**); however, since we want to be able to analyze arbitrary, possibly random (as in the B^2ASIL algorithm in Section 4.2) compositions of the operations, we are required to keep a record of the incremental variances disengaged by each one. Still, the conditions (**AN2–6**) are nested intricately in the sense that for a given operation, one or several conditions play active roles for the propagation of another. In this way, the condition (**AN5**), which does not regulate directly the increase of asymptotic variance for any of the operations, bridges the mutation and island resampling operations in the sense that it regulates the limit (**AN2**) in the case of mutation.*

4. Applications.

4.1. *Feynman-Kac models.* For a sequence of unnormalized transition kernels $\{Q_n\}_{n\in\mathbb{N}}$ defined on some common measurable space $(\mathsf{X},\mathcal{X})$ and some probability distribution $\eta_0 \in \mathsf{M}_1(\mathcal{X})$, a sequence $\{\eta_n\}_{n\in\mathbb{N}}$ of *Feynman-Kac measures* is defined by

$$\eta_n h \triangleq \frac{\gamma_n h}{\gamma_n \mathbb{1}_{\mathsf{X}}}, \quad n \in \mathbb{N},\ h \in \mathsf{F}_{\mathrm{b}}(\mathcal{X}), \tag{4.1}$$

where

$$\gamma_n h \triangleq \int \cdots \int h(x_n)\, \eta_0(\mathrm{d}x_0) \prod_{p=0}^{n-1} Q_p(x_p, \mathrm{d}x_{p+1}) \quad (h \in \mathsf{F}_{\mathrm{b}}(\mathcal{X}))$$

(with the usual convention $\prod_{p=m}^{n} a_p = 1$ when $m > n$). We may express recursively the sequences of unnormalized and normalized Feynman-Kac measures as, for $h \in \mathsf{F}_{\mathrm{b}}(\mathcal{X})$ and $(m,n) \in \mathbb{N}$ with $m \le n$,

Downloaded from informs.org by [216.73.217.172] on 15 November 2026, at 06:26 . For personal use only, all rights reserved.

$$\gamma_n h = \gamma_m Q_m \cdots Q_{n-1} h \quad \text{and}$$
$$\eta_n h = \frac{\gamma_m Q_m \cdots Q_{n-1} h}{\gamma_m Q_m \cdots Q_{n-1} \mathbb{1}_{\mathsf{X}}} = \frac{\eta_m Q_m \cdots Q_{n-1} h}{\eta_m Q_m \cdots Q_{n-1} \mathbb{1}_{\mathsf{X}}},$$

respectively, with the convention $Q_m \cdots Q_\ell(x, h) = h(x)$ if $m > \ell$. In particular,

$$\eta_{n+1} h = \frac{\eta_n Q_n h}{\eta_n Q_n \mathbb{1}_{\mathsf{X}}} \quad (h \in \mathsf{F}_{\mathrm{b}}(\mathcal{X}), n \in \mathbb{N}), \tag{4.2}$$

which means that we may cast the model into the framework considered in Section 3.1.3.

EXAMPLE 1. *A special instance of the previous framework is formed naturally by specifying, first, a sequence* $\{M_n\}_{n\in\mathbb{N}}$ *of normalized (Markov) transition kernels on* $(\mathsf{X}, \mathcal{X})$ *with an associated initial distribution* χ *and, second,* potential functions $\{g_n\}_{n\in\mathbb{N}^*}$, *where* $g_n : \mathsf{X} \to \mathbb{R}_+^*$ *for all* $n \in \mathbb{N}^*$, *and letting* $Q_n h(x) \triangleq M_n(g_{n+1} h)(x)$ *for all* $n \in \mathbb{N}^*$, $x \in \mathsf{X}$, *and* $h \in \mathsf{F}_{\mathrm{b}}(\mathcal{X})$. *In addition,* $\eta_0 \triangleq \chi$. *This setup covers a large variety of important models in probability and statistics, such as* optimal filtering *in hidden Markov models (or* state-space models*; see, e.g., [4]) and models for the analysis of* rare events *[11, 5]. We will return to this setting in Section 4.3.*

Using a Feynman-Kac model in practice is typically non-trivial as neither the distribution flow $\{\gamma_n\}_{n\in\mathbb{N}}$ nor $\{\eta_n\}_{n\in\mathbb{N}}$ can be computed in a closed form in general (with the exception of the very specific cases of optimal filtering in linear state-space models, in which case the solution is provided by the *Kalman filter*, or hidden Markov models with finite state space).

4.2. *The double bootstrap algorithm with adaptive selection.* In this section, our aim is to form online a sequence of archipelagos targeting the Feynman-Kac flow $\{\eta_n\}_{n\in\mathbb{N}}$ by using sequentially the operations described in Section 3. A special feature of the approach that we consider is that the SIL operation is not performed systematically at every iteration of the algorithm, but only when the island weights fail to satisfy some appropriately defined skewness criterion. In this way we avoid adding unnecessary variance to the estimator. More specifically, we will analyze an algorithm proposed in [31, Algorithm 3], where SIL is executed on the basis of the so-called *coefficient of variation* (CV; see [21] and [24]) given by $\mathrm{CV}_N^2(\{\Omega_N(i)\}_{i=1}^{N_1})$, where

Downloaded from informs.org by [216.73.217.172] on 15 November 2026, at 06:26 . For personal use only, all rights reserved.

$$\mathrm{CV}_N^2 : (\mathbb{R}_+^*)^{N_1} \ni \{a(i)\}_{i=1}^{N_1} \mapsto N_1 \sum_{i=1}^{N_1} \left(\frac{a(i)}{\sum_{i'=1}^{N_1} a(i')} \right)^2 - 1. \tag{4.3}$$

The CV is closely related to the *efficient sample size* (ESS, proposed in [23]), which is the criterion used in [31]; nevertheless, since the ESS can be expressed as $N_1/[1 + \mathrm{CV}_N^2(\{a(i)\}_{i=1}^{N_1})]$, the two criteria are equivalent. Note that the CV is minimal (zero) when all island weights are perfectly equal and maximal $(N_1 - 1)$ in the situation of maximal skewness, i.e., when the total mass of the system is carried by a single island (a situation which is however not possible in our framework, as we always assume the island weights to be strictly positive). More specifically, as long as the CV stays below a specified threshold $\tau > 0$, we let the N_1 islands evolve without interaction according to mutation and selection on the individual level. However, when the island weights get too dispersed as measured by the CV criterion, the islands are rejuvenated by SIL. The scheme, referred to by us as the *double bootstrap with adaptive selection on the island level* (B^2ASIL), is described in Algorithm 4, where we have added the iteration index p to the weighted archipelagos returned by the algorithm.

Using the theoretical results obtained in Section 3 we may prove the following result, establishing that exponential deviation and asymptotic normality are preserved through one iteration of the B^2ASIL algorithm. As a by product we obtain the incremental asymptotic variance caused by an iteration. Since focus is set on asymptotic normality, we provide recursive formulas describing precisely the evolution of the functionals and measures involved in (**AN1**–**5**), while leaving the derivation of the analogous formulas for the constants of the exponential deviation bound (**D**) to the reader. The proof of this result provides a nice illustration of the efficiency by which the theoretical results obtained in Section 3, despite appearing somewhat involved at a first sight, can be applied for analyzing sequences of archipelagos produced by executing alternatingly the SIL, SiL, and Mutation operations in an arbitrary order.

THEOREM 4.2. *Assume* (***S***) *and suppose that* $\{(\Omega_N^{(n)}(i), \{(\xi_N^{(n)}(i,j), \omega_N^{(n)}(i,j))\}_{j=1}^{N_2})\}_{i=1}^{N_1}$ *satisfies exponential deviation and is asymptotically normal for* $(\eta_n, \sigma_n^2, \nu_n^2, \{\mu_\ell^{(n)}\}_{\ell=1}^3)$, $n \in \mathbb{N}^*$. *Then the archipelago* $\{(\Omega_N^{(n+1)}(i), \{(\xi_N^{(n+1)}(i,j), \omega_N^{(n+1)}(i,j))\}_{j=1}^{N_2})\}_{i=1}^{N_1}$ *generated through one iteration of Algorithm 4 is asymptotically normal for* $(\eta_{n+1}, \sigma_{n+1}^2, \nu_{n+1}^2, \{\mu_\ell^{(n+1)}\}_{\ell=1}^3)$, *where* η_{n+1} *is given by* (4.2) *and for all* $h \in \mathsf{F}_{\mathrm{b}}(\mathcal{X})$,

Downloaded from informs.org by [216.73.217.172] on 15 November 2026, at 06:26 . For personal use only, all rights reserved.

$$\begin{cases} \sigma_{n+1}^2(h) = \dfrac{\sigma_n^2\{Q_n(h-\eta_{n+1}h)\} + \varepsilon_n\nu_n^2\{Q_n(h-\eta_{n+1}h)\}}{(\eta_n Q_n \mathbb{1}_{\mathsf{X}})^2} \\ \qquad\qquad + \dfrac{\varepsilon_n\eta_n R_n\{w_n^2(h-\eta_{n+1}h)^2\} + (1-\varepsilon_n)\mu_1^{(n)} R_n\{w_n^2(h-\eta_{n+1}h)^2\}}{(\eta_n Q_n \mathbb{1}_{\mathsf{X}})^2}, \\ \nu_{n+1}^2(h) = \dfrac{\nu_n^2\{Q_n(h-\eta_{n+1}h)\} + \eta_n R_n\{w_n^2(h-\eta_{n+1}h)^2\}}{(\eta_n Q_n \mathbb{1}_{\mathsf{X}})^2}, \\ \mu_1^{(n+1)}h = (1-\varepsilon_n)\dfrac{\mu_1^{(n)} Q_n h}{\eta_n Q_n \mathbb{1}_{\mathsf{X}}} + \varepsilon_n\eta_{n+1}h, \\ \mu_2^{(n+1)}h = (1-\varepsilon_n)\dfrac{\mu_1^{(n)} R_n(w_n^2 h)}{(\eta_n Q_n \mathbb{1}_{\mathsf{X}})^2} + \varepsilon_n\mu_3^{(n+1)}h, \\ \mu_3^{(n+1)}h = \dfrac{\eta_n R_n(w_n^2 h)}{(\eta_n Q_n \mathbb{1}_{\mathsf{X}})^2}, \end{cases}$$

where $\varepsilon_n \triangleq \mathbb{1}_{\{\mu_1^{(n)}\mathbb{1}_{\mathsf{X}} > \tau+1\}}$.

Proof. First, note that since the input archipelago satisfies (**AN3**), it holds that

$$\mathrm{CV}_N^2(\{\Omega_N^{(n)}(i)\}_{i=1}^{N_1}) \xrightarrow{\mathbb{P}} \mu_1^{(n)}\mathbb{1}_{\mathsf{X}} - 1,$$

which implies

$$\mathbb{1}_{\{\mathrm{CV}_N^2(\{\Omega_N^{(n)}(i)\}_{i=1}^{N_1}) > \tau\}} \xrightarrow{\mathbb{P}} \varepsilon_n,$$

where ε_n is defined in the statement of the theorem. Consequently, after the **if-else** statement in Algorithm 4, the resulting archipelago $\{(\tilde{\Omega}_N^{(p)}(i), \{(\tilde{\xi}_N^{(p)}(i,j), \tilde{\omega}_N^{(p)}(i,j))\}_{j=1}^{N_2}\}_{i=1}^{N_1}$ satisfies, by Theorem 3.2 and Theorem 3.3, exponential deviation and asymptotic normality, the latter for

$$\begin{cases} (\eta_n, \sigma_n^2, \nu_n^2, \{\mu_\ell^{(n)}\}_{\ell=1}^3) & \text{if } \varepsilon_n = 0, \\ (\eta_n, \sigma_n^2 + \nu_n^2, \nu_n^2, \eta_n, \mu_3^{(n)}, \mu_3^{(n)}) & \text{if } \varepsilon_n = 1. \end{cases}$$

Thus, the archipelago $\{(\tilde{\Omega}_N^{(p)}(i), \{(\tilde{\xi}_N^{(p)}(i,j), 1)\}_{j=1}^{N_2})\}_{i=1}^{N_1}$ obtained after additional SiL satisfies, by Theorem 3.5 and Theorem 3.6, exponential deviation as well as asymptotic normality, the latter for

$$\begin{cases} (\eta_n, \sigma_n^2(\cdot) + \mu_1^{(n)}\{(\cdot - \eta_n\cdot)^2\}, \nu_n^2(\cdot) + \eta_n\{(\cdot - \eta_n\cdot)^2\}, \mu_1^{(n)}, \mu_1^{(n)}, \eta_n) \\ \quad \text{if } \varepsilon_n = 0, \\ (\eta_n, \sigma_n^2(\cdot) + \nu_n^2(\cdot) + \eta_n\{(\cdot - \eta_n\cdot)^2\}, \nu_n^2(\cdot) + \eta_n\{(\cdot - \eta_n\cdot)^2\}, \eta_n, \eta_n, \eta_n) \\ \quad \text{if } \varepsilon_n = 1. \end{cases}$$

Downloaded from informs.org by [216.73.217.172] on 15 November 2026, at 06:26 . For personal use only, all rights reserved.

Downloaded from informs.org by [216.73.217.172] on 15 November 2026, at 06:26 . For personal use only, all rights reserved.

Data: $\{R_p\}_{p=0}^{n-1}$, τ
Result: $\{(\Omega_N^{(p)}(i), \{(\xi_N^{(p)}(i,j), \omega_N^{(p)}(i,j))\}_{j=1}^{N_2})\}_{i=1}^{N_1}$, $p \in [\![0, n]\!]$
/* initialization */
for $i \leftarrow 1$ **to** N_1 **do**
  for $j \leftarrow 1$ **to** N_2 **do**
    $\xi_N^{(0)}(i,j) \sim \eta_0$;
    $\omega_N^{(0)}(i,j) \leftarrow 1$;
  end
  $\Omega_N^{(0)}(i) \leftarrow 1$;
end
$\{(\Omega_N^{(1)}(i), \{(\xi_N^{(1)}(i,j), \omega_N^{(1)}(i,j))\}_{j=1}^{N_2})\}_{i=1}^{N_1} \leftarrow$
$\mathsf{Mut}\langle Q_0\rangle\left(\{(1, \{(\xi_N^{(0)}(i,j), 1)\}_{j=1}^{N_2})\}_{i=1}^{N_1}, R_0\right)$;
/* main loop */
for $p \leftarrow 1$ **to** $n-1$ **do**
  /* checking island weight skewness */
  if $\mathrm{CV}_N^2(\{\Omega_N^{(p)}(i)\}_{i=1}^{N_1}) > \tau$ **then**
    /* selection on the island level */
    $\{(\tilde{\Omega}_N^{(p)}(i), \{(\tilde{\xi}_N^{(p)}(i,j), \tilde{\omega}_N^{(p)}(i,j))\}_{j=1}^{N_2})\}_{i=1}^{N_1} \leftarrow$
    $\mathsf{SIL}\left(\{(\Omega_N^{(p)}(i), \{(\xi_N^{(p)}(i,j), \omega_N^{(p)}(i,j))\}_{j=1}^{N_2})\}_{i=1}^{N_1}\right)$;
  else
    /* no action */
    $\{(\tilde{\Omega}_N^{(p)}(i), \{(\tilde{\xi}_N^{(p)}(i,j), \tilde{\omega}_N^{(p)}(i,j))\}_{j=1}^{N_2})\}_{i=1}^{N_1} \leftarrow$
    $\{(\Omega_N^{(p)}(i), \{(\xi_N^{(p)}(i,j), \omega_N^{(p)}(i,j))\}_{j=1}^{N_2})\}_{i=1}^{N_1}$;
  end
  /* selection on the individual level */
  $\{(\tilde{\Omega}_N^{(p)}(i), \{(\check{\xi}_N^{(p)}(i,j), 1)\}_{j=1}^{N_2})\}_{i=1}^{N_1} \leftarrow$
  $\mathsf{SIL}\left(\{(\tilde{\Omega}_N^{(p)}(i), \{(\tilde{\xi}_N^{(p)}(i,j), \tilde{\omega}_N^{(p)}(i,j))\}_{j=1}^{N_2})\}_{i=1}^{N_1}\right)$;
  /* mutation */
  $\{(\Omega_N^{(p+1)}(i), \{(\xi_N^{(p+1)}(i,j), \omega_N^{(p+1)}(i,j))\}_{j=1}^{N_2}\}_{i=1}^{N_1} \leftarrow$
  $\mathsf{Mut}\langle Q_p\rangle\left(\{(\tilde{\Omega}_N^{(p)}(i), \{(\check{\xi}_N^{(p)}(i,j), 1)\}_{j=1}^{N_2})\}_{i=1}^{N_1}, R_p\right)$;
end

Algorithm 4: The B^{2}ASIL algorithm

Finally, considering also the final Mutation operation in Algorithm 4, and propagating, for the two different values of ε_n, the quantities of the previous display through the updating formulas of Theorem 3.9, establishes, together with Theorem 3.8, the statement of the theorem. □

COROLLARY 4.3. *Assume (**S**). Then all archipelagos* $\{(\Omega_N^{(n)}(i), \{(\xi_N^{(n)}(i,j), \omega_N^{(n)}(i,j))\}_{j=1}^{N_2})\}_{i=1}^{N_1}$, $n \in \mathbb{N}$, *produced by the B^{2}ASIL algorithm satisfy ex-*

Downloaded from informs.org by [216.73.217.172] on 15 November 2026, at 06:26 . For personal use only, all rights reserved.

ponential deviation and asymptotic normality, where for $h \in \mathsf{F}_{\mathrm{b}}(\mathcal{X})$ *and* $n \in \mathbb{N}^*$,

$$\begin{cases} \sigma_n^2(h) = \displaystyle\sum_{\ell=0}^{n-1} \left(1 + \sum_{p=\ell+1}^{n-1} \varepsilon_p\right) \frac{\eta_\ell R_\ell\{w_\ell^2 Q_{\ell+1} \cdots Q_{n-1}(h - \eta_n h)^2\}}{(\eta_\ell Q_\ell \cdots Q_{n-1} \mathbb{1}_{\mathsf{X}})^2}, \\ \nu_n^2(h) = \displaystyle\sum_{\ell=0}^{n-1} \frac{\eta_\ell R_\ell\{w_\ell^2 Q_{\ell+1} \cdots Q_{n-1}(h - \eta_n h)^2\}}{(\eta_\ell Q_\ell \cdots Q_{n-1} \mathbb{1}_{\mathsf{X}})^2}, \\ \mu_1^{(n)} h = \eta_n h \end{cases}$$

(under the standard conventions that $\prod_{\ell=m}^n a_\ell = 1$, $\sum_{\ell=m}^n a_\ell = 0$, *and* $Q_m \cdots Q_n = \mathrm{id}$ *if* $m > n$*), and* $\{\varepsilon_n\}_{n \in \mathbb{N}^*}$ *is given in Theorem 4.2. In addition,* $\mu_1^{(0)} = \eta_0$ *and*

$$\sigma_0^2(h) = \nu_0^2(h) = \eta_0\{(h - \eta_0 h)^2\} \quad (h \in \mathsf{F}_{\mathrm{b}}(\mathcal{X})).$$

Proof. The non-recursive expressions above are verified using induction. More specifically, one assumes that the given expressions of $(\sigma_n^2, \nu_n^2, \mu_1^{(n)})$ hold true for some $n \in \mathbb{N}$ (and for all $h \in \mathsf{F}_{\mathrm{b}}(\mathcal{X})$) and plug the same into the recursive expressions established in Theorem 4.2 under repeated use of the identities

$$Q_\ell \cdots Q_{n-1}\{Q_n(h - \eta_{n+1} h) - \eta_n Q_n(h - \eta_{n+1} h)\} = Q_\ell \cdots Q_n(h - \eta_{n+1} h) \quad (h \in \mathsf{F}_{\mathrm{b}}(\mathcal{X}), \ell \in \mathbb{N}),$$

and

$$\eta_\ell Q_\ell \cdots Q_{n-1} \mathbb{1}_{\mathsf{X}} \times \eta_n Q_n \mathbb{1}_{\mathsf{X}} = \eta_\ell Q_\ell \cdots Q_n \mathbb{1}_{\mathsf{X}} \quad (\ell \in \mathbb{N}).$$

We leave this to the reader. To verify the base case $n = 1$, note that the initial archipelago $\{(1, \{(\xi_N^{(0)}(i,j), 1)\}_{j=1}^{N_2})\}_{i=1}^{N_1}\}$ is, by the standard CLT and law of large numbers of for independent random variables, asymptotically normal for $(\eta_0, \sigma_0^2, \sigma_0^2, \eta_0, \eta_0, \eta_0)$, where $\sigma_0^2(h) = \eta_0\{(h - \eta_0 h)^2\}$, $h \in \mathsf{F}_{\mathrm{b}}(\mathcal{X})$, and satisfies, by Hoeffding's inequality, exponential deviation for $(\eta_0, 1, 2, 1/2)$. Now, by Theorem 3.8 and Theorem 3.9 also the weighted archipelago $\{(\Omega_N^{(1)}(i), \{(\xi_N^{(1)}(i,j), \omega_N^{(1)}(i,j))\}_{j=1}^{N_2})\}_{i=1}^{N_1}$, obtained by mutating the initial archipelago, satisfies exponential deviation and asymptotic normality for $\mu_1^{(1)} = \eta_1$ and

$$\sigma_1^2(h) = \nu_1^2(h) = \frac{\eta_0 R_0\{w_0^2(h - \eta_1 h)^2\}}{(\eta_0 Q_0 \mathbb{1}_{\mathsf{X}})^2} \quad (h \in \mathsf{F}_{\mathrm{b}}(\mathcal{X})).$$

Under the standard conventions, this is however in agreement with the formula in the statement of the theorem. This completes the proof. □

Of special interest is of course the special case where SIL is applied systematically at every iteration, corresponding to $\tau = 0$. This yields the standard B^2 algorithm, in which case the asymptotic variance is given by the following corollary.

COROLLARY 4.4. *Assume (**S**). Then all archipelagos* $\{(\Omega_N^{(n)}(i), \{(\xi_N^{(n)}(i,j), \omega_N^{(n)}(i,j))\}_{j=1}^{N_2})\}_{i=1}^{N_1}$, $n \in \mathbb{N}$, *produced by the* B^2 *algorithm satisfy exponential deviation and asymptotic normality, where for* $h \in \mathsf{F}_{\mathrm{b}}(\mathcal{X})$ *and* $n \in \mathbb{N}^*$,

$$\sigma_n^2(h) = \sum_{\ell=0}^{n-1} (n-\ell) \frac{\eta_\ell R_\ell \{w_\ell^2 Q_{\ell+1} \cdots Q_{n-1}(h - \eta_n h)^2\}}{(\eta_\ell Q_\ell \cdots Q_{n-1} \mathbb{1}_{\mathsf{X}})^2}, \tag{4.4}$$

and

$$\sigma_0^2(h) = \eta_0\{(h - \eta_0 h)^2\}.$$

PROOF. The result is an immediate consequence of Corollary 4.3, as $\tau = 0$ implies that $\varepsilon_n = 1$ for all $n \in \mathbb{N}^*$. □

On the other hand, letting $\varepsilon_n = 0$ for all $n \in \mathbb{N}^*$ in Corollary 4.3, corresponding to the case where SIL is never applied, yields the variance

$$\sigma_n^2(h) = \sum_{\ell=0}^{n-1} \frac{\eta_\ell R_\ell \{w_\ell^2 Q_{\ell+1} \cdots Q_{n-1}(h - \eta_n h)^2\}}{(\eta_\ell Q_\ell \cdots Q_{n-1} \mathbb{1}_{\mathsf{X}})^2}, \tag{4.5}$$

which we recognize as the well-known formula for the asymptotic variance of the standard SMC algorithm (more specifically, the *sequential importance sampling with resampling*, SISR, algorithm). This is completely in line with our expectations, as such an algorithm would simply propagate N_1 independent (non-interacting) islands, each island evolving as a standard SMC algorithm based on N_2 particles.

4.3. *Long-term stability of the double bootstrap algorithm.* As a last part of our study we establish the long-term numerical stability of the B^2 algorithm by providing a time uniform bound on the asymptotic variance of its output. Throughout this section we will, in the spirit of Example 1, assume that each unnormalized transition kernel Q_p, $p \in \mathbb{N}$, can be decomposed into a normalized transition kernel $M_p : \mathsf{X} \times \mathcal{X} \to [0,1]$ and a nonnegative potential potential function $g_{p+1} : \mathsf{X} \to \mathbb{R}_+$, i.e., for all $h \in \mathsf{F}_{\mathrm{b}}(\mathcal{X})$ and $x \in \mathsf{X}$,

$$Q_p h(x) = M_p(g_{p+1} h)(x). \tag{4.6}$$

Downloaded from informs.org by [216.73.217.172] on 15 November 2026, at 06:26 . For personal use only, all rights reserved.

Downloaded from informs.org by [216.73.217.172] on 15 November 2026, at 06:26 . For personal use only, all rights reserved.

In this setting, given a sequence $\{R_p\}_{p\in\mathbb{N}}$ of proposal kernels such that $M_p(x,\cdot) \ll R_p(x,\cdot)$ for all $x \in \mathsf{X}$ and $p \in \mathbb{N}$, the importance weight function is given by

$$w_p(x,x') = g_{p+1}(x')\frac{\mathrm{d}M_p(x,\cdot)}{\mathrm{d}R_p(x,\cdot)} \quad (x,x') \in \mathsf{X}^2.$$

REMARK 4.5. *Instead of letting the Feynman-Kac distribution flow be generated by the unnormalized kernel* (4.6)*, one could, as in [31], consider an alternative model with a flow* $\{\tilde{\eta}_p\}_{p\in\mathbb{N}}$ *generated by*

$$\tilde{Q}_p h(x) = g_p(x) M_p h(x) \quad (h \in \mathsf{F}_{\mathrm{b}}(\mathcal{X}), x \in \mathsf{X}, p \in \mathbb{N}^*), \tag{4.7}$$

with $\tilde{Q}_0 = M_0$ *and* $\tilde{\eta}_0 = \chi$*. In [10] the two models* (4.6) *and* (4.7) *are referred to as* updated *and* prediction *Feynman-Kac models, respectively. For the prediction model, it is, in the case of the* B^2 *algorithm, possible to achieve* full adaptation *(borrowing the terminology of [26]) of the algorithm, i.e., to generate archipelagos* $\{(1,\{(1,\xi_N^{(p)}(i,j))\}_{j=1}^{N_2})\}_{i=1}^{N_1}$*,* $p \in \mathbb{N}$*, with uniformly weighted islands and individuals targeting the distribution sequence of interest, by letting* $R_p = M_p$ *for all* $p \in \mathbb{N}$ *and decomposing the dynamics* (4.7) *into the product*

$$\tilde{Q}_p = G_p M_p, \tag{4.8}$$

where $G_p h(x) = g_p(x)h(x)$*,* $(x,h) \in \mathsf{X} \times \mathsf{F}_{\mathrm{b}}(\mathcal{X})$*, is the* Boltzmann multiplicative operator *associated with the potential* g_p*. Now* (4.8) *allows also the Feynman-Kac transition according to* $\tilde{Q}_p$ *to be decomposed into* two *subsequent Feynman-Kac sub-transitions, the first according to* G_p *and the other according to* M_p*. The former corresponds to the Mutation operation*

$$\begin{aligned}\{(\check{\Omega}_N^{(p)}(i),\{(\check{\omega}_N^{(p)}(i,j),\xi_N^{(p)}(i,j))\}_{i=1}^{N_2})\}_{i=1}^{N_1} \\ \leftarrow \mathsf{Mut}\langle G_p\rangle\left(\{(1,\{(1,\xi_N^{(p)}(i,j))\}_{j=1}^{N_2})\}_{i=1}^{N_1},\mathrm{id}\right),\end{aligned} \tag{4.9}$$

which simply assigns each particle and island the weights $\check{\omega}_N(i,j) = g_p(\xi_N^{(p)}(i,j))$ *and* $\check{\Omega}_N^{(p)}(i) = \sum_{j=1}^{N_2} g_p(\xi_N^{(p)}(i,j))/N_2$*, respectively (where we assumed that we start with uniformly weighted islands and individuals). After this* weighing operation*, the output* (4.9) *is, in accordance with Algorithm 4 (with* $\tau = 0$*), subjected to the SIL and SiL operations followed by the Mutation operation*

$$\begin{aligned}\{(1,\{(1,\xi_N^{(p+1)}(i,j))\}_{i=1}^{N_2})\}_{i=1}^{N_1} \leftarrow \mathsf{Mut}\langle M_p\rangle \\ \left(\{(1,\{(1,\tilde{\xi}_N^{(p)}(i,j))\}_{j=1}^{N_2})\}_{i=1}^{N_1},M_p\right),\end{aligned}$$

yielding an archipelago with perfectly uniform island and individual weights approximating the Feynman-Kac distribution $\tilde{\eta}_{p+1}$ at the next time point. Also this algorithm may be analyzed straightforwardly using our results, and carrying through this analysis retrieves exactly the variance expression obtained in [31, Equation 43]. We leave this as an exercise to the interested reader.

The previous way of obtaining an archipelago with uniformly weighted islands and individuals approximating the prediction Feynman-Kac distribution flow can be viewed as a special instance of a general auxiliary double bootstrap algorithm *(extending the so-called* auxiliary particle filter *proposed in [26]) based on the decomposition*

$$Q_p = T_p \check{Q}_p,$$

where $T_p h(x) = t_p(x)h(x)$, $(x,h) \in \mathsf{X} \times \mathsf{F}_{\mathrm{b}}(\mathcal{X})$, is a Boltzmann multiplicative operator associated with some positive auxiliary importance weight function $t_p \in \mathsf{F}_{\mathrm{b}}(\mathcal{X})$, and

$$\check{Q}_p(x,h) \triangleq t_p^{-1}(x) Q_p h(x) \quad (x \in \mathsf{X}, h \in \mathsf{F}_{\mathrm{b}}(\mathcal{X})).$$

In analogy with the previous, we may thus construct an alternative algorithm approximating $\{\eta_p\}_{p\in\mathbb{N}}$ by furnishing the main loop of the B^2 algorithm with a prefatory weighing operation

$$\begin{aligned}(4.10)\quad &\{(\check{\Omega}_N^{(p)}(i), \{(\check{\omega}_N^{(p)}(i,j), \xi_N^{(p)}(i,j))\}_{i=1}^{N_2})\}_{i=1}^{N_1} \\ &\qquad \leftarrow \mathsf{Mut}\langle T_p\rangle \left(\{(\tilde{\Omega}_N^{(p)}(i), \{(\tilde{\omega}_N^{(p)}(i,j), \xi_N^{(p)}(i,j))\}_{i=1}^{N_2})\}_{i=1}^{N_1}, \mathrm{id}\right),\end{aligned}$$

and, after intermediate SIL and SiL operations, a terminating Mutation operation

$$\begin{aligned}&\{(\Omega_N^{(p+1)}(i), \{(\omega_N^{(p+1)}(i,j), \xi_N^{(p+1)}(i,j))\}_{i=1}^{N_2})\}_{i=1}^{N_1} \\ &\qquad \leftarrow \mathsf{Mut}\langle \check{Q}_p\rangle \left(\{(1, \{(1, \tilde{\xi}_N^{(p)}(i,j))\}_{j=1}^{N_2})\}_{i=1}^{N_1}, R_p\right),\end{aligned}$$

where, consequently, the all weights are given by the importance weight function

$$\check{w}_p(x,x') = t_p^{-1}(x) \frac{\mathrm{d}Q_p(x,\cdot)}{\mathrm{d}R_p(x,\cdot)}(x') \quad ((x,x') \in \mathsf{X}^2).$$

Thus, choosing $t_p(x)$ as some prediction of the value of the derivative $\mathrm{d}Q_p(x,\cdot)/\mathrm{d}R_p(x,\cdot)$ in the support of $R_p(x,\cdot)$ yields close to uniformly weighted islands and individuals (i.e., a close to fully adapted algorithm);

Downloaded from informs.org by [216.73.217.172] on 15 November 2026, at 06:26 . For personal use only, all rights reserved.

Downloaded from informs.org by [216.73.217.172] on 15 November 2026, at 06:26 . For personal use only, all rights reserved.

for instance, following [26], a possible design is $t_p(x) = \mathrm{d}Q_p(x,\cdot)/\mathrm{d}R_p(x,\cdot)(R_p\,\mathrm{id}(x))$. *Of course, also this algorithm can be analyzed easily using our results (we refer to [16] for such an analysis of the standard auxiliary particle filter).*

4.4. *Time uniform convergence under the strong mixing assumption.* When studying the numerical stability of the B^2 algorithm we will first work under the following *strong mixing condition.*

(M) (i) There exist constants $0 < \sigma_- < \sigma_+ < \infty$ and $\varphi \in \mathsf{M}_1(\mathcal{X})$ such that for all $p \in \mathbb{N}$, $x \in \mathsf{X}$, and $\mathsf{A} \in \mathcal{X}$,

$$\sigma_-\varphi(\mathsf{A}) \leq M_p(x,\mathsf{A}) \leq \sigma_+\varphi(\mathsf{A}).$$

(ii) It holds that $w_+ \triangleq \sup_{p\in\mathbb{N}} \|w_p\|_\infty < \infty$.

(iii) It holds that $c_- \triangleq \inf_{(p,x)\in\mathbb{N}\times\mathsf{X}} Q_p\mathbb{1}_{\mathsf{X}}(x) > 0$.

The assumption **(M)**(i), implying that each M_p allows the whole state space X as a 1-*small set*, is rather restrictive and requires typically the state space X to be a compact set. Still, it plays a vital role in the literature of SMC analysis [see, e.g., 13, 10, 4, 16, 25, 18]. On the other hand, the weaker assumption **(M)**(ii) is satisfied for most applications and **(M)**(iii) does not require the potential functions to be uniformly bounded from below; the latter is a condition that appears frequently in the literature. Under **(M)**, denote

$$\rho \triangleq 1 - \frac{\sigma_-}{\sigma_+}; \tag{4.11}$$

then the previous assumptions allow the following explicit time uniform bound to be derived.

COROLLARY 4.6. *Suppose* ***(M)****. Then the sequence of asymptotic variances of the* B^2 *algorithm (see Corollary 4.4) satisfies, for all* $n \in \mathbb{N}$ *and* $h \in \mathsf{F}_{\mathrm{b}}(\mathcal{X})$,

$$\sigma_n^2(h) \leq w_+ \frac{\mathrm{osc}^2(h)}{(1-\rho)^2(1-\rho^2)^2 c_-}, \tag{4.12}$$

where ρ *is defined in* (4.11).

The proof is found in Section 5.10. In addition, by comparing the formulas of the asymptotic variances of the B^2ASIL and B^2 algorithms provided by Corollary 4.3 and Corollary 4.4, respectively, we conclude that at each time

step, the asymptotic variance of the latter algorithm is always bounded by the that of the former (as the indicator variables $\{\varepsilon_p\}_{p\in\mathbb{N}^*}$, determining the asymptotic island selection schedule of the B^{2}ASIL algorithm, are either zero or one for all p). The following corollary is hence immediate.

COROLLARY 4.7. *Suppose (**M**). Then also the asymptotic variances of the B^{2}ASIL algorithm (see Corollary 4.3) satisfy the bound* (4.12).

4.5. *Time uniform convergence under a local Doeblin condition.* The explicitness and simplicity of the variance bound in Corollary 4.6 are obtained at the cost of restrictive model assumptions that are rarely satisfied in real-world applications. Thus, in this section we will discuss how the assumptions of (**M**) can be lightened considerably and turned into easily verifiable conditions, satisfied for many models of interest, by considering assumptions under which the asymptotic variance is *stochastically bounded* (tight) rather than bounded by a deterministic constant. Since the asymptotic variance (4.4) of the B^2 algorithm differs only from that of the SISR algorithm (see (4.5)) by the factors $n-\ell$, the results obtained in this section will rely heavily on similar results obtained in [17] for the standard bootstrap particle filter. For this purpose, assume that each potential function depends on time through some random parameter only, i.e., for all $p\in\mathbb{N}^*$, $g_p = g\langle Z_p\rangle$, where $\{Z_p\}_{p\in\mathbb{N}}$ is some stochastic process taking values in some state space $(\mathsf{Z},\mathcal{Z})$ and $g\langle z\rangle \in \mathsf{F}_{\mathrm{b}}(\mathcal{X})$ for all $z\in\mathsf{Z}$. Moreover, we assume that the normalized transition kernels of the model are time homogeneous, i.e., $M_p = M$ for all $p\in\mathbb{N}$, and that Mutation is based on the underlying dynamics of the model, i.e., $R_p = R = M$, and, consequently, $w_p(x,x') = w_p\langle Z_{p+1}\rangle(x,x') = g\langle Z_{p+1}\rangle(x')$ for all $p\in\mathbb{N}$ and $(x,x')\in\mathsf{X}^2$. Thus, in this case the model generates a parameter dependent Feynman-Kac flow $\{\eta_p\langle Z_{0:p}\rangle\}_{p\in\mathbb{N}}$. (For instance, in the case of a hidden Markov model, the sequence $\{Z_p\}_{p\in\mathbb{N}}$ plays the role of noisy observations of some Markov chain (the *state process*) $\{X_p\}_{p\in\mathbb{N}}$ with transition kernel M on $(\mathsf{X},\mathcal{X})$. Conditionally on the state process, the observations are assumed to be independent and such that the conditional density $\mathsf{Z}\ni z\mapsto g\langle z\rangle(x)$ of each Z_p depends on the corresponding state $X_p = x\in\mathsf{X}$ only. In this important framework, $\eta_p\langle Z_{0:p}\rangle$ is the so-called *filter distribution* at time p, i.e., the conditional distribution of the latent state X_p given the observations $Z_{0:p}$.) In this case, the asymptotic variance $\sigma_n^2(h)$ of the B^2 algorithm is a function of the random vector $Z_{0:n}$, and we write $\sigma_n^2\langle Z_{0:n}\rangle(h)$ to emphasize this fact. We will replace the condition (**M**) (i) by a considerably weaker condition of the following type.

Downloaded from informs.org by [216.73.217.172] on 15 November 2026, at 06:26 . For personal use only, all rights reserved.

Downloaded from informs.org by [216.73.217.172] on 15 November 2026, at 06:26 . For personal use only, all rights reserved.

DEFINITION 4.8. *A set* $\mathsf{C} \in \mathcal{X}$ *is* local Doeblin *with respect to* M *if there is* $\varphi_{\mathsf{C}} \in \mathsf{M}_1(\mathcal{X})$ *with* $\varphi_{\mathsf{C}}(\mathsf{C}) = 1$ *and constants* $0 < \sigma_{\mathsf{C}}^- < \sigma_{\mathsf{C}}^+$ *such that for all* $x \in \mathsf{C}$ *and* $\mathsf{A} \in \mathcal{X}$,

$$\sigma_{\mathsf{C}}^- \varphi_{\mathsf{C}}(\mathsf{A}) \leq M(x, \mathsf{A} \cap \mathsf{C}) \leq \sigma_{\mathsf{C}}^+ \varphi_{\mathsf{C}}(\mathsf{A}).$$

Now, impose the following assumption.

(L) The process $\{Z_p\}_{p\in\mathbb{N}}$ is strictly stationary and ergodic. Moreover, there exists a set $\mathsf{K} \in \mathcal{Z}$ such that the following holds.

(i) $\mathbb{P}(Z_0 \in \mathsf{K}) > 2/3$.

(ii) For all $\varepsilon > 0$ there exists a local Doeblin set C such that for all $z \in \mathsf{Z}$,

$$\sup_{x\in\mathsf{C}^c} g\langle z\rangle(x) \leq \varepsilon \left\| g\langle z\rangle \right\|_\infty < \infty.$$

(iii) There exists a set $\mathsf{D} \in \mathcal{X}$ such that $\inf_{x\in\mathsf{D}} M(x, \mathsf{D}) > 0$ and

$$\mathbb{E}\left[\ln^- \inf_{x\in\mathsf{D}} g\langle Z_0\rangle(x)\right] < \infty.$$

In **(L)** (iii), $\ln^-$ denotes the negative part of the natural logarithm. The condition **(L)** can be checked easily for a large variety of models; see [17, Section 4] for examples.

REMARK 4.9. *The condition* ***(L)*** *can be weakened further by requiring the local Doeblin condition to hold only for some iterate* $Q\langle z_1\rangle \cdots Q\langle z_r\rangle$, $z_{1:r} \in \mathsf{Z}^r$, *with a minorizing measure* φ_{C} *and constants* σ_{C}^-, σ_{C}^+ *possibly depending on the block* $z_{1:r}$*; we refer to [17] for details. In this paper we have however chosen to state the most basic version of the condition (corresponding to* $r = 1$*) for simplicity.*

Under **(L)**, define $\mathsf{M}_1(\mathcal{X}, \mathsf{D}) \subset \mathsf{M}_1(\mathcal{X})$ as the set of all $\chi \in \mathsf{M}_1(\mathcal{X})$ for which there exists $\mathsf{D}' \in \mathcal{X}$ such that (i) $\inf_{x\in\mathsf{D}'} M(x, \mathsf{D}) > 0$, (ii) $\mathbb{E}[\ln^- \inf_{x\in\mathsf{D}'} g\langle Z_0\rangle(x)] < \infty$, and (iii) $\chi(\mathsf{D}') > 0$. Then the following holds true.

COROLLARY 4.10. *Assume* ***(L)*** *and suppose in addition that* $\eta_0 \in \mathsf{M}_1(\mathcal{X}, \mathsf{D})$. *Then for all* $h \in \mathsf{F}_{\mathrm{b}}(\mathcal{X})$, *the sequence* $\{\sigma_n^2\langle Z_{0:n}\rangle(h)\}_{n\in\mathbb{N}}$ *of asymptotic variances of the output of the* B^2 *algorithm is tight, i.e., it satisfies*

$$\lim_{\lambda\to\infty} \sup_{n\in\mathbb{N}} \mathbb{P}\left(\sigma_n^2\langle Z_{0:n}\rangle(h) \geq \lambda\right) = 0.$$

The previous result is obtained by inspecting the proof of [17, Theorem 11]; see Section 5.11 for some details.

Downloaded from informs.org by [216.73.217.172] on 15 November 2026, at 06:26 . For personal use only, all rights reserved.

5. Proofs. As mentioned in the introduction, our consistency and asymptotic normality proofs rely on limit theorems for triangular arrays developed in [15]. More specifically, [15] developed Theorem A.1 and Theorem A.2, which are re-stated in Appendix A for completeness, for the purpose of proving consistency and asymptotic normality for weighted samples of *particles*, whereas we use the same results for establishing these properties for weighted samples of particle *islands*. However, whereas the elements of the arrays considered in [15] correspond mainly to single particles, the arrays defined by us will be considerably more complex with each element being generally itself a weighted average of particles associated with an island. In this section we will use repeatedly the same notation $\{U_N(i)\}_{i=1}^{N_1}$ to denote triangular arrays (in the sense of Theorem A.1 and Theorem A.2), even though the roles of these arrays change throughout the proofs.

5.1. *Proof of Theorem 3.1.* We apply Theorem A.1. For this purpose, define the triangular array and filtration

$$\begin{aligned} U_N(i) &\triangleq \sum_{j=1}^{N_2} \frac{\omega_N(I_N(i),j)}{N_1 \sum_{j'=1}^{N_2} \omega_N(I_N(i),j')} h(\xi_N(I_N(i),j)) \\ &\qquad (i \in [\![1, N_1(N)]\!], N \in \mathbb{N}^*), \\ \mathcal{F}_N &\triangleq \sigma\left(\{(\Omega_N(i), \{(\xi_N(i,j), \omega_N(i,j))\}_{j=1}^{N_2})\}_{i=1}^{N_1}\right) \quad (N \in \mathbb{N}^*), \end{aligned} \tag{5.1}$$

respectively. Now, since the island indices $\{I_N(i)\}_{i=1}^{N_1}$ are, conditionally on $\mathcal{F}_N$, i.i.d. with common distribution $\mathsf{P}(\{\Omega_N(i')\}_{i'=1}^{N_1})$ it holds, as the ancestor sample is assumed to be consistent,

$$\begin{aligned} &\sum_{i=1}^{N_1} \mathbb{E}\left[U_N(i) \mid \mathcal{F}_N\right] \\ &\quad = N_1 \mathbb{E}\left[U_N(1) \mid \mathcal{F}_N\right] \\ &\quad = \sum_{i=1}^{N_1} \frac{\Omega_N(i)}{\sum_{i'=1}^{N_1} \Omega_N(i')} \sum_{j=1}^{N_2} \frac{\omega_N(i,j)}{\sum_{j'=1}^{N_2} \omega_N(i,j')} h(\xi_N(i,j)) \xrightarrow{\mathbb{P}} \eta h. \end{aligned} \tag{5.2}$$

Thus, since $|U_N(i)| \le \|h\|_\infty / N_1 < \infty$ for all $i \in [\![1, N_1]\!]$, it is enough to check the conditions (**A1**) and (**A2**) in Theorem A.1. The tightness condition (**A1**) is straightforwardly satisfied as sequences that converge in probability are tight. Moreover, to check (**A2**) we may apply Lemma A.3 with $V_N = \|h\|_\infty / N_1$ and $Y_N = X_N = 0$. Thus, the limits, in probability as $N \to \infty$, of the series $\sum_{i=1}^{N_1} U_N(i)$ and $\sum_{i=1}^{N_1} \mathbb{E}\left[U_N(i) \mid \mathcal{F}_N\right]$ coincide, which completes the proof.

Downloaded from informs.org by [216.73.217.172] on 15 November 2026, at 06:26 . For personal use only, all rights reserved.

5.2. *Proof of Theorem 3.2.* Trivially, since $\{I_N(i)\}_{i=1}^{N_1} \subset [\![1, N_1]\!]$ it holds for all $\varepsilon > 0$,

$$\mathbb{P}\left(\max_{i\in[\![1,N_1]\!]}\left|\frac{1}{N_2}\sum_{j=1}^{N_2}\omega_N(I_N(i),j)h(\xi_N(I_N(i),j)) - \varrho\times\eta h\right| \geq \varepsilon\right)$$
$$\leq \mathbb{P}\left(\max_{i\in[\![1,N_1]\!]}\left|\frac{1}{N_2}\sum_{j=1}^{N_2}\omega_N(i,j)h(\xi_N(i,j)) - \varrho\times\eta h\right| \geq \varepsilon\right),$$

where the right hand side has an exponential bound by assumption. This completes the proof.

5.3. *Proof of Theorem 3.3.* In order to check (**AN1**) using Theorem A.2, define the array

$$U_N(i) \triangleq \sqrt{\frac{N_2}{N_1}}\sum_{j=1}^{N_2}\frac{\omega_N(I_N(i),j)}{\sum_{j'=1}^{N_2}\omega_N(I_N(i),j')}\{h(\xi_N(I_N(i),j)) - \eta h\} \tag{5.3}$$
$$(i \in [\![1, N_1]\!], N \in \mathbb{N}^*),$$

and let $\{\mathcal{F}_N\}_{N\in\mathbb{N}^*}$ be the filtration (5.1). We first note that $\mathbb{E}[U_N^2(i) \mid \mathcal{F}_N] \leq 4N_2\|h\|_\infty^2/N_1 < \infty$ for all $i \in [\![1, N_1]\!]$. Along the lines of (5.2) (note however that the definition (5.1) of the triangular array in (5.2) differs slightly from that of the array (5.3) considered here),

$$\sum_{i=1}^{N_1} U_N(i) = \sum_{i=1}^{N_1}\{U_N(i) - \mathbb{E}[U_N(i) \mid \mathcal{F}_N]\} \tag{5.4}$$
$$+ \sqrt{N}\sum_{i=1}^{N_1}\frac{\Omega_N(i)}{\sum_{i'=1}^{N_1}\Omega_N(i')}\sum_{j=1}^{N_2}\frac{\omega_N(i,j)}{\sum_{j'=1}^{N_2}\omega_N(i,j')}\{h(\xi_N(i,j)) - \eta h\}.$$

By assumption, the second term on the right hand side of (5.4) converges in distribution to a Gaussian random variable with zero mean and variance $\sigma^2(h)$. To treat the first term using Theorem A.2 we first consider

$$\sum_{i=1}^{N_1}\mathbb{E}\left[U_N^2(i) \mid \mathcal{F}_N\right] \tag{5.5}$$
$$= N_2\sum_{i=1}^{N_1}\frac{\Omega_N(i)}{\sum_{i'=1}^{N_1}\Omega_N(i')}\left(\sum_{j=1}^{N_2}\frac{\omega_N(i,j)}{\sum_{j'=1}^{N_2}\omega_N(i,j')}\{h(\xi_N(i,j)) - \eta h\}\right)^2$$
$$\xrightarrow{\mathbb{P}} \nu^2(h),$$

where the limit holds as the ancestor archipelago is assumed to satisfy (**AN2**). In addition,

$$\sum_{i=1}^{N_1} \mathbb{E}^2 \left[U_N(i) \mid \mathcal{F}_N \right] = \frac{1}{N_1} \left(\sqrt{N} \sum_{i=1}^{N_1} \frac{\Omega_N(i)}{\sum_{i'=1}^{N_1} \Omega_N(i')} \sum_{j=1}^{N_2} \frac{\omega_N(i,j)}{\sum_{j'=1}^{N_2} \omega_N(i,j')} \{h(\xi_N(i,j)) - \eta h\} \right)^2 \xrightarrow{\mathbb{P}} 0,$$

as the right hand side tends, as the ancestor archipelago satisfies (**AN1**), in distribution to a scaled χ^2-distributed random variable as $N \to \infty$. Combining the two previous displays shows that the condition (**B1**) in Theorem A.2 holds with limit $\varsigma^2(h) = \nu^2(h)$. To check the condition (**B2**) in the same lemma, we note that $\max_{i \in [\![1, N_1]\!]} |U_N(i)| \leq V_N + X_N Y_N^2$, with $X_N = Y_N = 0$ and

(5.6)
$$V_N = \max_{i \in [\![1, N_1]\!]} \sqrt{\frac{N_2}{N_1}} \left| \sum_{j=1}^{N_2} \frac{\omega_N(i,j)}{\sum_{j'=1}^{N_2} \omega_N(i,j')} \{h(\xi_N(i,j)) - \eta h\} \right| \quad (N \in \mathbb{N}^*).$$

Note that the sequence $\{V_N\}_{N \in \mathbb{N}^*}$ is $\mathcal{F}_N$-adapted and vanishes in probability as $N \to \infty$ since the ancestor archipelago is assumed to satisfy (**D**) and thus (2.1). We may then apply Lemma A.3 to check the condition (**B2**) in Theorem A.2, implying that for all $u \in \mathbb{R}$,

$$\mathbb{E} \left[\exp \left(\mathrm{i}\, u \sum_{i=1}^{N_1} \{ U_N(i) - \mathbb{E} \left[U_N(i) \mid \mathcal{F}_N \right] \} \right) \mid \mathcal{F}_N \right] \xrightarrow{\mathbb{P}} \exp(-u^2 \nu^2(h)/2).$$

Now, using this limit, the decomposition (5.4), and the hypothesis that the ancestor archipelago satisfies (**AN1**), we conclude, via Lemma A.5, that for all $u \in \mathbb{R}$,

$$\mathbb{E} \left[\exp \left(\mathrm{i}\, u \sum_{i=1}^{N_1} U_N(i) \right) \right] \xrightarrow{\mathbb{P}} \exp(-u^2 \{ \sigma^2(h) + \nu^2(h) \}/2),$$

which concludes the proof of (**AN1**).

We establish Assumption (**AN2**) by applying Theorem A.1, this time to the array $U_N'(i) \triangleq U_N^2(i)$, $i \in [\![1, N_1]\!], N \in \mathbb{N}^*$, where $\{U_N(i)\}_{i=1}^{N_1}$ is defined by (5.3), and the filtration $\{\mathcal{F}_N\}_{N \in \mathbb{N}^*}$ is defined by (5.1). To prove that

Downloaded from informs.org by [216.73.217.172] on 15 November 2026, at 06:26 . For personal use only, all rights reserved.

$\sum_{i=1}^{N_1} U'_N(i)$ converges in probability, we first note that the sum $\sum_{i=1}^{N_1} \mathbb{E}[U'_N(i) \mid \mathcal{F}_N]$ converges in probability to $\nu^2(h)$ by (5.5). Moreover, the two conditions (**A1**) and (**A2**) in Theorem A.1 are straightforwardly satisfied for the array under consideration, the latter condition by Lemma A.3 as $\max_{i\in[\![1,N_1]\!]} |U'_N(i)| \leq V_N^2$, where V_N is defined in (5.6) and V_N^2 vanishes in probability by Lemma 2.3. Consequently, (**AN2**) is satisfied with $\tilde{\nu}^2 = \nu^2$.

In the case of multinomial island selection, (**AN3**) coincides with the consistency property, which is implied by (**AN1**); thus, $\tilde{\mu}_1 = \eta$.

We preface the proof of (**AN4**) by the following lemma.

LEMMA 5.1. *Assume (**D**) and (**S**). Then*

$$\lim_{\lambda\to\infty} \sup_{N\in\mathbb{N}^*} \mathbb{P}\left(\max_{(i,j)\in[\![1,N_1]\!]\times[\![1,N_2]\!]} N_2 \frac{\omega_N(i,j)}{\sum_{j'=1}^{N_2} \omega_N(i,j')} \geq \lambda \right) = 0.$$

PROOF. Using the boundedness of the particle weights,

$$\max_{(i,j)\in[\![1,N_1]\!]\times[\![1,N_2]\!]} N_2 \frac{\omega_N(i,j)}{\sum_{j'=1}^{N_2} \omega_N(i,j')} \leq |\omega|_\infty \max_{i\in[\![1,N_1]\!]} \left(\frac{1}{N_2} \sum_{j=1}^{N_2} \omega_N(i,j) \right)^{-1},$$

where the quantity on right hand side is tight as it converges in probability to $|\omega|_\infty/\varrho$ by (**D**), (**S**) and Lemma A.4. □

Now, to check (**AN4**) we apply Theorem A.1 to the array

$$U_N(i) \triangleq \frac{N_2}{N_1} \sum_{j=1}^{N_2} \left(\frac{\omega_N(I_N(i),j)}{\sum_{j'=1}^{N_2} \omega_N(I_N(i),j')} \right)^2 h(\xi_N(I_N(i),j))$$
$$(i \in [\![1,N_1]\!], N \in \mathbb{N}^*).$$

associated with the same filtration as previously. Since the ancestor archipelago satisfies (**AN5**),

(5.7)
$$\sum_{i=1}^{N_1} \mathbb{E}\left[U_N(i) \mid \mathcal{F}_N\right] = N_2 \sum_{i=1}^{N_1} \frac{\Omega_N(i)}{\sum_{i'=1}^{N_1} \Omega_N(i')} \sum_{j=1}^{N_2} \left(\frac{\omega_N(i,j)}{\sum_{j'=1}^{N_2} \omega_N(i,j')} \right)^2 h(\xi_N(i,j)) \xrightarrow{\mathbb{P}} \mu_3 h,$$

Downloaded from informs.org by [216.73.217.172] on 15 November 2026, at 06:26 . For personal use only, all rights reserved.

and we show that $\sum_{i=1}^{N_1} U_N(i)$ tends to the same limit by using Theorem A.1. First, $U_N(i) \le {N_2}^2 \|h\|_\infty / N_1 < \infty$ for all $i \in [\![1, N_1]\!]$ and $N \in \mathbb{N}^*$; moreover, be reusing (5.7) for $|h|$ we check (**A1**). To check the Lindeberg condition (**A2**), we bound, using $\{I_N(i)\}_{i=1}^{N_1} \subset [\![1, N_1]\!]$,

$$\max_{i \in [\![1,N_1]\!]} |U_N(i)| \le \max_{(i,j) \in [\![1,N_1]\!] \times [\![1,N_2]\!]} N_2 \frac{\omega_N(i,j)}{\sum_{j'=1}^{N_2} \omega_N(i,j')} \times \max_{i \in [\![1,N_1]\!]} \frac{1}{N_1} \left| \sum_{j=1}^{N_2} \frac{\omega_N(i,j)}{\sum_{j'=1}^{N_2} \omega_N(i,j')} h(\xi_N(i,j)) \right|,$$

where the first factor on the right hand side is tight by Lemma 5.1 and the second term is bounded by $\|h\|_\infty / N_1$, which tends to zero when N tends to infinity. Thus, Lemma A.4 can be applied for checking (**A2**), which establishes that $\tilde{\mu}_2 = \mu_3$.

Finally, in the case of selection on the island level, (**AN5**) coincides with (**AN4**) and (**AN6**) is trivially satisfied.

5.4. *Proof of Theorem 3.4.* We first note that (**C2**) is trivially satisfied. In order to check (**C1**) we apply Theorem A.1 to the array

$$U_N(i) \triangleq \frac{\Omega_N(i)}{N_2 \sum_{i'=1}^{N_1} \Omega_N(i')} \sum_{j=1}^{N_2} h(\xi_N(i, J_N(i,j))) \quad (i \in [\![1, N_1]\!], N \in \mathbb{N}^*)$$

associated with $\{\mathcal{F}_N\}_{N \in \mathbb{N}}$ given by (5.1). Note that all indices $\{J_N(i,j) \in [\![1, N_1]\!] \times [\![1, N_2]\!]\}$ are conditionally independent given $\mathcal{F}_N$. Moreover, for all $i \in [\![1, N_1]\!]$ it holds that $\{J_N(i,j)\}_{j=1}^{N_2} \sim \mathsf{P}(\{\omega_N(i,j')\}_{j'=1}^{N_2})^{\otimes N_2}$. Hence,

$$\sum_{i=1}^{N_1} \mathbb{E}\left[U_N(i) \mid \mathcal{F}_N\right] = \sum_{i=1}^{N_1} \frac{\Omega_N(i)}{\sum_{i'=1}^{N_1} \Omega_N(i')} \sum_{j=1}^{N_2} \frac{\omega_N(i,j)}{\sum_{j'=1}^{N_2} \omega_N(i,j')} h(\xi_N(i,j)) \xrightarrow{\mathbb{P}} \eta h,$$

where convergence holds by assumption. First, note that $U_N(i) \le \|h\|_\infty < \infty$ for all $i \in [\![1, N_1]\!]$ and $N \in \mathbb{N}^*$. Moreover, (**A1**) is trivially satisfied. Thus, consistency is established by showing that (**A2**) is satisfied, which is an immediate implication of Lemma A.3 with $X_N = Y_N = 0$ and $V_N = \|h\|_\infty \max_{i \in [\![1,N_1]\!]} \Omega_N(i) / \sum_{i'=1}^{N_1} \Omega_N(i')$, which is $\mathcal{F}_N$-adapted and tends to zero in probability thanks to (**C2**). Hence, by Theorem A.1, the series $\sum_{i=1}^{N_1} U_N(i)$ and $\sum_{i=1}^{N_1} \mathbb{E}\left[U_N(i) \mid \mathcal{F}_N\right]$ have the same limit ηh in probability. This completes the proof.

Downloaded from informs.org by [216.73.217.172] on 15 November 2026, at 06:26 . For personal use only, all rights reserved.

Downloaded from informs.org by [216.73.217.172] on 15 November 2026, at 06:26 . For personal use only, all rights reserved.

5.5. *Proof of Theorem 3.5.* We may bound the quantity of interest according to

$$\max_{i\in[\![1,N_1]\!]}\left|\frac{1}{N_2}\sum_{j=1}^{N_2}h(\xi_N(i,J_N(i,j)))-\eta h\right|$$
$$\leq \max_{i\in[\![1,N_1]\!]}\frac{1}{N_2}\left|\sum_{j=1}^{N_2}\delta_N(i,j)\right|+\max_{i\in[\![1,N_1]\!]}\left|\sum_{j=1}^{N_2}\frac{\omega_N(i,j)}{\sum_{j'=1}^{N_2}\omega_N(i,j')}h(\xi_N(i,j))-\eta h\right|,$$

where we have set

$$\delta_N(i,j)\triangleq h(\xi_N(i,J_N(i,j)))-\sum_{j'=1}^{N_2}\frac{\omega_N(i,j')}{\sum_{j''=1}^{N_2}\omega_N(i,j'')}h(\xi_N(i,j')) \tag{5.8}$$
$$((i,j)\in[\![1,N_1]\!]\times[\![1,N_2]\!]).$$

By Lemma 2.3, the second term on the right hand side satisfies

$$\mathbb{P}\left(\max_{i\in[\![1,N_1]\!]}\left|\sum_{j=1}^{N_2}\frac{\omega_N(i,j)}{\sum_{j'=1}^{N_2}\omega_N(i,j')}h(\xi_N(i,j))-\eta h\right|\geq\varepsilon/2\right)$$
$$\leq 2c_1N_1\exp\left(-c_2N_2\frac{\varepsilon^2\varrho^2}{16\left\|h\right\|_\infty^2}\right).$$

For each $i\in[\![1,N_1]\!]$, the variables $\{\delta_N(i,j)\}_{j=1}^{N_2}$ are, conditionally on $\mathcal{F}_N$, independent and identically distributed with zero mean; moreover, as $|\delta_N(i,j)|\leq 2\left\|h\right\|_\infty$ for all $(i,j)\in[\![1,N_1]\!]\times[\![1,N_2]\!]$, Hoeffding's inequality implies that for all $\varepsilon>0$,

$$\mathbb{P}\left(\max_{i\in[\![1,N_1]\!]}\frac{1}{N_2}\left|\sum_{j=1}^{N_2}\delta_N(i,j)\right|\geq\varepsilon/2\mid\mathcal{F}_N\right)\leq 2N_1\exp\left(-N_2\frac{\varepsilon^2}{8\left\|h\right\|_\infty^2}\right). \tag{5.9}$$

Combining the previous two displays shows that (**D**) is satisfied with the choice of $\tilde{c}_1$ and $\tilde{c}_2$ given in the theorem.

5.6. *Proof of Theorem 3.6.* We start with (**AN1**). In order to apply Theorem A.2, define the array

$$U_N(i)\triangleq\sqrt{\frac{N_1}{N_2}}\frac{\Omega_N(i)}{\sum_{i'=1}^{N_1}\Omega_N(i')}\sum_{j=1}^{N_2}\{h(\xi_N(i,J_N(i,j)))-\eta h\} \tag{5.10}$$
$$((i,j)\in[\![1,N_1]\!]\times[\![1,N_2]\!]),$$

Downloaded from informs.org by [216.73.217.172] on 15 November 2026, at 06:26 . For personal use only, all rights reserved.

equipped with the usual filtration $\{\mathcal{F}_N\}_{N\in\mathbb{N}}$ given by (5.1). We first note that $U_N(i) \le 2\sqrt{N}\,\|h\|_\infty < \infty$ for all $i \in [\![1, N_1]\!]$ and $N \in \mathbb{N}^*$. In order to check (**B1**), write, following the arguments of the proof of Theorem 3.4,

$$\begin{aligned}
&\sum_{i=1}^{N_1} \mathbb{E}\left[U_N^2(i) \mid \mathcal{F}_N\right] \\
&\quad = N_1 \sum_{i=1}^{N_1} \left(\frac{\Omega_N(i)}{\sum_{i'=1}^{N_1}\Omega_N(i')}\right)^2 \sum_{j=1}^{N_2} \frac{\omega_N(i,j)}{\sum_{j'=1}^{N_2}\omega_N(i,j')}\{h(\xi_N(i,j)) - \eta h\}^2 \\
&\qquad + N_1(N_2-1)\sum_{i=1}^{N_1}\left(\frac{\Omega_N(i)}{\sum_{i'=1}^{N_1}\Omega_N(i')}\right)^2 \\
&\qquad \times \left(\sum_{j=1}^{N_2}\frac{\omega_N(i,j)}{\sum_{j'=1}^{N_2}\omega_N(i,j')}\{h(\xi_N(i,j)) - \eta h\}\right)^2.
\end{aligned}$$

Moreover, since

$$\begin{aligned}
&\sum_{i=1}^{N_1} \mathbb{E}^2\left[U_N(i) \mid \mathcal{F}_N\right] \\
&\quad = N \sum_{i=1}^{N_1}\left(\frac{\Omega_N(i)}{\sum_{i'=1}^{N_1}\Omega_N(i')}\right)^2 \left(\sum_{j=1}^{N_2}\frac{\omega_N(i,j)}{\sum_{j'=1}^{N_2}\omega_N(i,j')}\{h(\xi_N(i,j)) - \eta h\}\right)^2,
\end{aligned}$$

we obtain

(5.11)

$$\begin{aligned}
&\sum_{i=1}^{N_1}\left\{\mathbb{E}\left[U_N^2(i) \mid \mathcal{F}_N\right] - \mathbb{E}^2\left[U_N(i) \mid \mathcal{F}_N\right]\right\} \\
&\quad = N_1 \sum_{i=1}^{N_1}\left(\frac{\Omega_N(i)}{\sum_{i'=1}^{N_1}\Omega_N(i')}\right)^2 \sum_{j=1}^{N_2}\frac{\omega_N(i,j)}{\sum_{j'=1}^{N_2}\omega_N(i,j')}\{h(\xi_N(i,j)) - \eta h\}^2 \\
&\qquad - N_1 \sum_{i=1}^{N_1}\left(\frac{\Omega_N(i)}{\sum_{i'=1}^{N_1}\Omega_N(i')}\right)^2 \left(\sum_{j=1}^{N_2}\frac{\omega_N(i,j)}{\sum_{j'=1}^{N_2}\omega_N(i,j')}\{h(\xi_N(i,j)) - \eta h\}\right)^2.
\end{aligned}$$

Since the ancestor archipelago satisfies (2.1) and is consistent for η, we deduce that

$$N_1 \sum_{i=1}^{N_1}\left(\frac{\Omega_N(i)}{\sum_{i'=1}^{N_1}\Omega_N(i')}\right)^2 \left(\sum_{j=1}^{N_2}\frac{\omega_N(i,j)}{\sum_{j'=1}^{N_2}\omega_N(i,j')}\{h(\xi_N(i,j)) - \eta h\}\right)^2 \xrightarrow{\mathbb{P}} 0.$$

Downloaded from informs.org by [216.73.217.172] on 15 November 2026, at 06:26 . For personal use only, all rights reserved.

Then, since the ancestor archipelago also satisfies (**AN3**) we conclude that the variance (5.11) tends in probability to $\mu_1\{(h-\eta h)^2\}$. Consequently, the triangular array satisfies Assumption (**B1**) with limit $\mu_1\{(h-\eta h)^2\}$. In order to check Assumption (**B2**) we may apply Lemma A.3 by bounding

$$\max_{i\in[\![1,N_1]\!]} |U_N(i)| \leq V_N + X_N Y_N^2 \quad (N \in \mathbb{N}^*),$$

with, for $N \in \mathbb{N}^*$,

$$\begin{cases} V_N = \sqrt{N} \max\limits_{i\in[\![1,N_1]\!]} \dfrac{\Omega_N(i)}{\sum_{i'=1}^{N_1} \Omega_N(i')} \\ \qquad\qquad \times \max_{i\in[\![1,N_1]\!]} \left| \sum_{j=1}^{N_2} \dfrac{\omega_N(i,j)}{\sum_{j'=1}^{N_2} \omega_N(i,j')} \{h(\xi_N(i,j)) - \eta h\} \right|, \\ X_N = N_1 \max\limits_{i\in[\![1,N_1]\!]} \dfrac{\Omega_N(i)}{\sum_{i'=1}^{N_1} \Omega_N(i')}, \\ Y_N^2 = \sqrt{\dfrac{N_2}{N_1}} \max\limits_{i\in[\![1,N_1]\!]} \left| \dfrac{1}{N_2} \sum_{j=1}^{N_2} \delta_N(i,j) \right|, \end{cases}$$

where the δ_Ns are defined in (5.8). Here $\{V_N\}_{N\in\mathbb{N}^*}$ is $\mathcal{F}_N$-adapted and tends to zero in probability by (**AN6**) and Lemma 2.3. In addition, $\{X_N\}_{N\in\mathbb{N}^*}$ is $\mathcal{F}_N$-adapted and, by (**AN6**), tight. Moreover, for all $N \in \mathbb{N}^*$, Y_N has, by (5.9), a tail of the type

$$\mathbb{P}(Y_N \geq \varepsilon \mid \mathcal{F}_N) \leq 2N_1 \exp\left(-N_1 \frac{\varepsilon^4}{2\|h\|_\infty^2} \right).$$

Thus, Lemma A.3 applies, which establishes (**B2**). Finally, we may conclude, by using Lemma A.5, the proof of (**AN1**) to obtain that $\tilde{\sigma}^2(h) = \sigma^2(h) + \mu_1\{(h-\eta h)^2\}$.

To check (**AN2**) and prove that the series $\sum_{i=1}^{N_1} U_N(i)$, where

$$U_N(i) \triangleq \frac{\Omega_N(i)}{N_2 \sum_{i'=1}^{N_1} \Omega_N(i')} \left(\sum_{j=1}^{N_2} \{h(\xi_N(i, J_N(i,j))) - \eta h\} \right)^2 \quad (i \in [\![1,N_1]\!], N \in \mathbb{N}^*),$$

converges in probability as $N \to \infty$, let $\{\mathcal{F}_N\}_{N\in\mathbb{N}}$ be defined as in (5.1) and consider the sum

(5.12)

$$\sum_{i=1}^{N_1} \mathbb{E}\left[U_N(i) \mid \mathcal{F}_N\right]$$

$$= \sum_{i=1}^{N_1} \frac{\Omega_N(i)}{\sum_{i'=1}^{N_1} \Omega_N(i')} \sum_{j=1}^{N_2} \frac{\omega_N(i,j)}{\sum_{j'=1}^{N_2} \omega_N(i,j')} \{h(\xi_N(i,j)) - \eta h\}^2$$

$$+ (N_2 - 1) \sum_{i=1}^{N_1} \frac{\Omega_N(i)}{\sum_{i'=1}^{N_1} \Omega_N(i')} \left(\sum_{j=1}^{N_2} \frac{\omega_N(i,j)}{\sum_{j'=1}^{N_2} \omega_N(i,j')} \{h(\xi_N(i,j)) - \eta h\} \right)^2,$$

where we used, as previously, that for each $i \in [\![1, N_1]\!]$, the variables $\{h(\xi_N(i, J_N(i,j)))\}_{j=1}^{N_2}$ are, conditionally on $\mathcal{F}_N$, independent and identically distributed with common mean $\sum_{j=1}^{N_2} \omega_N(i,j) h(\xi_N(i,j)) / \sum_{j'=1}^{N_2} \omega_N(i,j')$. The first term of the right hand side of (5.12) tends in probability to $\eta\{(h - \eta h)^2\}$ by consistency, while the second term tends in probability to $\nu^2(h)$ by (**AN2**). Since this establishes the condition (**A1**) in Theorem A.1, the series $\sum_{i=1}^{N_1} U_N(i)$ and $\sum_{i=1}^{N_1} \mathbb{E}[U_N(i) \mid \mathcal{F}_N]$ have the same limit $\eta\{(h - \eta h)^2\} + \nu^2(h)$ in probability as soon as the condition (**A2**) in the same theorem can be checked for the array in question. However, write

$$\max_{i \in [\![1,N_1]\!]} |U_N(i)| \le V_N + X_N Y_N^2 \quad (N \in \mathbb{N}^*),$$

where, for $N \in \mathbb{N}^*$,

$$\begin{cases} V_N = 2N_2 \max\limits_{i \in [\![1,N_1]\!]} \dfrac{\Omega_N(i)}{\sum_{i'=1}^{N_1} \Omega_N(i')} \\ \qquad \times \left(\max_{i \in [\![1,N_1]\!]} \left| \sum_{j=1}^{N_2} \dfrac{\omega_N(i,j)}{\sum_{j'=1}^{N_2} \omega_N(i,j')} h(\xi_N(i,j)) \right| \right)^2, \\ X_N = 2 \max\limits_{i \in [\![1,N_1]\!]} N_1 \dfrac{\Omega_N(i)}{\sum_{i'=1}^{N_1} \Omega_N(i')}, \\ Y_N = \max\limits_{i \in [\![1,N_1]\!]} \dfrac{1}{\sqrt{N}} \left| \sum_{j=1}^{N_2} \delta_N(i,j) \right|, \end{cases}$$

and the δ_Ns are defined in (5.8); then, since V_N tends to zero in probability (by (**AN6**) and Lemma 2.3), X_N is tight, and Y_N has an exponential tail (by Equation 5.9), Lemma A.3 applies, establishing that the array satisfies Assumption (**A2**). Consequently, we obtain that $\tilde{\nu}^2(h) = \eta\{(h - \eta h)^2\} + \nu^2(h)$.

To verify (**AN3**) we retain to the previous machinery and study the array

$$U_N(i) \triangleq \frac{N_1}{N_2} \left(\frac{\Omega_N(i)}{\sum_{i'=1}^{N_1} \Omega_N(i')} \right)^2 \sum_{j=1}^{N_2} h(\xi_N(i, J_N(i,j))) \quad (i \in [\![1, N_1]\!], N \in \mathbb{N}^*)$$

Downloaded from informs.org by [216.73.217.172] on 15 November 2026, at 06:26 . For personal use only, all rights reserved.

Downloaded from informs.org by [216.73.217.172] on 15 November 2026, at 06:26 . For personal use only, all rights reserved.

associated with the filtration $\{\mathcal{F}_N\}_{N\in\mathbb{N}^*}$ defined in (5.1). To establish the convergence of $\sum_{i=1}^{N_1} U_N(i)$ we reapply Theorem A.1 and consider

(5.13)
$$\sum_{i=1}^{N_1} \mathbb{E}\left[U_N(i) \mid \mathcal{F}_N\right] = N_1 \sum_{i=1}^{N_1} \left(\frac{\Omega_N(i)}{\sum_{i'=1}^{N_1}\Omega_N(i')}\right)^2 \sum_{j=1}^{N_2} \frac{\omega_N(i,j)}{\sum_{j'=1}^{N_2}\omega_N(i,j')} h(\xi_N(i,j)) \xrightarrow{\mathbb{P}} \mu_1 h,$$

where convergence follows since the ancestor archipelago satisfies (**AN3**). By reusing (5.13) for $|h|$ we check that the condition (**A1**) in Theorem A.1 is satisfied. Moreover, since

$$\max_{i\in[\![1,N_1]\!]} |U_N(i)| \le \|h\|_\infty \left(\sqrt{N_1}\frac{\Omega_N(i)}{\sum_{i'=1}^{N_1}\Omega_N(i')}\right)^2,$$

where the right hand side vanishes in probability by (**AN6**), Lemma A.3 implies that the array satisfies (**A2**) as well. Thus, (**AN3**) holds true with $\tilde{\mu}_1 = \mu_1$.

In addition, since Assumption (**AN4**) coincides with (**AN3**) in the case of uniform particle weights, we obtain immediately that $\tilde{\mu}_2 = \mu_1$. Moreover, (**AN5**) coincides precisely with (**C1**), which is satisfied as the output satisfies the stronger condition (**AN1**), and we obtain $\tilde{\mu}_3 = \eta$. Finally, (**AN6**) holds trivially true.

5.7. *Proof of Theorem 3.7.* First, note that

(5.14)
$$\sum_{i=1}^{N_1} \frac{\tilde{\Omega}_N(i)}{\sum_{i'=1}^{N_1}\tilde{\Omega}_N(i')} \sum_{j=1}^{N_2} \frac{\tilde{\omega}_N(i,j)}{\sum_{j'=1}^{N_2}\tilde{\omega}_N(i,j')} h(\tilde{\xi}_N(i,j)) = \frac{\sum_{i'=1}^{N_1}\Omega_N(i')}{\sum_{i''=1}^{N_1}\tilde{\Omega}_N(i'')} \sum_{i=1}^{N_1} \frac{\Omega_N(i)}{\sum_{i'=1}^{N_1}\Omega_N(i')} \sum_{j=1}^{N_2} \frac{\tilde{\omega}_N(i,j)}{\sum_{j'=1}^{N_2}\omega_N(i,j')} h(\tilde{\xi}_N(i,j)),$$

using the definition of $\{\tilde{\Omega}_N(i)\}_{i=1}^{N_1}$ in Algorithm 3. In order to determine the limit in probability of this quantity we apply Theorem A.1 to the array

$$U_N(i) \triangleq \frac{\Omega_N(i)}{\sum_{i'=1}^{N_1}\Omega_N(i')} \sum_{j=1}^{N_2} \frac{\tilde{\omega}_N(i,j)}{\sum_{j'=1}^{N_2}\omega_N(i,j')} h(\tilde{\xi}_N(i,j)) \quad (i \in [\![1,N_1]\!], N \in \mathbb{N}^*)$$

Downloaded from informs.org by [216.73.217.172] on 15 November 2026, at 06:26 . For personal use only, all rights reserved.

associated with the filtration $\{\mathcal{F}_N\}_{N\in\mathbb{N}^*}$ given in (5.1). For each $(i,j) \in [\![1,N_1]\!] \times [\![1,N_2]\!]$, the conditional distribution of $\tilde{\xi}_N(i,j)$ given $\mathcal{F}_N$ is $R(\xi_N(i,j),\cdot)$; thus,

(5.15)
$$\begin{aligned}\mathbb{E}\left[\tilde{\omega}_N(i,j)h(\tilde{\xi}_N(i,j)) \mid \mathcal{F}_N\right] &= \omega_N(i,j)\int w(\xi_N(i,j),\tilde{x})h(\tilde{x})\,R(\xi_N(i,j),\mathrm{d}\tilde{x}) \\ &= \omega_N(i,j)Qh(\xi_N(i,j)),\end{aligned}$$

implying that

(5.16)
$$\begin{aligned}&\sum_{i=1}^{N_1}\mathbb{E}\left[U_N(i) \mid \mathcal{F}_N\right] \\ &\quad= \sum_{i=1}^{N_1}\frac{\Omega_N(i)}{\sum_{i'=1}^{N_1}\Omega_N(i')}\sum_{j=1}^{N_2}\frac{\omega_N(i,j)}{\sum_{j'=1}^{N_2}\omega_N(i,j')}Qh(\xi_N(i,j)) \xrightarrow{\mathbb{P}} \eta Qh,\end{aligned}$$

where convergence holds since the ancestor archipelago satisfies Assumption (**C1**). This implies (**A1**). To check also the condition (**A2**) we apply Lemma A.3 with $X_N = Y_N = 0$ and $V_N = \|w\|_\infty \|h\|_\infty \max_{i\in[\![1,N_1]\!]}\Omega_N(i)/\sum_{i'=1}^{N_1}\Omega_N(i')$, where V_N is $\mathcal{F}_N$-adapted and tends to zero in probability by the assumption (**C2**). Hence, Theorem A.1 ensures that the two series $\sum_{i=1}^{N_1}U_N(i)$ and $\sum_{i=1}^{N_1}\mathbb{E}\left[U_N(i) \mid \mathcal{F}_N\right]$ have the same limit ηQh in probability. Moreover, by setting h is equal to the constant function $\mathbb{1}_{\tilde{\mathsf{X}}}$ we deduce that

(5.17)
$$\frac{\sum_{i=1}^{N_1}\Omega_N(i)}{\sum_{i'=1}^{N_1}\tilde{\Omega}_N(i')} \xrightarrow{\mathbb{P}} \frac{1}{\eta Q\mathbb{1}_{\tilde{\mathsf{X}}}},$$

which allows us to complete the proof of (**C1**) using Slutsky's lemma.

Finally, Assumption (**C2**) is checked straightforwardly by just noting that

$$\max_{i\in[\![1,N_1]\!]}\frac{\tilde{\Omega}_N(i)}{\sum_{i'=1}^{N_1}\tilde{\Omega}_N(i')} \le \|w\|_\infty \max_{i\in[\![1,N_1]\!]}\frac{\Omega_N(i)}{\sum_{i'=1}^{N_1}\Omega_N(i')}\frac{\sum_{i'=1}^{N_1}\Omega_N(i')}{\sum_{i''=1}^{N_1}\tilde{\Omega}_N(i'')},$$

where the right hand side tends to zero in probability by (5.17) and the fact that the ancestor archipelago satisfies (**C1**).

5.8. *Proof of Theorem 3.8.* Note that $\tilde{\varrho}\times\tilde{\eta}h = \varrho\times\eta Qh$ and bound the quantity of interest according to

(5.18)
$$\max_{i\in[\![1,N_1]\!]}\left|\frac{1}{N_2}\sum_{j=1}^{N_2}\tilde{\omega}_N(i,j)h(\tilde{\xi}_N(i,j)) - \varrho\times\eta Qh\right|$$

$$\leq \max_{i\in[\![1,N_1]\!]} \frac{1}{N_2}\left|\sum_{j=1}^{N_2}\tilde{\delta}_N(i,j)\right| + \max_{i\in[\![1,N_1]\!]} \frac{1}{N_2}\left|\sum_{j=1}^{N_2}\omega_N(i,j)Qh(\xi_N(i,j)) - \varrho\times\eta Qh\right|,$$

where

$$\tilde{\delta}_N(i,j) \triangleq \tilde{\omega}_N(i,j)h(\tilde{\xi}_N(i,j)) - \omega_N(i,j)Qh(\xi_N(i,j)) \quad ((i,j)\in[\![1,N_1]\!]\times[\![1,N_2]\!]). \tag{5.19}$$

Since the input archipelago satisfies (**D**) it holds that

$$\mathbb{P}\left(\max_{i\in[\![1,N_1]\!]}\frac{1}{N_2}\left|\sum_{j=1}^{N_2}\{\omega_N(i,j)Qh(\xi_N(i,j)) - \varrho\times\eta Qh\}\right| \geq \varepsilon/2\right) \leq N_1 c_1 \exp\left(-c_2 N_2 \frac{\varepsilon^2}{4\left\|Q\mathbb{1}_{\mathsf{X}}\right\|_\infty^2 \left\|h\right\|_\infty^2}\right).$$

For each $i\in[\![1,N_1]\!]$, the random variables $\{\tilde{\delta}_N(i,j)\}_{j=1}^{N_2}$ are, conditionally on $\mathcal{F}_N$, independent and, by (5.15), zero mean. Moreover, since for all $(i,j)\in[\![1,N_1]\!]\times[\![1,N_2]\!]$, $|\tilde{\delta}_N(i,j)|\leq\delta\left\|h\right\|_\infty$, where δ is defined in the statement of theorem, Hoeffding's inequality implies that for all $\varepsilon>0$,

$$\mathbb{P}\left(\max_{i\in[\![1,N_1]\!]}\frac{1}{N_2}\left|\sum_{j=1}^{N_2}\tilde{\delta}_N(i,j)\right| \geq \varepsilon/2 \mid \mathcal{F}_N\right) \leq 2N_1\exp\left(-N_2\frac{\varepsilon^2}{2\delta^2\left\|h\right\|_\infty^2}\right). \tag{5.20}$$

By combining the two previous displays we may conclude that (**D**) is satisfied with $\tilde{c}_1$ and $\tilde{c}_2$ defined as in the theorem statement.

5.9. *Proof of Theorem 3.9.* We preface the proof by the following auxiliary result, which is obtained as a straightforward extension of the generalized hoeffding inequality in [14, Lemma 4].

LEMMA 5.2. *Let the assumptions of Theorem 3.8 hold. Then for all $N_1\in\mathbb{N}^*$, $N_2\in\mathbb{N}^*$, and $\varepsilon>0$,*

$$\mathbb{P}\left(\max_{i\in[\![1,N_1]\!]}\left|\frac{\tilde{\Omega}_N(i)}{\Omega_N(i)} - \eta Q\mathbb{1}_{\tilde{\mathsf{X}}}\right| \geq \varepsilon\right) \leq N_1\check{c}_1\exp\left(-\check{c}_2 N_2\varepsilon^2\right), \tag{5.21}$$

where $\check{c}_1\triangleq 2(c_1\vee\tilde{c}_1)$ and $\check{c}_2\triangleq\{(c_2/\left\|w\right\|_\infty^2)\wedge\tilde{c}_2\}\varrho^2/4$.

Downloaded from informs.org by [216.73.217.172] on 15 November 2026, at 06:26 . For personal use only, all rights reserved.

Downloaded from informs.org by [216.73.217.172] on 15 November 2026, at 06:26 . For personal use only, all rights reserved.

To check (**AN1**), take $h \in \mathsf{F}_{\mathrm{b}}(\mathcal{X})$ and assume without loss of generality that $\tilde{\eta}h = 0$ and, consequently, $\eta Q h = 0$. We again rewrite the estimator according to (5.14) and apply Theorem A.2 to the second factor. For this purpose, define the array

$$U_N(i) \triangleq \sqrt{N} \frac{\Omega_N(i)}{\sum_{i'=1}^{N_1} \Omega_N(i')} \sum_{j=1}^{N_2} \frac{\tilde{\omega}_N(i,j)}{\sum_{j'=1}^{N_2} \omega_N(i,j')} h(\tilde{\xi}_N(i,j)) \quad (i \in [\![1, N_1]\!], N \in \mathbb{N}^*), \tag{5.22}$$

and furnish the same with the filtration $\{\mathcal{F}_N\}_{N \in \mathbb{N}^*}$ defined in (5.1). We may now write

$$\begin{aligned} \sum_{i=1}^{N_1} U_N(i) &= \sum_{i=1}^{N_1} \{U_N(i) - \mathbb{E}\left[U_N(i) \mid \mathcal{F}_N\right]\} \\ &\quad + \sqrt{N} \sum_{i=1}^{N_1} \frac{\Omega_N(i)}{\sum_{i'=1}^{N_1} \Omega_N(i')} \sum_{j=1}^{N_2} \frac{\omega_N(i,j)}{\sum_{j'=1}^{N_2} \omega_N(i,j')} Q h(\xi_N(i,j)), \end{aligned} \tag{5.23}$$

where, by assumption, since $\|Qh\|_\infty \le \|h\|_\infty \|Q \mathbb{1}_{\tilde{\mathsf{X}}}\|_\infty$, the second term on the right hand side satisfies the CLT

$$\sqrt{N} \sum_{i=1}^{N_1} \frac{\Omega_N(i)}{\sum_{i'=1}^{N_1} \Omega_N(i')} \sum_{j=1}^{N_2} \frac{\omega_N(i,j)}{\sum_{j'=1}^{N_2} \omega_N(i,j')} Q h(\xi_N(i,j)) \xrightarrow{\mathcal{D}} \mathsf{N}(0, \sigma^2(Qh)).$$

Our main challenge will be to handle the first term on the right hand side of (5.23). Since all individuals of the mutated archipelago are conditionally independent given $\mathcal{F}_N$, we notice that for all $i \in [\![1, N_1]\!]$,

$$\begin{aligned} &\mathbb{E}\left[\left(\sum_{j=1}^{N_2} \frac{\tilde{\omega}_N(i,j)}{\sum_{j'=1}^{N_2} \omega_N(i,j')} h(\tilde{\xi}_N(i,j)) \right)^2 \mid \mathcal{F}_N \right] \\ &\quad = \sum_{j=1}^{N_2} \left(\frac{\omega_N(i,j)}{\sum_{j'=1}^{N_2} \omega_N(i,j')} \right)^2 R(w^2 h^2)(\xi_N(i,j)) \\ &\qquad + \left(\sum_{j=1}^{N_2} \frac{\omega_N(i,j)}{\sum_{j'=1}^{N_2} \omega_N(i,j')} Q h(\xi_N(i,j)) \right)^2 \\ &\qquad - \sum_{j=1}^{N_2} \left(\frac{\omega_N(i,j)}{\sum_{j'=1}^{N_2} \omega_N(i,j')} \right)^2 (Qh)^2(\xi_N(i,j)). \end{aligned}$$

Downloaded from informs.org by [216.73.217.172] on 15 November 2026, at 06:26 . For personal use only, all rights reserved.

Using this, we turn to the variance and deduce the expression

$$\begin{aligned}&\sum_{i=1}^{N_1}\left\{\mathbb{E}\left[U_N^2(i)\mid\mathcal{F}_N\right]-\mathbb{E}^2\left[U_N(i)\mid\mathcal{F}_N\right]\right\}\\&\quad=N\sum_{i=1}^{N_1}\left(\frac{\Omega_N(i)}{\sum_{i'=1}^{N_1}\Omega_N(i')}\right)^2\sum_{j=1}^{N_2}\left(\frac{\omega_N(i,j)}{\sum_{j'=1}^{N_2}\omega_N(i,j')}\right)^2R(w^2h^2)(\xi_N(i,j))\\&\qquad-N\sum_{i=1}^{N_1}\left(\frac{\Omega_N(i)}{\sum_{i'=1}^{N_1}\Omega_N(i')}\right)^2\sum_{j=1}^{N_2}\left(\frac{\omega_N(i,j)}{\sum_{j'=1}^{N_2}\omega_N(i,j')}\right)^2(Qh)^2(\xi_N(i,j)),\end{aligned}$$

which tends in probability to $\mu_2R(w^2h^2)-\mu_2(Qh)^2$ as the input archipelago satisfies Assumption (**AN4**) . This implies that Assumption (**B1**) in Theorem A.2 holds with the same limit. To verify the Lindeberg condition (**B2**) in Theorem A.2, note that proceeding as in (5.18) yields

$$\max_{i\in[\![1,N_1]\!]}|U_N(i)|\le V_N+X_NY_N^2\quad(N\in\mathbb{N}^*),\tag{5.24}$$

where, for $N\in\mathbb{N}^*$,

$$\begin{cases}V_N=\sqrt{N}\displaystyle\max_{i\in[\![1,N_1]\!]}\frac{\Omega_N(i)}{\sum_{i'=1}^{N_1}\Omega_N(i')}\max_{i\in[\![1,N_1]\!]}\left|\sum_{j=1}^{N_2}\frac{\omega_N(i,j)}{\sum_{j'=1}^{N_2}\omega_N(i,j')}Qh(\xi_N(i,j))\right|,\\X_N=N_1\displaystyle\max_{i\in[\![1,N_1]\!]}\frac{\Omega_N(i)}{\sum_{i=1}^{N_1}\Omega_N(i')},\\Y_N^2=\sqrt{\dfrac{N_2}{N_1}}\displaystyle\max_{i\in[\![1,N_1]\!]}\left|\frac{\sum_{j=1}^{N_2}\tilde{\delta}_N(i,j)}{\sum_{j'=1}^{N_2}\omega_N(i,j')}\right|.\end{cases}$$

Here $\{V_N\}_{N\in\mathbb{N}^*}$ is $\mathcal{F}_N$-adapted and tends to zero in probability by (**AN6**) and Lemma 2.3, $\{X_N\}_{N\in\mathbb{N}^*}$ is $\mathcal{F}_N$-adapted and tight by (**AN6**), and Y_N has, by (5.20), (**D**), and the extension of Hoeffding's inequality in [14, Lemma 4], a tail of the form (A.1) (with $\alpha=2$). Thus, by Lemma A.3, (**B2**) holds true, and we may conclude the proof of (**AN1**) using first Lemma A.5 and then Slutsky's lemma.

We turn to (**AN2**) and decompose the quantity under consideration according to

$$N_2\sum_{i=1}^{N_1}\frac{\tilde{\Omega}_N(i)}{\sum_{i'=1}^{N_1}\tilde{\Omega}_N(i')}\left(\sum_{j=1}^{N_2}\frac{\tilde{\omega}_N(i,j)}{\sum_{j'=1}^{N_2}\tilde{\omega}_N(i,j')}h(\tilde{\xi}_N(i,j))\right)^2\tag{5.25}$$

$$= N_2 \frac{\sum_{i''=1}^{N_1} \Omega_N(i'')}{\sum_{i'=1}^{N_1} \tilde{\Omega}_N(i')} \sum_{i=1}^{N_1} \frac{\Omega_N(i)}{\sum_{i''=1}^{N_1} \Omega_N(i'')} \left(\frac{\Omega_N(i)}{\tilde{\Omega}_N(i)} - \frac{1}{\eta Q \mathbb{1}_{\tilde{\mathsf{X}}}} \right)$$
$$\times \left(\sum_{j=1}^{N_2} \frac{\tilde{\omega}_N(i,j)}{\sum_{j'=1}^{N_2} \omega_N(i,j')} h(\tilde{\xi}_N(i,j)) \right)^2$$
$$+ N_2 \frac{1}{\eta Q \mathbb{1}_{\tilde{\mathsf{X}}}} \frac{\sum_{i''=1}^{N_1} \Omega_N(i'')}{\sum_{i'=1}^{N_1} \tilde{\Omega}_N(i')} \sum_{i=1}^{N_1} \frac{\Omega_N(i)}{\sum_{i''=1}^{N_1} \Omega_N(i'')}$$
$$\times \left(\sum_{j=1}^{N_2} \frac{\tilde{\omega}_N(i,j)}{\sum_{j'=1}^{N_2} \omega_N(i,j')} h(\tilde{\xi}_N(i,j)) \right)^2 .$$

The convergence in probability of the second term on the right hand side will now to be established using Theorem A.1. For this purpose, define the triangular array

$$U_N(i) \triangleq N_2 \frac{\Omega_N(i)}{\sum_{i'=1}^{N_1} \Omega_N(i')} \left(\sum_{j=1}^{N_2} \frac{\tilde{\omega}_N(i,j)}{\sum_{j'=1}^{N_2} \omega_N(i,j')} h(\tilde{\xi}_N(i,j)) \right)^2 \tag{5.26}$$
$$(i \in [\![1, N_1]\!], N \in \mathbb{N}^*),$$

and associate the same with the σ-field $\mathcal{F}_N$ defined in (5.1). We now apply the previous machinery and study the convergence of the series

$$\sum_{i=1}^{N_1} \mathbb{E}\left[U_N(i) \mid \mathcal{F}_N\right]$$
$$= N_2 \sum_{i=1}^{N_1} \frac{\Omega_N(i)}{\sum_{i'=1}^{N_1} \Omega_N(i')} \sum_{j=1}^{N_2} \left(\frac{\omega_N(i,j)}{\sum_{j'=1}^{N_2} \omega_N(i,j')} \right)^2 R(w^2 h^2)(\xi_N(i,j))$$
$$+ N_2 \sum_{i=1}^{N_1} \frac{\Omega_N(i)}{\sum_{i'=1}^{N_1} \Omega_N(i')} \left(\sum_{j=1}^{N_2} \frac{\omega_N(i,j)}{\sum_{j'=1}^{N_2} \omega_N(i,j')} Q h(\xi_N(i,j)) \right)^2$$
$$- N_2 \sum_{i=1}^{N_1} \frac{\Omega_N(i)}{\sum_{i'=1}^{N_1} \Omega_N(i')} \sum_{j=1}^{N_2} \left(\frac{\omega_N(i,j)}{\sum_{j'=1}^{N_2} \omega_N(i,j')} \right)^2 (Qh)^2(\xi_N(i,j)),$$

which tends in probability to $\nu^2(Qh) + \mu_3 R(w^2 h^2) - \mu_3(Q^2 h)$ as the ancestor archipelago satisfies (**AN2**) and (**AN5**). Thus, the condition (**A1**) in Theorem A.1 is checked. In addition, (**A2**) is checked using Lemma A.3, as

$$\max_{i \in [\![1, N_1]\!]} |U_N(i)| \leq V_N + X_N Y_N^2 \quad (N \in \mathbb{N}^*),$$

Downloaded from informs.org by [216.73.217.172] on 15 November 2026, at 06:26 . For personal use only, all rights reserved.

Downloaded from informs.org by [216.73.217.172] on 15 November 2026, at 06:26 . For personal use only, all rights reserved.

where for $N \in \mathbb{N}^*$, $V_N = 0$ and

$$\begin{cases} X_N = N_1 \max\limits_{i \in [\![1,N_1]\!]} \dfrac{\Omega_N(i)}{\sum_{i=1}^{N_1} \Omega_N(i')}, \\ Y_N = \sqrt{\dfrac{N_2}{N_1}} \max\limits_{i \in [\![1,N_1]\!]} \left| \displaystyle\sum_{j=1}^{N_2} \frac{\tilde{\omega}_N(i,j)}{\sum_{j'=1}^{N_2} \omega_N(i,j')} h(\tilde{\xi}_N(i,j)) \right|, \end{cases}$$

where $\{X_N\}_{N \in \mathbb{N}^*}$ is $\mathcal{F}_N$-adapted and tight by (**AN6**) and each Y_N has, by [14, Lemma 4], since the input and output archipelagos satisfy (**D**), a tail of the form (A.1) (with $\alpha = 1$). Thus, (**A1**) holds true, and we may conclude that the series $\sum_{i=1}^{N_1} U_N(i)$ and $\sum_{i=1}^{N_1} \mathbb{E}[U_N(i) \mid \mathcal{F}_N]$ tend to the same limit in probability.

We turn to the first term of (5.25) and show that this tends to zero in probability. Indeed, note that the absolute value of the same is, up to the factor $\sum_{i'=1}^{N_1} \Omega_N(i') / \sum_{i''=1}^{N_1} \tilde{\Omega}_N(i'')$, which converges in probability by (5.17), bounded by

$$\begin{aligned} &\max_{i' \in [\![1,N_1]\!]} \left| \frac{\Omega_N(i')}{\tilde{\Omega}_N(i')} - \frac{1}{\eta Q \mathbb{1}_{\tilde{\mathsf{X}}}} \right| N_2 \sum_{i=1}^{N_1} \frac{\Omega_N(i)}{\sum_{i''=1}^{N_1} \Omega_N(i'')} \\ &\quad \times \left(\sum_{j=1}^{N_2} \frac{\tilde{\omega}_N(i,j)}{\sum_{j'=1}^{N_2} \omega_N(i,j')} h(\tilde{\xi}_N(i,j)) \right)^2, \end{aligned}$$

where the first factor vanishes in probability by Lemma 5.2 and Lemma A.4, and the convergence of the second factor was established above. This establishes Assumption (**AN2**).

To check Assumption (**AN6**), consider the bound

$$\begin{aligned} &N_1 \max_{i \in [\![1,N_1]\!]} \frac{\tilde{\Omega}_N(i)}{\sum_{i'=1}^{N_1} \tilde{\Omega}_N(i')} \\ &\leq \|w\|_\infty N_1 \max_{i \in [\![1,N_1]\!]} \frac{\Omega_N(i)}{\sum_{i'=1}^{N_1} \Omega_N(i')} \left(\left| \frac{\sum_{i'=1}^{N_1} \Omega_N(i')}{\sum_{i''=1}^{N_1} \tilde{\Omega}_N(i'')} - \frac{1}{\eta Q \mathbb{1}_{\tilde{\mathsf{X}}}} \right| + \frac{1}{\eta Q \mathbb{1}_{\tilde{\mathsf{X}}}} \right), \end{aligned} \tag{5.27}$$

where the second factor on the right hand side is tight as the ancestor archipelago is assumed to satisfy (**AN6**). Moreover, as the third factor tends to $1/\eta Q \mathbb{1}_{\tilde{\mathsf{X}}}$ in probability by (5.17) we conclude that (**AN6**) holds true also for the output.

In order to check (**AN3**), pick $h \in \mathsf{F}_{\mathrm{b}}(\mathcal{X})$ and decompose the quantity of interest according to

(5.28)
$$
\begin{aligned}
&N_1 \sum_{i=1}^{N_1} \left(\frac{\tilde{\Omega}_N(i)}{\sum_{i'=1}^{N_1} \tilde{\Omega}_N(i')} \right)^2 \sum_{j=1}^{N_2} \frac{\tilde{\omega}_N(i,j)}{\sum_{j'=1}^{N_2} \tilde{\omega}_N(i,j')} h(\tilde{\xi}_N(i,j)) \\
&\quad = N_1 \left(\frac{\sum_{i=1}^{N_1} \Omega_N(i)}{\sum_{i'=1}^{N_1} \tilde{\Omega}_N(i')} \right)^2 \sum_{i=1}^{N_1} \left(\frac{\Omega_N(i)}{\sum_{i'=1}^{N_1} \Omega_N(i')} \right)^2 \left(\frac{\tilde{\Omega}_N(i)}{\Omega_N(i)} - \eta Q \mathbb{1}_{\tilde{\mathsf{X}}} \right) \\
&\qquad \times \sum_{j=1}^{N_2} \frac{\tilde{\omega}_N(i,j)}{\sum_{j'=1}^{N_2} \omega_N(i,j')} h(\tilde{\xi}_N(i,j)) \\
&\qquad + N_1 \eta Q \mathbb{1}_{\tilde{\mathsf{X}}} \left(\frac{\sum_{i=1}^{N_1} \Omega_N(i)}{\sum_{i'=1}^{N_1} \tilde{\Omega}_N(i')} \right)^2 \sum_{i=1}^{N_1} \left(\frac{\Omega_N(i)}{\sum_{i'=1}^{N_1} \Omega_N(i')} \right)^2 \\
&\qquad \times \sum_{j=1}^{N_2} \frac{\tilde{\omega}_N(i,j)}{\sum_{j'=1}^{N_2} \omega_N(i,j')} h(\tilde{\xi}_N(i,j)).
\end{aligned}
$$

In order to handle the second term of this decomposition, we apply Theorem A.1 to the array

$$
\begin{aligned}
U_N(i) &\triangleq N_1 \left(\frac{\Omega_N(i)}{\sum_{i'=1}^{N_1} \Omega_N(i')} \right)^2 \sum_{j=1}^{N_2} \frac{\tilde{\omega}_N(i,j)}{\sum_{j'=1}^{N_2} \omega_N(i,j')} h(\tilde{\xi}_N(i,j)) \\
&\qquad (i \in [\![1, N_1]\!], N \in \mathbb{N}^*)
\end{aligned}
$$

furnished with the filtration $\{\mathcal{F}_N\}_{N \in \mathbb{N}^*}$ given by (5.1). First, we observe that

$$
\begin{aligned}
&\sum_{i=1}^{N_1} \mathbb{E}\left[U_N(i) \mid \mathcal{F}_N \right] \\
&\quad = N_1 \sum_{i=1}^{N_1} \left(\frac{\Omega_N(i)}{\sum_{i'=1}^{N_1} \Omega_N(i')} \right)^2 \sum_{j=1}^{N_2} \frac{\omega_N(i,j)}{\sum_{j'=1}^{N_2} \omega_N(i,j')} Qh(\xi_N(i,j)) \xrightarrow{\mathbb{P}} \mu_1 Qh,
\end{aligned}
$$

as the ancestor archipelago satisfies Assumption (**AN3**). Thus, the condition (**A1**) in Theorem A.1 holds true. In addition, as

$$
\max_{i \in [\![1, N_1]\!]} |U_N(i)| \leq \|w\|_\infty \|h\|_\infty \left(\sqrt{N_1} \max_{i \in [\![1, N_1]\!]} \frac{\Omega_N(i)}{\sum_{i'=1}^{N_1} \Omega_N(i')} \right)^2,
$$

also (**A2**) is verified by Lemma A.3 (applied with $X_N = Y_N = 0$) and the fact that the input archipelago satisfies (**AN6**). Consequently, the also series $\sum_{i=1}^{N_1} U_N(i)$ tends in probability to the limit $\mu_1 Qh$, which, by (5.17), implies that the second term of (5.28) tends to $\mu_1 Qh / \eta Q \mathbb{1}_{\tilde{\mathsf{X}}}$. To treat the first term

Downloaded from informs.org by [216.73.217.172] on 15 November 2026, at 06:26 . For personal use only, all rights reserved.

Downloaded from informs.org by [216.73.217.172] on 15 November 2026, at 06:26 . For personal use only, all rights reserved.

of (5.28), note that this is, up to the factor $\sum_{i'=1}^{N_1}\Omega_N(i')/\sum_{i''=1}^{N_1}\tilde{\Omega}_N(i'')$, which converges in probability by (5.17), bounded by

$$N_1 \max_{i\in[\![1,N_1]\!]}\left|\frac{\tilde{\Omega}_N(i)}{\Omega_N(i)}-\eta Q\mathbb{1}_{\tilde{\mathsf{X}}}\right|\sum_{i=1}^{N_1}\left(\frac{\Omega_N(i)}{\sum_{i'=1}^{N_1}\Omega_N(i')}\right)^2 \times\sum_{j=1}^{N_2}\frac{\tilde{\omega}_N(i,j)}{\sum_{j'=1}^{N_2}\omega_N(i,j')}|h|(\tilde{\xi}_N(i,j)),$$

which tends to zero in probability by the previous computation and Lemma 5.2. This completes the proof of (**AN3**).

In order to prove (**AN4**), introduce the array

$$U_N(i)\triangleq N\left(\frac{\Omega_N(i)}{\sum_{i'=1}^{N_1}\Omega_N(i')}\right)^2\sum_{j=1}^{N_2}\left(\frac{\tilde{\omega}_N(i,j)}{\sum_{j'=1}^{N_2}\omega_N(i,j')}\right)^2 h(\tilde{\xi}_N(i,j))$$
$$(i\in[\![1,N_1]\!],N\in\mathbb{N}^*)$$

and equip the same with usual filtration $\{\mathcal{F}_N\}_{N\in\mathbb{N}^*}$. With this notation, the quantity of interest in (**AN4**) can be written as

$$\left(\frac{\sum_{i=1}^{N_1}\Omega_N(i)}{\sum_{i'=1}^{N_1}\tilde{\Omega}_N(i')}\right)^2\sum_{i=1}^{N_1}U_N(i),$$

where the first factor tends to $1/(\eta Q\mathbb{1}_{\tilde{\mathsf{X}}})^2$ by Lemma 5.2. Thus, it is enough to show that the second factor tends to $\mu_2 R(w^2h)$ in probability, and for this purpose we use Theorem A.1. As the ancestor archipelago satisfies (**AN4**), the quantity

$$\sum_{i=1}^{N_1}\mathbb{E}\left[U_N(i)\mid\mathcal{F}_N\right] = N\sum_{i=1}^{N_1}\left(\frac{\Omega_N(i)}{\sum_{i'=1}^{N_1}\Omega_N(i')}\right)^2\sum_{j=1}^{N_2}\left(\frac{\omega_N(i,j)}{\sum_{j'=1}^{N_2}\omega_N(i,j')}\right)^2 R(w^2h)(\xi_N(i,j))$$

tends in probability to the desired limit $\mu_2 R(w^2h)$. This implies the condition (**A1**) in Theorem A.1. In addition, (**A2**) is checked using Lemma A.3; indeed,

$$\max_{i\in[\![1,N_1]\!]}|U_N(i)| \le|\omega|_\infty\|w\|_\infty^2\|h\|_\infty\left(\sqrt{N_1}\max_{i\in[\![1,N_1]\!]}\frac{\Omega_N(i)}{\sum_{i'=1}^{N_1}\Omega_N(i')}\right)^2\max_{i\in[\![1,N_1]\!]}\frac{N_2}{\sum_{j=1}^{N_2}\omega_N(i,j)},$$

where the the right hand side is adapted to $\{\mathcal{F}_N\}_{N\in\mathbb{N}^*}$ and vanishes in probability by Lemma A.4, as the ancestor archipelago satisfies (**AN6**) and (**D**). This shows (**AN4**).

Finally, in order to prove (**AN5**) we decompose the quantity of interest according to

$$
\begin{aligned}
(5.29)\qquad & N_2 \sum_{i=1}^{N_1} \frac{\tilde{\Omega}_N(i)}{\sum_{i'=1}^{N_1}\tilde{\Omega}_N(i')} \sum_{j=1}^{N_2} \left(\frac{\tilde{\omega}_N(i,j)}{\sum_{j'=1}^{N_2}\tilde{\omega}_N(i,j')}\right)^2 h(\tilde{\xi}_N(i,j)) \\
&= N_2 \left(\frac{\sum_{i'=1}^{N_1}\Omega_N(i')}{\sum_{i''=1}^{N_1}\tilde{\Omega}_N(i'')}\right) \sum_{i=1}^{N_1} \frac{\Omega_N(i)}{\sum_{i'=1}^{N_1}\Omega_N(i')} \left(\frac{\Omega_N(i)}{\tilde{\Omega}_N(i)} - \frac{1}{\eta Q \mathbb{1}_{\tilde{\mathsf{X}}}}\right) \\
&\quad \times \sum_{j=1}^{N_2} \left(\frac{\tilde{\omega}_N(i,j)}{\sum_{j'=1}^{N_2}\omega_N(i,j')}\right)^2 h(\tilde{\xi}_N(i,j)) \\
&\quad + N_2 \frac{1}{\eta Q \mathbb{1}_{\tilde{\mathsf{X}}}} \left(\frac{\sum_{i'=1}^{N_1}\Omega_N(i')}{\sum_{i''=1}^{N_1}\tilde{\Omega}_N(i'')}\right) \sum_{i=1}^{N_1} \frac{\Omega_N(i)}{\sum_{i'=1}^{N_1}\Omega_N(i')} \\
&\quad \times \sum_{j=1}^{N_2} \left(\frac{\tilde{\omega}_N(i,j)}{\sum_{j'=1}^{N_2}\omega_N(i,j')}\right)^2 h(\tilde{\xi}_N(i,j)).
\end{aligned}
$$

To deal with the second term we reapply Theorem A.1, this time to the array

$$
\begin{aligned}
U_N(i) &= N_2 \frac{\Omega_N(i)}{\sum_{i'=1}^{N_1}\Omega_N(i')} \sum_{j=1}^{N_2} \left(\frac{\tilde{\omega}_N(i,j)}{\sum_{j'=1}^{N_2}\omega_N(i,j')}\right)^2 h(\tilde{\xi}_N(i,j)) \\
&\qquad (i \in [\![1, N_1]\!], N \in \mathbb{N}^*).
\end{aligned}
$$

As usual, we study first the series

$$
\begin{aligned}
&\sum_{i=1}^{N_1} \mathbb{E}\left[U_N(i) \mid \mathcal{F}_N\right] \\
&\quad = N_2 \sum_{i=1}^{N_1} \frac{\Omega_N(i)}{\sum_{i'=1}^{N_1}\Omega_N(i')} \sum_{j=1}^{N_2} \left(\frac{\omega_N(i,j)}{\sum_{j'=1}^{N_2}\omega_N(i,j')}\right)^2 R(w^2 h)(\xi_N(i,j)) \\
&\quad \xrightarrow{\mathbb{P}} \mu_3 R(w^2 h),
\end{aligned}
$$

where the limit is a consequence of the fact that the ancestor archipelago satisfies (**AN5**). This establishes (**A1**) in Theorem A.1. To check also

Downloaded from informs.org by [216.73.217.172] on 15 November 2026, at 06:26 . For personal use only, all rights reserved.

Downloaded from informs.org by [216.73.217.172] on 15 November 2026, at 06:26 . For personal use only, all rights reserved.

Condition (**A2**), consider the upper bound

$$\max_{i\in[\![1,N_1]\!]}|U_N(i)| \leq |\omega|_\infty \|w\|_\infty^2 \|h\|_\infty \max_{i\in[\![1,N_1]\!]} \frac{\Omega_N(i)}{\sum_{i'=1}^{N_1}\Omega_N(i')} \max_{i\in[\![1,N_1]\!]} \frac{N_2}{\sum_{j=1}^{N_2}\omega_N(i,j)},$$

which is $\{\mathcal{F}_N\}_{N\in\mathbb{N}^*}$-adapted and tends to zero in probability by Lemma A.4, as the ancestor archipelago satisfies (**C2**) and (**D**). Now, Theorem A.1 guarantees that $\sum_{i=1}^{N_1}U_N(i)$ and $\sum_{i=1}^{N_1}\mathbb{E}[U_N(i)\mid\mathcal{F}_N]$ have the same limit $\mu_3 R(w^2h)$ in probability. Moreover, note that the second term of (5.29) is, up to the factor $\sum_{i'=1}^{N_1}\Omega_N(i')/\sum_{i''=1}^{N_1}\tilde{\Omega}_N(i'')$, bounded by

$$N_2 \max_{i\in[\![1,N_1]\!]}\left|\frac{\Omega_N(i)}{\tilde{\Omega}_N(i)}-\frac{1}{\eta Q\mathbb{1}_{\tilde{\mathsf{X}}}}\right|\sum_{i=1}^{N_1}\frac{\Omega_N(i)}{\sum_{i'=1}^{N_1}\Omega_N(i')} \times\sum_{j=1}^{N_2}\left(\frac{\tilde{\omega}_N(i,j)}{\sum_{j'=1}^{N_2}\omega_N(i,j')}\right)^2|h|(\tilde{\xi}_N(i,j)),$$

which tends to zero in probability by (5.21) and Lemma A.4. Thus, also (**AN5**) holds true.

5.10. *Proof of Corollary 4.6.* First, a prefatory lemma.

LEMMA 5.3. *Assume (**M**). Then for all $(\ell,n)\in\mathbb{N}^2$ such that $\ell\leq n$ and $h\in\mathsf{F}_{\mathrm{b}}(\mathcal{X})$,*

$$\left\|\frac{Q_\ell\cdots Q_{n-1}(h-\eta_n h)}{\eta_\ell Q_\ell\cdots Q_{n-1}\mathbb{1}_{\mathsf{X}}}\right\|_\infty \leq \frac{\rho^{n-\ell}}{1-\rho}\operatorname{osc}(h), \tag{5.30}$$

where ρ is defined in (4.11).

PROOF. For $x\in\mathsf{X}$, write

$$\begin{aligned}&\frac{Q_\ell\cdots Q_{n-1}(h-\eta_n h)(x)}{\eta_\ell Q_\ell\cdots Q_{n-1}\mathbb{1}_{\mathsf{X}}}\\ &=\frac{Q_\ell\cdots Q_{n-1}h(x)}{\eta_\ell Q_\ell\cdots Q_{n-1}\mathbb{1}_{\mathsf{X}}}-\frac{Q_\ell\cdots Q_{n-1}\mathbb{1}_{\mathsf{X}}(x)}{\eta_\ell Q_\ell\cdots Q_{n-1}\mathbb{1}_{\mathsf{X}}}\eta_n h\\ &=\frac{Q_\ell\cdots Q_{n-1}\mathbb{1}_{\mathsf{X}}(x)}{\eta_\ell Q_\ell\cdots Q_{n-1}\mathbb{1}_{\mathsf{X}}}\left[\frac{Q_\ell\cdots Q_{n-1}h(x)}{Q_\ell\cdots Q_{n-1}\mathbb{1}_{\mathsf{X}}(x)}-\frac{\eta_\ell Q_\ell\cdots Q_{n-1}h}{\eta_\ell Q_\ell\cdots Q_{n-1}\mathbb{1}_{\mathsf{X}}}\right].\end{aligned} \tag{5.31}$$

Note that since $Q_\ell \dots Q_{n-1} h(x) = \delta_x Q_\ell \dots Q_{n-1} h$ (where δ_x denotes the Dirac mass located at x) we may, under (**M**), apply [18, Proposition 10.20], yielding the uniform bound

$$\left| \frac{\delta_x Q_\ell \cdots Q_n h}{\delta_x Q_\ell \cdots Q_n \mathbb{1}_{\mathsf{X}}} - \frac{\eta_\ell Q_\ell \cdots Q_n h}{\eta_\ell Q_\ell \cdots Q_n \mathbb{1}_{\mathsf{X}}} \right| \leq \rho^{n-\ell} \operatorname{osc}(h) \quad (x \in \mathsf{X}). \tag{5.32}$$

Combining (5.31) and (5.32) with the uniform bound

$$\frac{Q_\ell \cdots Q_{n-1} \mathbb{1}_{\mathsf{X}}(x)}{\eta_\ell Q_\ell \cdots Q_{n-1} \mathbb{1}_{\mathsf{X}}} \leq \frac{\sigma_+}{\sigma_-} = \frac{1}{1-\rho} \quad (x \in \mathsf{X})$$

yields (5.30). □

For arbitrary $(\ell, n) \in \mathbb{N}^2$ with $\ell \leq n$, combining the identity

$$\eta_\ell Q_\ell \cdots Q_{n-1} \mathbb{1}_{\mathsf{X}} = \eta_\ell Q_\ell \mathbb{1}_{\mathsf{X}} \times \eta_{\ell+1} Q_{\ell+1} \cdots Q_{n-1} \mathbb{1}_{\mathsf{X}}$$

with the bound $\eta_\ell Q_{\ell-1} \mathbb{1}_{\mathsf{X}} \geq c_-$ (the latter implied by (**M**)(iii)) yields

$$\begin{aligned} &\frac{\eta_\ell R_\ell \{ w_\ell^2 Q_{\ell+1} \cdots Q_{n-1} (h - \eta_n h)^2 \}}{(\eta_\ell Q_\ell \cdots Q_{n-1} \mathbb{1}_{\mathsf{X}})^2} \\ &\quad \leq c_-^{-1} \left\| w_\ell \right\|_\infty \left\| \frac{Q_{\ell+1} \cdots Q_{n-1} (h - \eta_n h)}{\eta_{\ell+1} Q_{\ell+1} \cdots Q_{n-1} \mathbb{1}_{\mathsf{X}}} \right\|_\infty^2 . \end{aligned}$$

Now, using Lemma 5.3 we obtain

$$\frac{\eta_\ell R_\ell \{ w_\ell^2 Q_{\ell+1} \cdots Q_{n-1} (h - \eta_n h)^2 \}}{(\eta_\ell Q_\ell \cdots Q_{n-1} \mathbb{1}_{\mathsf{X}})^2} \leq w_+ \frac{\rho^{2(n-\ell-1)}}{(1-\rho)^2 c_-} \operatorname{osc}^2(h).$$

Finally, the proof of Corollary 4.6 is concluded by summing up the terms.

5.11. *Proof of Corollary 4.10.* Since the ℓth terms of the asymptotic variances (4.4) and (4.5) differ only by the multiplicative constant $n - \ell$, the proof follows straightforwardly by direct inspection of the proof of the analogous result for the standard bootstrap particle filter given in [17, Theorem 11] (which in turn is an adaptation of the proof of Theorem 10 in the same paper, providing the analogous result for the particle predictor). More specifically, the result is obtained by

- embedding, using a trivial extension of Kolmogorov's extension theorem, the stationary sequence $\{Z_p\}_{p \in \mathbb{N}}$ into a stationary process $\{Z_p\}_{p \in \mathbb{Z}}$ with doubly infinite time.

Downloaded from informs.org by [216.73.217.172] on 15 November 2026, at 06:26 . For personal use only, all rights reserved.

- bounding, for a given $n \in \mathbb{N}$, using [17, Equations 34–35], $\sigma_n^2\langle Z_{0:n}\rangle(h)$ by a quantity of form $c\sum_{\ell=0}^{n}(n-\ell)\Delta_{n-\ell}\langle h\rangle(Z_{-\infty:\ell-1}, Z_{\ell:n})$, where c is a $\mathbb{P}$-a.s. finite random variable and each function $\Delta_m\langle h\rangle : \mathsf{Z}^\infty \to \mathbb{R}_+$, $m \in \mathbb{N}$, is of the same type as the terms of the sum in [17, Equation 35].
- using the stationarity to conclude that $\sum_{\ell=0}^{n}(n-\ell)\Delta_{n-\ell}\langle h\rangle(Z_{-\infty:\ell-1}, Z_{\ell:n})$ has the same distribution as $\sum_{\ell=0}^{n}\ell\Delta_\ell\langle h\rangle(Z_{-\infty:-\ell-1}, Z_{-\ell:0})$.
- bounding, using [17, Equation 39], each term of the sum as $\Delta_\ell\langle h\rangle(Z_{-\infty:-\ell-1}, Z_{-\ell:0}) \leq d\beta^\ell$, $\mathbb{P}$-a.s., where d is a $\mathbb{P}$-a.s. finite random variable and $\beta < 1$ is a constant. This shows that $\sigma_n^2\langle Z_{0:n}\rangle(h) \leq cd\sum_{\ell=0}^{\infty}\ell\beta^\ell < \infty$, $\mathbb{P}$-a.s., which concludes the proof.

APPENDIX A: TECHNICAL RESULTS

We first recall two results, obtained in [15], which are essential for the developments of the present paper.

THEOREM A.1 ([15]). *Let $(\Omega, \mathcal{A}, \{\mathcal{F}_N\}_{N\in\mathbb{N}}, \mathbb{P})$ be a filtered probability space. In addition, let, for a given sequence $\{M_N\}_{N\in\mathbb{N}}$ of integers such that $M_N \to \infty$ as $N \to \infty$, $\{U_N(i)\}_{i=1}^{M_N}$, $N \in \mathbb{N}$, be a triangular array of random variables on $(\Omega, \mathcal{A}, \mathbb{P})$ such that for all $N \in \mathbb{N}$, the variables $\{U_N(i)\}_{i=1}^{M_N}$ are conditionally independent given $\mathcal{F}_N$ with $\mathbb{E}[|U_N(i)| \mid \mathcal{F}_N] < \infty$, $\mathbb{P}$-a.s., for all $i \in [\![1, M_N]\!]$. Moreover, assume that*

(A1) $$\lim_{\lambda\to\infty}\sup_{N\in\mathbb{N}}\mathbb{P}\left(\sum_{i=1}^{M_N}\mathbb{E}[|U_N(i)| \mid \mathcal{F}_N] \geq \lambda\right) = 0.$$

(A2) *For all $\varepsilon > 0$, as $N \to \infty$,*

$$\sum_{i=1}^{M_N}\mathbb{E}\left[|U_N(i)|\mathbb{1}_{\{|U_N(i)|\geq\varepsilon\}} \mid \mathcal{F}_N\right] \xrightarrow{\mathbb{P}} 0.$$

Then, as $N \to \infty$,

$$\max_{m\in[\![1,M_N]\!]}\left|\sum_{i=1}^{m}U_N(i) - \sum_{i=1}^{m}\mathbb{E}\left[U_N(i) \mid \mathcal{F}_N\right]\right| \xrightarrow{\mathbb{P}} 0.$$

THEOREM A.2 ([15]). *Let the assumptions of Theorem A.1 hold with $\mathbb{E}[U_N^2(i) \mid \mathcal{F}_N] < \infty$, $\mathbb{P}$-a.s., for all $i \in [\![1, M_N]\!]$, and (**A1**) and (**A2**) replaced by:*

Downloaded from informs.org by [216.73.217.172] on 15 November 2026, at 06:26 . For personal use only, all rights reserved.

(B1) *For some constant* $\varsigma^2 > 0$, *as* $N \to \infty$,

$$\sum_{i=1}^{M_N} \left(\mathbb{E}[U_N^2(i) \mid \mathcal{F}_N] - \mathbb{E}^2 \left[U_N(i) \mid \mathcal{F}_N \right] \right) \xrightarrow{\mathbb{P}} \varsigma^2.$$

(B2) *For all* $\varepsilon > 0$, *as* $N \to \infty$,

$$\sum_{i=1}^{M_N} \mathbb{E} \left[U_N^2(i) \mathbb{1}_{\{|U_N(i)| \geq \varepsilon\}} \mid \mathcal{F}_N \right] \xrightarrow{\mathbb{P}} 0.$$

Then, for all $u \in \mathbb{R}$, *as* $N \to \infty$,

$$\mathbb{E} \left[\exp \left(\mathrm{i}\, u \sum_{i=1}^{M_N} \{ U_N(i) - \mathbb{E}[U_N(i) \mid \mathcal{F}_N] \} \right) \mid \mathcal{F}_N \right] \xrightarrow{\mathbb{P}} \exp \left(-u^2 \varsigma^2 / 2 \right).$$

The following lemma is useful when verifying the tightness conditions (**A2**) and (**B2**).

LEMMA A.3. *Let the* $(\{M_N\}_{N\in\mathbb{N}}, \{U_N(i)\}_{i=1}^{M_N}, \{\mathcal{F}_N\}_{N\in\mathbb{N}})$ *be the triangular array given in Theorem A.1. Assume that there exist sequences* $\{V_N\}_{N\in\mathbb{N}}$, $\{X_N\}_{N\in\mathbb{N}}$, *and* $\{Y_N\}_{N\in\mathbb{N}}$ *of nonnegative random variables such that*

(i) for all $N \in \mathbb{N}$, $\mathbb{P}$*-a.s.,*

$$\max_{i \in [\![1, M_N]\!]} |U_N(i)| \leq V_N + X_N Y_N^2,$$

(ii) $\{V_N\}_{N\in\mathbb{N}}$ *and* $\{X_N\}_{N\in\mathbb{N}}$ *are* $\{\mathcal{F}_N\}_{N\in\mathbb{N}}$*-adapted and such that* $V_N \xrightarrow{\mathbb{P}} 0$ *as* $N \to \infty$ *and*

$$\lim_{\lambda \to \infty} \sup_{N \in \mathbb{N}} \mathbb{P} \left(X_N \geq \lambda \right) = 0,$$

(iii) for some $\alpha \in \{1, 2\}$, $\nu > 0$, $c > 0$, *and* $C > 0$, $\mathbb{P}$*-a.s.,*

$$\mathbb{P} \left(Y_N \geq y \mid \mathcal{F}_N \right) \leq C M_N \exp \left(-c M_N^{\nu} y^{2\alpha} \right). \tag{A.1}$$

Then for $p \in \{1, 2\}$,

$$\lim_{\lambda \to \infty} \sup_{N \in \mathbb{N}} \mathbb{P} \left(\sum_{i=1}^{M_N} \mathbb{E} \left[|U_N^p(i)| \mid \mathcal{F}_N \right] \geq \lambda \right) = 0$$
$$\Rightarrow \sum_{i=1}^{M_N} \mathbb{E} \left[|U_N^p(i)| \mathbb{1}_{\{|U_N(i)| \geq \varepsilon\}} \mid \mathcal{F}_N \right] \xrightarrow{\mathbb{P}} 0, \quad \forall \varepsilon > 0, \text{ as } N \to \infty.$$

Downloaded from informs.org by [216.73.217.172] on 15 November 2026, at 06:26 . For personal use only, all rights reserved.

Downloaded from informs.org by [216.73.217.172] on 15 November 2026, at 06:26 . For personal use only, all rights reserved.

PROOF. We start with the case $p = 1$. First, note that for all $\upsilon > 0$,

$$\begin{aligned}
&\mathbb{E}\left[Y_N^2 \mathbb{1}_{\{Y_N^2 \geq \upsilon\}} \mid \mathcal{F}_N\right] \\
&\quad = \int_{\upsilon}^{\infty} \mathbb{P}\left(Y_N \geq \sqrt{y} \mid \mathcal{F}_N\right) \mathrm{d}y + \upsilon \mathbb{P}\left(Y_N \geq \sqrt{\upsilon} \mid \mathcal{F}_N\right) \\
&\quad \leq C M_N \int_{\upsilon}^{\infty} \exp\left(-c M_N^{\nu} y^{\alpha}\right) \mathrm{d}y + C M_N \upsilon \exp\left(-c M_N^{\nu} \upsilon^{\alpha}\right),
\end{aligned}$$

where we used the condition (iii) in the second step. Thus,

(A.2)
$$\mathbb{E}\left[Y_N^2 \mathbb{1}_{\{Y_N^2 \geq \upsilon\}} \mid \mathcal{F}_N\right] \leq \begin{cases} \left(\upsilon + M_N^{-\nu}/c\right) C M_N \exp\left(-c M_N^{\nu} \upsilon\right) & \text{for } \alpha = 1, \\ \left(\upsilon + M_N^{-\nu}/(2c\upsilon)\right) C M_N \exp\left(-c M_N^{\nu} \upsilon^2\right) & \text{for } \alpha = 2, \end{cases}$$

using the standard upper tail bound for Gaussian distributions. In any case,

$$M_N \mathbb{E}\left[Y_N^2 \mathbb{1}_{\{Y_N^2 \geq \upsilon\}} \mid \mathcal{F}_N\right] \xrightarrow{\mathbb{P}} 0. \tag{A.3}$$

In addition, note that (ii) implies, for all $\varepsilon' > 0$ and all $\delta > 0$, the existence of a constant $\lambda_\delta < \infty$ such that for all $\lambda \geq \lambda_\delta$,

$$\sup_{N \in \mathbb{N}} \mathbb{P}\left(\mathbb{1}_{\{X_N \geq \lambda\}} \sum_{i=1}^{M_N} \mathbb{E}\left[|U_N(i)| \mid \mathcal{F}_N\right] \geq \varepsilon'\right) \leq \sup_{N \in \mathbb{N}} \mathbb{P}\left(X_N \geq \lambda\right) \leq \delta. \tag{A.4}$$

Now, for any $\varepsilon > 0$ and $\lambda > 0$ the quantity of interest may be bounded as

$$\begin{aligned}
\sum_{i=1}^{M_N} \mathbb{E}\left[|U_N(i)| \mathbb{1}_{\{|U_N(i)| \geq \varepsilon\}} \mid \mathcal{F}_N\right] &\leq M_N V_N \mathbb{P}\left(Y_N \geq \sqrt{\frac{\varepsilon}{2\lambda}} \mid \mathcal{F}_N\right) \\
&\quad + \left(\mathbb{1}_{\{X_N \geq \lambda\}} + \mathbb{1}_{\{V_N \geq \varepsilon/2\}}\right) \sum_{i=1}^{M_N} \mathbb{E}\left[|U_N(i)| \mid \mathcal{F}_N\right] \\
&\quad + \lambda M_N \mathbb{E}\left[Y_N^2 \mathbb{1}_{\{Y_N^2 \geq \varepsilon/(2\lambda)\}} \mid \mathcal{F}_N\right],
\end{aligned}$$

where the upper bound may, by (iii), (A.3), and (A.4), be made arbitrarily small in probability by increasing first λ and then N. This completes the proof in the case $p = 1$.

We turn to the case $p = 2$. However, by letting $\tilde{U}_N(i) \triangleq U_N^2(i)$, $i \in [\![1, M_N]\!]$, $N \in \mathbb{N}^*$, and noting that $\max_{i \in [\![1, M_N]\!]} \tilde{U}_N(i) \leq \tilde{V}_N + \tilde{X}_N \tilde{Y}_N^2$, where $\tilde{V}_N \triangleq 2V_N^2$, $\tilde{X}_N \triangleq 2X_N^2$, and $\tilde{Y}_N \triangleq Y_N^2$, we thus realise that the proof of the case $p = 1$ goes through if we can verify that (A.3) and (A.4) hold true when X_N, Y_N, and $\{U_N(i)\}_{i=1}^{M_N}$ are replaced by $\tilde{X}_N$, $\tilde{Y}_N$, and $\{\tilde{U}_N(i)\}_{i=1}^{M_N}$, respec-

Downloaded from informs.org by [216.73.217.172] on 15 November 2026, at 06:26 . For personal use only, all rights reserved.

tively. Nevertheless, (A.4) holds straightforwardly as tightness of $\{X_N\}_{N\in\mathbb{N}}$ implies tightness of $\{\tilde{X}_N\}_{N\in\mathbb{N}}$. Moreover, using condition (iii) one shows, along previous lines, that

$$\begin{aligned}
&\mathbb{E}\left[\tilde{Y}_N^2 \mathbb{1}_{\{\tilde{Y}_N^2 \geq \upsilon\}} \mid \mathcal{F}_N\right] \\
&\quad = \int_{\upsilon}^{\infty} \mathbb{P}\left(Y_N \geq \sqrt[4]{y} \mid \mathcal{F}_N\right) \mathrm{d}y + \upsilon \mathbb{P}\left(Y_N \geq \sqrt[4]{\upsilon} \mid \mathcal{F}_N\right) \\
&\quad \leq C M_N \int_{\upsilon}^{\infty} \exp\left(-c M_N^{\nu} y^{\alpha/2}\right) \mathrm{d}y + \upsilon C M_N \exp\left(-c M_N^{\nu} \upsilon^{\alpha/2}\right).
\end{aligned}$$

For $\alpha = 1$,

$$\begin{aligned}
&\mathbb{E}\left[\tilde{Y}_N^2 \mathbb{1}_{\{\tilde{Y}_N^2 \geq \upsilon\}} \mid \mathcal{F}_N\right] \\
&\quad \leq \left(2\sqrt{\upsilon} M_N^{-\nu}/c + 2M_N^{-2\nu}/c^2 + \upsilon\right) C M_N \exp\left(-c M_N^{\nu} \sqrt{\upsilon}\right).
\end{aligned}$$

while the case $\alpha = 2$ corresponds to the first case of (A.2). Consequently, as $N \to \infty$,

$$M_N \mathbb{E}\left[\tilde{Y}_N^2 \mathbb{1}_{\{\tilde{Y}_N^2 \geq \upsilon\}} \mid \mathcal{F}_N\right] \xrightarrow{\mathbb{P}} 0,$$

which completes the proof. □

LEMMA A.4. *Let $a \in \mathbb{R}$ be nonzero, $a \neq 0$, and let $\{X_N(i)\}_{i=1}^{N_1}$, $N_1 \in \mathbb{N}^*$, be random variables such that $X_N(i) \neq 0$ for all $i \in [\![1, N_1]\!]$. Assume that $\max_{i\in[\![1,N_1]\!]} |X_N(i) - a| \xrightarrow{\mathbb{P}} 0$ as $N \to \infty$. Then*

$$\max_{i\in[\![1,N_1]\!]} \left|X_N^{-1}(i) - a^{-1}\right| \xrightarrow{\mathbb{P}} 0.$$

PROOF. Pick $\varepsilon > 0$; then we may write for all $\eta > 0$,

$$\begin{aligned}
\mathbb{P}\left(\max_{i\in[\![1,N_1]\!]} \left|X_N^{-1}(i) - a^{-1}\right| \geq \varepsilon\right) &\leq \mathbb{P}\left(\max_{i\in[\![1,N_1]\!]} |X_N(i) - a| \geq \eta\right) \\
&\quad + \mathbb{P}\left(\max_{i\in[\![1,N_1]\!]} \left|X_N^{-1}(i) - a^{-1}\right| \geq \varepsilon, \max_{i\in[\![1,N_1]\!]} |X_N(i) - a| < \eta\right),
\end{aligned}$$

where the first term tends to zero as N tends to infinity for any η by assumption. For all $i \in [\![1, N_1]\!]$, there exists, by Taylor's formula, $\zeta_N(i) \in (X_N(i) \wedge a, X_N(i) \vee a)$ such that $|X_N^{-1}(i) - a^{-1}| = \zeta_N^{-2}(i)|X_N(i) - a|$. Thus, if $a > 0$ and $0 < \eta < a$,

$$\mathbb{P}\left(\max_{i\in[\![1,N_1]\!]} \left|X_N^{-1}(i) - a^{-1}\right| \geq \varepsilon, \max_{i\in[\![1,N_1]\!]} |X_N(i) - a| < \eta\right)$$

$$\leq \mathbb{P}\left(\max_{i\in[\![1,N_1]\!]} |X_N(i)-a| > \varepsilon\{a-\eta\}^2\right),$$

where the right hand side tends, by assumption, to zero as N tends to infinity. On the other hand, if $a<0$ and $0<\eta<-a$,

$$\mathbb{P}\left(\max_{i\in[\![1,N_1]\!]} \left|X_N^{-1}(i)-a^{-1}\right| \geq \varepsilon, \max_{i\in[\![1,N_1]\!]} |X_N(i)-a| < \eta\right)$$
$$\leq \mathbb{P}\left(\max_{i\in[\![1,N_1]\!]} |X_N(i)-a| > \varepsilon\{a+\eta\}^2\right),$$

where again the right hand side tends to zero. This concludes the proof. □

LEMMA A.5. *Let $\{Z_N\}_{N\in\mathbb{N}}$ be a sequence of random variables such that for some constant $z_\infty \in \mathbb{R}$, as $N\to\infty$,*

$$\mathbb{E}[Z_N] \to z_\infty$$

and is uniformly bounded by some constant $z^+ \in \mathbb{R}$. Let $\{X_N\}$ be a sequence of random variables that (i) converges in probability to some constant $x_\infty \in \mathbb{R}$ and (ii) is dominated by some integrable random variable. Then, as $N\to\infty$,

$$\mathbb{E}[X_N Z_N] \to x_\infty z_\infty.$$

PROOF. The result is obtained straightforwardly by writing

$$|\mathbb{E}[X_N Z_N] - x_\infty z_\infty| \leq z^+ \mathbb{E}[|X_N - x_\infty|] + |x_\infty|\,|\mathbb{E}[Z_N] - z_\infty|,$$

where the right hand side tends to zero as N tends to infinity by assumption and dominated convergence. □

Acknowledgment. The authors thank the editor and the anonymous referee for insightful comments that improved the presentation of the paper.

REFERENCES

[1] B. Balasingam, M. Bolić, P. M. Djurić, and J. Míguez. Efficient distributed resampling for particle filters. In *2011 IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP)*, pages 3772–3775, May 2011.

[2] M. Bolic, P. M. Djuric, and Sangjin Hong. Resampling algorithms and architectures for distributed particle filters. *IEEE Transactions on Signal Processing*, 53(7):2442–2450, July 2005.

[3] O. Cappé and E. Moulines. On the use of particle filtering for maximum likelihood parameter estimation. In *European Signal Processing Conference (EUSIPCO)*, Antalya, Turkey, September 2005.

Downloaded from informs.org by [216.73.217.172] on 15 November 2026, at 06:26 . For personal use only, all rights reserved.

[4] O. Cappé, E. Moulines, and T. Rydén. *Inference in Hidden Markov Models*. Springer, 2005. MR2159833
[5] F. Cérou, P. Del Moral, T. Furon, and A. Guyader. Sequential Monte Carlo for rare event estimation. *Stat. Comput.*, 22(3):795–808, 2012.
[6] N. Chopin. A sequential particle filter method for static models. *Biometrika*, 89:539–552, 2002.
[7] N. Chopin. Central limit theorem for sequential Monte Carlo methods and its application to Bayesian inference. *Ann. Statist.*, 32(6):2385–2411, 2004. MR2153989
[8] N. Chopin, P. Jacob, and O. Papaspiliopoulos. SMC2: A sequential Monte Carlo algorithm with particle Markov chain Monte Carlo updates. *J. Roy. Statist. Soc. B*, 75(3):397–426, 2013.
[9] D. Crisan and B. L. Rozovskii, editors. *The Oxford handbook of nonlinear filtering*. Oxford N.Y. Oxford University Press, 2011.
[10] P. Del Moral. *Feynman-Kac Formulae. Genealogical and Interacting Particle Systems with Applications*. Springer, 2004. MR2044973
[11] P. Del Moral and J. Garnier. Genealogical particle analysis of rare events. *Ann. Appl. Probab.*, 15(4):2496–2534, 2005. MR2187302
[12] P. Del Moral and A. Guionnet. Central limit theorem for nonlinear filtering and interacting particle systems. *Ann. Appl. Probab.*, 9(2):275–297, 1999.
[13] P. Del Moral and A. Guionnet. On the stability of interacting processes with applications to filtering and genetic algorithms. *Annales de l'Institut Henri Poincaré*, 37:155–194, 2001.
[14] R. Douc, A. Garivier, E. Moulines, and J. Olsson. Sequential Monte Carlo smoothing for general state space hidden Markov models. *Ann. Appl. Probab.*, 21(6):2109–2145, 2011. MR2895411
[15] R. Douc and E. Moulines. Limit theorems for weighted samples with applications to sequential Monte Carlo methods. *Ann. Statist.*, 36(5):2344–2376, 2008.
[16] R. Douc, É. Moulines, and J. Olsson. Optimality of the auxiliary particle filter. *Probab. Math. Statist.*, 29(1):1–28, 2009.
[17] R. Douc, E. Moulines, and J. Olsson. Long-term stability of sequential Monte Carlo methods under verifiable conditions. *Ann. Appl. Probab.*, 24(5):1767–1802, 2014. MR3226163
[18] R. Douc, E. Moulines, and D. Stoffer. *Nonlinear Time Series: Theory, Methods and Applications with R Examples*. Chapman & Hall/CRC Texts in Statistical Science, 2014.
[19] A. Doucet, N. De Freitas, and N. Gordon, editors. *Sequential Monte Carlo Methods in Practice*. Springer, New York, 2001.
[20] K. Heine and N. Whiteley. Fluctuations, stability and instability of a distributed particle filter with local exchange. *ArXiv e-prints*, May 2015.
[21] A. Kong, J. S. Liu, and W. Wong. Sequential imputation and Bayesian missing data problems. *J. Am. Statist. Assoc.*, 89(278-288):590–599, 1994.
[22] H. R. Künsch. Recursive Monte-Carlo filters: algorithms and theoretical analysis. *Ann. Statist.*, 33(5):1983–2021, 2005. MR2211077
[23] J. S. Liu. Metropolized independent sampling with comparisons to rejection sampling and importance sampling. *Stat. Comput.*, 6:113–119, 1996.
[24] J.S. Liu. *Monte Carlo Strategies in Scientific Computing*. Springer, New York, 2001.
[25] J. Olsson, O. Cappé, R. Douc, and E. Moulines. Sequential Monte Carlo smoothing with application to parameter estimation in non-linear state space models. *Bernoulli*, 14(1):155–179, 2008. arXiv:math.ST/0609514.

Downloaded from informs.org by [216.73.217.172] on 15 November 2026, at 06:26 . For personal use only, all rights reserved.

[26] M. K. Pitt and N. Shephard. Filtering via simulation: Auxiliary particle filters. *J. Am. Statist. Assoc.*, 94(446):590–599, 1999.
[27] B. Ristic, M. Arulampalam, and A. Gordon. *Beyond Kalman Filters: Particle Filters for Target Tracking*. Artech House, 2004.
[28] O. Rosen and A. Medvedev. Efficient parallel implementation of state estimation algorithms on multicore platforms. *IEEE Transactions on Control Systems Technology*, 21(1):107–120, Jan 2013.
[29] A. C. Sankaranarayanan, A. Srivastava, and R. Chellappa. Algorithmic and architectural optimizations for computationally efficient particle filtering. *IEEE Transactions on Image Processing*, 17(5):737–748, May 2008. MR2516598
[30] S. Sutharsan, T. Kirubarajan, T. Lang, and M. Mcdonald. An optimization-based parallel particle filter for multitarget tracking. *IEEE Transactions on Aerospace and Electronic Systems*, 48(2):1601–1618, April 2012.
[31] C. Vergé, C. Dubarry, P. Del Moral, and E. Moulines. On parallel implementation of sequential Monte Carlo methods: the island particle model. *Statistics and Computing*, 23, 2013.

Pierre Del Moral
School of Mathematics and Statistics
High Street, Kensington
NSW 2052 Australia
E-mail: p.del-moral@unsw.edu.au

Eric Moulines
Ecole Polytechnique,
Centre de Mathématiques Appliquée
INRIA XPOP
Route de Saclay
91128 Palaiseau, Cedex
France
E-mail: eric.moulines@polytechnique.edu

Jimmy Olsson
Department of Mathematics
KTH
SE-100 44 Stockholm
Sweden
E-mail: jimmyol@kth.se

Christelle Vergé
ONERA – The French Aerospace Lab
91761 Palaiseau
France
E-mail: christelle.verge@onera.fr

Downloaded from informs.org by [216.73.217.172] on 15 November 2026, at 06:26 . For personal use only, all rights reserved.