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Dynamic Ridesharing

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Abstract Recent technological developments such as GPS, mobile devices, and increases in data storage and computation capacity have greatly enhanced the communication capabilities of travelers, facilitating ridesharing in real time. This opportunity has created a new ridesharing industry built on making use of the unused vehicle capacity moving about on the roads. There are, however, still important barriers for widespread dynamic ridesharing adoption. It is therefore important to better understand the problems and challenges of ridesharing, thus creating the need for novel research that explicitly takes into account dynamic ridesharing. In this tutorial we review recent research on new vehicle routing models, cost sharing mechanisms, and planning models that incorporate ridesharing.

Keywords dynamic ridesharing; vehicle routing; mechanism design; traffic equilibrium

1. Introduction

With rapid population growth and urban development, traffic congestion has become an important issue, especially in large cities. The 2012 Annual Urban Mobility Report by the Texas Transportation Institute estimates that (a) the amount of delay endured by the average commuter was 38 hours, up from 16 hours in 1982, and (b) the cost of congestion is more than \$120 billion, nearly \$820 for every commuter in the United States (Schrank et al. [37]). At the same time, there is no public support for increased taxation to finance infrastructure capacity expansion, and thus transit agencies and cities turn to alternative approaches such as congestion pricing to raise funds (Brown [10]). These costs to the travelers can be significant, with yearly fares sometimes exceeding \$1,000. For example, on the 11-mile stretch of the 110 Freeway in Los Angeles, the peak fare can be \$15.4 per trip (Nelson [33]). Cities also use similar strategies to eliminate or reduce parking subsidies, thus increasing the overall trip cost (Elinson [14]). Thus, there is a need for innovative transportation modes that can be implemented to improve transportation conditions in a cost-efficient manner. Ridesharing appears as one such innovative transportation mode that could at least help mitigate the congestion increase, as it can tap into the significant amount of unused capacity in transportation networks.

Ridesharing occurs when individuals share a personal vehicle for a trip and split travel costs such as gas, toll, and parking fees with others who have similar itineraries and time schedules (Furuhata et al. [18]). Benefits of ridesharing include travel cost savings, travel time reduction, traffic congestion mitigation, fuel conservation, and air pollution reduction (Chan and Shaheen [12], Kelley [26], Morency [30]). However, ridesharing is still not a regular transportation alternative and is considered an informal and disorganized activity. The lack of efficient methods to coordinate itineraries and schedules is an important factor

that inhibits the wide adoption of ridesharing. Recently, technological advances including GPS and mobile devices have greatly enhanced the communication capabilities of travelers, facilitating the creation of ridesharing in real time. Taking advantage of this opportunity, a number of companies, such as Carma, Uber, Lyft, SideCar, flinc, CarpoolWorld, etc., have emerged to develop systems where travelers (including both drivers and passengers) can be matched in real time via web browsers and mobile apps (Furuhata et al. [18]). Uber and Lyft operate similar to taxi companies, whereas Carma and flinc provide matching services where the incentive is for the driver to partially recover the cost of his or her trip through ridesharing. In a sense, these companies are establishing a marketplace for drivers to offer up their empty seats to other travelers. In these newly developed systems, the drivers receive a compensation for participating, which can be in the form of smaller travel times, reduced tolls, or direct payment that help mitigate the travel costs. An essential difference of such ridesharing systems from traditional public transit systems is that the former systems function as a matching agency that pairs passengers with citizen? drivers. With real-time information to facilitate sharing system capacity, a distributed competitive transportation system has the potential to reduce congestion and meet the mobility needs of the modern decentralized American city.

Although the idea of carpooling/ridesharing dates back to the 1950s as a result of the need to conserve fuel during World War II, ridesharing today has yet to be fully embraced by the public. However, the emergence of shared mobility services such as Uber and Lyft that utilize the latest navigation technologies has had some effect in changing the travel behavior of individuals. This phenomenon as well as the significant potential of improving congestion and reducing costs partially explains the increased interest in the research community to develop models and tools to make the adoption of real-time ridesharing systems a realization. In a survey paper, Furuhata et al. [18] outline emerging research areas in ridesharing. They include routing of vehicles for the pickup and drop-off of passengers, the pricing of such services, incentives that government agencies may employ to encourage such systems, the matching of passengers with drivers, payment methods, and security concerns of such services.

The purpose of this tutorial is to present representative models and tools that address some of the above issues. In particular, Section 2 presents a routing approach for ridesharing. One of the primary differences between routing models for ridesharing and traditional routing problems is that in the ridesharing problem there may be an incentive to take a detour to pick up additional passengers in order to qualify for a high-occupancy vehicle (HOV) lane to reduce the travel time. In Section 3, we present a mechanism design approach to determine the cost share portion of each participant in the rideshare. The approach presented in this tutorial is different from an approach where the driver determines a price to maximize his or her revenue. This latter approach may be more suitable for companies such as Uber and Lyft, but for ridesharing, the objective is typically to share the cost of the trip, not profit maximization. In Section 4, we present an equilibrium model that determines how the adoption of ridesharing will impact traffic congestion. These types of models can be used by government agencies to develop appropriate incentive structures to increase the adoption of ridesharing.

This tutorial is a summary of research conducted as part of a Department of Transportation (DOT)-funded grant and that has appeared in Furuhata et al. [18, 19], Wang et al. [38], and Xu et al. [41, 42]. For a review on other possible research areas motivated by dynamic ridesharing, we refer the reader to the survey paper by Furuhata et al. [18] and the paper by Agatz et al. [3] for a good review of the matching problem in ridesharing.

2. A Ridesharing Routing Problem

In this section, we present an optimization model specifically tailored to ridesharing. The main distinction between this model and the traditional routing problem is that the travel

time and/or travel cost on a road link depends on the load of the vehicle. That is, we consider a vehicle pickup and delivery problem with time windows; its objective is to minimize the total travel distance and passenger travel time under congestion and load-dependent toll cost. Passenger travel time under congestion means that we want to consider the reduced ride time of taking HOV lanes if the required number of people are in the vehicle. Load-dependent toll cost refers to a toll rate based on the number of people in the vehicle. Although there are a number of studies and published methods for the pickup and delivery problem with time windows (see, for example, the survey papers by Berbeglia et al. [7, 6]), there has been little work on routing problems that specifically consider HOV lanes and the policy of reduced toll rates on high-occupancy vehicles that will save both travel time and cost by having more passengers riding in the vehicle for each trip. That is, there may be an incentive to take the detour to pick up additional passengers to qualify to ride in the HOV lanes or discounted toll rates. We first present an optimization model for ridesharing and then present some experimental results to show the behavior of the trip tour as a function in the savings in trip time as a result of the use of HOV lanes. We note that the material presented in this section appears in Wang et al. [38].

2.1. Mathematical Formulation

We formulate a 0-1 integer programming model for optimally solving a vehicle pickup and delivery problem with time windows considering time savings and discounted toll rates based on the number of people in the vehicle. Assume there is one ridesharing vehicle serving n requests, and each request has a pickup and delivery location. Both the pickup and delivery have time windows. The time window of a request is always feasible, which means a taxi-like service that is not part of the ridesharing vehicles can always be used to satisfy the request. For simplicity, we assume a single vehicle, but the model can easily be extended to consider multiple vehicles.

We follow the formulation of Lu and Dessouky [29] as the basis for our formulation. The problem can be defined on a directed graph, $G = (N, A)$. Let N be the node set, $N = \{1, 2, \dots, 2n + 2\}$, where node i is:

$$\begin{aligned} i = 1, \dots, n & \quad \text{request } i\text{'s pickup location,} \\ i = n + 1, \dots, 2n & \quad \text{request } (i - n)\text{'s delivery location,} \\ i = 2n + 1 & \quad \text{driver's start location, and} \\ i = 2n + 2 & \quad \text{driver's end location.} \end{aligned}$$

Let N_P denote the set of pickup nodes; $N_P = \{1, 2, \dots, n\}$. Let N_D denote the set of delivery nodes; $N_D = \{n + 1, n + 2, \dots, 2n\}$. Let A be the arc set. The time and cost associated to each arc $(i, j) \in A$ depends on the number of people in the vehicle.

Parameters:

R_i pickup demand (number of passengers) of request i , $i \in N_P$;

$$G_i = \begin{cases} R_i & i \in N_P, \\ -R_{i-n} & i \in N_D, \\ 0 & \text{otherwise;} \end{cases}$$

Ca vehicle capacity;

$S_{i, n+i}$ travel time from node i to $n + i$ using a taxi;

E_i the earliest time customer can be picked up or delivered at node i ;

L_i the latest time customer can be picked up or delivered at node i ;

O the number of people on the vehicle at the drivers start location;

Y_{ij} minimum travel time from node i to j ;

T_{ijk} travel time from node i to j if there are k people on the vehicle;

C_{ijk} toll cost from node i to j if there are k people on the vehicle;

D_{ijk} travel distance from node i to j if there are k people on the vehicle.

In our model, the route from node i to j is determined by the number of people in the vehicle, and the weights on the distance, the travel time, and the toll cost in the objective function are given parameters. For example, if there is a larger weight on the travel time, it means that people might be willing to take a detour and pay more toll fees to shorten the travel time. Therefore, given an actual road network map, T_{ijk} , C_{ijk} , and D_{ijk} are preprocessed to represent their best values according to the weights on the distance, travel time, and toll cost that give the smallest total cost in the objective function with k people in the vehicle. As a result, Y_{ij} may not necessarily be equal to T_{ijk} when $k = Ca$ since the selected route between i and j when the vehicle is full may not be the fastest route since it depends on the weights of the other factors, such as distance and toll cost.

Variables:

$$x_{ij} = \begin{cases} 1 & \text{if the vehicle travels from node } i \text{ to node } j, \\ 0 & \text{otherwise;} \end{cases}$$

$$u_i = \begin{cases} 1 & \text{if node } i \text{ is visited by a taxi,} \\ 0 & \text{otherwise;} \end{cases}$$

$$b_{ij} = \begin{cases} 1 & \text{if node } i \text{ is before node } j \text{ in the tour of the vehicle,} \\ 0 & \text{otherwise;} \end{cases}$$

v_i the time at which a customer is picked up or delivered at node i ;

z_i the number of people in the vehicle after serving node i ;

t_{ij} actual time from node i to j ;

c_{ij} actual toll cost from node i to j ;

d_{ij} actual distance from node i to j .

The parameters β , γ , μ , and λ represent the weights for the different objective components representing total ride time, distance, toll fee, and whether the request was serviced by taxi, respectively. Note that the weight λ could be indexed in the pickup node to represent the actual taxi cost for that trip. In this work, however, we consider a uniform, large, weight λ to prioritize servicing as many requests as possible by the ridesharing vehicles instead of the use of a taxi-like service.

Problem formulation:

$$\min \left\{ \sum_{i \in N_P} \beta(v_{n+1} - v_i) + \sum_{(i,j) \in A} (\gamma d_{i,j} + \mu c_{ij}) + \sum_{(i,n+i) \in A} \lambda u_i \right\}$$

subject to

$$\sum_{j \in N} x_{ij} + u_i = 1 \quad i \in N \setminus \{2n+2\}, \tag{1}$$

$$\sum_{i \in N} x_{ij} + u_j = 1 \quad j \in N \setminus \{2n+1\}, \tag{2}$$

$$u_i = u_{n+i} \quad i \in N_P, \tag{3}$$

$$b_{ki} \leq b_{kj} + (1 - x_{ij}) \quad (i,j) \in A \setminus \{(2n+2, 2n+1)\}, k \in N \setminus \{i\}, \tag{4}$$

$$b_{kj} \leq b_{ki} + (1 - x_{ij}) \quad (i,j) \in A \setminus \{(2n+2, 2n+1)\}, k \in N \setminus \{i\}, \tag{5}$$

$$b_{ki} + u_i \leq 1 \quad i \in N \setminus \{2n+1, 2n+2\}, k \in N \setminus \{i\}, \tag{6}$$

$$b_{ik} + u_i \leq 1 \quad i \in N \setminus \{2n+1, 2n+2\}, k \in N \setminus \{i\}, \tag{7}$$

$$x_{ij} \leq b_{ij} \quad (i,j) \in A, \tag{8}$$

$$b_{ii} = 0 \quad i \in N, \tag{9}$$

$$b_{n+i,i} = 0 \quad i \in N_P, \tag{10}$$

$$b_{i,n+i} + u_i = 1 \quad i \in N_P, \tag{11}$$

$$b_{i,2n+1} = 0 \quad i \in N, \tag{12}$$

$$b_{2n+2,i} = 0 \quad i \in N, \tag{13}$$

$$z_i = (G_i + O)(1 - u_i) + \sum_{m \in N} (b_{mi} G_m) \quad i \in N, \quad (14)$$

$$z_i \leq Ca \quad i \in N, \quad (15)$$

$$v_i + S_{i,n+i} \leq v_{n+i} + (1 - u_i)M \quad i \in N_P, \quad (16)$$

$$v_i + t_{ij} \leq v_j + (1 - x_{ij})M \quad i \in N, j \in N, \quad (17)$$

$$E_i \leq v_i \leq L_i \quad i \in N, \quad (18)$$

$$t_{ij} \geq T_{ijk} - |z_i - k|M - (1 - x_{ij})M \quad (i, j) \in A, k = 1, 2 \dots Ca, \quad (19)$$

$$c_{ij} \geq C_{ijk} - |z_i - k|M - (1 - x_{ij})M \quad (i, j) \in A, k = 1, 2 \dots Ca, \quad (20)$$

$$d_{ij} \geq D_{ijk} - |z_i - k|M - (1 - x_{ij})M \quad (i, j) \in A, k = 1, 2 \dots Ca, \quad (21)$$

$$x_{ij} = \{0, 1\} \quad (i, j) \in A, \quad (22)$$

$$u_i = \{0, 1\} \quad i \in N_P, \quad (23)$$

$$b_{ij} = \{0, 1\} \quad (i, j) \in A, \quad (24)$$

$$z_i \in Z \quad i \in N, \quad (25)$$

$$v_i \geq 0 \quad i \in N, \quad (26)$$

$$t_{ij} \geq 0 \quad (i, j) \in A, \quad (27)$$

$$c_{ij} \geq 0 \quad (i, j) \in A, \quad (28)$$

$$d_{ij} \geq 0 \quad (i, j) \in A, \quad (29)$$

$$v_{n+i} - v_i \geq Y_{ij} \quad i \in N_P. \quad (30)$$

The objective is to minimize the total customer ride time, total travel distance of the vehicle, the total toll fee, and the total taxi service cost if there is some customer that cannot be served by the ridesharing vehicle. In the objective function, the first term is the total customer ride time with a weight of β . The value $(v_{n+i} - v_i)$ is the ride time of customer i . The second term is the total cost of travel distance and toll fee. The summation of d_{ij} on all edges is the total travel distance for the vehicle with a weight of γ . The summation of c_{ij} on all edges is the total toll fee with a weight of μ . The third term is the total taxi cost with a weight of λ since the customers that cannot be served by the vehicles will be served by a taxi.

Constraints (1) and (2) ensure that each location is visited only once either by the vehicle or by the taxi. Constraint (3) ensures that if the pickup location of the request is visited by a taxi, the delivery location of the request would also be visited by a taxi. Constraints (4) and (5) force $b_{ki} = b_{kj}$ when $x_{ij} = 1$. It means that if node i comes immediately before node j , all other nodes before i would also be before j and all other nodes before j except i would also be before i . Constraints (6) and (7) enforce that $b_{ki} = b_{ik} = 0$ when $u_i = 1$. It means if the node is visited by a taxi, it would not be before or after any nodes in the tour of the vehicle. Constraint (8) enforces $b_{ij} = 1$ when $x_{ij} = 1$. Constraint (9) defines $b_{ii} = 0$. When $x_{ij} = 0$, constraints (4), (5), and (8) are redundant.

Constraints (10) and (11) ensure that if the request is served by the vehicle, the pickup is done before delivery. Constraints (12) and (13) ensure that in the tour of the vehicle, no other nodes would be before the drivers start location and no other nodes would be after the drivers end location. Constraint (14) sets z_i to the number of people on the vehicle after serving node i . Note that $\sum_{m \in N} (b_{mi} G_m)$ is the number of people picked up and who have not been dropped off before node i , G_i is the number of people who need to be either picked up or dropped off at node i , and O is the number of people on the vehicle at the drivers start location. Constraint (15) is to ensure that the capacity constraint of the vehicle is not violated. Constraint (16) ensures the consistency of the time variables if the node is visited by a taxi; M is a big number to ensure that the inequality will always hold when $u_i = 0$. Constraint (17) ensures the consistency of the time variables if the node is

visited by the vehicle; M is a big number to make sure that the inequality will always hold when $x_{ij} = 0$. Constraint (18) is the time window constraint for each node. When $x_{ij} = 0$, constraints (19)–(21) will always hold. When $x_{ij} = 1$, constraint (19) corresponds to the following set of constraints:

$$\begin{aligned} t_{ij} &\geq T_{ij1} - |z_i - 1|M, \\ t_{ij} &\geq T_{ij2} - |z_i - 2|M, \\ t_{ij} &\geq T_{ij3} - |z_i - 3|M. \\ &\dots \end{aligned}$$

If z_i is not equal to k , then t_{ij} will be larger than a very small number. If z_i is equal to k , t_{ij} will have a tight constraint, which is $t_{ij} \geq T_{ijk}$. Since we want to minimize $v_{n+i} - v_i$, then it would make $t_{ij} = T_{ijk}$. Constraints (20) and (21) are similar to constraint (19). If z_i is not equal to k or if x_{ij} is equal to 0, then c_{ij} will be larger than a very small number. If z_i is equal to k and x_{ij} is equal to 1, c_{ij} will have a tight constraint, which is $c_{ij} \geq C_{ijk}$. Since we want to minimize c_{ij} , then it would make $c_{ij} = C_{ijk}$. In the case of constraint (21), if there are k people on the vehicle from node i to node j , then d_{ij} will be equal to D_{ijk} . Constraints (22), (23), and (24) are the binary constraints for the variables x_{ij} , u_i , and b_{ij} , respectively. Constraints (25) set z_i to an integer value. Constraints (26), (27), (28), and (29) are the nonnegativity constraints for the variables v_i , t_{ij} , c_{ij} , and d_{ij} , respectively. Constraint (30) is the minimum travel time constraint since the actual travel time from node i to j will always be larger than the minimum travel time from node i to j . Since the computing time is limited, this constraint could help provide a better lower bound.

2.2. Experimental Results

The pickup and delivery problem with time windows is a NP-hard problem, and optimization algorithms are only able to solve small-sized problems. Heuristics are thus necessary to solve larger-sized problems. To study the sensitivity of the solution to changes in time savings to HOV lanes for large-sized problems, we used a series of heuristics to solve the above model. The first is an insertion heuristic, which is used repetitively in the construction of the initial routes. The second part is the adjust pickup time algorithm, which is used to reduce the customer ride time by adjusting the pickup time of the customers. That is, because of the time windows, the vehicle might have to wait at some customers pickup location while having other customers waiting in the vehicle. Since the customers were picked up as soon as possible, the adjust pickup time algorithm postpones these customers pickup time so that their ride time will be reduced. The third part is a tabu search, which is used to improve the solution quality.

We do simulations on 100 requests and 15 vehicles. The simulations are run on a map constructed as follows. Since we consider the existence of HOV lanes and toll roads in our problem, we set up a map that contains the necessary information. The map is a 16×10 grid with 160 nodes that are used as the start and end locations of the drivers and the pickup and delivery locations of the requests and with 294 edges that connect the 160 nodes as a grid. All the edges are undirected. We set the length of each edge to be 10 kilometers. Of the 294 edges, 50 are randomly chosen to be toll roads that charge \$9 to travel on. The toll rate information was derived from the Highway Performance Monitoring System of the Federal Highway Administration (Office of Highway Policy Information, 2011). We used fees that represented the average of the toll rate data for California. The other 244 edges are freeways that do not charge toll fees. We randomly set 147 out of the remaining 244 edges to contain HOV lanes. We set the time on each edge to be 10 minutes for the general-purpose lanes (non-HOV lanes).

Our set of results consists of a sensitivity analysis for the time savings in HOV lanes. We ran nine simulations at intervals of 10% reduction in travel time for using HOV lanes,

starting with a 10% reduction. The calculations of the distance ratio and the ride time ratio are shown below. Here, $RD_{i,n+i}$ is the actual distance traveled from node i to node $n+i$ (the actual distance request i traveled from its pickup location to its drop-off location on the vehicle), $D_{i,n+i,1}$ is the distance if request i traveled alone, and n is the total number of requests. The distance ratio is the average actual distance divided by the drive-alone distance for each request. The higher the distance ratio, the more detours the request takes. The quantity $v_{n+i} - v_i$ indicates the actual ride time that request i spends on the vehicle from its pickup location to its drop-off location. If request i traveled alone, $T_{i,n+i,1}$ is the travel time. The ride time ratio is the average actual ride time divided by the drive-alone ride time for each request. The lower the ride time ratio, the more ride time is saved by participating in ridesharing:

$$\text{distance ratio} = \frac{1}{n} \sum_{i \in N_P} \frac{RD_{i,n+i}}{D_{i,n+i,1}},$$

$$\text{ride time ratio} = \frac{1}{n} \sum_{i \in N_P} \frac{v_{n+i} - v_i}{T_{i,n+i,1}}.$$

Figures 1 and 2 demonstrate the sensitivity of the distance ratio and ride time ratio to the time savings in HOV lanes for the different map settings, respectively. “NO HOV” lanes means there are no HOV lanes in the map, “HOV2” means all HOV lanes require at least two individuals in the vehicle, and “HOV3” requires three. The results shown in the figures are averages of 10 instances. These results show that as HOV lanes become more attractive (there is a larger time savings by traveling on them), rides become longer (as seen by the increase in distance ratio in Figure 1) and travel time becomes shorter (as seen by the decrease in ride time ratio in Figure 2). Note the two possible reasons to take detours in distance are (1) to pick up additional passengers for the vehicle and (2) to travel the extra distance to make use of the faster travel time of HOV lanes. Even when there is no time savings for using HOV lanes, in the NO HOV case, there still is some ridesharing occurring as evidenced by having both a distance ratio and ride time ratio slightly bigger than 1. For the cases of HOV2 and HOV3, the first type of detour tends to dominate when the time savings in HOV lanes is not significant, and the second type of detour tends to dominate otherwise.

When the map contains HOV lanes, Figures 1 and 2 indicate that vehicles are more willing to take detours to save ride time as time savings in HOV lanes increases. Furthermore, we observe that the ride time ratio is always larger when the map is all HOV3 lanes than when

FIGURE 1. Distance ratio sensitivity to time savings in HOV lanes.

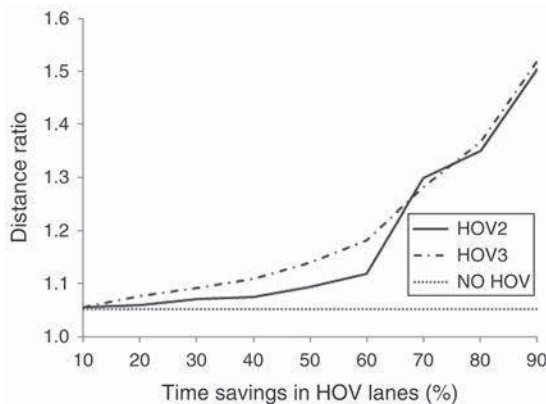
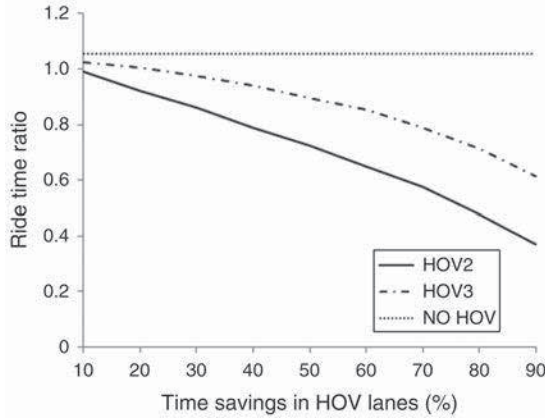
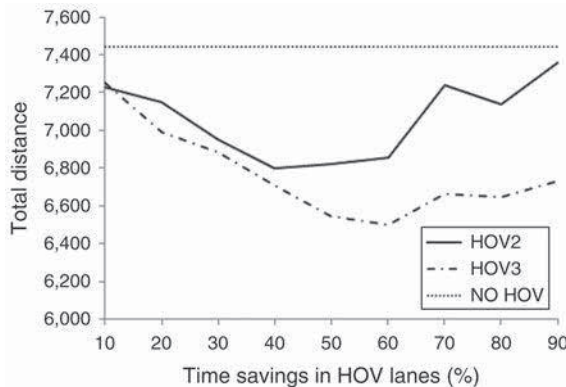


FIGURE 2. Ride time sensitivity to time savings in HOV lanes.



they are only HOV2 lanes. That is, it is more difficult to use HOV lanes to save ride time when there are only HOV3 lanes since the vehicle has to take a detour to have one extra pickup and the ride time of the passenger already in the vehicle is increased. However, the distance ratio is less for HOV2 lanes than it is for all HOV3 lanes when the time savings in HOV lanes is less than 70% and is indifferent otherwise. To analyze the change in distance ratio, we can look at Figure 3, which shows the total distance traveled by the vehicles, and the behavior between the various HOV scenarios is in the reverse order for that of the distance ratio (from Figure 1) since this is from the perspective of the vehicle instead of the passenger. Thus, the NO HOV case has the highest total distance traveled by the vehicles but the lowest distance ratio. From Figure 3, the total distance initially decreases but increases roughly at the same time savings point when the distance ratios for the HOV2 and HOV3 cases are about the same. The decrease in ride time is more significant than the increase in total distance when the time savings in HOV lanes is large. The distance ratio is larger for the map with all HOV3 lanes than it is for all HOV2 lanes since detours are necessary to pick up additional customers to qualify for HOV lane use. And for the map of HOV3 lanes, more detours than that of HOV2 lanes are necessary. When the time savings in HOV lanes equals to or is larger than 70%, with increasing time savings, the total distances for both HOV2 and HOV3 cases are, in general, getting significantly larger. That is, the customers are kept in the vehicle while the vehicle is driving around. It means that the detours are mainly taken to capture the faster paths since the time savings in HOV lanes is significant. In summary, we evaluated the sensitivity to HOV lanes using two different

FIGURE 3. Total Distance sensitivity to time savings in HOV lanes.



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measures: a distance ratio and a ride time ratio. From the results, we see that when time savings in HOV lanes becomes more significant, the distance ratio will increase while the ride time ratio will decrease. This indicates that people will be more willing to take a detour to share a ride with others or to capture faster paths if the ride time can be decreased significantly.

3. Cost Sharing Mechanism for Ridesharing

An important research issue in ridesharing is how to split the cost of the routes to service travel requirements of travelers among the passengers. This cost sharing problem is highly interrelated with the routing problem described in the previous section, since the routes and schedules of the vehicles determine the operating cost that needs to be shared. In turn, the cost sharing mechanism imposes constraints on the routes and schedules that need to be optimized, for example, because the fares of passengers should not exceed their fare quotes. The cost sharing problem has largely been neglected in the literature since there are subsidies for dial-a-ride services. Without subsidies, the fares would substantially increase, and passengers would then be more concerned about how the operating cost is shared among them in a fair manner.

How passengers should share the operating cost in an online setting, where knowledge of future ride requests is missing, is a nontrivial problem for the following reasons: Passengers do not submit their ride requests at the same time but should be given incentives to submit their ride requests as early as possible to allow the ridesharing systems more time to find routing solutions that can offer all passengers lower fares due to synergies among all ride requests. Passengers have different start and end locations and thus cause different amounts of inconvenience to the other passengers, which should be reflected in the fares. Passengers should be quoted fares immediately after submitting their ride requests. This gives passengers certainty about the cost of service and allows the ridesharing system to plan routes better knowing which passengers have committed to participate. This requires ridesharing systems to make instantaneous and irreversible decisions despite having no knowledge of future ride requests (Brenner and Schäfer [9]).

Here, we define the online cost sharing problem for ridesharing and then present properties of cost sharing mechanisms for ridesharing systems that we believe are attractive to both vehicles and passengers—namely, online fairness, immediate response, individual rationality, budget balance, and ex post incentive compatibility. We propose a novel cost sharing mechanism, called proportional online cost sharing, and show that it satisfies all five properties, provided that the routes and schedules of the vehicles satisfy some restrictions (for example, minimize the operating cost).

3.1. Problem Description

In this section, we describe the online cost sharing problem for ridesharing systems. Passengers arrive (that is, submit their ride requests) sequentially by specifying their desired pickup and drop-off locations. The arrival time of a passenger is the time when it submits its ride request. We assume, for simplicity, that all passengers arrive before the vehicles start to service passengers. We also assume, without loss of generality, that exactly one passenger arrives at each time, where an arrival order is a function that maps arrival times to passengers.

The demand of the requested ride of each passenger is quantified by the alpha value; that is, α_p corresponds to the transport resources requested by passenger p . We assume that the alpha values are positive and independent of the arrival times of the passengers. These assumptions are, for example, satisfied by the shortest point-to-point travel distance from the pickup to the drop-off location of a passenger, which is the quantity that we use in this chapter for its alpha value.

The total cost at any given time under any arrival order is the operating cost required to service all the passengers arrived so far. We assume that (1) the total cost is normalized (the total cost is zero at time 0), (2) the total cost is nondecreasing in time, and (3) the total cost at any given time is independent of the arrival order of passengers. These assumptions are, for example, satisfied by the minimal operating cost required to service all passengers, which is the quantity that we use for the total cost. The ridesharing system can accommodate advanced features, such as operating times and capacities of vehicles and time constraints of passengers, as long as it can determine total costs that satisfy the assumptions.

The marginal cost of each passenger is the increase in total cost as a result of its arrival. The shared cost of each passenger is its share of the total cost determined by a cost sharing mechanism. (The sum of the shared costs of all passengers does not necessarily equal the total cost.) The ridesharing system provides (myopic) fare quotes to passengers immediately after their arrivals. The fare is quoted to a passenger after its arrival. (A fare quote of infinity means that the passenger cannot be serviced.) The fare limit w_p of passenger p is the maximum amount that it is willing to pay for its requested ride. A passenger drops out and is not serviced if its fare limit is lower than its fare quote. In this case, the ridesharing system simply pretends that the passenger never arrived, which explains why we assume that all passengers accept their fare quotes. Otherwise, the passenger accepts its fare quote and is serviced.

3.2. Desirable Properties

We now describe five desirable properties of cost sharing mechanisms (Fiestras-Janeiro et al. [15], Jain and Mahdian [25]) adapted to ridesharing systems.

Property 1. *Online fairness: The shared costs per alpha value of serviced passengers are never higher than the shared costs per alpha value of all passengers that arrive after them.*

Property 2. *Budget balance: The total cost equals the sum of the shared costs of all serviced passengers.*

Property 3. *Individual rationality: The shared costs of serviced passengers never exceed their fare limits.*

Property 4. *Immediate response: The shared costs of serviced passengers are monotonically nonincreasing in time.*

Property 5. *Ex post incentive compatibility: No passenger has an incentive to delay the submission of travel requests.*

Cost sharing mechanisms *should be fair*. There are several notions of fairness in the cost sharing literature. Proportional fairness, a widely accepted notion of fairness, implies that the shared cost per alpha value of passengers never exceeds the total cost per alpha value. Equitable fairness is a special case of proportional fairness that implies equality. *Budget balance* means that the total cost is shared by all serviced passengers. There are no profits or subsidies. The fares of serviced passengers should not exceed the fare quotes that they were provided at their arrival (to prevent them from dropping out later if they accept the fare quote). *Immediate response* guarantees that the shared costs of serviced passengers are monotonically nonincreasing in time. The best strategy of every passenger is to submit its ride request as early as possible, provided that all other passengers do not change their submit times and whether they accept or decline their fare quotes, because a passenger cannot decrease its shared cost by delaying its ride request submission.

3.3. Proportional Online Cost Sharing

We now describe a novel online cost sharing mechanism, called proportional online cost sharing (POCS), which satisfies all the properties listed above and is thus a first step toward addressing some of the problems raised by the missing knowledge of the future arrivals of passengers. The idea behind POCS is the following: POCS partitions passengers into coalitions, where coalitions contain all passengers that arrive within given time intervals (rather than, for example, all passengers served by the same vehicle). Initially, each newly arriving passenger forms its own coalition. However, passengers can choose to form coalitions with passengers that arrive directly after them to decrease their shared costs per alpha value, which implies the online fairness, immediate response, and ex post incentive compatibility properties. For example, the immediate response property is satisfied because passengers add other passengers to their coalitions only when this decreases their shared costs per alpha value and thus also their shared costs (since the alpha values are positive).

We start by defining the coalition cost per alpha value to be able to describe formally how POCS calculates the shared costs. Let $mc_{\pi(j)}$ be the marginal cost of the j th customer in arrival order π .

Definition 1. For all times k_1 , k_2 , and t and all arrival orders π with $k_1 \leq k_2 \leq t$, the coalition cost per alpha value of passengers $\pi(k_1), \dots, \pi(k_2)$ at time t under arrival order π is

$$ccpa_{\pi(k_1, k_2)} := \frac{\sum_{j=k_1}^{k_2} mc_{\pi(j)}}{\sum_{j=k_1}^{k_2} \alpha_{\pi(j)}}.$$

We now describe how POCS calculates the shared costs.

Definition 2. For all times k and t with $k \leq t$ and all arrival orders π , the shared cost of passenger $\pi(k)$ at time t under arrival order π is

$$cost_{\pi(k)}^t := \alpha_{\pi(k)} \min_{k \leq j \leq t} \max_{1 \leq i \leq j} ccpa_{\pi(i, j)}.$$

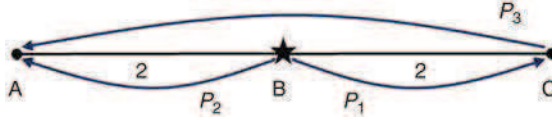
We now explain POCS in terms of a water flow model to provide intuition for its workings. Whenever a new passenger $\pi(t)$ arrives, it is allocated a new large container whose cross-sectional area is equal to its alpha value. The container is then filled with a volume of water that is equal to its marginal cost. The new container is connected via a one-way valve to the container of passenger $\pi(t-1)$ (if $t > 1$). The one-way valve allows water to flow only from (the) earlier container (of passenger) $\pi(t-1)$ to container $\pi(t)$. Finally, the water is allowed to reach equilibrium. We use $h_{\pi(k)}^t$ and

$$v_{\pi(k)}^t = \alpha_{\pi(k)} h_{\pi(k)}^t$$

to denote the level and volume, respectively, of the water in the container of passenger $\pi(t)$ at that point in time.

We demonstrate how POCS works based on the following example. There is one vehicle that can transport up to three passengers, starts at the star and incurs an operating cost of 10 for each unit of distance traveled, and does not need to return to its initial location. There are three passengers with arrival orders P_1 , P_2 , and P_3 . Figure 4 shows the ride requests of all the passengers, and Table 1 shows their alpha values and fare limits. Table 1 also shows the total costs after the arrival of each passenger and the marginal costs of all passengers under the assumption that all passengers are serviced. For example, the alpha value of passenger P_2 is the point-to-point travel distance from its pickup location B to its drop-off location A (B-A), and the total cost after the arrival of passenger P_2 is 10 times the minimal travel distance of the vehicle required to service passengers P_1 and P_2 from its initial location (B-C-B-A or B-A-B-C). Table 2 shows the water level and volume of

FIGURE 4. Ridesharing example.



all containers for the ridesharing example from Figure 4, and Figure 5 visualizes them for times 2 and 3. Clearly, the total volume of water in the containers of all passengers always corresponds to the total cost. The volume in each passengers containers corresponds to its shared cost according to POCS. A coalition is formed between all passengers that have the same water level.

In Furuhata et al. [19] we prove that POCS satisfies the online fairness, immediate response, individual rationality, and budget balance properties. We also show that it satisfies an ex post incentive compatibility property—that is, no passenger has an incentive to delay submission of its travel request, provided that other passengers do not change their submit times and whether they accept or decline their fare quotes.

3.4. Experimental Analysis

The cost sharing mechanism POCS requires the ability to compute the minimal operating cost for servicing a set of passengers. Calculating the minimal operating cost is typically an NP-hard problem and thus time consuming. However, ridesharing systems need to calculate the minimal operating cost every time a ride request is submitted, which would prevent them from operating in real time. This issue can be avoided by calculating quickly an operating cost that is not guaranteed to be minimal (Moulin [31]). We thus present an experimental study with a transport simulation where the ridesharing system uses a heuristic to compute a low operating cost that is not guaranteed to be minimal. In this case, the assumption that the total cost is independent of the arrival order of passengers (which implies that the decisions of passengers to accept their fare quotes or drop out, and thus also their fare quotes themselves, do not depend on the arrival order of passengers) is not satisfied. This assumption is used (only) to prove that POCS satisfies the ex post incentive compatibility property. We thus investigate whether the best strategy of every passenger remains to arrive truthfully—for example, because the likelihood of transport capacity still available tends to decrease over time.

To conduct our experimental analysis, we simulate the behavior of a system that uses POCS to assign costs to passengers and passengers that accept the fare quote to vehicles. This simulator first generates a given number of vehicles and passengers. Each vehicle is next characterized by its capacity, start location, end location, operating time window, and operating cost for each unit of distance traveled. Then each passenger is characterized by its truthful arrival time, start location, end location, pickup time window, drop-off time window, and fare limit. The settings of our simulator are slightly more general than what we have used in the ridesharing examples because operating time windows of vehicles and

TABLE 1. Ridesharing values.

	Passenger		
	P_1	P_2	P_3
Alpha value	2	2	4
Fare limit	60	60	60
Total cost	20	60	60
Marginal cost	20	40	0

TABLE 2. Water flow model values.

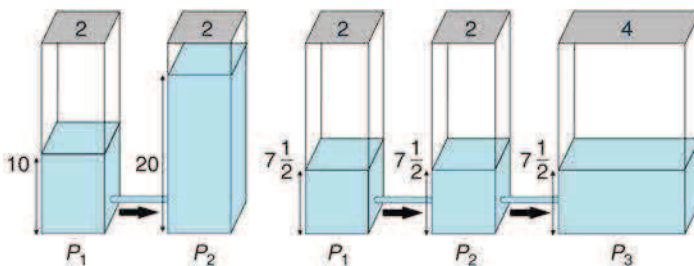
Time	Water level			Water volume		
	P_1	P_2	P_3	P_1	P_2	P_3
1	10			20		
2	10	20		20	40	
3	$7\frac{1}{2}$	$7\frac{1}{2}$	$7\frac{1}{2}$	15	15	30

pickup and drop-off time windows of passengers are taken into account. The simulation then simulates the arrival of each passenger. When a new passenger submits a ride request, the transport simulator requests from each vehicle the operating cost increase from adding the passenger to all passengers previously assigned to it, selects a vehicle with the lowest operating cost increase, and then uses POCS to calculate a fare quote for the passenger under the assumption that the passenger is assigned to the selected vehicle. If the fare limit of the passenger is lower than this fare quote, then the passenger drops out, and the transport simulator does not service it. Otherwise, the passenger accepts the fare quote, and the transport simulator adds it to all passengers previously assigned to the selected vehicle and then updates the shared costs of all the passengers previously assigned. Once a passenger is assigned to a vehicle, it is never reassigned to a different vehicle.

Each vehicle has to calculate its route and operating cost increase (or, equivalently, operating cost) when adding a new passenger to all the passengers previously assigned to it. The vehicle maintains an itinerary for all the passengers assigned to it—in the form of a sequence of locations. This includes its start location, its end location, and the start locations and end locations of all the passengers assigned to it. It calculates its travel distance as the shortest travel distance needed to visit all locations in the order given in its itinerary, and it calculates its operating cost as the product of its travel distance and its operating cost for each unit of distance traveled. We use a heuristic scheduling method (Nanry and Barnes [32], Xiang et al. [40]). In the construction phase of the heuristic, the vehicle uses a cheapest insertion method to construct a (feasible) itinerary by inserting the start location and end location of the new passenger into the cached itinerary for the passengers previously assigned to it. In the subsequent improvement phase of the heuristic, the shuttle uses tabu search (Gendreau and Potvin [22], Reeves [36]), a form of hill climbing, to improve the itinerary from the construction phase by reducing the operating cost of the vehicle. This heuristic solution is not guaranteed to equal the minimal operating cost and thus is not guaranteed to be independent of the arrival order of passengers.

3.4.1. Experiment 1. In Experiment 1, we evaluate the probability that passengers accept their fare quotes and, in case they do, how their fares depended on their arrival times. We perform 10,000 simulations with the transport simulator in a grid city of size 11×11 (that is, with 121 locations) and report average results. There are 25 vehicles that can each

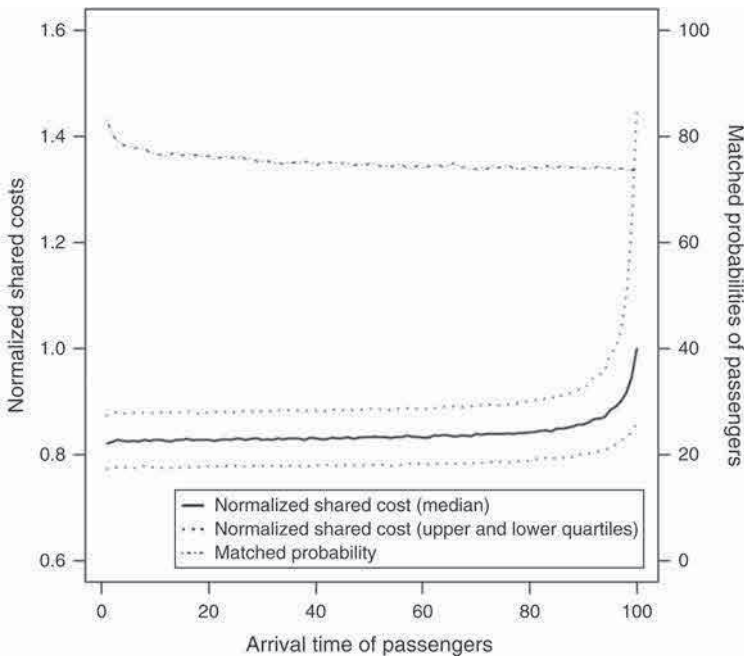
FIGURE 5. Water flow model visualization.



transport up to 10 passengers and operate the same hours from dawn (time 101) to dusk (time 1,440). We aim to set identical experimental conditions to focus on the effect of POCS on the shared costs of the passengers. We assume that passengers submit their ride requests before dawn (the departure time of the vehicles), since the total cost is not independent of their arrival orders otherwise. We also assume that vehicles have sufficient time to service all the passengers before dusk. The vehicles start at a depot in the center of the city. Each vehicle incurs an operating cost of 1 for each unit of distance traveled and needs to return to its initial location at dusk. There are 100 passengers that all arrive truthfully one at a time (that is, their arrival times range from time 1 to time 100). The start location of 20% of the passengers is the depot. The start locations of the other passengers and the end locations of all passengers are randomly selected from all locations with uniform probability. The pickup and drop-off time windows are identical for each passenger but might be different from passenger to passenger. Their lower bounds are dawn, and the differences between their upper and lower bounds are randomly selected from being 2.5 to 3.0 times higher than their alpha values (that is, the shortest point-to-point travel distances from their start locations to their end locations). Thus, passengers do not have tight schedules, resulting in low fare quotes. The fare limits of passengers are randomly selected from being 1.5 to 3.0 times higher than their alpha values. Thus, passengers have high fare limits. For both of these reasons, the fare quotes often do not exceed the fare limits. Many passengers therefore accept their fare quotes and are serviced.

Figure 6 shows the probability that passengers accept their fare quotes (“Matched probabilities of passengers”) as a function of their arrival times k —that is, the percentage of simulations with $cost_{\pi(k)}^k \leq w_{pi(k)}$. The probability that passengers accept their fare quotes is about 75%. It decreases with their arrival times (since their fare quotes tend to increase as their arrival times increase) but only very slowly. Figure 6 also shows the fares per alpha value of all passengers that accepted their fare quotes (“Normalized shared costs”) as a function of their arrival times k —that is, $cost_{\pi(k)}^{100}$ averaged over all simulations with $cost_{\pi(k)}^k \leq w_{pi(k)}$. The fares per alpha value of passengers increase as their arrival times

FIGURE 6. Results of experiment 1.



increase (as suggested by the online fairness property) but only very slowly. The only exception is the sharp increase for arrival times close to 100 since passengers that arrive then can no longer share their costs with a high number of passengers that arrive after them. Thus, Experiment 1 demonstrates that passengers have an incentive to arrive truthfully since their fare quotes and fares tend to increase as their arrival times increase. Thus, it is more likely that they accept their fare quotes and are serviced for low fares if they arrive as early as possible.

3.4.2. Experiment 2. The definition of ex post incentive compatibility states that the best strategy of every passenger is to arrive truthfully provided that all other passengers arrive truthfully and do not change whether they accept or decline their fare quotes: two assumptions are not guaranteed to be satisfied in practice. In Experiment 2, we therefore evaluate how likely it is that passengers can decrease their fares by delaying their arrivals if the second condition is removed. Experiment 2 is similar to Experiment 1, except that we distinguish four scenarios with different flexibilities of vehicles and passengers, and we use experimental parameters that decrease the scale of the experiment since each simulation is now more time consuming. We perform 1,000 simulations in a grid city of size 5×5 and report average results. Each simulation consists of at most 45 runs in addition to a run where passengers $P_1 \dots P_{10}$ arrive truthfully in order $P_1 \dots P_{10}$ —namely, runs where all passengers arrive truthfully except that passenger P_i delays its arrival and arrives only immediately after passenger P_j for all i and j with $1 \leq i < j \leq 10$, where passenger P_i accepts its fare quote when all passengers arrive truthfully. There are either 2 or 10 vehicles (for two scenarios) that can each transport up to 3 passengers, operate the same hours from dawn to dusk, and start at a depot in the center of the city. Each vehicle incurs an operating cost of 1 for each unit of distance traveled and needs to return to its initial location at dusk. There are 10 passengers that arrive one at a time (that is, their arrival times range from time 1 to time 10) before the vehicles start to service them. The start and end locations of all passengers are randomly selected from all locations with uniform probability. The pickup and drop-off time windows are identical for each passenger but might be different from passenger to passenger. Their lower bounds are dawn, and the differences between their upper and lower bounds are either 3.0 or 4.0 times (for two scenarios) higher than their alpha values. The fare limits of passengers are 3.0 times higher than their alpha values.

Table 3 shows, for each scenario, both the number of runs and the probabilities that passengers that delay their arrivals improve (since their fares decrease), do not change (since their fares remain unchanged), or worsen (since either their fare quotes increase sufficiently for them to drop out or if they do not drop out their fares increase). Experiment 2 demonstrates that passengers have an incentive to arrive truthfully since, in all scenarios, the probability that passengers that delay their arrivals improve their situations is lower than 20% while the probability that they worsen their situation is higher than 50%. Notice that Experiment 2 does not measure one advantage of passengers that delay their arrivals—namely, passengers that originally drop out because their fare quotes exceeded their fare limits, by delaying their arrivals, improve their fare quotes so they no longer have to drop

TABLE 3. Results of experiment 2.

Scenario	No. of shuttles	Time window	No. of runs	Situation improves (%)	No change (%)	Situation worsens (%)	
						Not drop out	Drop out
1	2	3.0	33,116	11	32	24	33
2	2	4.0	37,047	15	31	39	15
3	10	3.0	36,975	16	31	51	2
4	10	4.0	37,911	17	29	51	3

out. Also, Experiment 2 assumes that passengers delay their arrivals randomly (rather than strategically) because of missing knowledge of future arrivals of passengers. The probability that the situations for passengers that delay their arrivals worsen is zero if passengers are able to delay their arrivals strategically since they can always decide to arrive truthfully instead, in which case their situations do not change. We thus expect the probability that their situations improve to increase.

We now analyze Experiment 2 in Table 3 in more depth. We select 50 simulations randomly where passenger P_1 accepts its fare quote when all passengers arrive truthfully and, for each simulation, consider the nine runs where all passengers arrive truthfully except that passenger P_1 delays its arrival and arrives only immediately after passenger P_j for all j with $1 < j \leq 10$. The situation of passenger P_1 improves in 70 of the resulting 450 runs. The reason for 3 of these runs is that other passengers drop out when passenger P_1 delays its arrival, which reduces its fare. The reason for the remaining 67 runs is that the operating cost is not independent of the arrival order of passengers, both because passengers are never reassigned to different shuttles and because the scheduling method is nonoptimal—in particular, because its construction phase does not find feasible itineraries even when they exist (in which case the improvement phase is ineffective). We therefore expect that smarter scheduling methods will reduce the number of cases where passengers improve their situation by delaying their arrivals.

In summary, these simulation experiments show that an implementable version of POCS, where total operating costs are computed with efficient heuristics, still exhibits behavior in line with ex post incentive compatibility. That is, passengers have no incentive to postpone the announcement of their trip request. The first experiment shows that the shared cost of passengers increases as a function of the passenger arrival time, and the second experiment shows that it is unlikely that delaying a trip request reduces shared cost, as this only happens 15% of the time.

4. Traffic Equilibrium with Ridesharing

We now consider a transportation system where ridesharing has the ability to capture a significant portion of travel demand via dynamic matching. In this system we assume that the passengers pay drivers for the ridesharing services to share the travel cost. The ridesharing price is an abstraction to represent compensation that drivers take into account in their decision to participate in ridesharing, such as a reduction in travel time or toll costs that results from being able to use HOV lanes. We assume that the system operates as an open marketplace, and thus the ridesharing price will be determined by the market as well as congestion conditions.

The purpose of this line of research is to determine how attractive ridesharing will be to travelers in a given city. The key decision factors include the complex interaction between traffic congestion, the ridesharing price, and its adoption (i.e., the number of drivers and passengers that participate in ridesharing). For instance, to decide whether to participate in ridesharing, drivers may weigh the inconvenience, such as loss of privacy, against the compensation they may earn for taking on passengers. In turn, passengers would trade off the inconvenience, such as security concerns and loss of freedom, against the travel time and cost of a shared ride. These trade-offs would balance in an equilibrium that determines congestion, the ridesharing price, and the number of drivers and passengers that participate in ridesharing. For example, an increase in ridesharing could lead to a reduction in congestion and possibly an increase in the ridesharing price, which in turn would make driving more attractive. On the other hand, an increase in the ridesharing price would reduce the number of potential passengers, leading to an increase in potential drivers and congestion. Understanding how ridesharing would influence traffic congestion is fundamental in the evaluation of a ridesharing enterprise or in assessing the convenience of regulatory policies or incentives to promote ridesharing.

Here, we propose a new static traffic assignment model, based on a user equilibrium assumption, that takes into consideration the unique characteristics of ridesharing. Such a model could allow us to analyze how ridesharing and traffic congestion interact with each other and also to determine the impact that different regulatory interventions could have on ridesharing, and hence on traffic. Therefore the goal of this paper is to establish a static traffic assignment model that can determine (1) the ridesharing price, (2) how ridesharing will impact the traffic congestion conditions, and (3) the number of travelers that participate in ridesharing. Existing traffic assignment models have to be extended to consider the specific characteristics of ridesharing, where (1) the cost/price of ridesharing is determined by the number of people participating, and (2) the offer for shared rides (capacity of the transportation mode) varies with congestion and price. The approach to accomplish this is to combine two equilibrium models: a market pricing model that we refer to as the economic equilibrium system and a traditional traffic equilibrium system. Through common price and congestion parameters, the two systems interact with each other and thus become one integrated system that determines the prices and the congestion levels simultaneously.

Classic traffic assignment problems (TAPs) evaluate the distribution of travelers among different routes and origin–destination (OD) pairs. There are many methods to assign traffic to paths, a standard assumption in TAP is the user equilibrium assumption, also known as Wardrop’s user traffic equilibrium law (Wardrop [39]). According to this assumption, the travel times in all the used paths are equal to or less than those on unused paths. There could exist several fastest paths in this assumption, as long as they have the same travel (time) cost. By introducing a “utility” (or “disutility”) function, we obtain the elastic demand TAP, which decides not only how people will choose their paths but also how many people would travel given certain congestion conditions (Babonneau and Vial [4], Florian and Nguyen [16], Fukushima [17], Gartner [20, 21], Hearn and Yildirim [23], LeBlanc and Farhangian [28], Yildirim and Hearn [43]). When there are two or more transit nodes, we consider a multimode TAP, where two or more transit modes—say, private vehicle and public transit—are being studied simultaneously (Aashtiani and Magnanti [1], Abdulaal and LeBlanc [2], Beckmann et al. [5], Boyce and Bar-Gera [8], LeBlanc and Farhangian [28]).

4.1. Mathematical Model

To describe the interactions between traffic congestion and ridesharing activities in the market, we consider the elastic demand traffic assignment model under the user equilibrium assumption. Suppose drivers and passengers are travelling according to their own decisions. Drivers can decide whether or not to travel depending on both traffic congestion and ridesharing conditions (prices, number of travelers, etc.). Passengers may also decide whether or not to take a shared ride according to the traffic congestion and the ridesharing price.

The following model captures the above decision activities and the interactive impacts among traffic congestion, the number of travelers, and the price paid for ridesharing services. Such interactions include situations where (a) traffic congestion impacts the number of travelers and the ridesharing price; (b) the number of travelers (or more specifically, the number of drivers) determines traffic congestion, and also influences the price; and (c) the ridesharing price impacts the number of travelers and also the traffic congestion.

4.2. Problem Description

Consider a transportation network represented by a graph with nodes and arcs, where nodes could be origins, destinations, or intermediate stops, and arcs are direct roads that connect two nodes. Each individual travels from an origin to a destination, which is called an OD pair. For each OD pair, there exist multiple paths that start from the same origin and end at the same destination. The congestion cost of each path is evaluated by the travel time

along that path, which is a summation of travel times on each arc that builds up the path; *the travel time of each arc is determined by the number of vehicles (drivers) traveling on that link/arc*. Therefore, each arc may be shared by multiple paths (though not necessarily from the same OD pair), and conversely, each path of a certain OD pair may encounter drivers from other OD pairs. This fact is essential since a slight change in the number of drivers on one arc may impact the travel times or congestion costs of many paths. A user equilibrium is a state where for each OD pair, the travel times of all used paths are equal to or less than those that would be experienced by a single vehicle/driver on any unused path (Wardrop [39]).

In addition to user equilibrium, to include the specific features of ridesharing, we also assume the following.

- *Elastic demand*: The number of drivers (respectively, passengers) of each OD pair is not fixed. It is determined by drivers' (respectively, passengers') willingness to travel, i.e., the utility function. Drivers and passengers have different utility functions.

- *Unified driver utility function*: The utility function of drivers is identical for all drivers across OD pairs. For this model, one utility function is employed for all drivers, including both solo and ridesharing drivers. It is determined by traffic congestion and ridesharing prices.

- *Same OD pair*: Ridesharing drivers would only be willing to take on passengers that travel between the same OD pair, i.e., drivers would not pick up or drop off any passenger in the middle of their route, even if no detour is required. Not restricting ridesharing to the same OD pair would require more variables to keep track of how drivers may pick up or drop off a passenger in the middle of their trips. Ridesharing among travelers of the same OD pair can represent situations where each node in the graph represents a neighborhood (area) and there still exist a number of people traveling between the same two neighborhoods (areas). Such simplification helps us grasp the most essential features of ridesharing activities.

- *Congestion cost*: The travel time or the congestion cost is calculated only by the total number of vehicles/drivers (including both solo and ridesharing drivers). That is to say, the number of passengers in each vehicle does not contribute to the congestion cost.

- *Inconvenience cost*: The passenger pickup and drop-off times will be treated as part of the inconvenience cost of drivers.

- *The ridesharing prices*: Passengers would pay drivers some fee for the trip. The price is determined by the availability of vehicles and requests of passengers for each OD pair. The fee can be seen as a form of compensation to the drivers for the additional cost and inconvenience, and it turns out to limit the number of passengers in the network.

- *Unlimited capacity*: The vehicle capacity (i.e., the number of passengers per vehicle) is unlimited. Under this assumption, we do not distinguish different types of drivers in the model, since all passengers can be squeezed into one vehicle or may be distributed evenly among all available vehicles. It is reasonable from the perspective of the total travel cost: a driver can be either driving alone or sharing a ride only when the costs of the two are equal. In other words, suppose the travel cost of a solo driver is only the congestion cost, while the travel cost of a ridesharing driver is the congestion cost plus the inconvenience cost minus the ridesharing income. In this case, the inconvenience cost of ridesharing will be canceled by the income (or profit) of sharing a ride. Otherwise, all drivers would prefer the lower total cost: either driving alone or taking on passengers. In this work we ignore how passengers are assigned to vehicles/drivers since other behavioral considerations (security, environmental conscience, economics) influence these decisions.

- *Aggregate form*: On the basis of the above assumptions, we are not treating drivers or passengers individually. We are considering an aggregate model per OD pair; i.e., all the drivers of each OD pair are collaborating, and so too do all the passengers. Such simplification helps us understand how ridesharing activities would impact the traffic congestion at a system level.

Given the above assumptions, the main interactions between travelers, ridesharing prices and traffic congestion can be summarized as below:

- The traffic congestion cost is calculated by the total number of drivers.
- The total number of drivers is determined by traffic congestion and the ridesharing prices according to the drivers' willingness to travel on the road. Similarly, the number of passengers is also determined by traffic congestion and the ridesharing prices according to the passengers' willingness to participate in ridesharing.
- The ridesharing prices are determined by the interaction between ridesharing drivers and passengers, and they are also dependent on the congestion cost.

Our goal is to derive a model that reflects the above interactions and provides the number of travelers (both drivers and passengers), ridesharing prices, and congestion cost (travel time) for each OD pair at the equilibrium.

Below is a list of notations that we use to formulate an elastic demand TAP model that includes ridesharing. Note that we use an arc-based formulation of the traffic assignment user equilibrium model such as in Babonneau and Vial [4] or Nie [34]:

- \mathcal{N} = set of nodes.
- \mathcal{A} = set of arcs.
- $\mathcal{O} \subseteq \mathcal{N}$ = set of origins.
- $\mathcal{D} \subseteq \mathcal{N}$ = set of destinations.
- \mathcal{K} = set of OD pairs; $\mathcal{K} \subseteq \mathcal{O} \times \mathcal{D}$.
- $k \in \mathcal{K}$ = OD pair, where $k = (o_k, d_k)$, $o_k \in \mathcal{O}$, $d_k \in \mathcal{D}$.
- $a \in \mathcal{A}$ = arc.
- p_k = ridesharing price for each passenger of OD pair $k \in \mathcal{K}$.
- q_k = number passengers for OD pair $k \in \mathcal{K}$.
- $\lambda_k^{(0)}$ = free flow time (fixed) for OD pair $k \in \mathcal{K}$.
- λ_k = congestion cost for OD pair $k \in \mathcal{K}$; $\lambda_k \geq \lambda_k^{(0)}$.
- δ_k = total number of drivers for OD pair k .
- x_a^k = amount of flow (number of drivers) for OD pair $k \in \mathcal{K}$ on arc $a \in \mathcal{A}$.
- y_a = total amount of flow on arc $a \in \mathcal{A}$, $y_a = \sum_{k \in \mathcal{K}} x_a^k$.
- $\boldsymbol{\delta}$ = vector with components δ_k , $\boldsymbol{\delta} \in \mathbb{R}^{|\mathcal{K}|}$.
- \mathbf{x}^k = vector with components x_a^k with respect to k , $\mathbf{x}^k \in \mathbb{R}^{|\mathcal{A}|}$.
- \mathbf{x} = vector with components x_a^k , $\mathbf{x} \in \mathbb{R}^{|\mathcal{K}| \times |\mathcal{A}|}$.
- \mathbf{y} = vector with components y_a , $\mathbf{y} \in \mathbb{R}^{|\mathcal{A}|}$.

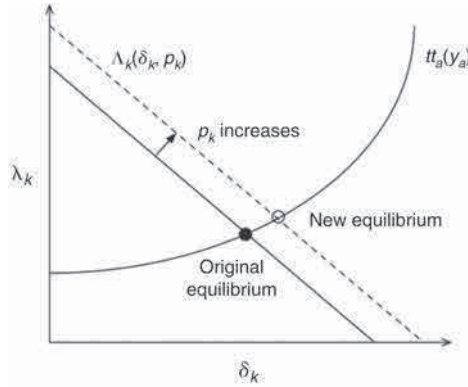
4.3. Adding Ridesharing Prices to an Elastic Demand TAP

In the elastic demand traffic assignment problem, the objective function consists of two components: the sum of the integrals of the congestion cost over all the arcs and the sum of the integrals of the utility function over all OD pairs.

We consider that each arc a has a congestion function $tt_a(y_a)$ to represent the travel cost/time of traversing arc a when there is an arc flow of y_a on that arc. We assume that this is a strictly increasing function of y_a . The classic BPR (Bureau of Public Roads) function Dafermos and Sparrow [13] is one of the most widely used congestion functions.

The utility function of drivers is denoted by $\Lambda_k(\delta_k, p_k)$, which is a function of the total number of drivers δ_k and ridesharing price p_k for OD pair k . It provides the worst congestion cost the drivers could endure given the number of vehicles and the ridesharing price. Therefore it represents an aggregate utility for all drivers (including both solo drivers and ridesharing drivers) according to certain traffic congestion and ridesharing prices. The above utility function looks similar to one in the standard elastic demand model, except that it includes the ridesharing price as a second variable. In elastic demand models, this driver utility function decreases with δ_k as having more drivers (more congestion) makes a trip less

FIGURE 7. Traffic equilibrium.



appealing. The dependence on the ridesharing price will capture the fact that a payment for taking on passengers can be a form of compensation to drivers. We therefore assume that the utility function increases with the ridesharing price; i.e., drivers may accept worse traffic conditions if there is an increase in their compensation of providing ridesharing services while the number of drivers stays the same.

From the above definitions, we have the following relationship between the number of drivers and the traffic congestion (see Figure 7): when the number of drivers (or the amount of traffic flow) δ_k (or y_a) increases, the congestion cost $tt_a(y_a)$ will also increase, whereas the utility $\Lambda_k(\delta_k, p_k)$ will decrease. Also, the utility $\Lambda_k(\delta_k, p_k)$ increases if the ridesharing price p_k goes up. The traffic equilibrium is attained balancing these two relationships.

The ridesharing price for every OD pair should be determined by the balance between supply and demand for shared rides in the market. The economic equilibrium system (Hubbard and O'Brien [24]) is adopted to describe the trade-off between the ridesharing price and the number of drivers and passengers, where drivers are the supply and passengers are the demand. Let us denote by $\mathcal{S}_k(q_k)$ the aggregate supply function and by $\mathcal{D}_k(q_k)$ the aggregate demand function for each OD pair k . These functions represent for the supply (respectively, demand) the price at which drivers are willing to offer (respectively, passengers are willing to pay) given the number of available ridesharing seats (respectively, the number of passengers) q_k . The price p_k and the number of passengers q_k is determined when these two functions are equal; that is, $\mathcal{S}_k(q_k) = \mathcal{D}_k(q_k) = p_k$ for each OD pair k .

In the next subsection we will give an explicit form to these supply and demand functions. In particular, we include the fact (Burris and Winn [11], Kelley [26], LeBlanc [27]) that the willingness of drivers to participate in ridesharing is increasing with the congestion cost, making it possible for drivers to ask for a lower price. Therefore the supply function is decreasing in the congestion level λ_k , giving a supply function of the form $\mathcal{S}_k(q_k, \lambda_k)$. We note that these aggregate supply and demand functions should be such that ridesharing is only possible when there are drivers traveling on that OD pair.

In sum, we are trying to evaluate the ridesharing market by modifying an elastic demand traffic assignment model and incorporating the balance between supply and demand for shared rides in every OD pair. This can be expressed as the following model:

$$\min_{x, y, \delta, p} \sum_{a \in \mathcal{A}} \int_0^{y_a} tt_a(s) ds - \sum_{k \in \mathcal{K}} \int_0^{\delta_k} \Lambda_k(r, p_k) dr \quad (31)$$

$$\text{s.t. } \mathbf{N}\mathbf{x}^k - \mathbf{\Delta}^k \delta_k = 0, \quad \forall k \in \mathcal{K}, \quad (32)$$

$$\sum_{k \in \mathcal{K}} x_a^k - y_a = 0, \quad \forall a \in \mathcal{A}, \quad (33)$$

$$\mathcal{S}_k(q_k, \lambda_k) = \mathcal{D}_k(q_k) = p_k, \quad \forall k \in \mathcal{K}, \quad (34)$$

$$x_a^k \geq 0, \quad \forall a \in \mathcal{A}, \forall k \in \mathcal{K}. \quad (35)$$

Constraints (33) represent the flow decomposition constraint (i.e., the total amount of flow on each arc equals the sum of flows over all OD pairs on that arc). Constraint (34) describes the demand–supply balancing constraint from above, where λ_k equals the congestion that δ_k drivers create. Constraint (35) is the nonnegativity constraint of variables.

Constraint (32) depicts the flow conservation constraint in a compact form, where \mathbf{N} and Δ^k are the coefficient matrix and vector; $\mathbf{N} = [(i, a)]_{i \in \mathcal{N}, a \in \mathcal{A}}$ is an $|\mathcal{N}| \times |\mathcal{A}|$ matrix with element

$$(i, a) = \begin{cases} 1 & \text{node } i \in \mathcal{N} \text{ is the tail of arc } a \in \mathcal{A} \text{ (i.e., } a = (j, i)), \\ -1 & \text{node } i \in \mathcal{N} \text{ is the head of arc } a \in \mathcal{A} \text{ (i.e., } a = (i, j)), \\ 0 & \text{otherwise,} \end{cases}$$

and $\Delta^k = (\Delta_i^k)_{i \in \mathcal{N}}$ is a vector in $\mathbb{R}^{|\mathcal{N}|}$ for any $k \in \mathcal{K}$, with element

$$\Delta_i^k = \begin{cases} 1 & i = d_k \in \mathcal{D}, \\ -1 & i = o_k \in \mathcal{O}, \\ 0 & \text{otherwise.} \end{cases}$$

In other words, for each OD pair k , constraint (32) is a compact form of $|\mathcal{N}|$ constraints, each of which is a flow conservation constraint at each node: (a) if it is a demand (destination) node, all incoming flows minus all outgoing flows should be equal to the demand of OD pair k ; (b) if it is a supply (origin) node, the difference should be the negative value of the demand; or (c) if otherwise, the difference must be zero.

Note that in this model we are focusing on the congestion cost caused by the drivers *only*; therefore the objective (31) is defined *only* with respect to the drivers' total congestion cost and total utility cost. The passengers' utility will be taken into account when determining the ridesharing prices (i.e., in constraint (34)). Model (31) ~ (35) shows how the ridesharing price influences the drivers' willingness to travel and the traffic congestion.

4.4. Computational Results

The traffic assignment model (31)–(35) above can be solved for specific driver utility, ridesharing supply, and ridesharing demand functions; see Xu et al. [41]. We now present computational results that show how modifying problem parameters influences ridesharing price and participation (of both drivers and passengers). The instances solved and the Frank–Wolfe method used to solve them are described in detail in Xu et al. [41].

We conduct sensitivity with respect to three parameters of the utility functions for the players:

β : This parameter influences the congestion function. As β increases, the congestion decreases.

ϵ : This parameter influences the passenger utility function. As ϵ increases, the passenger utility function increases.

σ : This parameter influences the driver inconvenience function. As σ increases, the driver inconvenience function increases.

We present results for high congestion ($\beta = 1$) and low congestion ($\beta = 10$) cases. Furthermore, for each β setting, we consider nine combinations of the parameters ϵ and σ . That is, ϵ and σ can each take values 1, 2, and 4. This means that we consider cases where passenger utility is low, medium, or high (varying ϵ) and cases where a driver can be inconvenienced a little, moderately, or a lot (varying σ). In the results that follow, these nine instances are labeled by $\#\epsilon\sigma$; in other words, instance #24 means that ϵ was set to 2 and σ to 4.

FIGURE 8. Sensitivity analysis of disutilities.

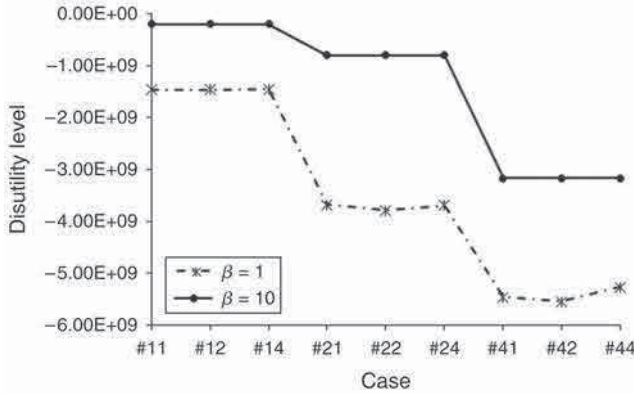


Figure 8 shows how the elastic demand part of the objective function $\sum_{k \in \mathcal{K}} \int_0^{\delta_k} \Lambda_k(r, p_k) dr$ or disutility changes as we vary these parameters. We conclude that both β and ϵ have a significant impact on the disutility levels. At $\beta = 1$, when the congestion scale is high, the disutility levels are lower than for $\beta = 10$, when the congestion scale is much smaller. In other words, the utility levels (negative values of disutility values) are higher when the congestion levels are higher because the traffic equilibrium is a balance between the congestion and the utility functions. Consequently, there must be enough of a benefit to attract travelers using this mode of transportation; otherwise, people will choose alternative modes as a result of high congestion.

Figure 9 shows one example of the distribution of price among all the 528 OD pairs and their corresponding free flow times. It can be seen in Figure 9 that the ridesharing price is close to half of the free flow time for each OD pair. While the free flow times are sorted in ascending order, the price is not strictly increasing with them. This serves to illustrate the impact of congestion on the prices. The closer congestion λ_k is to the free flow time $\lambda_k^{(0)}$, the higher the price p_k is above the half free flow time.

Figure 10 plots the number of drivers for all 18 cases. The number of drivers is consistent with the congestion levels discussed in the previous section, since the congestion is a quartic function of arc flows, which are linear combinations of the number of drivers. From Figure 10, we find that β mainly decides the scale of the number of drivers. A smaller β corresponds to more drivers.

FIGURE 9. Price distribution.

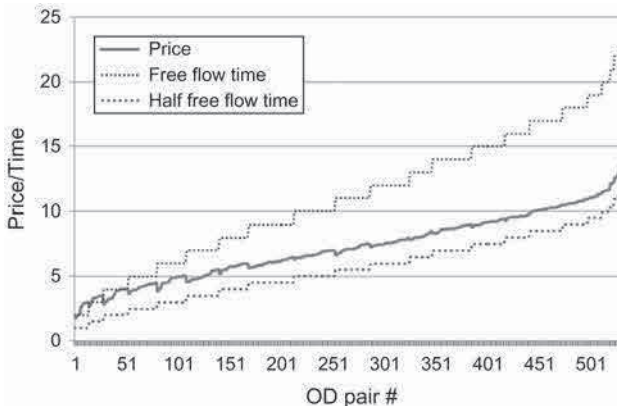
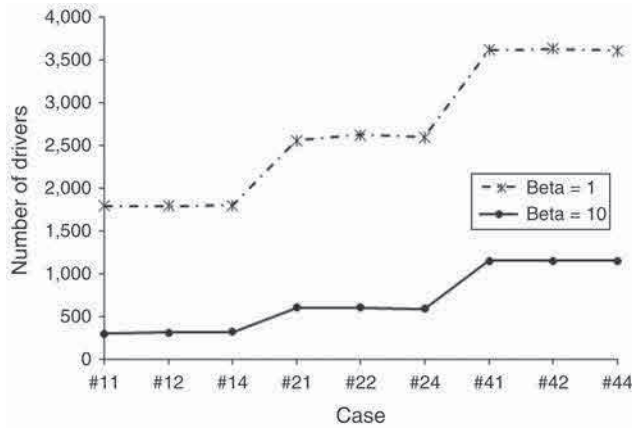


FIGURE 10. Sensitivity analysis of the number of drivers.



5. Conclusions

This tutorial covers three different research lines motivated by the sudden surge in dynamic ridesharing systems enabled by new communication and computation technologies. These lines investigate new vehicle routing problems, cost sharing mechanisms, and traffic assignment models that explicitly consider ridesharing.

Today's transportation systems consider load-dependent travel time and route cost, in the form of HOV lanes, high-occupancy toll (or HOT) lanes, or reduced tolls. In this scenario a driver must take these load-dependent costs and travel time into account to properly quantify the cost of participating in ridesharing. A 0-1 integer programming model is presented that can be solved efficiently with heuristic methods. Computational results show that, as a participant in ridesharing becomes more flexible in time, the less one should have to pay for the trip. Also, there is significant benefit to considering the toll cost savings and time savings with additional pickups. We evaluated the sensitivity to HOV lanes using two different measures: a distance ratio and a ride time ratio. From the results, we see that, when time savings in HOV lanes become more significant, the distance ratio will increase while the ride time ratio will decrease. This indicates that people will be more willing to take a detour to share a ride with others or to capture faster paths if the ride time can be decreased significantly. However, if it is too difficult to be qualified to use HOV lanes (for example, HOV4 lanes), it becomes less attractive to take detours to share a ride.

Next, we described properties of cost sharing mechanisms that make ridesharing systems attractive to both vehicle providers and passengers—namely, online fairness, immediate response, individual rationality, budget balance, and ex post incentive compatibility. We proposed the POCS, or proportional online cost sharing, mechanism, that provides passengers with upper bounds on their fares immediately after they submit ride requests, allowing them to accept their fare quotes or drop out. One challenge is that POCS requires computing the optimal operating cost, which can be computationally challenging. The computational results presented show that even if this optimal operational cost is only approximated, the resulting approximate POCS still shows that ride cost increases with the request submission time. Furthermore, the percentage of passengers that improves their trip cost by delaying is only 14%.

Finally, we developed analytic and planning tools to understand the effect of ridesharing on traffic conditions. Such knowledge is crucial to evaluate the ridesharing system proposed and to create mechanisms to design, regulate, and operate such a market. We present a model that describes the complex interactions among the traffic congestion, the ridesharing price, and the number of travelers (including solo drivers, ridesharing drivers, and passengers). We assumed that all the drivers and passengers who are sharing the same car (or ride) must travel on the same OD pairs. The model is formulated as a traffic assignment problem

with elastic demands. The computational results show that (1) the ridesharing base price influences the congestion level, (2) within a certain price range, an increase in price may reduce the traffic congestion, and (3) the utilization of ridesharing increases as the congestion increases.

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