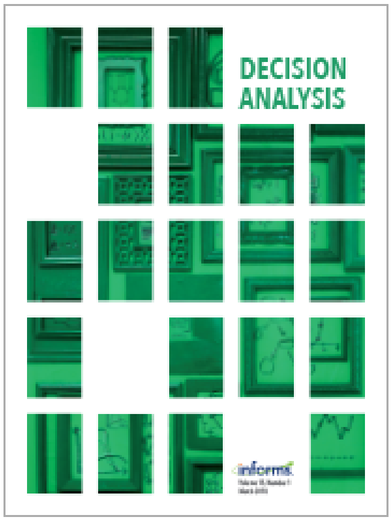

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On the Value of the Tail Event Information

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Abstract. This paper analyzes the buying price of the tail event information that is generated by the outcome of a simple event that identifies whether the value of a random prospect exceeds a critical threshold value. Our discussion begins with the analysis of perfect tail event information. We determine how the maximum amount a risk-averse decision maker is willing to pay changes as a function of this threshold value, and we discuss whether quick financial comparisons can be made between these information alternatives. We also provide results on the value of information calculus to measure how the buying price behaves as we increase our information content through acquisition of two or more tail event information alternatives. Our theoretical results indicate the correlation between the buying prices of simpler tail event information alternatives and more complex versions of event information. The focus of the article then shifts to modeling of an additional risk that may be encountered in information acquisition. In particular, we analyze imperfect tail event information and discuss how the accuracy of the information source affects the corresponding buying price. Finally, we demonstrate our findings through examples and provide insights into our results.

Keywords: value of information • multiaction decision problems • event information • perfect information • imperfect information

1. Introduction

In an environment in which decisions are made under uncertainty, information has a serious potential to improve your decisions. It might make a significant difference financially when the decision maker collects more information before committing to an action. From an economic perspective, perfect information that provides the full resolution of uncertainty is clearly the most valuable information alternative though it may not be available. Therefore, the decision maker may search for other forms of information that offer only a partial resolution of uncertainty yet hold value comparable to the potential financial benefits of making an informed decision.

In this paper, we consider a simple static investment problem in which a decision maker faces a choice among multiple investment alternatives. If the decision maker chooses a risky alternative, the decision maker makes the commitment and faces the eventual outcome of the investment. The do-nothing alternative is also present, rendering no change in the decision maker's wealth. Suppose the decision maker is interested in information motivated by a simple question: does the financial outcome of this investment alternative exceed some critical threshold amount? The answer is important if some minimum attractive return on an investment is desired. For example, in the oil industry, the concept of minimum economic field size, which is introduced in Rose (2001) is defined as the minimum amount of extractable oil and gas necessary to offset the exploratory costs and establish the initial investment as economically viable. As we demonstrate in this article, information collected to resolve this question imposes a unique mathematical structure that can be used as a building block to evaluate more complex information alternatives in finite decision problems under uncertainty.

The type of information alternative that we analyze is quantified as a simple collection of two complementary events and is called tail event information. It is a special type of event information first introduced and discussed in Bakır and Klutke (2014). After paying to acquire tail event information, the decision maker finds out whether the outcome of the investment alternative is greater than some predetermined threshold. Tail event information does not resolve completely the outcome of the underlying investment alternative. In this sense, tail event information is perfect partial information. We show that it is possible to construct more complex information alternatives using tail event information offering the same expected reward as the perfect information in finite decision problems. The idea may be traced back to the seminal work by Merkhofer (1977), which mathematically articulates how the value of information relates to the flexibility to act on it maximizing financial returns. Our contribution to this line of argument is to introduce a relatively coarse information structure that is generated by one or more simple questions (or events). This structure may offer an equivalent value to finer information alternatives in decision environments with a finite number of actions.

Before we delve into the technical details, a brief literature review on the value of information is presented in the next section.

2. Literature Review

The value of information has been widely explored in decision problems involving multiple courses of action. Portfolio decision analysis literature presents prime examples of such work, and we present only a few here. Most studies consider perfect information on one or more risky investment prospects or on some financial signal that is correlated with these prospects. In a preliminary study, Mehrez and Stulman (1984) formulate the project selection problem with an objective of maximizing the expected reward subject to an information-gathering budget constraint. This study concludes that selecting projects with the most rewarding perfect information content is equivalent to maximizing the portfolio's expected value. In a follow-up study, Mehrez and Sethi (1989) extend this mathematical formulation to cover the case of perfect information on a financial signal that is probabilistically relevant to the projects' outcomes. Keisler (2004) discusses several randomized strategies in the portfolio selection problem under various levels of information (perfect and imperfect) collection on project returns and investigates when information adds value to the process under risk neutrality. The risk-neutrality assumption is retained in Zan and Bickel (2013), who extend Keisler (2004) to identify how information gathered through signals brings value in relation to prioritization rules used in risky project selection. Models with dependency between alternatives have been studied extensively as well (see, for example, Bhattacharjya et al. 2013, Evangelou and Eidsvik 2017). Value of information methods have been widely applied also in healthcare decision making (for instance, Claxton 1999, Eckermann and Willan 2007), oil exploration (Bickel et al. 2006, Morosov and Bratvold 2022), and earth sciences (Bates et al. 2014, Eidsvik et al. 2015, among others).

The underlying motivation for this article is markedly different from prior work. Given the most basic assumptions of the classic multiaction decision problem under uncertainty, we consider a collection of information alternatives with a notably coarse structure, constructed based on revelation of simple events that are relevant to the basic rules of thumb that may be developed in a risky decision. We begin with the assumption that information is perfect (though partial) and analyze different tail information alternatives comparatively. We then proceed to identify the economic value relation between the elementary tail event information and more complex information generated by disjoint events. Research that investigates the relationship between structurally linked information alternatives is limited with a greater focus on analyzing perfect information acquired on different sources of uncertainty. For instance, Merkhofer (1977) presents a price–quantity problem in which the expected value of perfect information on two sources of uncertainty may be less than the sum of values of information acquired individually if there is no independence between these sources. Samson et al. (1989) study the conditions in which perfect information value is additive for risk-neutral decision makers when information is acquired on discrete random variables. Keisler (2005) extends this work to include decisions on multiple sources of uncertainty with normally distributed payoffs and derive conditions under which information value is superadditive. Results in these studies and Zan (2013) indicate that dependence between multiple sources of uncertainty has significant impact on additivity of information value.

Our study offers a distinct perspective on the relationship between structurally similar information alternatives, diverging in several important ways from the outlined past work. We focus on risk-averse decision makers and acquisition of perfect partial information. Additionally, our analysis examines different forms of information about one source of uncertainty rather than on a single form of information about multiple sources of uncertainty. No distributional assumptions are made. Finally, our methodology employs the buying price approach rather than the expected utility increase to evaluate information because it assigns a financial value, which is easier to interpret in practical applications.

A vital contribution of our article is the exploration of the degree to which imperfection leads to a loss in the value of information. Imperfect information is widely studied (as sample information). However, theoretical or empirical comparisons against perfect information are rare. Among a few studies that explore this relation are Bickel (2008) in a two-action decision problem, Canessa et al. (2015) in the context of evolutionary ecology, and Koski et al. (2020) in the context of environmental management. In practice, realizations of perfect information are rare; hence, we also analyze how sensitive information value is to deviations from perfect information.

The rest of this paper is organized as follows. Section 3 provides the notation and definitions. In Section 4, we examine the relationship between the value of tail event information and the critical threshold that generates it. The technical significance of this study is highlighted in Section 5, in which our main results are stated. Tail event information is not only of interest for its practical validity, but also for its basic structure. An analytical argument is made to illustrate this and to show the relation between the buying price of structurally more complex information and tail event information. In Section 6, we discuss the case in which information is not necessarily perfect. We state results that link the degree of imperfection in tail event information to its value. Finally, Section 7 provides an overview of the implications of all the results and concludes. The proofs of the propositions stated throughout the paper can be found in the appendix.

3. Formulation of the Decision Problem

The decision maker is an expected utility maximizer ranking alternatives according to a continuous and monotonically increasing utility function $u : \mathbb{R} \to \mathbb{R}$ that has wealth as its only argument. The initial wealth level is w. A one-time decision should be made to choose one among n independent investment alternatives, each of which can be conveniently modeled as a lottery in the collection $\mathbb{L} = \{\Pi_i : \Omega \to \mathbb{R}, i = 1, \ldots, n\}$. If no alternative is financially attractive, the decision maker may choose to do nothing; in this case, the terminal utility remains $u(w)$. Otherwise, a risky alternative $\Pi_i \in \mathbb{L}$ may be chosen, which, as a result, brings the terminal utility to $u(w + \Pi_i)$. The probability assessment on each Π_i is represented with a cumulative distribution function F_i and a density function f_i over the financial outcomes.

Suppose the decision maker acquires tail event information $\mathcal{T}_i(c)$ on one of the investment alternatives Π_i in $\mathbb{L}$ generated by the event $\{\Pi_i > c\}$. We call c the critical threshold that generates tail event information. After learning the outcome of the tail event information (i.e., whether $\Pi_i > c$ or $\Pi_i \leq c$), the decision maker reduces uncertainty regarding Π_i, but the precise value of Π_i remains unknown. We assume that all investment alternatives are probabilistically independent; hence, there is no updating of the prior probability assessments of the other alternatives. However, using the Bayesian rules, the distribution of Π_i is revised given the outcome of $\mathcal{T}_i(c)$. Finally, the decision is made to maximize expected utility.

Multiple approaches exist for evaluating information: expected utility increase, selling price, probability price, certainty equivalent, and buying price. They all account for the risky nature of information acquisition and the risk preferences of the decision maker; however, they do not agree in comparative evaluation of information alternatives (see Hazen and Sounderpandian 1999, Bakır and Klutke 2011 for preference reversals). Among these, the buying and selling prices are expressed in monetary terms. Notably, the buying price is based on an expected utility formulation that assumes an a priori payment for information. It considers changes in risk preferences after an a priori payment for information is made, which is not captured by the expected utility increase approach. In addition, monetary evaluation of information under risk aversion provides a more objective measurement of relative preferences and, therefore, is more suitable for applications. This is the primary basis for the use of buying price to evaluate information.

The buying price $B_i(w, c, u)$ of $\mathcal{T}_i(c)$ for a decision maker with utility function u and wealth w is the maximum amount that the decision maker is willing to pay to acquire information. The notation can be simplified: we use a shorthand form $b_i(c) = B_i(w, c, u)$ throughout the paper unless a more explicit notation is needed. The buying price b_i is calculated using the following equation (adopted from La Valle 1968):

$$\begin{aligned}
&\max\left\{\max_{k=1,\ldots,n} \mathbb{E}[u(w+\Pi_k)], u(w)\right\} \\
&= P\{\Pi_i > c\}\cdot \max\left\{\max_{\substack{j=1,\ldots,n,\\ j\neq i}} \mathbb{E}[u(w+\Pi_j - b_i(c))],\right. \\
&\qquad\qquad \left. u(w-b_i(c)), \mathbb{E}[u(w+\Pi_i - b_i(c))\,|\,\Pi_i > c]\right\} \\
&\quad + P\{\Pi_i \le c\}\cdot \max\left\{\max_{\substack{j=1,\ldots,n,\\ j\neq i}} \mathbb{E}[u(w+\Pi_j - b_i(c))],\right. \\
&\qquad\qquad \left. u(w-b_i(c)), \mathbb{E}[u(w+\Pi_i - b_i(c))\,|\,\Pi_i \le c]\right\}.
\end{aligned} \tag{1}$$

In Equation (1), the buying price $b_i(c)$ is determined to make the expected utility of the optimal decision without information at the wealth level w and the expected utility of the decision with information at the reduced wealth level $w - b_i(c)$ equal. The right side of the equation is where the benefit of information acquisition is evaluated; the main benefit is the flexibility to change the uninformed decision given one of the two events generating tail event information. Because all investment alternatives are assumed independent, there are two possible ways a decision might change. If the information is acquired on the uninformed optimal, then the uninformed second best may become the best alternative if the decision maker learns that uninformed optimal returns do not exceed c. In case the information is on some other alternative, then there could be a change of decision from the uninformed optimal to the investment alternative upon which information is acquired.

Utilizing this formulation of the decision problem at hand, we analyze in the next section how different tail event information alternatives compare as a function of the critical threshold amount c.

4. Comparative Analysis of Tail Event Information

It is well-established in the value of information literature that information is valuable when it offers a potential decision change. In this regard, the initial uninformed decision has a significant influence on how much the decision maker pays for information. Studies that demonstrate this fact are many. For example, in a two-action, accept–reject decision environment, Mehrez (1985) shows that a risk-neutral decision maker is willing to pay more for perfect information if the initial decision is to reject the lottery. Again for a risk-neutral decision maker, Eeckhoudt and Godfroid (2000) present an illustrative example, a variant of the classic newsboy problem, which shows that the amount a decision maker is willing to pay for information depends on the initial decision. In Delquié (2008), the reject decision outcome is also uncertain, and it is shown for certain information alternatives that their value is maximized when the decision maker is initially indifferent between two lotteries. Abbas et al. (2013) investigate the relationship between the decision maker's risk preferences and the value of information and concludes that presence of a monotonic relationship depends on the uninformed decision. Other studies that highlight the significance of the initial decision include Fatti et al. (1987) and Bickel (2008).

Against this background, it should come as no surprise that the sensitivity of the value of tail information depends on the initial decision. Recall that we are in a multiaction decision environment as described in Section 3. Tail information is acquired on only one investment alternative $\Pi_i \in \mathbb{L}$. Define the functional $\mathbb{B}_{-i}(b) = \max\left\{\max_{s=1,\ldots,n,\, s\neq i} \mathbb{E}[u(w+\Pi_s - b)], u(w-b)\right\}$. $\mathbb{B}_{-i}(b)$ is the expected utility of the best alternative other than Π_i at the wealth level $w - b$. If $\mathcal{T}_i(c)$ is valuable, Equation (1) reduces to

$$\begin{aligned}
&\max\left\{\max_{k=1,\ldots,n} \mathbb{E}[u(w+\Pi_k)], u(w)\right\} \\
&= P\{\Pi_i > c\}\cdot \mathbb{E}[u(w+\Pi_i - b_i(c))\,|\,\Pi_i > c] \\
&\quad + P\{\Pi_i \le c\}\cdot \mathbb{B}_{-i}(b_i(c)).
\end{aligned} \tag{2}$$

Equation (2) holds because, irrespective of the initial decision, $\mathcal{T}_i(c)$ is valuable only when investment Π_i is preferred upon the occurrence of $\{\Pi_i > c\}$ and is rejected otherwise. Any other scenario implies $b_i(c) = 0$, which is not of much interest. Using Equation (2), we first compare the value of tail event information generated by different values of c.

4.1. Comparing the Value of Tail Event Information Alternatives

In this multiaction investment decision problem, consider tail event information $\mathcal{T}_i(c_1)$, $\mathcal{T}_i(c_2), \ldots, \mathcal{T}_i(c_m)$, where $c_1 < c_2 < \cdots < c_m$ and, without loss of generality, $c_1 < 0$, $c_m > 0$. We define $c_k = \max_{i=1,\ldots,m}\{c_i : c_i < 0\}$; then, it naturally follows that $c_{k+1} = \min_{i=1,\ldots,m}\{c_i : c_i > 0\}$. The first result, which we state below, is an extension of proposition 3 in Bakır and Klutke (2014).

Corollary 1. *Suppose a decision is made among n independent risky investment alternatives and the do-nothing alternative by a decision maker with initial wealth w, and tail event information is collected on investment alternative Π_i. If the initial decision is to do nothing, and $c_a < c_b < 0$ or $c_a > c_b > 0$, then $b_i(c_a) \le b_i(c_b)$.*

Proof. When the initial decision is to do nothing, our problem resembles the information acquisition scenario discussed in Bakır and Klutke (2014) because of the independence between alternatives. We prove the first case here and skip the arguments for the other case on grounds of similarity. If $c_a < c_b < 0$, then $\{\Pi_i \le c_a\} \subset \{\Pi_i \le c_b\}$, and both $\{\Pi_i \le c_a\}$ and $\{\Pi_i \le c_b\}$ include outcomes of the same sign (negative). We apply proposition 3 of Bakır and Klutke (2014) and arrive at the conclusion. □

The immediate implication of Corollary 1 is an ordering between $b_i(c_1), \ldots, b_i(c_k)$ and also between $b_i(c_{k+1}), \ldots, b_i(c_m)$: $b_i(c_1) \le b_i(c_2) \le \cdots \le b_i(c_k)$, and $b_i(c_{k+1}) \ge b_i(c_{k+2}) \ge \cdots \ge b_i(c_m)$ when the initial decision is to do nothing. However, Corollary 1 does not clarify how we order buying prices of two tail event information alternatives generated by critical thresholds having different signs (i.e., $b_i(c_k)$ and $b_i(c_{k+1})$). These comparisons are not straightforward, though, and depend on the risk attitude, uninformed initial decision, and conditional decision about the investment alternative for which we collect information. If do nothing is the uninformed decision, then a workable result can be derived. In the opposite case with a risky investment choice, a practical result with readily verifiable conditions is elusive.

Proposition 1. *Suppose a risk-averse decision maker is presented with a decision among n independent risky investment alternatives and the do-nothing alternative at initial wealth w. Furthermore, tail event information on investment alternative Π_i is collected. If the uninformed decision is do nothing, then for $c_a < 0$ and $c_b > 0$, the following results hold:*

a. *For a nonincreasingly risk-averse utility function, if do nothing is preferred over Π_i given $\Pi_i \in (c_a, c_b]$, then $b_i(c_b) \ge b_i(c_a)$.*

b. *For a nondecreasingly risk-averse utility function, if Π_i given $\Pi_i \in (c_a, c_b]$ is preferred over do nothing, then $b_i(c_b) \le b_i(c_a)$.*

Proof. We begin with the proof of (a). The condition in (a) states do nothing $\ge \Pi_i 1_{(c_a,c_b]}$, where $1_{(c_a,c_b]}$ is the indicator function. This implies do nothing $\ge \Pi_i 1_{(-\infty,c_a]}$, and do nothing $\ge \Pi_i 1_{(-\infty,c_b]}$. Since $c_b > 0$, $\Pi_i 1_{(c_b,\infty)} \ge$ do-nothing. These observations indicate that $b_i(c_b) > 0$. There is no indication how do nothing and $\Pi_i 1_{(c_a,\infty)}$ compare, but if c_a is small enough a number so that do nothing $\ge \Pi_i 1_{(c_a,\infty)}$, then $b_i(c_a) = 0$, and the result follows automatically. Therefore, we handle the remaining and interesting case in which do nothing $\preceq \Pi_i 1_{(c_a,\infty)}$, which yields the below equality:

$$u(w - b_i(c_a)) \cdot P\{\Pi_i \le c_a\} + \int_{c_a}^{\infty} u(w + x - b_i(c_a)) \cdot f_i(x)dx$$
$$= u(w - b_i(c_b)) \cdot P\{\Pi_i \le c_b\} + \int_{c_b}^{\infty} u(w + x - b_i(c_b)) \cdot f_i(x)dx.$$

If $b_i(c_a) > b_i(c_b)$,

$$u(w - b_i(c_b)) \cdot P\{\Pi_i \le c_a\} + \int_{c_a}^{\infty} u(w + x - b_i(c_b)) \cdot f_i(x)dx$$
$$\ge u(w - b_i(c_b)) \cdot P\{\Pi_i \le c_b\} + \int_{c_b}^{\infty} u(w + x - b_i(c_b)) \cdot f_i(x)dx.$$

After a little arrangement, we obtain

$$\int_{c_a}^{c_b} u(w + x - b_i(c_b)) \cdot f_i(x)dx \ge u(w - b_i(c_b)) \cdot P\{c_a < \Pi_i \le c_b\}. \qquad (3)$$

Note that (3) is a comparison between a risky alternative and a certain alternative at a lower wealth level than w. The condition in (a) is such that do nothing is preferred over the risky alternative at wealth level w. This contradicts the nonincreasingly risk-averse nature of the utility function, and therefore, (a) follows. The proof of (b) is similar. This time, we know do nothing $\preceq \Pi_i 1_{(c_a,c_b]}$, which brings the implication: do nothing $\preceq \Pi_i 1_{(c_a,\infty]}$. Then, assuming $b_i(c_b) > b_i(c_a)$, and going through similar steps as in the proof of (a), we obtain

$$\int_{c_a}^{c_b} u(w + x - b_i(c_b)) \cdot f_i(x)dx \le u(w - b_i(c_b)) \cdot P\{c_a < \Pi_i \le c_b\}. \quad (4)$$

Inequality (4) is a contradiction for a nondecreasingly risk-averse u because the decision maker should retain preference for a risky alternative at a lower wealth. This concludes the proof of (b). □

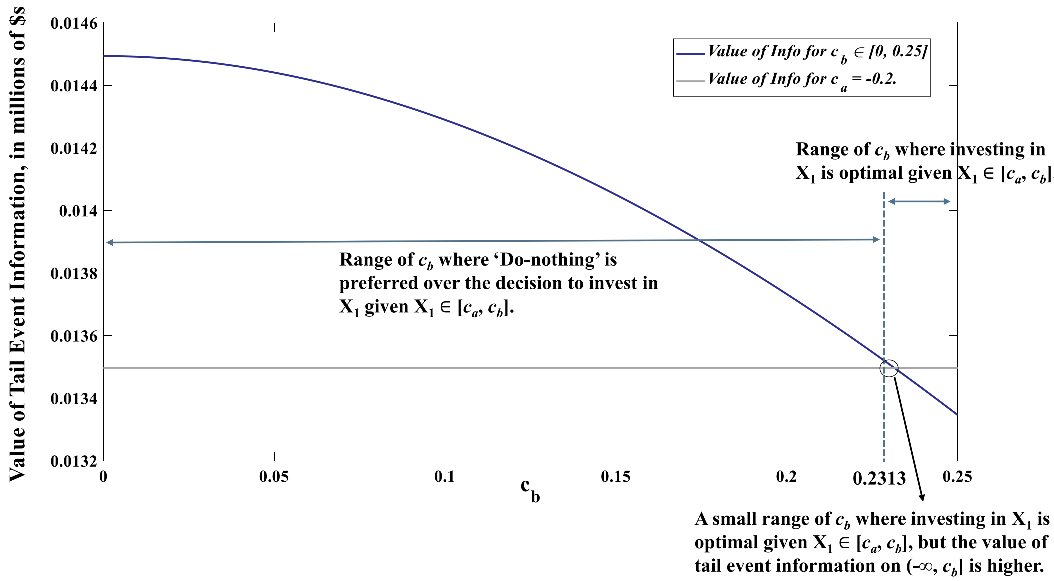
The results in Proposition 1 are not extended to cover cases in which the direction of preference given $\Pi_i \in (c_a, c_b]$ is reversed for both types of utility functions. However, our numerical experiments indicate that, when conditional preferences on $(c_a, c_b]$ are reversed in parts (a) and (b) of Proposition 1, an intuitive relation is observed between $b_i(c_b)$ and $b_i(c_a)$ in most cases. For example, for a nonincreasingly risk-averse linear plus exponential utility function, $b_i(c_b)$ is less than $b_i(c_a)$ in most cases when Π_i is preferred over do nothing given $\Pi_i \in (c_a, c_b]$. Exceptions are when Π_i is only slightly better than do nothing on $(c_a, c_b]$ to the degree that we observe a decision switch (i.e., do nothing becomes more preferable than Π_i on $(c_a, c_b]$) even with a tiny reduction in the wealth level of the decision maker. Such theoretical violations for marginal decision settings are common in buying price value of information analysis. This is because evaluation of decisions made after paying the monetary amount for information is performed under a different risk preference than the evaluation before the payment is finalized.

Proposition 1 provides insights into information value when c_a and c_b have opposite signs. The condition that the do nothing alternative is the best alternative over $(c_a, c_b]$ implicitly states that, in absolute terms, c_a is large and c_b is small. Under the sign restrictions for c_a and c_b, we may also refer to $\Pi_i \in (-\infty, c_a]$ as a sure-loss event and $\Pi_i \in (c_b, \infty)$ as a sure-gain event. The result in part (a) of the proposition is a reflection of the relative risk-taking attitude of a nonincreasingly risk-averse decision maker at higher wealth levels. This is because when both the uninformed decision is to do nothing and the relative absolute magnitude of c_a is large compared with c_b, a nonincreasingly risk-averse decision maker pays more for tail event information generated by the sure-gain event possibly to seize the potential financial benefits of Π_i. In fact, a small value for c_b makes the associated tail event information more valuable to the extent that, when $c_b = 0$, information on Π_i becomes as valuable as perfect information. In part (b) of the proposition, it is the nondecreasingly risk-averse decision maker who is willing to pay for tail event information on the sure-loss event. For example, within a mean-variance utility framework, information generated by the sure-loss event is more valuable because of the potential for a decision reversal after collecting information. Below is a short example to illustrate Proposition 1.

Example 1. Consider a decision maker with a linear plus exponential utility function $u(x) = x - e^{-0.1x}$, where x is in millions of dollars. Accordingly, all monetary quantities are also in millions of dollars. For simplicity, we assume the initial wealth level of the decision maker is $w = 0$. The decision maker faces a decision between three risky prospects and the do-nothing alternative. Returns from these scaled beta distributed financial prospects are denoted by X_1, X_2, X_3, where $X_1 \sim Beta(1, 3, -4.5, 2)$, $X_2 \sim Beta(1.5, 4, -4.5, 2)$, $X_3 \sim Beta(2, 5, -4.5, 2)$. The decision maker initially selects the do nothing alternative. We compare the value of the tail event information generated by events $X_1 \in (-\infty, c_a = -0.2]$ and $X_1 \in (-\infty, c_b]$, $c_b > 0$. Because $c_a < 0$ and $c_b > 0$, the comparative evaluation of tail event information alternatives generated by $X_1 \in (-\infty, c_a]$ and $X_1 \in (-\infty, c_b]$ is consistent with the assumptions in Proposition 1. For a fixed value of $c_a = -0.2$ (i.e., c_a is −\$200 thousand), we observe $b_1(c_a) \simeq \$13,497.31$.

We also plot the value of tail event information generated by the events $X_1 \in (-\infty, c_b]$ for values of $c_b \in [0, 0.25]$ in Figure 1. Predictably, the value of information is decreasing in c_b. Tail event information for $X_1 \in (-\infty, 0]$ is equal to perfect information in value. As c_b diverges from zero, the associated tail event information becomes less similar to perfect information. For validation of Proposition 1, it is critical that we know the decision between the do-nothing alternative and X_1 given $X_1 \in (c_a, c_b]$. To keep things simple, c_a is fixed at −0.2, whereas we sensitize c_b in [0, 0.25]. The conditional decision on $X_1 \in (c_a, c_b]$ is to do nothing between $c_b = 0$ and $c_b \sim 0.2312$. The decision reversal to invest in X_1 occurs when $c_b \in (0.2312, 0.2313)$. Despite this change, the value of information generated by the event $X_1 \in (-\infty, c_b]$ remains above the value of information generated by $X_1 \in (-\infty, -0.2]$. Conversely, when $c_b > 0.2314$, the value of information generated by $X_1 \in (-\infty, -0.2]$ exceeds the value of information generated by $X_1 \in (-\infty, c_b]$. This is a demonstration of how results in Proposition 1 fail to hold in both directions. □

Figure 1. (Color online) Comparison of the Buying Price of Perfect Tail Event Information Generated by $X_1 \in (-\infty, c_b]$ for Values of $c_b \in [0, 0.25]$ Against the Information Generated by $X_1 \in (-\infty, -0.2]$ in Example 1



In the next section, we shift our attention to the structural relationship between the tail event information and other complex information alternatives and how this relation influences the buying price of information. In a generic setting under risk aversion and without distributional restrictions, we seek to identify the changes in the value of perfect partial information as the content of information on one risky prospect changes in a simple multiaction investment problem.

5. Buying Price for the Acquisition of Multiple Tail Event Information Alternatives

Our view of information agrees with the finance and economics literature, in which it is quantified by an algebra (or in many cases σ-algebra; see Allen 1983) generated by events. An algebra is a finite collection of events. This representation brings information calculus into the probability domain. Basic rules of probability suggest that more complex information alternatives can be generated by multiple tail events.

To elaborate on this argument, consider two tail event information alternatives, $\mathcal{T}_i(c_1)$ and $\mathcal{T}_i(c_2)$, acquired on the same investment Π_i. Without loss of generality, let us assume that $c_1 > c_2$. If the decision maker pays a total sum to acquire both $\mathcal{T}_i(c_1)$ and $\mathcal{T}_i(c_2)$, then the decision maker learns whether one of the three following disjoint events occur: $\{\Pi_i > c_1\}$, $\{c_2 < \Pi_i \leq c_1\}$, and $\{\Pi_i \leq c_2\}$. In other words, acquisition of $\mathcal{T}_i(c_1)$ and $\mathcal{T}_i(c_2)$ together is technically equivalent to acquisition of a more complex information alternative generated by the events $\{\Pi_i > c_1\}$, $\{c_2 < \Pi_i \leq c_1\}$, and $\{\Pi_i \leq c_2\}$. This is a finer partition of the outcome space than the one imposed by individual tail event information alternatives, so acquisition of $\mathcal{T}_i(c_1)$ and $\mathcal{T}_i(c_2)$ together increases information content in probabilistic terms as well. This finer partition can be represented by $\mathcal{T}_i(c_1) \cup \mathcal{T}_i(c_2)$.

Our discussion in this section begins with the behavior of the buying price based on the above structural relationship. The buying price is expected to increase with the informational content. However, as we argue next, the total amount that the decision maker should pay to acquire two tail event information alternatives is not necessarily greater than the individual buying prices of those tail event information alternatives.

5.1. Union of Two Tail Event Information Alternatives

Consider the acquisition of $\mathcal{T}_i(c_1)$ and $\mathcal{T}_i(c_2)$ with associated buying prices $b_{i,1}$, and $b_{i,2}$, respectively. We retain the assumption $c_1 > c_2$ without loss of generality. The equations to determine the buying prices of the tail event information alternatives $\mathcal{T}_i(c_1)$ and $\mathcal{T}_i(c_2)$ for the interesting case with $b_i(c_1), b_i(c_2) > 0$ are

$$\max\left\{\max_{k=1,\ldots,n} \mathbb{E}[u(w+\Pi_k)], u(w)\right\} = \int_{c_1}^{\infty} u(w+x-b_i(c_1))\cdot f_i(x)dx + P\{\Pi_i \le c_1\}\cdot \mathbb{B}_{-i}(b_i(c_1)),$$

$$\max\left\{\max_{k=1,\ldots,n} \mathbb{E}[u(w+\Pi_k)], u(w)\right\} = \int_{c_2}^{\infty} u(w+x-b_i(c_2))\cdot f_i(x)dx + P\{\Pi_i \le c_2\}\cdot \mathbb{B}_{-i}(b_i(c_2)).$$

We can rewrite the above equations as

$$\max\left\{\max_{k=1,\ldots,n} \mathbb{E}[u(w+\Pi_k)], u(w)\right\} = \int_{c_1}^{\infty} u(w+x-b_i(c_1))\cdot f_i(x)dx + \left(P\{c_2 < \Pi_i \le c_1\} + P\{\Pi_i \le c_2\}\right)\cdot \mathbb{B}_{-i}(b_i(c_1)),$$

$$\max\left\{\max_{k=1,\ldots,n} \mathbb{E}[u(w+\Pi_k)], u(w)\right\} = \int_{c_1}^{\infty} u(w+x-b_i(c_2))\cdot f_i(x)dx + \int_{c_2}^{c_1} u(w+x-b_i(c_2))\cdot f_i(x)dx + P\{\Pi_i \le c_2\}\cdot \mathbb{B}_{-i}(b_i(c_2)). \tag{5}$$

Equation (5) illustrates the decisions given $\{\Pi_i > c_1\}$, $\{c_2 < \Pi_i \le c_1\}$, and $\{\Pi_i \le c_2\}$. This representation is useful to make a comparison with the buying price of acquiring $\mathcal{T}_i(c_1)$ and $\mathcal{T}_i(c_2)$ together. In particular, if the decision maker acquires $\mathcal{T}_i(c_1) \cup \mathcal{T}_i(c_2)$, then the buying price $b_{i,1\cup 2}$ is found by solving

$$\max\left\{\max_{k=1,\ldots,n} \mathbb{E}[u(w+\Pi_k)],\, u(w)\right\} = \int_{c_1}^{\infty} u(w+x-b_{i,1\cup 2})\cdot f_i(x)dx + P\{\Pi_i \le c_2\}\cdot \mathbb{B}_{-i}(b_{i,1\cup 2}) + \max\left\{\int_{c_2}^{c_1} u(w+x-b_{i,1\cup 2})\cdot f_i(x)dx,\; P\{c_2 < \Pi_i \le c_1\}\cdot \mathbb{B}_{-i}(b_{i,1\cup 2})\right\}. \tag{6}$$

A careful analysis of (5) and (6) reveals that, depending on the decision maker's investment decision given $\{c_2 < \Pi_i \le c_1\}$, Equation (6) is either identical to the first equation in (5) or the second. This follows because the ranking of investment alternatives other than Π_i remains unchanged given any piece of information on Π_i by the independence of investment alternatives in $\mathbb{L}$. With this observation, we now state our first proposition of this section.

Proposition 2. *Suppose a decision is made among n independent risky investment alternatives and the do-nothing alternative by a risk-averse decision maker with utility function u and initial wealth w. Let $b_i(c_1)$ and $b_i(c_2)$ be the buying prices of the tail event information alternatives $\mathcal{T}_i(c_1)$ and $\mathcal{T}_i(c_2)$, respectively, and let $b_{i,1\cup 2}$ be the value of acquiring both $\mathcal{T}_i(c_1)$ and $\mathcal{T}_i(c_2)$ together. Then, $b_{i,1\cup 2} = \max\{b_i(c_1), b_i(c_2)\}$.*

Proof. The line of arguments that we present so far shows that $b_{i,1\cup 2}$ is either equal to $b_i(c_1)$ or $b_i(c_2)$. Suppose first that $b_{i,1\cup 2} = b_i(c_1)$. This occurs if $P\{c_2 < \Pi_i \le c_1\}\cdot u(w - b_{i,1\cup 2}) \ge \int_{c_2}^{c_1} u(w+x-b_{i,1\cup 2})\cdot f_i(x)dx$. If we modify the second equation in (5) and substitute $b_i(c_1)$ for $b_i(c_2)$, then the left-hand side must be greater. Therefore, $b_i(c_2) < b_i(c_1) = b_{i,1\cup 2}$ to reestablish the inequality. This shows in this case that $b_{i,1\cup 2} = \max\{b_i(c_1), b_i(c_2)\}$. The result for the other case is handled similarly, so the proof is complete for the interesting case $b_i(c_1), b_i(c_2) > 0$. The other cases, in which at least one of $b_i(c_1)$ or $b_i(c_2)$ is zero, are handled in the appendix. □

This result is significant for one reason: tail event information can be used to construct other information alternatives generated by multiple events. Proposition 2 is the first step toward the establishment of the relation between the buying price of tail event information and more complex forms of information. In the next section, we extend Proposition 2 to compute the buying price of more generic information alternatives.

5.2. Value of the Union of Multiple Tail Event Information Alternatives

We now consider multiple tail information alternatives $\mathcal{T}_i(c_1), \ldots, \mathcal{T}_i(c_n)$. Without loss of generality, we assume $c_1 > c_2 > \cdots > c_n$. Suppose the decision maker acquires $\mathcal{T}_i(c_1), \ldots, \mathcal{T}_i(c_n)$ altogether (i.e., $\mathcal{T}_i(c_1) \cup \cdots \cup \mathcal{T}_i(c_n)$, which, for brevity, we call $\mathcal{F}_i$ in what follows). From a structural point of view, this is equivalent to acquiring information generated by the disjoint intervals $(-\infty, c_n]$, $(c_n, c_{n-1}], \ldots, (c_1, \infty)$. Let the buying price of this information bundle be $b_{\mathcal{F}_i}$ and define $E_{i,n} = \{\Pi_i \le c_n\}, E_{i,s} = \{c_{s+1} < \Pi_i \le c_s\}$ for $s = 1, \ldots, n-1$, and $E_{i,0} = \{\Pi_i > c_1\}$.

The buying price solves the below equation:

$$\max\left\{\max_{k=1,\ldots,n}\mathbb{E}[u(w+\Pi_k)],\,u(w)\right\} = \sum_{s=0}^{n}P\{E_{i,s}\}\cdot\max\left\{\max_{\substack{j=1,\ldots,n,\\ j\neq i}}\mathbb{E}[u(w+\Pi_j-b_{\mathcal{F}_i})],\,u(w-b_{\mathcal{F}_i}),\right.$$
$$\left.\mathbb{E}[u(w+\Pi_i-b_{\mathcal{F}_i})\,|\,E_{i,s}]\right\}. \quad (7)$$

We rely on the independence assumption to determine the conditional decisions on $E_{i,s}, s=0,\ldots,n$. Assume that the uninformed decision is an alternative other than Π_i and let $\mathrm{CE}(w-b_{\mathcal{F}_i})$ be the certainty equivalent of this uninformed optimal. Also, define $\hat{s}$ as

$$\hat{s}=\begin{cases}=0 & \text{if } \mathrm{CE}(w-b_{\mathcal{F}_i})\in(c_1,\infty),\\ =s & \text{if } \mathrm{CE}(w-b_{\mathcal{F}_i})\in(c_{s+1},c_s],\ s=1,\ldots,n-1,\\ =n & \text{if } \mathrm{CE}(w-b_{\mathcal{F}_i})\in(-\infty,c_n].\end{cases}$$

Then, for interim values of $\hat{s}=1,\ldots,n-1$, Equation (7) is rewritten as

$$\max\left\{\max_{k=1,\ldots,n}\mathbb{E}[u(w+\Pi_k)],u(w)\right\} = \sum_{s=0}^{\hat{s}-1}P\{E_{i,s}\}\cdot\mathbb{E}[u(w+\Pi_i-b_{\mathcal{F}_i})\,|\,E_{i,s}]+\sum_{s=\hat{s}+1}^{n}P\{E_{i,s}\}$$
$$\cdot\max\left\{\max_{\substack{j=1,\ldots,n,\\ j\neq i}}\mathbb{E}[u(w+\Pi_j-b_{\mathcal{F}_i})],u(w-b_{\mathcal{F}_i})\right\}+P\{E_{i,\hat{s}}\}$$
$$\cdot\max\left\{\max_{\substack{j=1,\ldots,n,\\ j\neq i}}\mathbb{E}[u(w+\Pi_j-b_{\mathcal{F}_i})],u(w-b_{\mathcal{F}_i}),\right.$$
$$\left.\mathbb{E}[u(w+\Pi_i-b_{\mathcal{F}_i})\,|\,E_{i,\hat{s}}]\right\}, \quad (8)$$

or

$$\max\left\{\max_{k=1,\ldots,n}\mathbb{E}[u(w+\Pi_k)],u(w)\right\} = P\{\cup_{s=0}^{\hat{s}-1}E_{i,s}\}\cdot\mathbb{E}[u(w+\Pi_i-b_{\mathcal{F}_i})\,|\,\cup_{s=0}^{\hat{s}-1}E_{i,s}]+P\{\cup_{s=\hat{s}+1}^{n}E_{i,s}\}$$
$$\cdot\max\left\{\max_{\substack{j=1,\ldots,n,\\ j\neq i}}\mathbb{E}[u(w+\Pi_j-b_{\mathcal{F}_i})],u(w-b_{\mathcal{F}_i})\right\}+P\{E_{i,\hat{s}}\}$$
$$\cdot\max\left\{\max_{\substack{j=1,\ldots,n,\\ j\neq i}}\mathbb{E}[u(w+\Pi_j-b_{\mathcal{F}_i})],u(w-b_{\mathcal{F}_i}),\right.$$
$$\left.\mathbb{E}[u(w+\Pi_i-b_{\mathcal{F}_i})\,|\,E_{i,\hat{s}}]\right\}. \quad (9)$$

Equation (9) has the same structure as the buying price equation in (6). To see this, consider two tail event information alternatives, $\mathcal{T}_i(c_{\hat{s}})$ and $\mathcal{T}_i(c_{\hat{s}+1})$ generated by $\{\Pi_i\le c_{\hat{s}}\}$ and $\{\Pi_i\le c_{\hat{s}+1}\}$. If the decision maker acquires both $\mathcal{T}_i(c_{\hat{s}})$ and $\mathcal{T}_i(c_{\hat{s}+1})$, then the buying price equation is

$$\max\left\{\max_{k=1,\ldots,n}\mathbb{E}[u(w+\Pi_k)],u(w)\right\} = \int_{c_{\hat{s}}}^{\infty}u(w+x-b_{i,\hat{s}\cup\hat{s}+1})\cdot f_i(x)dx+P\{\Pi_i\le c_{\hat{s}+1}\}$$
$$\cdot\mathbb{B}_{-i}(b_{i,\hat{s}\cup\hat{s}+1})+\max\left\{\int_{c_{\hat{s}+1}}^{c_{\hat{s}}}u(w+x-b_{i,\hat{s}\cup\hat{s}+1})\cdot f_i(x)dx,\right.$$
$$\left.P\{c_{\hat{s}+1}<\Pi_i\le c_{\hat{s}}\}\cdot\mathbb{B}_{-i}(b_{i,\hat{s}\cup\hat{s}+1})\right\}, \quad (10)$$

where $b_{i,\hat{s}\cup\hat{s}+1}$ is the buying price of $\mathcal{T}_i(c_{\hat{s}})\cup\mathcal{T}_i(c_{\hat{s}+1})$. Because $\cup_{s=0}^{\hat{s}-1}E_{i,s}=(c_{\hat{s}},\infty)$ and $\cup_{s=\hat{s}+1}^{n}E_{i,s}=(-\infty,c_{\hat{s}+1})$, Equations (9) and (10) are identical. Hence, $b_{\mathcal{F}_i}=b_{i,\hat{s}\cup\hat{s}+1}$, that is, the buying price of the complex information alternative generated by the tail event information alternatives $\mathcal{T}_i(c_1),\ldots,\mathcal{T}_i(c_n)$ is linked to the buying price of a much simpler information alternative generated by two tail event information alternatives. In Proposition 3, we take one more step and state that the information alternatives that are generated by a finite partition of the outcome space have the same value with a simple tail event information alternative providing a coarser partition.

Proposition 3. *Suppose a decision maker with initial wealth w and a risk-averse utility function u chooses between n independent risky investment alternatives and the do-nothing alternative. This decision maker is also presented with perfect partial information $\mathcal{F}_i$ on some alternative Π_i as a finite partition of its outcome space generated by interval events of the form $(-\infty,c_n],(c_n,c_{n-1}],\ldots,(c_1,\infty)$. Define $\hat{s}$ such that $\mathrm{CE}(w-b_{\mathcal{F}_i})\in(c_{\hat{s}+1},c_{\hat{s}}]$, where $b_{\mathcal{F}_i}$ is the buying price of $\mathcal{F}_i$ and $\mathrm{CE}(w-b_{\mathcal{F}_i})$ is the certainty equivalent of the best investment alternative other than Π_i at the wealth level $w-b_{\mathcal{F}_i}$. Then,*

i. *If $\hat{s}\in\{1,\ldots,n-1\}$, $b_{\mathcal{F}_i}=\max\{b_i(c_{\hat{s}}),b_i(c_{\hat{s}+1})\}$.*
ii. *If $\hat{s}=0$, $b_{\mathcal{F}_i}=b_i(c_1)$.*
iii. *If $\hat{s}=n$, $b_{\mathcal{F}_i}=b_i(c_n)$.*

Proof. Here, we finish the proof of the case when the uninformed optimal is an alternative other than Π_i and when $\hat{s}\in\{1,2,\ldots,n-1\}$. The other cases are handled in the appendix. The buying price equation is already presented in (9), and we have already argued that it is identical to the buying price equation of the information alternative generated by the union of $\mathcal{T}_i(c_{\hat{s}})$ and $\mathcal{T}_i(c_{\hat{s}+1})$. We now apply Proposition 2 and conclude that $b_{\mathcal{F}_i} = \max\{b_i(c_{\hat{s}}), b_i(c_{\hat{s}+1})\}$, where $b_i(c_{\hat{s}})$, and $b_i(c_{\hat{s}+1})$ are the buying prices of $\mathcal{T}_i(c_{\hat{s}})$ and $\mathcal{T}_i(c_{\hat{s}+1})$, respectively. □

Proposition 3 establishes an equivalence between information generated by a finite partition of the outcome space and tail event information. Such an equivalence is not only significant for making computational work easier, but also for its potential simplification in data collection. In particular, because tail event information alternatives are based on a two-piece partition of the outcome space, they may be less expensive and easier to collect. In practice, they may be based on expert opinion, and the elicitation process may, therefore, be simpler. An example for illustrating Proposition 3 is provided below.

Example 2. The background setup and the basic assumptions in this example are similar to Example 1 in Section 4.1 except that we discuss two separate cases. These cases are differentiated by the distribution parameters of the three risky prospects and an initial wealth level of $w = \$50$ million. In the first case, $X_1 \sim Beta(3, 1, -4.5, 2)$, $X_2 \sim Beta(4, 1.5, -4.5, 2)$, $X_3 \sim Beta(5, 2, -4.5, 2)$. The most preferred alternative prior to information collection is X_1; the expected utilities are 49.99326 for the do-nothing alternative and 50.36844, 50.22064, and 50.13618 for the risky investment alternatives X_1, X_2 and X_3, respectively.

Suppose three tail event information alternatives generated by the events $X_1 \in (-\infty, -0.2]$, $X_1 \in (-\infty, 0]$, and $X_1 \in (-\infty, 0.15]$ and denoted, respectively, by $\mathcal{T}_1(-0.2)$, $\mathcal{T}_1(0)$, and $\mathcal{T}_1(0.15)$ are available. If these alternatives are collected altogether, then a fourth, more complex, information alternative denoted by $\mathcal{F}$ is produced. The buying price of this bundle is numerically calculated as $b_{\mathcal{F}} = \$0.453956$ million. At the wealth level $\tilde{w} = w - b_{\mathcal{F}} \simeq \$49.54604$, the most preferred alternative is still X_1, but the second best is investing in X_2. The certainty equivalent of X_2 at the wealth level $\tilde{w}$ is \$0.227223 million. Because this quantity is greater than the threshold value for $\mathcal{T}_1(-0.2)$, $\mathcal{T}_1(0)$, and $\mathcal{T}_1(0.15)$, by Proposition 3, the buying price of $\mathcal{F}$ is equal to the buying price of $\mathcal{T}_1(0.15)$. Our numerical computation for the buying price of $\mathcal{T}_1(0.15)$ yields \$0.453949 million. To put this into perspective, the buying prices of $\mathcal{T}_1(0)$ and $\mathcal{T}_1(-0.2)$ are \$0.448755 million and \$0.434976 million, respectively. The difference between the financial values of $\mathcal{F}$ and $\mathcal{T}_1(0.15)$ is on the order of magnitude of 10^{-5}, which is negligible and can be reduced further by employing more precise computational methods.

In the second case, two of the distribution parameters are modified to make the risky prospects financially less appealing. As such, we have $X_1 \sim Beta(3, 2, -4.5, 2)$, $X_2 \sim Beta(4, 3, -4.5, 2)$, $X_3 \sim Beta(5, 4, -4.5, 2)$. The do-nothing alternative is better than all the risky alternatives at the initial wealth level $w = \$50$ million. The expected utilities are 49.39278, 49.20695, and 49.10371 for investing in X_1, X_2 and X_3, respectively. Note that the supports of the distributions are the same as the first case. Therefore, we use the same three threshold values for the tail information alternatives. Consequently, $\mathcal{T}_1(-0.2)$, $\mathcal{T}_1(0)$ and $\mathcal{T}_1(0.15)$ can be collected. When they are collected altogether, the monetary value is $b_{\mathcal{F}} = \$0.272965$ million. Because linear plus exponential utility functions are nonincreasingly risk averse, the do-nothing alternative is surely preferred at the wealth level $\tilde{w} = w - b_{\mathcal{F}} \simeq 49.72704$ with a certainty equivalent of \$0. Then, by Proposition 3, the buying price of the combined information alternative is equal to the maximum value of $\mathcal{T}_1(0)$ and $\mathcal{T}_1(0.15)$. The numerical computations yield $b_{\mathcal{T}_1(0)} = \$0.272924$ million and $b_{\mathcal{T}_1(0.15)} = \$0.26988$ million, which indicate that the tail information generated by $X_1 \in (-\infty, 0]$ has greater value. The value of $\mathcal{T}_1(-0.2)$ is lower: 0.267457. As in the first case, whereas $b_{\mathcal{F}}$ and $b_{\mathcal{T}_1(0)}$ are theoretically expected to be identical, a numerical difference on the order of magnitude of 10^{-5} is observed between $b_{\mathcal{T}_1(0)}$ and $b_{\mathcal{F}}$ in our computation. The discrepancy is within the acceptable tolerance for this analysis and can be lowered by more precise root approximation algorithms. □

Up to this point, we assume that information is perfect and, hence, fully accurate. Information sources are not always reliable, however, and therefore, information may be imperfect. Imperfect information is a common occurrence in various decision contexts. In the next section, we relax the perfect information assumption and examine how the imperfect nature of tail event information affects its value.

6. Imperfect Tail Event Information

In reality, perfect information is uncommon regardless of the decision context. Despite its intuitive connection to a simple question that is fundamental to the underlying decision problem and its partial nature, tail event information shares the same drawback. In this section, we, therefore, assume that our information source is not perfectly reliable. As such, there is a probability $1 - \pi > 0$ such that incoming information is wrong. In particular, for the imperfect tail event information $\widetilde{\mathcal{T}}_i(c)$, this new assumption implies

$$P\{``\Pi_i \in (-\infty, c]" \mid \Pi_i \in (-\infty, c]\} = \pi,$$

and

$$P\{``\Pi_i \in (-\infty, c]" \mid \Pi_i \in (c, \infty)\} = 1 - \pi,$$

where statement of information in quotation marks represents the imperfect information. Likewise, $P\{``\Pi_i \in (c,\infty)" \mid \Pi_i \in (-\infty, c]\} = 1-\pi$ and $P\{``\Pi_i \in (c,\infty)" \mid \Pi_i \in (c,\infty)\} = \pi$ hold as well. In this case, a simple application of the standard Bayesian calculus yields

$$P\{\Pi_i \in (-\infty, c] \mid ``\Pi_i \in (-\infty, c]"\} = \frac{\pi F(c)}{\pi F(c) + (1-\pi)(1-F(c))},$$

and

$$P\{\Pi_i \in (c, \infty) \mid ``\Pi_i \in (c, \infty)"\} = \frac{\pi(1-F(c))}{(1-\pi)F(c) + \pi(1-F(c))}.$$

To express the buying price equation for $\widetilde{\mathcal{T}}_i(c)$, we also need to derive the conditional density function to calculate the conditional expected utility of Π_i. Following the similar derivation steps for the probabilities above, we first calculate $f(x \mid ``(-\infty, c]")$ and $f(x \mid ``(c,\infty)")$ and then use them to calculate $\mathbb{E}[u(w+\Pi_i)]$ under the conditions imposed by imperfect information:

$$\begin{aligned}\mathbb{E}&[u(w+\Pi_i) \mid ``(-\infty, c]"] \\ &= \frac{\pi}{\pi F(c) + (1-\pi)(1-F(c))} \cdot \int_{-\infty}^{c} u(w+x)\cdot f(x)dx \\ &\quad + \frac{1-\pi}{\pi F(c) + (1-\pi)(1-F(c))} \cdot \int_{c}^{\infty} u(w+x)\cdot f(x)dx, \end{aligned} \tag{11}$$

and

$$\begin{aligned}\mathbb{E}&[u(w+\Pi_i) \mid ``(c, \infty)"] \\ &= \frac{1-\pi}{(1-\pi)F(c) + \pi(1-F(c))} \cdot \int_{-\infty}^{c} u(w+x)\cdot f(x)dx \\ &\quad + \frac{\pi}{(1-\pi)F(c) + \pi(1-F(c))} \cdot \int_{c}^{\infty} u(w+x)\cdot f(x)dx. \end{aligned} \tag{12}$$

Because the value of π is critical in writing the buying price equation, we now analyze the comparative preferences between Π_i and other alternatives as a function of the accuracy of information. We observe that, as π increases, the weight of the first term in (11) increases. Conversely, an opposite relationship holds for the first term in (12). In the interesting case when the buying price of perfect $\mathcal{T}_i$ is positive, a sufficiently high value for π makes (11) less than the expected utility of the best alternative in the original decision problem, but a high value of π makes (12) greater. By continuity of the conditional expectation term, there should be a threshold value $\pi^*(w)$ such that $\Pi_i 1_{``(-\infty,c]"}$ is preferred over any other alternative if $\pi < \pi^*(w)$ and vice versa if $\pi > \pi^*(w)$. Recall the earlier definition $\mathbb{B}_{-i}(b) = \max\left\{\max_{\substack{s=1,\ldots,n,\\ s\neq i}} \mathbb{E}[u(w+\Pi_s - b)], u(w-b)\right\}$. Mathematically, $\pi^*(w)$ satisfies

$$\begin{aligned}&\frac{\pi^*(w)}{\pi^*(w)F(c) + (1-\pi^*(w))(1-F(c))} \cdot \int_{-\infty}^{c} u(w+x)\cdot f(x)dx \\ &+ \frac{1-\pi^*(w)}{\pi^*(w)F(c) + (1-\pi^*(w))(1-F(c))} \cdot \int_{c}^{\infty} u(w+x)\cdot f(x)dx \\ &= \mathbb{B}_{-i}(0). \end{aligned} \tag{13}$$

Note that the weighted sum term in (13) is the same as (11) (after substituting $\pi^*(w)$ in there). Similarly, there should be a $\widetilde{\pi}(w)$ that we can substitute in (12) such that

$$\begin{aligned}&\frac{1-\widetilde{\pi}(w)}{(1-\widetilde{\pi}(w))F(c) + \widetilde{\pi}(w)(1-F(c))} \cdot \int_{-\infty}^{c} u(w+x)\cdot f(x)dx \\ &+ \frac{\widetilde{\pi}(w)}{(1-\widetilde{\pi}(w))F(c) + \widetilde{\pi}(w)(1-F(c))} \cdot \int_{c}^{\infty} u(w+x)\cdot f(x)dx \\ &= \mathbb{B}_{-i}(0). \end{aligned} \tag{14}$$

It turns out, $\pi^*(w)$ and $\widetilde{\pi}(w)$ are not only closely related, but also there is a wealth-independent linear relation between them. This relation is stated below.

Lemma 1. *Suppose* $\pi^*(w)$ and $\widetilde{\pi}(w)$ *satisfy Equations* (13) and (14), *respectively. Then,* $\pi^*(w) = 1 - \widetilde{\pi}(w)$.

Proof. See appendix. □

Lemma 1 is useful for identifying decisions given imperfect information in this problem. This is critical for formulating the buying price equations. At this point, we are ready to state the main results of this section. First, we state that imperfect tail event information is less valuable than perfect tail event information.

We also add that, for the utility functions that satisfy the one- or zero-switch property (see Bell 1988), the buying price of tail event information exhibits a monotonic behavior as a function of π, the probability of information accuracy.

Proposition 4. *Suppose a risk-averse decision maker with a utility function u and initial wealth w makes a decision among n investment alternatives and the do-nothing alternative. Then, the buying price of perfect tail event information on Π_i, $b_i(c)$, is greater than the value of imperfect tail event information. Moreover, if u is either one- or zero-switch and if $\pi > \frac{1}{2}$, then $\widetilde{b}_i(c) > 0$ is nondecreasing in π. In the opposite case, it is nonincreasing in π.*

Proof. See appendix. □

In expected utility theory, wealth plays a crucial role in risk preferences except for zero-switch utility functions in which ranking of two risky alternatives does not depend on wealth. For the case of one-switch utility functions, the decision maker may change preference between two risky alternatives only once as a function of wealth. Accordingly, for two risky alternatives Π_1 and Π_2, if there is a change of preferences, there is a wealth level w^* such that, without loss of generality, the decision maker prefers Π_1 over Π_2 if a decision is made at a wealth level below w^* and the reverse at a level above w^*. Proposition 4 confirms that a conjecture based on intuition holds for one- or zero-switch utility functions: the amount the decision maker is willing to pay to acquire imperfect tail event information is increasing in its accuracy. The maximum value of information is attained at the boundary values of π, specifically $\pi = 1$ or $\pi = 0$. Perfect tail event information refers to the case $\pi = 1$; however, imperfect tail event information with $\pi = 0$ is also equivalent in informational content to perfect tail event information. As anticipated, imperfect information with completely random accuracy, that is, when $\pi = 1/2$, does not bring an additional benefit to improve decision making. Therefore, resulting information is not worth paying a significant price.

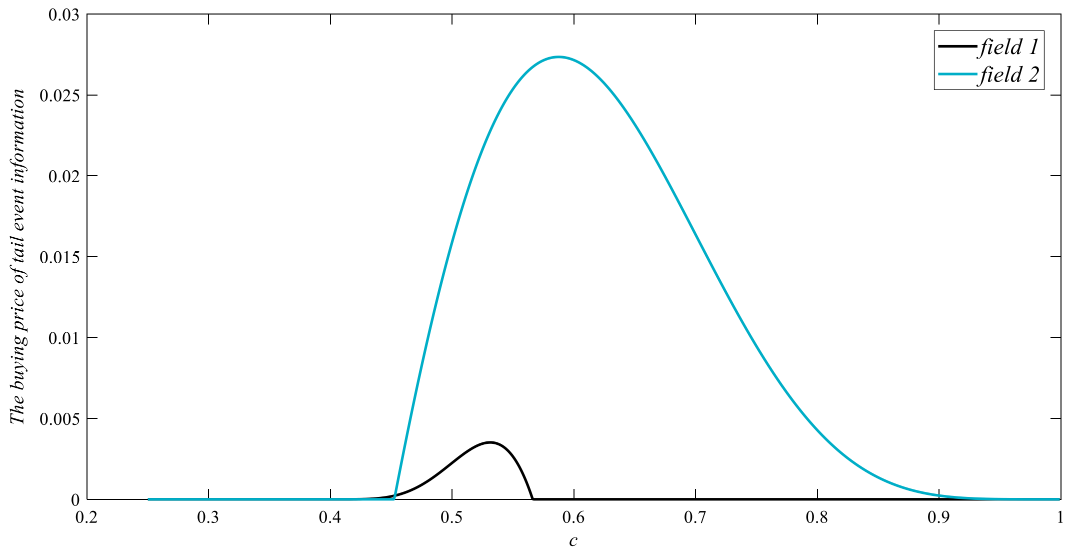
In this article, we discuss the theory behind the value of tail event information. As we highlight in the introduction, analysis of tail event information may be beneficial for the study of the financial outcomes of preliminary mineral or oil exploration activities. The next example is presented to illustrate the behavior of tail event information buying price in a numerical problem inspired from a practical application of an information-collection activity for mineral or oil exploration. The backstory originates from Bakır and Klutke (2014), but the information structure is modified to fit the tail event information framework.

Example 3. A mineral producer is faced with the decision whether to drill a field to extract natural gas. The minimum field size required for economic profitability corresponds to a production rate of 30 million cubic feet per day. This rate must be sustained for 30 years. There are two alternative fields in which the company may invest. In field 1, company experts believe that the amount of natural gas extracted might be approximately 37 to 65 million cubic feet per day. The degree of variation in this initial assessment is significant for the company; therefore, company executives are eager to perform a geological investigation. The alternative field, field 2, has even more variation in potential reservoir size: it is initially believed to yield approximately 23 to 92.5 million cubic feet per day. Suppose that the company makes an average of $1 for each 1,000 cubic feet. This translates to a daily profit of approximately $30,000 and a monthly profit of around $900,000 at the minimum field size. We assume that the company decisions can be evaluated by a linear plus exponential utility function, $u(x) = x - 10e^{-x}$, where x is in billions of dollars. Current assets of the mineral company are estimated to be worth $100 billion. The company may choose to invest in and drill one of these two fields or opt to do nothing.

In light of the information above, ignoring the time value of money, we may estimate the 30-year profits from field 1, X_1, to be approximately in the range of $400 to $700 million (consistent with the daily production rates mentioned in the previous paragraph). Similarly, X_2 for field 2 ranges between $250 million and $1 billion. The prior distribution of X_1 is scaled Beta with parameters $r = 5$ and $k = 3$, that is, $X_1 \sim Beta(5, 3, 400M, 700M)$, whereas X_2 is distributed according to scaled Beta with parameters $r = 3$ and $k = 5$, that is, $X_2 \sim Beta(3, 5, 250M, 1B)$. The initial decision, without information collection, is to drill field 1. The expected utilities for the three decision alternatives drill field 1, drill field 2, and do nothing are 100.5871, 100.5308 and 99.9995, respectively. The field 1 yield is less variable and is sure to bring profit higher than the minimum field size.

Figure 2 illustrates the value of perfect tail event information on the eventual profitability of both fields. In reality, the mineral producer can collect information through remote surveys, exploratory drilling, or geological mapping, which may provide more precise information than

Figure 2. (Color online) The Buying Price of Tail Event Information for Different Values of c, c in Billions of Dollars



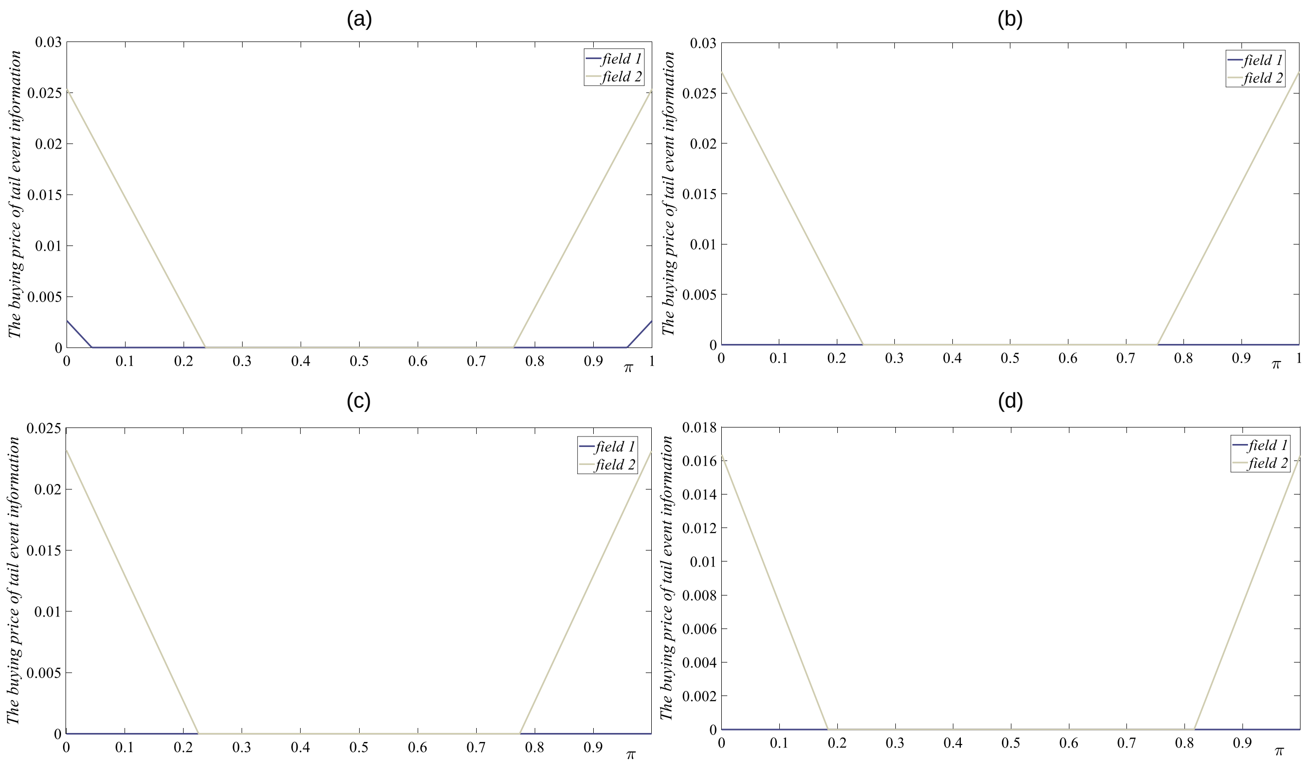
what is conveyed by typical tail event information. Nevertheless, as we contend in the article, when decisions involve a finite set of risky alternatives, tail event information can yield a financial benefit comparable, or even equivalent, to that obtained from more complex information structures. With this in mind, we illustrate the value of tail event information generated by the events $\{X_1 \leq c\}$ and $\{X_2 \leq c\}$ for various values of $c \in \{250$ Million, 1 Billion$\}$. The tail event information on X_1 has much less value than the tail event information on X_2 for two main reasons: (1) the uninformed decision is to develop field 1, and (2) the prior on the yield in field 1 has a much smaller variance, making field 1 development more appealing to the mineral producer unless the producer learns that the maximum yield is less than roughly \$550M. The maximum value of information on X_1 is around \$3 million. The mineral producer is more inclined to uncover the potential of field 2, as it can generate a higher yield than field 1, although the greater variability in financial outcomes reduces its priority against field 1. The value of information on X_2 can go up to just below \$30 million, which is 10 times more than the maximum for field 1.

The behavior of the value of imperfect tail information on both fields is demonstrated in Figure 3. The plots in this figure illustrate how the buying price of tail event information varies as a function of π. Each plot showcases information value across different values of c. Values of c were chosen so that the value of perfect tail event information on X_2 is positive (because the values of c for a positive value of perfect tail information on X_1 constitute a very small interval). All plots look very similar despite the significant differences in the value of the corresponding perfect tail information. First, it is already established that, if the buying price for perfect information is zero, the same holds true for its imperfect counterpart. Hence, plots for the information on X_1 are flat at zero value except in Figure 3(a), in which perfect information is valuable. Second, the value of information exhibits a sharp decreasing trend as a function of π at both ends of the plot. The value of imperfect partial information on X_2 becomes zero when the probability on the accuracy of information is slightly less than 0.8. There is an intuitive symmetry when information accuracy is either with probability π *or* $1 - \pi$. Accordingly, the marginal case of $\pi = 0$ is just as informative as the perfect information. □

Our theoretical analysis of imperfect tail event information has assumed so far a constant probability of accuracy although this may not hold in practical decision-making contexts. Information accuracy may vary nonlinearly with changes in the threshold parameter of tail event information. Often, a full functional description of this relationship is unavailable with only limited observations at specific threshold parameter values. The following example examines a decision-making scenario of this kind, demonstrating how the value of tail event information changes in relation to the threshold parameter.

Example 4. We continue with the experimental setup of Example 3. Natural gas yields follow the same beta distributions; $X_1 \sim Beta(5, 3, 400M, 700M)$ and $X_2 \sim Beta(3, 5, 250M, 1B)$. Suppose now that the probabilities of

Figure 3. (Color online) Buying Price of Imperfect Tail Event Information for Different Values of π: The Probability That Information Is Accurate



Notes. (a) Case 1: $c = 0.55$. (b) Case 2: $c = 0.60$. (c) Case 3: $c = 0.65$. (d) Case 4: $c = 0.70$.

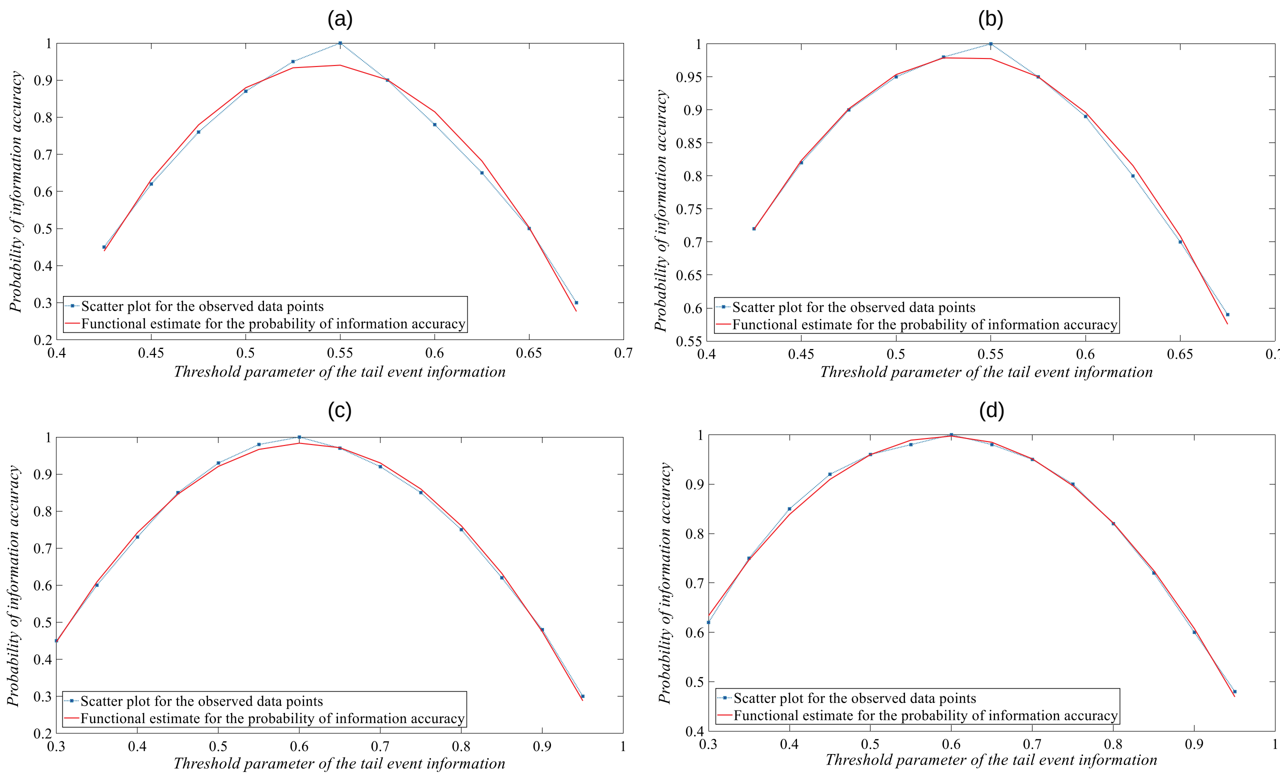
accuracy of the imperfect tail event information collected on the potential of both fields depend on the threshold parameter. The functional relationship between the probability of accuracy and the threshold parameter can be defined as $g_i(c) = P\{``X_i \in (-\infty, c]" | X_i \in (-\infty, c]\}$ and $h_i(c) = P\{``X_i \in (c, \infty)" | X_i \in (c, \infty)\}$ for $i = 1, 2$. Complete elicitation of these probability of accuracy functions is not possible for all values of c within the support range of the distributions. Nevertheless, we assume that expert opinion and past data provide realizations of both functions at a finite number of points, which are listed in Table 1. To estimate both functions, we rely on these data points and employ a quadratic polynomial fitting method. Accordingly, we obtain the following functional estimates using MATLAB Statistics and Machine Learning Toolbox: $\hat{g}_1(c) = -37.2774c^2 + 40.3542c - 9.9782$, $\hat{h}_1(c) = -21.1469c^2 + 22.687c - 5.1033$, $\hat{g}_2(c) = -5.8104c^2 + 7.0182c - 1.1356$, $\hat{h}_2(c) = -4.1868c^2 + 4.9808c - 0.4838$. The plots of these functional estimates against the scatterplot of data points in Table 1 are illustrated in Figure 4.

We use the functional estimates for the probabilities of accuracy for various levels of c to calculate the buying price of imperfect tail event information on both fields. The results are shown in Figure 5. Two of the plots represent the values of imperfect information for both fields. A third plot is included to provide a comparative demonstration against the value of perfect tail information on

Table 1. Realizations of the Probability of Accuracy Functions, $g_1(c)$, $g_2(c)$, $h_1(c)$, $h_2(c)$, for Various Values of the Threshold Parameter c in Example 4

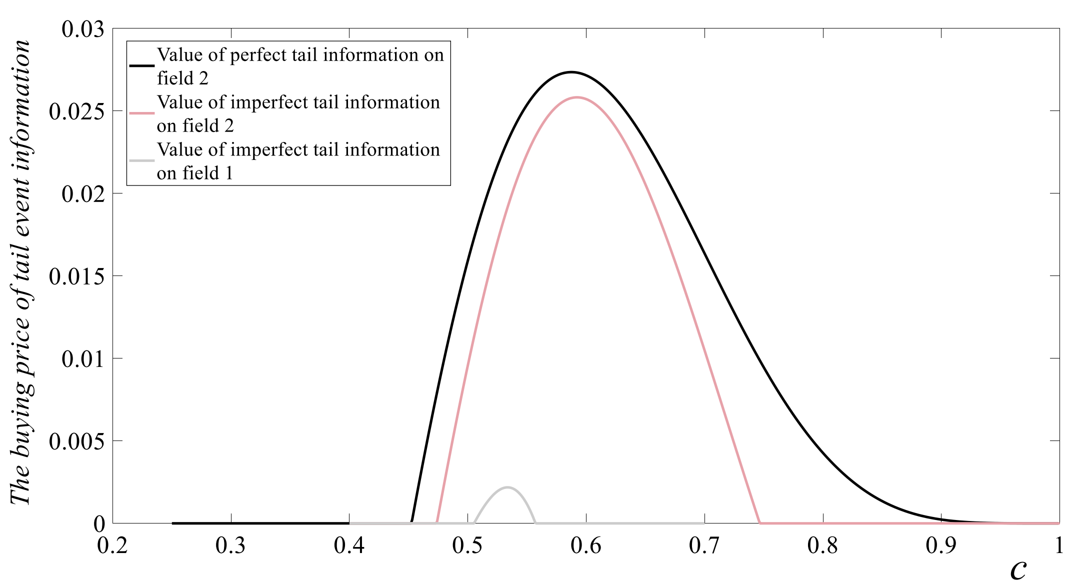
c	$g_1(c)$	$h_1(c)$	c	$g_2(c)$	$h_2(c)$
0.675	0.3	0.59	0.95	0.3	0.48
0.65	0.5	0.7	0.9	0.48	0.6
0.625	0.65	0.8	0.85	0.62	0.72
0.6	0.78	0.89	0.8	0.75	0.82
0.575	0.9	0.95	0.75	0.85	0.9
0.55	1	1	0.7	0.92	0.95
0.525	0.95	0.98	0.65	0.97	0.98
0.5	0.87	0.95	0.6	1	1
0.475	0.76	0.9	0.55	0.98	0.98
0.45	0.62	0.82	0.5	0.93	0.96
0.425	0.45	0.72	0.45	0.85	0.92
			0.4	0.73	0.85
			0.35	0.6	0.75
			0.3	0.45	0.62

Figure 4. (Color online) Plots for the Estimated Probability of Accuracy Functions Against the Scatterplots of Data Points in Table 1



Notes. (a) $\hat{g}_1$. (b) $\hat{h}_1$. (c) $\hat{g}_2$. (d) $\hat{h}_2$.

Figure 5. (Color online) Buying Price of Imperfect Tail Event Information for Different Values of c in Example 4



field 2 as calculated in Example 3. The plots for the value of imperfect information fall below the plots graphing the value of perfect information for both fields; however, Figure 5 illustrates this specifically for field 2. Imperfect information holds value for the decision maker only when it is perceived as highly accurate with an accuracy probability of approximately 0.90 or higher. Example 3 also shows that imperfect information can be valuable even when it is known to be highly inaccurate. In this instance, however, the estimated probability of accuracy never reaches such low levels. □

7. Conclusion

Value in information collection lies in its potential to generate better outcomes than the uninformed decision. In most decision-making situations, information is not available in its perfect form; therefore, the decision maker's investment of time and money for information collection yields only partial information. This is not entirely bad news, though, because, as in many strategic decision settings, only a finite set of alternatives is available. Under these conditions, partial information such as those generated by events or intervals of outcomes may be just as good as perfect information. In this context, examining partial information, its characteristics, and the relationships among different partial information structures—in both probabilistic terms and the value they generate—is essential.

An example of a partial information structure is tail event information that originates from a simple question on consequences of decision making: whether the stochastic outcomes exceed a critical threshold or not. This type of information is technically significant for numerous scenarios in which decisions are binary. Binary decisions are made in basic, invest-or-not types of financial investment problems. Tail event information makes practical sense in binary decision contexts in which simple decision rules can be developed. From a theoretical standpoint, tail event information has a simple structure. This convenience can be an advantage for making quick comparisons of buying price in many instances based on the critical threshold value that forms the outcome partition rendered by the tail event information. Furthermore, tail event information is of theoretical interest because of its relation to other more complex partial information alternatives. It is a standard measure theory exercise to show that multiple tail event information alternatives generate more complex information structures. We investigate whether that mathematical connection has any implications for the value of information calculus in multiaction decision environments. Consequently, we exploit this tractable relationship to calculate the value of more complex information alternatives (which we may call interval information) as a function of the buying price of tail event information alternatives that generate them.

A fundamental assumption that we adopt in our introductory analysis is that information may be partial, but it is perfect in the sense that there is absolute certainty about the occurrence of the event we learn about. However, information may not be entirely reliable, and hence, one should consider the possibility of error. We extend our analysis to the case of imperfect information under this motivation. As such, we probe into the relation between imperfect and perfect tail event information and investigate how the value of imperfect information behaves as a function of the probability that it is accurate. An example that highlights an application on the value of oil exploration analysis shows that, even when information deviates slightly from perfect accuracy, the value drops to zero. Theoretical results are obtained under the assumption that the probability of information accuracy is constant. As shown in Example 4, however, it may be nonlinear as a function of the critical threshold generating the tail event information. A valid direction for future research is to explore how the value of imperfect information behaves when the probability of accuracy is variable rather than constant.

Another benefit of the study of tail event information is its natural relation to extreme outcomes in a decision problem that could have profound impacts on the consequences. Prediction of extreme events is inarguably becoming a necessity in weather science, public health, security, finance, and environment studies. Data collected to address this problem rarely provide a perfect outlook. Therefore, the extension in Section 6 to imperfect partial information should be evaluated with the potential applicability of the results to these critical domains of research. A future research direction along this line is to explore how we can measure the value of data collection to uncover the likelihood of extremes and to evaluate the risk mitigation efforts against such extremes. This could be an instrumental contribution to the study of extreme events.

There are other potential areas of research on this subject. First, we do not directly explore the maximum buying price of tail event information as a function of the critical threshold. Second, an implicit assumption is made in this and many other value of information–centered research that we know exactly the mathematical structure of information alternatives. In the context of tail event information, the decision maker is assumed to know the value of the critical threshold (albeit not the information outcome) beforehand although this may not be the case. Finally, extensions can be made into multiaction decision problems in which investment alternatives are correlated or tail events from multiple, distinct sources are collected at the same time. In sum, theoretical development and results suggest a promising avenue for further investigation into tail event information.

Appendix

Proof of Proposition 2. We analyze multiple cases based on the initial decision. Suppose the initial decision is to choose an alternative other than Π_i (i.e., do nothing or some other risky alternative). If $b_i(c_1) = 0$, this implies that Π_i is not preferred even when $\Pi_i > c_1$. Therefore, there is no decision change given $\{\Pi_i > c_2\}$ or in any other event critical to the determination of $b_i(c_2)$ or $b_{i,1\cup 2}$. This implies $b_i(c_2) = b_{i,1\cup 2} = 0$, and the result holds in this case. If $b_i(c_2) = 0$, the decision given $\{\Pi_i > c_2\}$ is not to prefer Π_i; however, the decision given $\{\Pi_i > c_1\}$ cannot be inferred. The implication of a decision to keep Π_i second to some other alternative is already discussed (because that implies $b_i(c_1) = 0$), so we can argue that assuming $\{\Pi_i > c_1\}$ implies a decision change favoring Π_i. Then, both $b_i(c_1)$ and $b_{i,1\cup 2}$ are positive, but the decision on $\{c_2 < \Pi_i \le c_1\}$ should still be to reject Π_i because, otherwise, we observe contradiction to the decision given $\{\Pi_i > c_2\}$ (which is not to prefer Π_i). Therefore, $b_{i,1\cup 2} = b_i(c_1)$, confirming the conclusion of the proposition.

Next, we discuss the case in which the initial (uninformed) decision is to choose Π_i. In this case, $b_i(c_2) = 0$ implies that Π_i is attractive even when $\Pi_i \le c_2$. As a result, we immediately conclude that Π_i remains the best alternative given $\{\Pi_i > c_1\}$ or $\{\Pi_i > c_2\}$ or $\{\Pi_i \le c_1\}$. All these inferences help us recognize that $b_i(c_1)$ and $b_{i,1\cup 2}$ are both zero. On the other hand, if we begin our arguments with $b_i(c_1) = 0$, then Π_i is the most preferred alternative even when $\{\Pi_i \le c_1\}$ occurs. Nothing can be said about the decision on $\{\Pi_i \le c_2\}$. As in the previous case of the above paragraph, acceptance of Π_i given $\{\Pi_i \le c_2\}$ renders $b_i(c_2) = 0$, but we already discussed this case. Accordingly, we may consider that Π_i is no longer preferred given $\{\Pi_i \le c_2\}$. This consideration yields $b_i(c_2) > 0$, and also $b_{i,1\cup 2} > 0$. Moreover, $b_{i,1\cup 2} > 0$ must be equal to $b_i(c_2)$; note that the decision on $\{c_2 < \Pi_i \le c_1\}$ is not reversed from the uninformed decision. This is because a reversal generates a contradiction to the initial assumption $b_i(c_1) = 0$. Consequently, $b_{i,1\cup 2}$ is the maximum of $b_i(c_1)$ and $b_i(c_2)$, and the proof is complete. □

Proof of Proposition 3. The next case is when the uninformed optimal is other than Π_i but $\hat{s}$ is either zero or n. Because these two marginal cases are handled similarly, we only present the proof for $\hat{s} = 0$. When $\hat{s} = 0$, the buying price equation becomes

$$\begin{aligned}
&\max\left\{\max_{k=1,\ldots,n} \mathbb{E}[u(w+\Pi_k)], u(w)\right\} \\
&= P\{\cup_{s=1}^{n} E_{i,s}\}\cdot \max\left\{\max_{\substack{j=1,\ldots,n,\\ j\neq i}} \mathbb{E}[u(w+\Pi_j - b_{\mathcal{F}_i})], u(w-b_{\mathcal{F}_i})\right\} \\
&\quad + P\{E_{i,0}\}\cdot \max\left\{\max_{\substack{j=1,\ldots,n,\\ j\neq i}} \mathbb{E}[u(w+\Pi_j - b_{\mathcal{F}_i})], u(w-b_{\mathcal{F}_i}),\right. \\
&\qquad\left. \mathbb{E}[u(w+\Pi_i - b_{\mathcal{F}_i}) \mid E_{i,0}]\right\}. \qquad \text{(A.1)}
\end{aligned}$$

Note that $E_{i,0} = (c_1, \infty)$ and $\cup_{s=1}^{n} E_{i,s} = (-\infty, c_1]$, which immediately implies that (A.1) is, in fact, the buying price equation for the tail event information $\mathcal{T}_i(c_1)$, and therefore, $b_{\mathcal{F}_i} = b_i(c_1)$. Similar lines of argument establish $b_{\mathcal{F}_i} = b_i(c_n)$ when $\hat{s} = n$ as well. □

Proof of Lemma 1. For brevity, define $\ell(w,c) = \int_{-\infty}^{c} u(w+x)\cdot f(x)dx$, $g(w,c) = \int_{c}^{\infty} u(w+x)\cdot f(x)dx$, $w_{\text{“}(-\infty,c]\text{”}}(\pi) = \frac{\pi}{\pi F(c)+(1-\pi)(1-F(c))}$, and $w_{\text{“}(c,\infty)\text{”}}(\pi) = \frac{1-\pi}{(1-\pi)F(c)+\pi(1-F(c))}$. Note that the right-hand sides of (13) and (14) are the same. Then, we can merge them to write

$$\begin{aligned}
&w_{\text{“}(-\infty,c]\text{”}}(\pi^*(w))\cdot \ell(w,c) + (1 - w_{\text{“}(-\infty,c]\text{”}}(\pi^*(w)))\cdot g(w,c) \\
&= w_{\text{“}(c,\infty)\text{”}}(\tilde{\pi}(w))\cdot \ell(w,c) + (1 - w_{\text{“}(c,\infty)\text{”}}(\tilde{\pi}(w)))\cdot g(w,c).
\end{aligned}$$

After a quick arrangement, we obtain

$$\begin{aligned}
&[w_{\text{“}(-\infty,c]\text{”}}(\pi^*(w)) - w_{\text{“}(c,\infty)\text{”}}(\tilde{\pi}(w))]\cdot \ell(w,c) \\
&= [w_{\text{“}(-\infty,c]\text{”}}(\pi^*(w)) - w_{\text{“}(c,\infty)\text{”}}(\tilde{\pi}(w))]\cdot g(w,c)
\end{aligned}$$

which holds if and only if $w_{\text{“}(-\infty,c]\text{”}}(\pi^*(w)) = w_{\text{“}(c,\infty)\text{”}}(\tilde{\pi}(w))$. This requires $\pi^*(w) = 1 - \tilde{\pi}(w)$. The proof is now complete. □

Proof of Proposition 4. We first establish that imperfect information has zero value when the buying price of perfect information is zero. Perfect information has zero value either if Π_i is the preferred alternative (among the remaining n alternatives) even after we learn $\Pi_i \in (-\infty, c]$ or if Π_i is not preferred upon learning $\Pi_i \in (c, \infty)$. In both cases, there is no solution $\pi^*(w)$ to Equations (13) and (14) (recall that, here, $\tilde{\pi}(w) = 1 - \pi^*(w)$ by Lemma 1). In this case, imperfect information does not bring a decision change; hence, its buying price is zero as well.

Next, we continue with the interesting case when the value of perfect tail event information $b_i(c) > 0$. In that case, $\pi^*(\cdot)$ exists at various wealth levels critical to the buying price equation. There are many cases that we should carefully analyze depending on the value of $\pi^*(w - \widetilde{b}_i(c))$ and how π of the imperfect tail event information compares with $\pi^*(w - \widetilde{b}_i(c))$. Those cases are listed below. In order to write the buying price equations in a concise manner, we employ the quantities defined in the proof of Lemma 1. We present the proof in its extensive form for one case and only sketch the steps in other cases for brevity.

Case A.1 ($[\pi^*(w - \widetilde{b}_i(c)) < \frac{1}{2}$ and $\pi < \pi^*(w - \widetilde{b}_i(c))]$). The buying price $\widetilde{b}_i(c)$ satisfies

$$\max\{\max_{k=1,\ldots,n} \mathbb{E}[u(w + \Pi_k)], u(w)\}$$
$$= P\{``(-\infty, c]"\} \cdot \{w_{``(-\infty,c]"}(\pi) \cdot \ell(w - \widetilde{b}_i(c), c)(1 - w_{``(-\infty,c]"}(\pi)) \cdot \mathscr{g}(w - \widetilde{b}_i(c), c)\} + P\{(c, \infty)\} \cdot \mathbb{B}_{-i}(\widetilde{b}_i(c)),$$

which after a quick arrangement becomes

$$\max\left\{\max_{k=1,\ldots,n} \mathbb{E}[u(w + \Pi_k)], u(w)\right\}$$
$$= \pi \cdot \left[\int_{-\infty}^{c} u(w + x - \widetilde{b}_i(c)) \cdot f(x)dx + (1 - F(c)) \cdot \mathbb{B}_{-i}(\widetilde{b}_i(c))\right]$$
$$+ (1 - \pi) \cdot \left[\int_{c}^{\infty} u(w + x - \widetilde{b}_i(c)) \cdot f(x)dx + F(c) \cdot \mathbb{B}_{-i}(\widetilde{b}_i(c))\right]. \quad \text{(A.2)}$$

We observe that the second term in the right-hand side of the above equation in brackets is structurally identical to the right-hand side of the buying price equation for the perfect tail information (except that $b_i(c)$ is substituted). In fact, we have

$$\pi \cdot \left[\int_{-\infty}^{c} u(w + x - \widetilde{b}_i(c)) \cdot f(x)dx + (1 - F(c)) \cdot \mathbb{B}_{-i}(\widetilde{b}_i(c))\right]$$
$$+ (1 - \pi) \cdot \left[\int_{c}^{\infty} u(w + x - \widetilde{b}_i(c)) \cdot f(x)dx + F(c) \cdot \mathbb{B}_{-i}(\widetilde{b}_i(c))\right]$$
$$= \int_{c}^{\infty} u(w + x - b_i(c)) \cdot f(x)dx + F(c) \cdot \mathbb{B}_{-i}(b_i(c)).$$

Assume $\widetilde{b}_i(c) > b_i(c)$, and substitute $b_i(c)$ for $\widetilde{b}_i(c)$ in the above equation, and we obtain

$$\pi \cdot \left[\int_{-\infty}^{c} u(w + x - b_i(c)) \cdot f(x)dx + (1 - F(c)) \cdot \mathbb{B}_{-i}(b_i(c))\right]$$
$$+ (1 - \pi) \cdot \left[\int_{c}^{\infty} u(w + x - b_i(c)) \cdot f(x)dx + F(c) \cdot \mathbb{B}_{-i}(b_i(c))\right]$$
$$> \int_{c}^{\infty} u(w + x - b_i(c)) \cdot f(x)dx + F(c) \cdot \mathbb{B}_{-i}(b_i(c)).$$

This is a contradiction because, at the wealth level, $w - b_i(c)$, $\int_c^\infty u(w + x - b_i(c)) \cdot f(x)dx + F(c) \cdot \mathbb{B}_{-i}(b_i(c)) \geq \int_{-\infty}^c u(w + x - b_i(c)) \cdot f(x)dx + (1 - F(c)) \cdot \mathbb{B}_{-i}(b_i(c))$. This proves the relation between the buying prices of perfect and imperfect information.

Now, we assume u is one- or zero-switch. A positive buying price for the perfect tail event information implies that the decision maker's optimal preferences at wealth levels w and $w - b_i(c)$ are such that the investment alternative Π_i is better than all the other alternatives if $\{\Pi_i > c\}$ but is worse if $\{\Pi_i \leq c\}$. Because u is one- or zero-switch, these preferences should be preserved at all wealth levels between $w - b_i(c)$ and w. Then, because $b_i(c) \geq \widetilde{b}_i(c)$, we should have

$$\int_{-\infty}^{c} u(w + x - \widetilde{b}_i(c)) \cdot f(x)dx + (1 - F(c)) \cdot \mathbb{B}_{-i}(\widetilde{b}_i(c))$$
$$\leq \int_{c}^{\infty} u(w + x - \widetilde{b}_i(c)) \cdot f(x)dx + F(c) \cdot \mathbb{B}_{-i}(\widetilde{b}_i(c)).$$

If we increase the value of π in (A.2), the right-hand side decreases. In order to reestablish the inequality, the value of $\widetilde{b}_i(c)$ must be decreased. As $\pi < 1/2$ gets closer to $1/2$, imperfect tail information becomes less valuable in Case A.1.

Case A.2 ($[\pi^*(w - \widetilde{b}_i(c)) < \frac{1}{2}$ and $\pi \in (\pi^*(w - \widetilde{b}_i(c)), 1 - \pi^*(w - \widetilde{b}_i(c)))]$). In this case, the decision after acquiring the imperfect tail event information remains as an alternative other than Π_i. Therefore, $\widetilde{b}_i(c) = 0$, and the result automatically follows.

Case A.3 ($[\pi^*(w - \widetilde{b}_i(c)) < \frac{1}{2}, \pi > \pi^*(w - \widetilde{b}_i(c))$ and $\pi > 1 - \pi^*(w - \widetilde{b}_i(c))]$). In this case, the buying price equation for the imperfect tail event information is formulated based on the following conditional decisions: given "$(-\infty, c]$", an alternative other than Π_i is preferred; in the opposite case, Π_i is preferred. After arrangements are made, the buying price equation becomes

$$\max\left\{\max_{k=1,\ldots,n} \mathbb{E}[u(w + \Pi_k)], u(w)\right\}$$
$$= \pi \cdot \left[\int_{c}^{\infty} u(w + x - \widetilde{b}_i(c)) \cdot f(x)dx + F(c) \cdot \mathbb{B}_{-i}(\widetilde{b}_i(c))\right]$$
$$+ (1 - \pi) \cdot \left[\int_{-\infty}^{c} u(w + x - \widetilde{b}_i(c)) \cdot f(x)dx + (1 - F(c)) \cdot \mathbb{B}_{-i}(\widetilde{b}_i(c))\right]. \quad \text{(A.3)}$$

Note that, the part in brackets of the first term on the right-hand side of the above equation is similar to the informed side of the buying price equation for the perfect tail event information except for different buying prices. We can exploit this similarity and follow the steps at the end of the proof of Case A.1 to complete the proof of this case.

Next, we assume u is one- or zero-switch. Note that $\pi > 1/2$ in this case. As we did in Case A.1, we can now argue that the expected utility term multiplied by π in Equation (A.3) is greater than the similar term multiplied by $1-\pi$. Then, an increase in π increases the right-hand side of (A.3). As a result, an increase in the value of $\widetilde{b}_i(c)$ reestablishes the equality. In other words, the buying price of imperfect tail information in Case A.3 is increasing in π as desired.

Case A.4 ($[\pi^*(w-\widetilde{b}_i(c)) > \frac{1}{2}, \pi \in (1-\pi^*(w-\widetilde{b}_i(c)), \pi^*(w-\widetilde{b}_i(c))]$). As in Case A.2, the buying price of the imperfect tail event information is zero. This is because a decision change does not occur under imperfect information. The result follows.

Case A.5 ($[\pi^*(w-\widetilde{b}_i(c)) > \frac{1}{2}, \pi < \pi^*(w-\widetilde{b}_i(c))$ and $\pi < 1-\pi^*(w-\widetilde{b}_i(c))]$). When π satisfies the conditions of Case A.5, the preferences reflected to the buying price equation of the imperfect tail event information are (i) given "$(-\infty, c]$", Π_i is preferred, and (ii) given (c,∞), an alternative other than Π_i is preferred. Hence, this case is identical to Case A.1, so the result follows for the relation between the values of perfect and imperfect tail event information.

To see that the result stated in the proposition for the relation between tail event information accuracy and its buying price holds, we first observe that $\pi < 1/2$ under Case A.5. As we argue in the previous paragraph, this case has the identical buying price equation as in Case A.1. In our proof of Case A.1, we have already shown that $\widetilde{b}_i(c)$ is decreasing in π for one- or zero-switch u.

Case A.6 ($[\pi^*(w-\widetilde{b}_i(c)) > \frac{1}{2}$, and $\pi > \pi^*(w-\widetilde{b}_i(c))]$). One can immediately verify that Cases A.3 and A.6 require identical imperfect tail event information buying price equations. Furthermore, $\pi > \frac{1}{2}$ clearly holds. Observing the similarity of this case to Case A.3, we may jump to the conclusion we seek: $\widetilde{b}_i(c)$ is increasing in π for one- or zero-switch u. Because we are finished with all possible cases, the proof of the proposition is complete. □

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