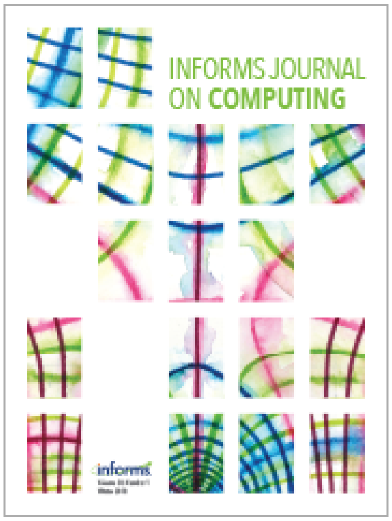

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Efficient Convexification Strategy for Generalized Geometric Programming Problems

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Abstract. Generalized geometric programming (GGP) problems consist of a signomial being minimized in the objective function subject to signomial constraints, and such problems have been utilized in various fields. After modeling numerous applications as GGP problems, solving them has become a significant requirement. A convex underestimator is considered an important concept to solve GGP problems for obtaining the global minimum. Among convex underestimators, variable transformation is one of the most popular techniques. This study utilizes an estimator to solve the difficulty of selecting an appropriate transformation between the exponential transformation and power convex transformation techniques and considers all popular types of transformation techniques to develop a novel and efficient convexification strategy for solving GGP problems. This proposed convexification strategy offers a guide for selecting the most appropriate transformation techniques on any condition of a signomial term to obtain the tightest convex underestimator. Several numerical examples in the online supplement are presented to investigate the effects of different convexification strategies on GGP problems and demonstrate the effectiveness of the proposed convexification strategy with regard to both solution quality and computation efficiency.

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Keywords: convex programming • generalized geometric programming • convexification strategy • variables transformation technique • global optimization

1. Introduction

The generalized geometric programming (GGP) problem, first introduced by Passy and Wilde (1967), seeks an approximate global optimum to a signomial objective function subject to a set of signomial constraints. A GGP problem can be expressed as the following model (the P1 model):

$$\text{Minimize } \sum_{p=1}^{G} C_p(\mathbf{x}) + \sum_{p=1}^{H} S_p(\mathbf{x})$$

$$\text{subject to } \sum_{p=1}^{G_w} C_{w,p}(\mathbf{x}) + \sum_{p=1}^{H_w} S_{w,p}(\mathbf{x}) \le l_w,\ w = 1, \dots, s,$$

where $\mathbf{x} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^T$, $0 < \underline{x}_i \le x_i \le \overline{x}_i$, $\underline{x}_i$, and $\overline{x}_i$ are the lower and upper bounds of x_i for $i = 1, 2, \dots, n$, respectively; $S_p(\mathbf{x}) = c_p \prod_{i=1}^{n} x_i^{\alpha_{p,i}}$ and $S_{w,p}(\mathbf{x}) = c_{w,p} \prod_{i=1}^{n} \cdot x_i^{\alpha_{w,p,i}}$ are nonconvex signomial terms; $C_p(\mathbf{x})$ and $C_{w,p}(\mathbf{x})$ are convex functions or linear functions; and $\alpha_{p,i}$, $\alpha_{w,p,i}$, c_p, $c_{w,p}$, and l_w are constants.

GGP problems have been utilized in various fields, including analog circuitry (Daems et al. 2003, Dam and Mandal 2012), digital circuitry (Boyd et al. 2005), nonconvex portfolio optimization (Kallrath 2003), wireless circuit design (Li 2009), and inventory management (Jung and Klein 2001). After the modeling of numerous applications as GGP problems, solving them has become a significant requirement. A large number of local algorithms for solving GGP problems exist in the current literature. Compared with local optimization algorithms, global optimization algorithms are relatively scarce. An efficient global approach also remains lacking. This study focuses on developing an efficient convexification strategy that considers all popular types of transformation techniques to obtain tighter convex underestimators.

Pardalos and Romeijn (2002) provided an impressive overview of global approaches. Among those approaches, convex underestimator is one of the most popular methods to solve a nonconvex function for obtaining global optimization. Adjiman et al. (1998) proposed the αBB type underestimator, which can be applied to any twice differentiable function. Tardella (2003) studied the class of functions with which convex envelope on a polyhedron coincides. Akrotirianakis and Floudas (2004a, 2004b) introduced a new class of convex underestimators for twice continuously differentiable nonlinear functions and proved that the resulting convex relaxation is better than the αBB one. Floudas et al. (2005) presented various convex underestimators. Meyer and Floudas (2005) described the structure of the polyhedral convex envelopes of edge-concave functions over polyhedral domains using geometric arguments and showed the improvements over the classical αBB convex underestimators for box-constrained optimization problems. Caratzoulas and Floudas (2005) developed a convex underestimator for trigonometric functions. Gounaris and Floudas (2008) presented a piecewise application of the αBB method that produces a tight convex underestimator for multivariate function.

Among convex underestimators, variable transformation is one of the most frequently used techniques. By applying variable transformations, nonconvex signomial terms $S_p(\mathbf{x})$ and $S_{w,p}(\mathbf{x})$ in the P1 model are replaced with convex underestimators $S_p^C(\mathbf{x}, \hat{\mathbf{z}})$ and $S_{w,p}^C(\mathbf{x}, \hat{\mathbf{z}})$, respectively. An additional variable vector $\mathbf{z}$ is linked to original variable vector $\mathbf{x}$ through the nonlinear transformation function ($\mathbf{z} = T(\mathbf{x})$), and $\hat{\mathbf{z}}$ in the convex underestimators is the piecewise linear function of $T(\mathbf{x})$ as follows:

$$\hat{\mathbf{z}} = L(T(\mathbf{x})). \tag{1}$$

Therefore, the replaced P1 model will become a convex program for achieving global optimization.

Given that the application of variable transformations to the nonconvex signomial term will move the nonconvexities from the original one to the nonconvex transformation function ($T(\mathbf{x})$), removing the nonconvexities of the nonconvex transformation function by the piecewise linearization technique is necessary. The piecewise linear functions in Equation (1) are important. Through the convex underestimators and piecewise linear functions, the entire replaced P1 model becomes a mixed-integer convex problem. Floudas (2000) indicated that this model is guaranteed to reach finite ε-convergence to global optimization.

The transformation techniques applied to positive signomial terms can be briefly classified into negative power transformation (NPT) (Pörn et al. 2008, Lu et al. 2010), exponential transformation (ET) (Maranas and Floudas 1997, Pörn et al. 2008), and power convex transformation (PCT) (Björk et al. 2003, Lundell et al. 2009, Lu 2012). In contrast, the transformation technique for negative signomial terms only involves power transformation (PT) (Björk et al. 2003, Pörn et al. 2008, Li and Lu 2009). Lundell (2009) and Lundell and Westerlund (2012) provided a clear overview of the aforementioned techniques. Such techniques and their varied forms are widely used to obtain a convex underestimator for developing numerous convexification strategies.

Björk et al. (2003) developed a convexification strategy for signomial terms and introduced the concept of PCT to obtain a convex underestimator of a signomial term. Li and Tsai (2005) developed a convexification strategy with NPT, ET, and PT techniques to obtain the global optimization of the signomial term with free-sign variables. Pörn et al. (2008) specified how to obtain a convex underestimator of a signomial term without necessarily transforming variables through the ET technique under specific conditions. Lundell et al. (2009) proposed a convexification strategy for signomial terms, especially with PCT and ET techniques. Li and Lu (2009) utilized the NPT and PT techniques to develop a convexification strategy for GGP problems with mixed free-sign variables. Tsai and Lin (2011) proposed a global approach for solving GGP problems. They utilized the NPT and ET techniques, as well as the superior piecewise linear functions proposed by Li et al. (2009), to improve convexification efficiency. Lu (2012) used a beta parameter to improve the efficiency of PCT in its convexification strategy. Lundell and Westerlund (2009a) proved in theorem that the PCT and ET techniques always result in a tighter convex underestimator compared with the NPT technique. However, Lundell and Westerlund (2009b) showed that the PCT technique only presents a tighter convex underestimator in parts of the domain compared with the ET technique. At this stage, explicit conditions for selecting ET or PCT remain unknown.

Although numerous convexification strategies have been proposed in prior studies, an efficient convexification strategy that considers all types of transformation techniques has not yet been developed. This study first utilizes an estimator to solve the difficulty pointed by Lundell and Westerlund (2009b) and then proposes an efficient convexification strategy that considers all types of transformation techniques for solving GGP problems. The proposed convexification strategy has the following general rules:

General rule 1. The amount of transformations should be kept to a minimum.

General rule 2. The underestimator should be as tight as possible when using the same number of transformations in a nonconvex term.

General rule 3. The most efficient piecewise linearization technique should always be used to remove the nonconvexities from the nonconvex transformation function $T(\mathbf{x})$ in Equation (1).

These general rules are the fundamentals of the proposed convexification strategy. Prior general rules favor the higher priority than the posterior general rule. For general rules 1 and 2, this study offers an estimator for selecting an appropriate transformation between the ET and PCT techniques and proposes a convexification strategy for selecting the most appropriate transformation techniques on any condition of a signomial term to obtain the tightest convex underestimator. For general rule 3, efficient special ordered sets of type 2 (SOS2) proposed by Vielma and Nemhauser (2011) are used to represent the piecewise linear functions with a logarithmic number of binary variables and constraints.

The remainder of this study is organized as follows. Section 2 discusses all types of variable transformation techniques for obtaining a convex underestimator and recommends the most efficient one for each type of variable transformation technique. Section 3 presents the analysis of the gap between nonconvex functions and convex underestimators as well as the proposed convexification strategy, which yields an excellent solution quality and exhibits a good computational efficiency. Section 4 concludes this study. Several numerical examples in the online supplement demonstrate the advantages of the proposed convexification strategy.

2. Variable Transformation Techniques for Convex Underestimators

To discuss the transformation techniques clearly, describing the convexity characterizations of a signomial term is necessary. All variables in this study are positive. The following propositions are considered.

Proposition 1 (Lu 2012). *For a twice-differentiable signomial term $S(\mathbf{x}) = c\prod_{i\in I} x_i^{\alpha_i}$ with the set $I = \{1, \dots, n\}$, $S(\mathbf{x})$ will be convex if and only if the constant c and the exponent $\alpha_i (i \in I)$ satisfy one of the following conditions:*

(i) $c > 0, \alpha_i < 0 \forall i \in I$;

(ii) $c > 0$, $\sum_{i=1}^{n} \alpha_i \geq 1$, $\alpha_j > 1, \alpha_i < 0 \forall i \in I \setminus \{j\}$, $j \in I$; *or*

(iii) $c < 0, \alpha_i > 0 \forall i \in I$, $0 < \sum_{i=1}^{n} \alpha_i \leq 1$. □

Proposition 2 (Lu 2012). *Given a positive signomial term $S(\mathbf{x}) = c\prod_i x_i^{\alpha_i}$, where $c > 0$, if the exponents of $S(\mathbf{x})$ satisfy $\sum_i \alpha_i^{-1} < 0$, the convexity of f can be measured by Equation (2):*

$$p = 1/\textstyle\sum_i \alpha_i. \tag{2}$$

$S(\mathbf{x})$ is a convex function if and only if $p \leq 1$, whereas f is a 1-convex function when $p = 1$. □

Proposition 1 provides a clear way for verifying the convexity of a signomial term. Proposition 2 adopts the concept of pth power convex functions proposed by Lindberg (1981) and provides an appropriate indication for measuring the convexity of positive signomial terms. A smaller p value in Equation (2) indicates that the function f is more convex, whereas the maximal value of p is limited to 1. Function f is 1-convex, which signifies a convex function that is barely convex when the p value equals 1. Björk et al. (2003) showed that a 1-convex function is more efficient than other pth power convex functions for a convex underestimator in a convexification strategy. Therefore, given a nonconvex signomial term $S(\mathbf{x})$, transforming $S(\mathbf{x})$ into a convex underestimator $S^C(\mathbf{x}, \hat{\mathbf{z}})$ satisfying condition (ii) in Proposition 1 is more efficient than $S^C(\mathbf{x}, \hat{\mathbf{z}})$ satisfying condition (i) when the number of variables to be transformed is the same in both conditions. In contrast to condition (i), condition (ii) in Proposition 1 can obtain a 1-convex function when $\sum_{i\in I}\alpha_i = 1$. Proposition 2 can be a cost indication of convexifying a signomial term as a convex underestimator when the sign of the signomial term is positive.

For example, in convexifying $S_1 = x_1 x_2^{-1}$ into a –0.5-convex underestimator $S_1' = z_{1a}^{-1} x_2^{-1}$, the inverse transformation function $z_{1a} = x_1^{-1}$ is required. In another method, the required transformation function for convexifying S_1 into a 1-convex underestimator $S_1'' = z_{1b}^2 x_2^{-1}$ is $z_{1b} = x_1^{0.5}$. Thus, S_1' (–0.5-convex underestimator) is more convex than S_1''(1-convex underestimator), while z_{1a} is more concave than z_{1b}. Consequently, convexifying S_1 into a smaller –0.5-convex underestimator (S_1') is unnecessary if it can be convexified into a larger 1-convex underestimator (S_1'') because the cost of convexifying S_1 into S_1'' is the lowest.

2.1. Variable Transformation Techniques for Positive Signomial Terms

Denote $S(\mathbf{x})$ in Equation (3) as an example of a nonconvex positive signomial term. The current variable transformation techniques for positive signomial terms are described in the following context:

$$S(\mathbf{x}) = c\prod_{i=1}^{n} x_i^{\alpha_i},\ c > 0,\ \alpha_1 \leq \cdots \leq \alpha_m < 0 < \alpha_{m+1} \leq \cdots \leq \alpha_n,$$
$$0 < \underline{x}_i \leq x_i \leq \overline{x}_i \forall i. \tag{3}$$

2.1.1. ET Technique. The ET technique was first proposed by Maranas and Floudas (1997), and it is the most popular convexification technique in global optimization. An exponential function is always a convex function irrespective of the variable domain, and the Napierian logarithm function can easily decompose the signomial term into a set of separate power functions. This technique applies variable transformation $x_k = \exp(z_k)$ when the inverse transformation function $z_k = \ln x_k$ is approximated by piecewise linear function $\hat{z}_k$. Pörn et al. (2008) efficiently applied the ET technique to obtain a convex underestimator of a signomial term. This approach is expressed in the following proposition.

Proposition 3 (ET Technique (Pörn et al. 2008)). *The convex underestimator of $S(\mathbf{x})$ in Equation (3) with $c > 0$ is expressed as follows:*

$$S_{ET}^C(\mathbf{x}, \hat{\mathbf{z}}) = c\prod_{j=1}^{m} x_j^{\alpha_j} \cdot \exp\Big(\sum_{k=m+1}^{n} (\alpha_k \hat{z}_k)\Big), \tag{4}$$

$$\hat{z}_k = L(\ln(x_k)),\ k = m+1, \dots, n, \tag{5}$$

where $L(\ln(x_k))$ in Equation (5) are the piecewise linear functions of $\ln(x_k)$. □

2.1.2. NPT Technique. This technique applies transformation $x_k = z_k^{-1/\beta}$, where $\beta > 0$ when the inverse transformation function $z_k = x_k^{-\beta}$ is approximated by piecewise linear function $\hat{z}_k$. Lu et al. (2010) proposed the NPT technique with a small value of β to improve the tightness of the convex underestimator. However, the method of evaluating the β value in Lu et al. (2010) does not consider the influence of the number of breakpoints. The current study refines the evaluation of β in the NPT technique as follows.

Proposition 4 (Refined NPT Technique). *The convex underestimator of $S(\mathbf{x})$ in Equation (3) with $c > 0$ is expressed as follows:*

$$S_{NPT}^{C}(\mathbf{x}, \hat{\mathbf{z}}) = c\prod_{j=1}^{m} x_j^{\alpha_j} \cdot \prod_{k=m+1}^{n} \hat{z}_k^{-\frac{\alpha_k}{\beta}}, \tag{6}$$

$$\hat{z}_k = L(x_k^{-\beta}), \quad k = m+1, \ldots, n, \tag{7}$$

where β is a positive tiny constant, and $L(x_k^{-\beta})$ in Equation (7) is the piecewise linear function of $x_k^{-\beta}$. For each piecewise linear function $L(x_k^{-\beta})$, there are r breakpoints given as follows: $a_{k,1}, a_{k,2}, \ldots, a_{k,r}$. Considering the floating-point round-off error in the computer and letting ε be the feasibility tolerance, the minimal value of β can be found by Constraints (8) and (9):

$$\beta \geq a_{k,w+1}^{2}\frac{\varepsilon}{a_{k,w+1} - a_{k,w}}, \quad k = m+1, \ldots, n, \quad w = 1, \ldots, r-1, \tag{8}$$

$$\frac{\alpha_k}{\beta} \in \mathbb{Z}, k = m+1, \ldots, n. \tag{9}$$

Proof. (i) The convex underestimator constructed by the NPT technique (Equations (6) and (7)) can be referred to Lu et al. (2010). (ii) The difference value of $f(a_{k,w})$ and $f(a_{k,w+1})$ is $a_{k,w}^{-\beta} - a_{k,w+1}^{-\beta}(w = 1, \ldots, r-1)$. Since $f(x_k) = x_k^{-\beta}$ is a strictly monotonically decreasing function, there must exist a value $x_k^* \in (a_{k,w}, a_{k,w+1})$ for satisfying Equation (10):

$$a_{k,w}^{-\beta} - a_{k,w+1}^{-\beta} = -f'(x_k^*)(a_{k,w+1} - a_{k,w}). \tag{10}$$

Equation (10) must exceed the feasibility tolerance ε as follows:

$$a_{k,w}^{-\beta} - a_{k,w+1}^{-\beta} = -f'(x_k^*)(a_{k,w+1} - a_{k,w}) > \varepsilon. \tag{11}$$

Since $-f'(x_k^*) > -f'(a_{k,w+1})$, Equation (11) can be simplified as follows:

$$\begin{aligned} a_{k,w}^{-\beta} - a_{k,w+1}^{-\beta} &= -f'(x_k^*)(a_{k,w+1} - a_{k,w}) > \\ &\quad -f'(a_{k,w+1})(a_{k,w+1} - a_{k,w}) \\ &= \beta a_{k,w+1}^{-\beta-1}(a_{k,w+1} - a_{k,w}) \geq \varepsilon, \\ \beta &\geq a_{k,w+1}^{\beta+1}\frac{\varepsilon}{(a_{k,w+1} - a_{k,w})}. \end{aligned} \tag{12}$$

Since $0 < \beta \leq 1$, Equation (12) can be simplified as Constraint (8).

(iii) Given that β should be small as possible, then α_k/β in Equation (6) will become a large and rational number. In floating computation, a power function with a large and rational exponent will exacerbate the total floating-point round-off error. Constraint (9) will prevent this problem in the convex function in Equation (6). □

The term $S_{NPT}^{C}(\mathbf{x}, \hat{\mathbf{z}})$ in Equation (6) satisfies condition (i) in Proposition 1 and is a p-convex function, where $p = 1/(\sum_{j=1}^{m}\alpha_j + \sum_{k=m+1}^{n} -\alpha_k/\beta)$. The tightness of $S_{NPT}^{C}(\mathbf{x}, \hat{\mathbf{z}})$ increases when the value of β decreases near to zero. Lundell and Westerlund (2009a) showed that $S_{ET}^{C}(\mathbf{x}, \hat{\mathbf{z}})$ in Equation (4) always yields tighter convex underestimations than $S_{NPT}^{C}(\mathbf{x}, \hat{\mathbf{z}})$ in Equation (6). They also proved that the tightness of $S_{NPT}^{C}(\mathbf{x}, \hat{\mathbf{z}})$ in Equation (6) is almost equal to that of $S_{ET}^{C}(\mathbf{x}, \hat{\mathbf{z}})$ in Equation (4) when $Q_k \to -\infty$.

2.1.3. PCT Technique. The PCT technique is based on the concept of power convex functions introduced by Lindberg (1981). Björk et al. (2003) first convexified a nonconvex signomial term into a 1-convex function. Lundell and Westerlund (2009a) and Lundell et al. (2009) used the PCT technique in their convexification strategies. Lu (2012) used a β parameter to improve the efficiency of the PCT technique in its convexification strategy. Lu (2012) showed that an appropriate small value of β can obtain a tight convex underestimator for accelerating the convergence process to achieve global optimization and prevent the floating-point round-off error in the computation. However, the method of evaluating the β value in Lu (2012) does not fully consider the influence of the number of breakpoints. The present study refines the evaluation of β in the PCT technique as follows.

Proposition 5 (Refined PCT Technique). *The convex underestimator of $S(\mathbf{x})$ in Equation (3) with $c > 0$ is expressed as follows:*

$$\begin{aligned} S_{PCT}^{C}(\mathbf{x}, \hat{\mathbf{z}}) &= c\prod_{j=1}^{m} x_j^{\alpha_j} \cdot \prod_{k=m+1}^{n} \hat{z}_k^{\gamma_k}, \ \gamma_k = -\frac{\alpha_k}{\beta} \ \textit{for} \\ & k = m+1, \ldots, n-1, \ \gamma_n = 1 - \sum_{j=1}^{m}\alpha_j - \sum_{k=m+1}^{n-1}\gamma_k, \end{aligned} \tag{13}$$

$$\hat{z}_k = L(x_k^{\alpha_k/\gamma_k}), k = m+1, \ldots, n, \tag{14}$$

where β is a positive tiny constant, γ_k is an integer, and $L(x_n^{\alpha_n/\gamma_n})$ in Equation (14) is the piecewise linear function of $x_n^{\alpha_n/\gamma_n}$. For each piecewise linear function $L(x_n^{\alpha_n/\gamma_n})$, there are r breakpoints given as follows: $a_{k,1}, a_{k,2}, \ldots, a_{k,r}$. Considering the floating-point round-off error in the computer and letting

ε be the feasibility tolerance, the minimal value of β can be found by Constraints (15)–(17):

$$\beta \geq \frac{\sum_{k=m+1}^{n-1} \alpha_k}{\sum_{j=1}^{m} \alpha_j - 1 + \frac{\alpha_n(a_{n,w+1} - a_{n,w})}{\varepsilon a_{n,w+1}}}, w = 1, \ldots, r-1, \tag{15}$$

$$\beta \geq a_{k,w+1}^2 \frac{\varepsilon}{a_{k,w+1} - a_{k,w}}, k = m+1, \ldots, n-1, w = 1, \ldots, r-1, \tag{16}$$

$$\gamma_k \in \mathbb{Z}, k = m+1, \ldots, n. \tag{17}$$

Proof. (i) According to Proposition 2, $S_{PCT}^C(\mathbf{x}, \hat{\mathbf{z}})$ in Equation (13) is a 1-convex function since $\sum_{j=1}^{m}\alpha_j + \sum_{k=m+1}^{n}\gamma_k = 1$, and $S_{PCT}^C(\mathbf{x}, \hat{\mathbf{z}})$ satisfies the condition (ii) of Proposition 1. (ii) Constraint (15) can be referred to Lu (2012). (iii) Constraints (16) and (17) are similar to Proposition 4. □

2.2. Variable Transformation Technique for Negative Signomial Terms

Only the PT technique can be used for convexifying negative signomial terms. Björk et al. (2003) proposed the PT technique to convexify the negative signomial term by transforming all variables in this signomial term, and the PT technique is based on condition (iii) of Proposition 1. Lu (2012) introduced a threshold T to determine which variables should be transformed from the original term. Given that the approach of Lu (2012) is complete and concise, it is considered in the present study and expressed in the following proposition.

Proposition 6 (PT Technique (Lu 2012)). *Denote $S(\mathbf{x})$ in Equation (18) as a nonconvex signomial term with negative sign,*

$$S(\mathbf{x}) = c\prod_{i=1}^{n} x_i^{\alpha_i}, \; c < 0, \; \alpha_1 \leq \cdots \leq \alpha_m < 0 < \alpha_{m+1} \leq \cdots \leq \alpha_s \leq \cdots \leq \alpha_n, \; \sum_{j=m+1}^{s} \alpha_j \leq 1 - T, \tag{18}$$

where $0 < \underline{x}_i \leq x_i \leq \overline{x}_i \forall i$, and T is a positive constant and is the threshold that determines which variables are needed to be transformed from $S(\mathbf{x})$. Let $I = \{1, 2, \ldots, n\}$ and $J = \{j | \sum_{j=m+1}^{s} \alpha_j \leq 1 - T\}$; then the convex underestimator of $S(\mathbf{x})$ is expressed as follows:

$$S_{PT}^C(\mathbf{x}, \hat{\mathbf{z}}) = c\prod_{j\in J} x_j^{\alpha_j} \cdot \prod_{k \in I\backslash J} \hat{z}_k^{\gamma_k}, \; \gamma_k = \Big(1 - \sum_{j\in J} \alpha_j\Big) \frac{|\alpha_k|}{\sum_{s\in I\backslash J} |\alpha_s|}, \tag{19}$$

$$\hat{z}_k = L\big(x_k^{\frac{\alpha_k}{\gamma_k}}\big), k \in I\backslash J, \tag{20}$$

where $L(x_n^{\alpha_n/\gamma_n})$ in Equation (20) is the piecewise linear function of the power function $x_n^{\alpha_n/\gamma_n}$. The value of T in Equation (18) is small, and the requiring number of variables needed to be transformed is less. However, smaller T will increase the convexities of the transformed variables and will require more breakpoints in the piecewise linear function ($\hat{z}_k$ in Equation (20)). □

Regarding the value of T in Proposition 6, we consider term $S = -x_1^{0.4}x_2^{0.5}x_3^2$. If $T = 0.1$, then $J = \{1, 2\}$ since $0.4 + 0.5 \leq 1 - 0.1$ and $I = \{3\}$. The term S is convexified as $S_{PT}^C = -x_1^{0.4}x_2^{0.5}\hat{z}_3^{0.1}$, where $\hat{z}_3 = L(x_3^{20})$. If $T = 0.5$, then $J = \{1\}$ since $0.4 \leq 1 - 0.5$ and $I = \{2, 3\}$. The term S is convexified as $S_{PT}^C = -x_1^{0.5}\hat{z}_2^{3/25}\hat{z}_3^{12/25}$, where $\hat{z}_2 = L(x_2^{25/6})$ and $\hat{z}_3 = L(x_3^{25/6})$. Whether the value of T is small or large, the convex underestimator ($S_{PT}^C(\mathbf{x}, \hat{\mathbf{z}})$) of S must satisfy condition (iii) in Proposition 1.

3. Proposed Convexification Strategy

For a nonconvex signomial term, different variable domains and exponent parameters form varying nonconvexities. Therefore, several variable transformation techniques are proposed to convexify these varying conditions. Four variable transformation (ET, NPT, PCT, and PT) techniques are currently used, but no variable transformation technique can perfectly fulfill all conditions of arbitrary nonconvex signomial terms. The core job of the proposed convexification strategy is to both select an appropriate variable transformation technique depending on the conditions of a nonconvex signomial term and improve the efficiency and accuracy of the convexification strategy.

For example, a nonconvex signomial term $x_1^{-1}x_2^{-2}x_3^2$ can be convexified into $x_1^{-1}x_2^{-2}\hat{z}_3^{-2}$ by the NPT technique. This term can be further convexified through PCT for a tighter underestimator as $x_1^{-1}x_2^{-2}\hat{z}_3^4$. However, an efficient convexification strategy that considers all variable transformation techniques discussed in Section 2 has not been created. Therefore, this study develops an efficient convexification strategy that not only considers all variable transformation techniques but also follows the general rules in Section 1. Before developing an efficient convexification strategy, the comparison of variable transformation techniques is discussed.

For a nonconvex positive signomial term, Pörn et al. (2008) and Lundell and Westerlund (2009a) proved that the ET technique always yields a tighter convex underestimator than the NPT technique when the number of transformed variables is the same in both techniques. Björk et al. (2003) and Lundell and Westerlund (2009b) found that PCT is more efficient than ET on positive signomial terms with only one variable needed to be transformed. However, the computational efficiency and solution quality of the PCT technique are worse than those of the ET technique in numerous scenarios. Conversely, ET outperforms PCT when the number of transformed variables increases. Lundell and Westerlund (2009b) proved that applying PCT to a positive signomial term only yields a tighter convex underestimator than ET in parts of the target domain. Considering that the explicit conditions for selecting ET or PCT have not been identified, Proposition 7 presents an effective estimator for determining the ET or PCT technique for positive signomial terms.

Proposition 7. *For a twice-differentiable nonconvex positive signomial term $S(\mathbf{x}) = c\prod_{i=1}^{n} x_i^{\alpha_i}$, where $c > 0$, $\alpha_1 \le \ldots \le \alpha_m < 0 < \alpha_{m+1} \le \ldots \le \alpha_n$, and $0 < \underline{x}_i \le x_i \le \overline{x}_i \forall i$, the function $S(\mathbf{x})$ should be transformed into a convex underestimation by PCT in Proposition 5 when the result of Equation (21) is greater than or equal to zero; otherwise, $S(\mathbf{x})$ should be transformed by ET in Proposition 3:*

$$\prod_{k=m+1}^{n} PCT_k - \prod_{k=m+1}^{n} ET_k, \tag{21}$$

where PCT_k

$$= \begin{cases} \dfrac{1}{m_k(\gamma_k+1)}\left(\overline{x}_k^{\frac{\alpha_k(\gamma_k+1)}{\gamma_k}} - \underline{x}_k^{\frac{\alpha_k(\gamma_k+1)}{\gamma_k}}\right) \text{ if } \gamma_k \ne -1, \\ \qquad\qquad m_k = \dfrac{\overline{x}_k^{\alpha_k/\gamma_k} - \underline{x}_k^{\alpha_k/\gamma_k}}{\overline{x}_k - \underline{x}_k}, \text{ and} \\ \dfrac{\alpha_k}{m_k\gamma_k}\ln\left(\dfrac{\overline{x}_k}{\underline{x}_k}\right) \text{ if } \gamma_k = -1 \end{cases}$$

$$ET_k = \frac{\overline{x}_k - \underline{x}_k}{\alpha_k(\ln \overline{x}_k - \ln \underline{x}_k)}\left(\overline{x}_k^{\alpha_k} - \underline{x}_k^{\alpha_k}\right).$$

Proof. (i) The convex underestimations of $S(\mathbf{x})$ can be derived by Equations (4) and (5) (ET in Proposition 3) and Equations (13) and (14) (PCT in Proposition 5), respectively. For simplicity, only two breakpoints ($\underline{x}_k$ and $\overline{x}_k$) are used for the piecewise linear functions $L(\ln(x_k))$ in Equation (5) and $L(x_k^{\alpha_k/\gamma_k})$ in Equation (14). Those piecewise linear functions can be expressed as follows:

$$L(\ln(x_k)) = \ln \underline{x}_k + m_k'(x_k - \underline{x}_k), m_k' = \frac{\ln \overline{x}_k - \ln \underline{x}_k}{\overline{x}_k - \underline{x}_k}, \tag{22}$$

$$L(x_k^{\alpha_k/\gamma_k}) = \underline{x}_k^{\alpha_k/\gamma_k} + m_k(x_k - \underline{x}_k), m_k = \frac{\overline{x}_k^{\alpha_k/\gamma_k} - \underline{x}_k^{\alpha_k/\gamma_k}}{\overline{x}_k - \underline{x}_k}. \tag{23}$$

(ii) Based on ET technique and Equation (22), the difference between $S(\mathbf{x})$ and its convex underestimator can be calculated by the following definite integral:

$$\begin{aligned} ET_{diff} &= \int_{\underline{x}_n}^{\overline{x}_n}\cdots\int_{\underline{x}_1}^{\overline{x}_1}(S(\mathbf{x}) - S_{ET}^C(\mathbf{x},\hat{\mathbf{z}}))dx_1\cdots dx_n, \\ ET_{diff} &= \left(\int_{\underline{x}_n}^{\overline{x}_n}\cdots\int_{\underline{x}_1}^{\overline{x}_1} S(\mathbf{x})dx_1\cdots dx_n\right) - c\int_{\underline{x}_n}^{\overline{x}_n}\cdots\int_{\underline{x}_1}^{\overline{x}_1}\prod_{j=1}^{m} x_j^{\alpha_j} \\ &\quad\cdot \exp\left(\sum_{k=m+1}^{n}(\alpha_k L(\ln x_k))\right)dx_1\cdots dx_n. \end{aligned} \tag{24}$$

Let $ET_k = \int_{\underline{x}_k}^{\overline{x}_k}\exp(\alpha_k L(\ln x_k))dx_k = \int_{\underline{x}_k}^{\overline{x}_k}\exp(\alpha_k(\ln \underline{x}_k + m_k'(x_k - \underline{x}_k))dx_k$ for $k = m+1,\ldots,n$; and let $u_k' = \alpha_k(\ln \underline{x}_k + m_k'(x_k - \underline{x}_k))$, $du_k' = \alpha_k m_k' dx_k$. Then, ET_k can be evaluated as follows:

$$\begin{aligned} ET_k &= \frac{1}{\alpha_k m_k'}\left(\exp(\alpha_k(\ln \underline{x}_k + m_k'(x_k - \underline{x}_k))\Big|_{\underline{x}_k}^{\overline{x}_k}\right) \\ &= \frac{1}{\alpha_k m_k'}\left(\overline{x}_k^{\alpha_k} - \underline{x}_k^{\alpha_k}\right) = \frac{\overline{x}_k - \underline{x}_k}{\alpha_k(\ln \overline{x}_k - \ln \underline{x}_k)}\left(\overline{x}_k^{\alpha_k} - \underline{x}_k^{\alpha_k}\right). \end{aligned}$$

Equation (24) can be expressed as $ET_{diff} = \left(\int_{\underline{x}_n}^{\overline{x}_n}\cdots\int_{\underline{x}_1}^{\overline{x}_1} \cdot S(\mathbf{x})dx_1\cdots dx_n\right) - c\prod_{j=1}^{m}\frac{\overline{x}_i^{\alpha_i+1} - \underline{x}_i^{\alpha_i+1}}{\alpha_i+1}\prod_{k=m+1}^{n} ET_k$.

(iii) Based on PCT technique and Equation (23), the difference between $S(\mathbf{x})$ and its convex underestimator can be calculated by the following definite integral:

$$\begin{aligned} PCT_{diff} &= \int_{\underline{x}_n}^{\overline{x}_n}\cdots\int_{\underline{x}_1}^{\overline{x}_1}(S(\mathbf{x}) - S_{PCT}^C(x,\hat{z}))dx_1\cdots dx_n, \\ PCT_{diff} &= \left(\int_{\underline{x}_n}^{\overline{x}_n}\cdots\int_{\underline{x}_1}^{\overline{x}_1} S(\mathbf{x})dx_1\cdots dx_n\right) \\ &\quad - c\left(\int_{\underline{x}_n}^{\overline{x}_n}\cdots\int_{\underline{x}_1}^{\overline{x}_1}\prod_{j=1}^{m} x_j^{\alpha_j}\prod_{k=m+1}^{n} L(x_k^{\alpha_k/\gamma_k})dx_1\cdots dx_n\right). \end{aligned} \tag{25}$$

Let $PCT_k = \int_{\underline{x}_k}^{\overline{x}_k} L(x_k^{\alpha_k/\gamma_k})dx_k = \int_{\underline{x}_k}^{\overline{x}_k}(\underline{x}_k^{\alpha_k/\gamma_k} + m_k(x_k - \underline{x}_k))^{\gamma_k}dx_k$ for $k = m+1,\ldots,n$; and let $u_k = \underline{x}_k^{\alpha_k/\gamma_k} + m_k(x_k - \underline{x}_k)$, $du_k = m_k dx_k$. Then, PCT_k can be evaluated as follows:

$$PCT_k = \begin{cases} \dfrac{1}{m_k(\gamma_k+1)}\left((\underline{x}_k^{\alpha_k/\gamma_k} + m_k(x_k - \underline{x}_k))^{\gamma_k+1}\Big|_{\underline{x}_k}^{\overline{x}_k}\right) \\ \quad = \dfrac{1}{m_k(\gamma_k+1)}\left(\overline{x}_k^{\frac{\alpha_k(\gamma_k+1)}{\gamma_k}} - \underline{x}_k^{\frac{\alpha_k(\gamma_k+1)}{\gamma_k}}\right), \gamma_k \ne -1. \\ \dfrac{1}{m_k}\left(\ln(\underline{x}_k^{\alpha_k/\gamma_k} + m_k(x_k - \underline{x}_k))\Big|_{\underline{x}_k}^{\overline{x}_k}\right) \\ \quad = \dfrac{1}{m_k}\left(\ln \overline{x}_k^{\alpha_k/\gamma_k} - \ln \underline{x}_k^{\alpha_k/\gamma_k}\right) = \dfrac{\alpha_k}{m_k\gamma_k}\ln\left(\dfrac{\overline{x}_k}{\underline{x}_k}\right), \\ \quad \gamma_k = -1. \end{cases}$$

Equation (25) can be expressed as:

$$PCT_{diff} = \left(\int_{\underline{x}_n}^{\overline{x}_n}\cdots\int_{\underline{x}_1}^{\overline{x}_1} S(\mathbf{x})dx_1\cdots dx_n\right) - c\prod_{i=1}^{m}\frac{\overline{x}_i^{\alpha_i+1} - \underline{x}_i^{\alpha_i+1}}{\alpha_i+1}\prod_{k=m+1}^{n} PCT_k.$$

(iv) Therefore, it is clear to obtain Equation (26):

$$ET_{diff} - PCT_{diff} = c\prod_{i=1}^{m}\frac{\overline{x}_i^{\alpha_i+1} - \underline{x}_i^{\alpha_i+1}}{\alpha_i+1}\left(\prod_{k=m+1}^{n} PCT_k - \prod_{k=m+1}^{n} ET_k\right). \tag{26}$$

If Equation (26) is greater than or equal to zero, it means that the error generated by ET is much greater than that generated by PCT. Therefore, the convex underestimator derived by PCT is tighter than the convex underestimator derived by ET. Because $c > 0$ and $(\overline{x}_i^{\alpha_i+1} - \underline{x}_i^{\alpha_i+1})/(\alpha_i + 1) > 0$ for $i = 1, \ldots, m$, the sign of Equation (26) can be simplified as follows:

$$ET_{diff} - PCT_{diff} = \prod_{k=m+1}^{n} PCT_k - \prod_{k=m+1}^{n} ET_k. \quad \square$$

According to the preceding discussion, the ET, NPT, and PCT techniques produce different tightness of convex underestimators for a nonconvex positive signomial term. If the number of transformed variables in these techniques is the same and the value of estimator in Equation (21) is nonnegative, the tightness of each technique is ranked as follows:

PCT technique > ET technique > NPT technique.

Otherwise, if the value of estimator in Equation (21) is negative, the tightness of each technique is ranked as follows:

ET technique > PCT technique > NPT technique.

A signomial term with three positive variables, $S(\mathbf{x}) = cx_1^{\alpha_1}x_2^{\alpha_2}x_3^{\alpha_3}$, where $\alpha_3 \geq \alpha_2 \geq \alpha_1$, is considered to describe the proposed convexification strategy on the basis of the general rules in Section 1. The convexification strategies for positive and negative signomial terms are listed in Tables 1 and 2, respectively. In Tables 1 and 2, the convexification rules specify how to obtain a tight convex underestimator. Extending the strategy to a signomial term containing any number of variables is significant.

In Tables 1 and 2, $L(x_i^{\alpha_i/\gamma_i})$ and $L(\ln x_i)$ are the piecewise linear functions of the nonlinear functions $x_i^{\alpha_i/\gamma_i}$ and $\ln x_i$, respectively. The accuracy of piecewise linear functions increases with a large number of breakpoints. The terms $S^C(\mathbf{x}, \hat{\mathbf{z}})$ and $S_w^C(\mathbf{x}, \hat{\mathbf{z}})$ are the convex underestimators of the objective function and the wth constraint in the P1 model, respectively. The factors affecting the computational efficiency of the convexification strategy include the tightness of the constructed convex underestimator and the efficiency of the piecewise linearization technique. Tables 1 and 2 indicate that the proposed convexification strategy can offer a guide for selecting the most appropriate variable transformation to construct the tightest convex underestimator. Beale and Tomlin (1970) proposed a formulation for expressing a piecewise linear function with SOS2 constraints. Vielma and Nemhauser (2011) proposed an efficient SOS2 constraint with a logarithmic number of binary variables and constraints and constructed a mixed-integer linear programming (MILP) to piecewise the nonlinear function. Computational results of Vielma and Nemhauser (2011) revealed that the piecewise linearization technique in Proposition 8 outperforms others.

Proposition 8 (Vielma and Nemhauser 2011). *$L(T(x))$ is denoted as the piecewise linear function of the nonlinear function $T(x)$ and $a_w(w = 1, \ldots, r; 0 \leq a_1 < a_2 < \ldots < a_r)$ as*

Table 1. Proposed Convexification Strategy for Positive Signomial Term ($c > 0$)

Rule	Condition	Convexification rule
1	$0 > \alpha_3 \geq \alpha_2 \geq \alpha_1$.	No need for convexification. $S(\mathbf{x})$ is already a convex function by condition (C1) in Proposition 1.
2	$\alpha_3 > 0 > \alpha_2 \geq \alpha_1$, $\alpha_3 \geq 1 - \alpha_1 - \alpha_2$.	No need for convexification. $S(\mathbf{x})$ is already a convex term by condition (C2) in Proposition 1.
3	$\alpha_3 > 0 > \alpha_2 \geq \alpha_1$, $\alpha_3 < 1 - \alpha_1 - \alpha_2$.	Use the PCT technique in Proposition 5 to obtain a convex underestimator, and the convex underestimator is a 1-convex function. $S_{PCT}^C(\mathbf{x}, \hat{\mathbf{z}}) = cx_1^{\alpha_1}x_2^{\alpha_2}\hat{z}_3^{\gamma_3}$, $\gamma_3 = 1 - \alpha_1 - \alpha_2$, $\hat{z}_3 = L(x_3^{\alpha_3/\gamma_3})$.
4	$\alpha_3 \geq \alpha_2 > 0 > \alpha_1$, $\alpha_3 \geq 1 - \alpha_1$.	Use the PCT technique in Proposition 5 to obtain the convex underestimator, and the convex underestimator is a 1-convex function. $S_{PCT}^C(\mathbf{x}, \hat{\mathbf{z}}) = cx_1^{\alpha_1}\hat{z}_2^{\gamma_2}x_3^{\alpha_3}$, $\gamma_2 = (1 - \alpha_1 - \alpha_3)$, $\hat{z}_2 = L(x_2^{\alpha_2/\gamma_2})$.
5	$\alpha_3 \geq \alpha_2 > 0 > \alpha_1$, $\alpha_3 < 1 - \alpha_1$.	Evaluate the value of the estimator in Equation (21). If the value is greater than or equal to zero, use the PCT technique in Proposition 5 to build the convex underestimator of $S(\mathbf{x})$ as follows: $S_{PCT}^C(\mathbf{x}, \hat{\mathbf{z}}) = cx_1^{\alpha_1}\hat{z}_2^{\gamma_2}\hat{z}_3^{\gamma_3}, \gamma_2 = -\alpha_2/\beta, \gamma_3 = 1 - \alpha_1 + \alpha_2/\beta\ \hat{z}_i = L(x_i^{\alpha_i/\gamma_i}), i = 2, 3$. If the value of the estimator in Equation (21) is less than zero, the convex underestimator of $S(\mathbf{x})$ can obtain by the ET technique in Proposition 3 as follows: $S_{ET}^C(\mathbf{x}, \hat{\mathbf{z}}) = cx_1^{\alpha_1}\exp(\alpha_2\hat{z}_2 + \alpha_3\hat{z}_3)$, $\hat{z}_i = L(\ln x_i), i = 1, 2$.
6	$\alpha_3 \geq \alpha_2 \geq \alpha_1 > 0$.	Evaluate the value of the estimator in Equation (21). If the value is greater than or equal to zero, use the PCT technique in Proposition 5 to build the convex underestimator of $S(\mathbf{x})$ as follows: $S_{PCT}^C(\hat{\mathbf{z}}) = c\hat{z}_1^{\gamma_1}\hat{z}_2^{\gamma_2}\hat{z}_3^{\gamma_3}, \gamma_1 = -\alpha_1/\beta, \gamma_2 = -\alpha_2/\beta, \gamma_3 = 1 - \alpha_1/\beta + \alpha_2/\beta\ \hat{z}_i = L(x_i^{\alpha_i/\gamma_i}), i = 1, 2, 3$. If the value of estimator in Equation (21) is less than zero, the convex underestimator of $S(\mathbf{x})$ can be obtained by ET technique in Proposition 3 as follows: $S_{ET}^C(\hat{\mathbf{z}}) = c\exp(\alpha_1\hat{z}_1 + \alpha_2\hat{z}_2 + \alpha_3\hat{z}_3)$, $\hat{z}_i = L(\ln x_i), i = 1, 2, 3$.

Table 2. Proposed Convexification Strategy for Negative Signomial Term ($c < 0$)

Rule	Condition	Convexification rule
7	$\alpha_3 \geq \alpha_2 \geq \alpha_1 > 0$, $\alpha_1 + \alpha_2 + \alpha_3 \leq 1$.	No need for convexification. $S(\mathbf{x})$ is already a convex term by condition (C3) in Proposition 1.
8	$\alpha_3 \geq \alpha_2 \geq \alpha_1 > 0$, $\alpha_1 + \alpha_2 + \alpha_3 > 1$, $\alpha_1 + \alpha_2 < 1$.	Use the PT technique in Proposition 6 to obtain the convex underestimator as follows: $S^C_{PT}(\mathbf{x}, \hat{\mathbf{z}}) = cx_1^{\alpha_1}x_2^{\alpha_2}\hat{z}_3^{\gamma_3}$, $\gamma_3 = (1 - \alpha_1 - \alpha_2)$, $\hat{z}_3 = L(x_3^{\alpha_3/\gamma_3})$.
9	$\alpha_3 \geq \alpha_2 \geq \alpha_1 > 0$, $\alpha_1 + \alpha_2 > 1$, $\alpha_1 < 1$.	Use the PT technique in Proposition 6 to obtain the convex underestimator as follows: $S^C_{PCT}(\mathbf{x}, \hat{\mathbf{z}}) = cx_1^{\alpha_1}\hat{z}_2^{\gamma_2}\hat{z}_3^{\gamma_3}$, $\gamma_i = \frac{(1-\alpha_1)\lvert\alpha_i\rvert}{\lvert\alpha_2\rvert+\lvert\alpha_3\rvert}$, $\hat{z}_i = L(x_i^{\alpha_i/\gamma_i}), i = 2, 3$.
10	$\alpha_3 \geq \alpha_2 > 0 > \alpha_1\alpha_3 + \alpha_2 < 1$.	Use the PT technique in Proposition 6 to obtain the convex underestimator as follows: $S^C_{PCT}(\mathbf{x}, \hat{\mathbf{z}}) = c\hat{z}_1^{\gamma_1}x_2^{\alpha_2}x_3^{\alpha_3}$, $\gamma_1 = (1 - \alpha_3 - \alpha_2)$, $\hat{z}_1 = L(x_1^{\alpha_1/\gamma_1})$.
11	$\alpha_3 \geq \alpha_2 > 0 > \alpha_1$, $\alpha_3 + \alpha_2 > 1, \alpha_2 < 1$.	Use the PT technique in Proposition 6 to obtain the convex underestimator as follows: $S^C_{PCT}(\mathbf{x}, \hat{\mathbf{z}}) = c\hat{z}_1^{\gamma_1}x_2^{\alpha_2}\hat{z}_3^{\gamma_3}$, $\gamma_i = \frac{(1-\alpha_2)\lvert\alpha_i\rvert}{\lvert\alpha_1\rvert+\lvert\alpha_3\rvert}$, $\hat{z}_i = L(x_i^{\alpha_i/\gamma_i}), i = 1, 3$.
12	$\alpha_3 > 0 > \alpha_2 \geq \alpha_1$, $\alpha_3 < 1$.	Use the PT technique in Proposition 6 to obtain the convex underestimator as follows: $S^C_{PCT}(\mathbf{x}, \hat{\mathbf{z}}) = c\hat{z}_1^{\gamma_1}\hat{z}_2^{\gamma_2}x_3^{\alpha_3}$, $\gamma_i = \frac{(1-\alpha_3)\lvert\alpha_i\rvert}{\lvert\alpha_1\rvert+\lvert\alpha_2\rvert}$, $\hat{z}_i = L(x_i^{\alpha_i/\gamma_i}), i = 1, 2$.
13	$\alpha_1, \alpha_2, \alpha_3 < 0$, or $\alpha_1, \alpha_2 < 0, \alpha_3 > 1$, or $\alpha_3 \geq \alpha_2 > 1, \alpha_1 < 0$, or $\alpha_3 \geq \alpha_2 \geq \alpha_1 > 1$.	Use the PT technique in Proposition 6 to obtain the convex underestimator as follows: $S^C_{PCT}(\hat{\mathbf{z}}) = c\hat{z}_1^{\gamma_1}\hat{z}_2^{\gamma_2}\hat{z}_3^{\gamma_3}$, $\gamma_i = \frac{\lvert\alpha_i\rvert}{\lvert\alpha_1\rvert+\lvert\alpha_2\rvert+\lvert\alpha_3\rvert}$, $\hat{z}_i = L(x_i^{\alpha_i/\gamma_i}), i = 1, 2, 3$.

the breakpoint of $T(x)$. Three sets are considered as follows: $W = \{1, \ldots, r\}$ is the indicator set of breakpoints, $W' = \{1, \ldots, r-1\}$ is the indicator set of the intervals of each two adjacent breakpoints, and $S = \{1, \ldots, l = \lceil \log_2(r-1) \rceil\}$ is the indicator set of the new introducing binary variables. Then, the new variables are introduced as follows:

$\lambda_w (w \in W)$: a sequence of continuous variables representing the weighting of each break point.

$u_s (s \in S)$: a sequence of binary variables. Through the logarithmic numbers of the binary variables and constraints in Equation (27), the sequence p_w will be forced to follow a SOS2 structure.

Then, $L(T(x))$ can be expressed by the following mixed-integer linear programming:

$$\begin{gathered} x = \sum_{w \in W} \lambda_w a_w, \; L(T(x)) = \sum_{w \in W} \lambda_w T(a_w), \\ \lambda_w \in [0, 1] \forall w \in W, \; \sum_{w \in W} \lambda_w = 1, \\ \sum_{w \in W_s^+} \lambda_w \leq u_s, \; \sum_{w \in W_s^-} \lambda_w \leq (1 - u_s) \; \forall s \in S, \\ u_s \in \{0, 1\} \; \forall s \in S, \end{gathered} \tag{27}$$

where

(i) *$W_s^+ = \{w \in W : C_w \subseteq V_s^+\}$, where $V_s^+ = \{w' \in W' : s \in \sigma(B(w'))\}$,*

(ii) *$W_s^- = \{w \in W : C_w \subseteq V_s^-\}$, where $V_s^- = \{w' \in W' : s \notin \sigma(B(w'))\}$,*

(iii) *C_w denotes the set of continuous variable λ_w, which is involved by interval $w' \in W'$, thus $C_1 = \{1\}$, $C_w = \{w-1, w\}$ for $w = 2, \ldots, r-1$, and $C_r = \{r-1\}$,*

(iv) *$B(w')$ is an injective function for the domain W' to the codomain $\{0, 1\}^l$, where $w' \in W' (B : W' \to \{0, 1\}^l)$, and the sequence of the range of $B(w')$ must follow the Gray code, and*

(v) *$\sigma(B(w'))$ is the support of vector $B(w')$.* □

4. Conclusion

To address the limitations of the currently popular convexification strategies for solving GGP problems, this study considers popular types of transformation techniques to develop an efficient convexification strategy that follows general rules 1–3 in Section 1. The proposed convexification strategy utilizes an estimator to solve the difficulty pointed by Lundell and Westerlund (2009b) (i.e., selecting an appropriate transformation between the ET and PCT techniques) and offers a guide for selecting the most appropriate transformation techniques on any condition of a signomial term to obtain the tightest convex underestimator. This study also employs an efficient SOS2 model proposed by Vielma and Nemhauser (2011) to represent piecewise linear functions for improving the efficiency of piecewise linearization. Numerical examples in the online supplement demonstrate that the proposed convexification strategy outperforms the reference convexification strategies in terms of both solution quality and computation efficiency.

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