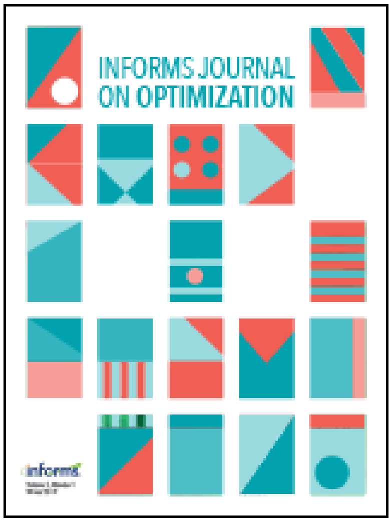

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On Robust Fractional 0-1 Programming

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Abstract. We study single- and multiple-ratio robust fractional 0-1 programming problems (RFPs). In particular, this work considers RFPs under a wide range of disjoint and joint uncertainty sets, where the former implies separate uncertainty sets for each numerator and denominator and the latter accounts for different forms of interrelatedness between them. We first demonstrate that unlike the deterministic case, a single-ratio RFP is nondeterministic polynomial-time hard under general polyhedral uncertainty sets. However, if the uncertainty sets are imbued with a certain structure, variants of the well-known budgeted uncertainty, the disjoint and joint single-ratio RFPs are polynomially solvable when the deterministic counterpart is. We also propose mixed-integer linear programming (MILP) formulations for multiple-ratio RFPs. We conduct extensive computational experiments using test instances based on real and synthetic data sets to evaluate the performance of our MILP reformulations as well as to compare the disjoint and joint uncertainty sets. Finally, we demonstrate the value of the robust approach by examining the robust solution in a deterministic setting and vice versa.

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Keywords: fractional 0-1 programming • robust optimization • nonlinear integer programming • robust assortment optimization

1. Introduction

Fractional objective functions arise naturally in a broad range of problems that involve optimization of efficiency measures, probabilities, averages, and percentages among others. A comprehensive overview of continuous fractional programming papers is given by Schaible (1982) and Stancu-Minasian (2017). In this paper, we focus on fractional 0-1 programming problem (FP; also referred to as *hyperbolic 0-1 programming*). Formally, the deterministic fractional 0-1 program is defined as

$$\max_{x \in X} \sum_{i \in I} \frac{a_{i0} + \sum_{j \in J} a_{ij} x_j}{b_{i0} + \sum_{j \in J} b_{ij} x_j}, \tag{FP}$$

where $I = \{1, \ldots, m\}$, $J = \{1, \ldots, n\}$, and $X \subseteq \mathbb{B}^n$ for $\mathbb{B} := \{0, 1\}$. If $m = 1$, then the problem is referred to as *single ratio*; otherwise, it is *multiple ratio*.

Many diverse applications can be readily formulated as FP. Such applications encompass scheduling (Saipe 1975), retail assortment (Subramanian and Sherali 2010, Davis et al. 2014, Méndez-Díaz et al. 2014), set covering (Amaldi et al. 2011, 2012), facility location (Tawarmalani et al. 2002), stochastic service systems (Elhedhli 2005, Han et al. 2013), biclustering (Busygin et al. 2005, Trapp et al. 2010), finding diverse solutions to binary linear programs (Trapp and Konrad 2015), cell formation problems (Bychkov et al. 2014, Pinheiro et al. 2018), and clinical trials (Bertsimas et al. 2019). Additionally, specialized techniques have been proposed for special cases of FP, including the minimum fractional spanning tree problem (Ursulenko et al. 2013), the minimum cost-to-time cycle problem (Dasdan et al. 1999), the maximum mean cut problem (McCormick and Ervolina 1994), the minimum fractional assignment problem (Shigeno et al. 1995), and the maximum clique ratio problem (Moeini 2015, Sethuraman and Butenko 2015). We refer the reader to a recent survey by Borrero et al. (2017) and the references therein for an overview of the literature, applications, and solution methods for fractional 0-1 programming.

In practice, the parameters of an optimization problem are often subject to uncertainty, and existing techniques for FP may not be adequate for problems with unknown parameters. Our approach to uncertain fractional 0-1 programming falls within the framework of robust optimization. Specifically, we assume that some or all of the coefficients a_{ij} and b_{ij} may not be known exactly but are modeled as bounded random variables $\widetilde{a}_{ij}$ and $\widetilde{b}_{ij}$, respectively. These coefficients are presumed to lie in some uncertainty set $\mathcal{U}$; that is, $(\widetilde{a}, \widetilde{b}) \in \mathcal{U}$. Then the robust counterpart of FP, robust fractional 0-1 programming problem (RFP), with respect to the uncertainty set $\mathcal{U}$ optimizes against the worst-case scenario

$$Z^{\star}_{\mathcal{U}} = \max_{x \in X} \min_{(\tilde{a}, \tilde{b}) \in \mathcal{U}} \sum_{i \in I} \frac{a_{i0} + \sum_{j \in J} \widetilde{a}_{ij} x_j}{b_{i0} + \sum_{j \in J} \widetilde{b}_{ij} x_j}. \qquad (\text{RFP}[\mathcal{U}])$$

Throughout this paper, we assume that the data satisfy the following assumption.

Assumption 1. *For all $x \in X$, $(\widetilde{a}, \widetilde{b}) \in \mathcal{U}$, and $i \in I$, $a_{i0} + \sum_{j \in J} \widetilde{a}_{ij} x_j \geq 0$, and $b_{i0} + \sum_{j \in J} \widetilde{b}_{ij} x_j > 0$.*

Most fractional programming problems typically have nonnegative data because such data represent probabilities, prices, weights, utilities, etc. (see, e.g., Borrero et al. 2017 and the applications described therein). The portion of Assumption 1 related to a strictly positive denominator is a commonly made assumption for the deterministic version (Hansen et al. 1991, Boros and Hammer 2002). Moreover, the nonnegative numerator assumption is not restrictive because, by adding a sufficiently large constant value to each ratio, we can transform its numerator into the one that takes only nonnegative values for any $(\widetilde{a}, \widetilde{b}) \in \mathcal{U}$ and $x \in X$.

In the following, we define $(t)^+ = \max\{0, t\}$ for any $t \in \mathbb{R}$ and let $A \times B$ denote the Cartesian product of sets A and B.

1.1. Relevant Literature

1.1.1. Fractional Programming. The single-ratio FP with a strictly positive denominator can be solved to optimality by repeatedly solving a sequence of optimization problems with a linear objective function over X via parametrization algorithms, such as Newton's method (Dinkelbach 1967) and binary search (Lawler 1976, Ahuja et al. 1993, Radzik 2013). Moreover, if solving such a linear optimization problem over X can be done in polynomial time, then single-ratio FP can be solved in polynomial time. Furthermore, Megiddo (1979) shows that if a nominal binary linear problem admits a polynomial time algorithm, then so does single-ratio FP.

Multiple-ratio FP is nondeterministic polynomial-time hard (NP-hard) even for two ratios and strictly positive denominators (Hansen et al. 1990; Prokopyev et al. 2005a, b). For solving multiple-ratio FP, there exist several mixed-integer linear programming (MILP) reformulations of FP. An early formulation was given by Williams (1974). A different formulation was proposed by Li (1994) and later studied by Wu (1997). Additionally, Tawarmalani et al. (2002) present six other formulations. A more recent formulation approach based on performing a base 2 expansion of certain integer-valued expressions was given by Borrero et al. (2016), which can reduce the number of bilinear terms that require linearization and may substantially reduce the number of variables with respect to the previous MILP reformulations.

1.1.2. Robust Optimization. The robust optimization approach of El Ghaoui and Lebret (1997), Ben-Tal and Nemirovski (1998, 1999, 2000), and El Ghaoui et al. (1998) presumes that the unknown data are within some uncertainty set. A critical modeling decision is selecting an appropriate uncertainty set that is close to the true representation of uncertainty in the data and also ensures tractability of the resulting robust counterpart. In particular, the budgeted uncertainty proposed in Bertsimas and Sim (2003, 2004) is widely used because it achieves a good balance between accuracy and tractability.

Continuous robust single-ratio fractional programming was introduced by Gorissen (2015). The author shows that under some convexity assumptions, the robust counterparts can be solved in polynomial time. Additionally, the Lagrange dual of a (continuous) robust single-ratio fractional problem was studied in Jeyakumar and Li (2011) and Jeyakumar et al. (2013) for disjoint uncertainty sets; that is, the uncertainty set $\mathcal{U}$ can be naturally decomposed as the Cartesian product of simpler sets.

In the case of fractional 0-1 programming, the work of Rusmevichientong and Topaloglu (2012) is the only study that explores robust formulations. The authors study a single-ratio assortment optimization problem under the multinomial logit choice model, where only customer preferences are uncertain. However, their results cannot be directly extended for more general classes of fractional problems, including the cases where the revenues are subject to uncertainty or the choice model is mixed multinomial logit.

1.2. Structure and Contributions of This Paper

To the best of our knowledge, this study is the first work that addresses robust fractional 0-1 programming in its general structure. We perform a comprehensive study of RFP[$\mathcal{U}$] that includes several types of the budgeted uncertainty sets and also encompasses single- and multiple-ratio cases. We also briefly explore the complexity of RFP[$\mathcal{U}$] for general polyhedral $\mathcal{U}$. The structure of this paper can be summarized as follows.

- In Section 2, we introduce the (disjoint and joint) generalizations of the budgeted uncertainty set for fractional 0-1 programs and discuss computational complexity of RFP.
- In Section 3, we propose an approach to find an optimal solution of single-ratio RFP by solving a polynomial number of linear optimization problems over X; in particular, if linear optimization over X is polynomial-time solvable, then so is RFP[$\mathcal{U}$].
- In Section 4, we extend classical MILP formulations for FP to tackle multiple-ratio RFP[$\mathcal{U}$] and exploit the binary expansion technique to improve the efficacy of the MILPs. We also provide some insights on selection of the appropriate level of uncertainty.
- In Section 5, we present computations with real and synthetic data. Additionally, we examine the price of robustness and evaluate the performance of the proposed MILPs via extensive computational experiments.

2. Model of Data Uncertainty

The selection of an appropriate uncertainty set can affect the tractability of a robust optimization problem. In this section, we describe the budgeted uncertainty set and several variations thereof for fractional 0-1 programming as considered in this paper, which lead to tractable (polynomial-time) methods for single-ratio RFP[$\mathcal{U}$] in Section 3. However, we also demonstrate that the robust counterpart of a polynomially solvable unconstrained single-ratio FP (with strictly positive denominator) is NP-hard for a general polyhedral uncertainty set $\mathcal{U}$.

In particular, following the convention introduced by Bertsimas and Sim (2003, 2004), each unknown coefficient $\widetilde{a}_{ij}$ and $\widetilde{b}_{ij}$ lies in a symmetric interval centered on the nominal value; that is, $\widetilde{a}_{ij} \in [a_{ij} - d^a_{ij}, a_{ij} + d^a_{ij}]$ and $\widetilde{b}_{ij} \in [b_{ij} - d^b_{ij}, b_{ij} + d^b_{ij}]$ with $d^a_{ij}, d^b_{ij} \geq 0$. The coefficients d^a_{ij} and d^b_{ij} denote the potential deviation from nominal values a_{ij} and b_{ij}, respectively, for each $i \in I, j \in J$.

Additionally, it is unlikely for all the coefficients to simultaneously change to their worst-case values. Hence, only a predetermined number of the unknown coefficients take values different from their nominal value. Given a ratio $i \in I$ and vectors $\widetilde{a}_i, \widetilde{b}_i \in \mathbb{R}^n$, let $S_i(\widetilde{a}_i) = \{j \in J \mid \widetilde{a}_{ij} \neq a_{ij}\}$ and $S_i(\widetilde{b}_i) = \{j \in J \mid \widetilde{b}_{ij} \neq b_{ij}\}$ be the set of indices of the uncertain parameters in which values are different from the nominal in the numerator and the denominator, respectively.

Uncertainty pertaining to linear 0-1 constraints is covered in the literature (Bertsimas and Sim 2003); thus, we assume that the constraint coefficients are fixed. Furthermore, we assume without loss of generality that the data are integral (otherwise, the rational coefficients can be scaled to satisfy this assumption). Hence, we make the following assumption.

Assumption 2. *All data are integer; that is,* $a_{i0}, b_{i0}, a_{ij}, b_{ij} \in \mathbb{Z}$, *and* $d^a_{ij}, d^b_{ij} \in \mathbb{Z}_+$ *for all* $i \in I, j \in J$.

2.1. Disjoint Uncertainty Set

Given $\Gamma^a_i, \Gamma^b_i \in \{0, 1, \ldots, n\}$ as the budget of uncertainty or the level of conservatism, for each $i \in I$, we define

$$\mathcal{U}^a_i = \left\{\widetilde{a}_i \in \mathbb{R}^n \,\middle|\, \widetilde{a}_{ij} \in \left[a_{ij} - d^a_{ij}, a_{ij} + d^a_{ij}\right] \text{ for } j \in J,\ |S_i(\widetilde{a}_i)| \leq \Gamma^a_i\right\} \text{ and} \tag{1}$$

$$\mathcal{U}^b_i = \left\{\widetilde{b}_i \in \mathbb{R}^n \,\middle|\, \widetilde{b}_{ij} \in \left[b_{ij} - d^b_{ij}, b_{ij} + d^b_{ij}\right], \text{for } j \in J,\ \left|S_i\left(\widetilde{b}_i\right)\right| \leq \Gamma^b_i\right\}. \tag{2}$$

Note that $\mathcal{U}^a_i$ and $\mathcal{U}^b_i$ correspond to the budgeted uncertainty sets studied in Bertsimas and Sim (2003, 2004) and that Γ^a_i and Γ^b_i are the numbers of coefficients allowed to vary from their nominal value in the numerator and denominator of the ith ratio, respectively. Then the *disjoint uncertainty set* for fractional programming is

$$\mathcal{U}^{ab} = \left\{\left(\widetilde{a}, \widetilde{b}\right) \in \mathbb{R}^{m \times n} \times \mathbb{R}^{m \times n} \,\middle|\, \left(\widetilde{a}_i, \widetilde{b}_i\right) \in \mathcal{U}^a_i \times \mathcal{U}^b_i, \text{for all } i \in I\right\}.$$

We refer to $\mathcal{U}^{ab}$ as disjoint because uncertainty of the coefficients of each numerator/denominator is independent of the rest of the data. Also, observe that in the ith ratio by setting $\Gamma^a_i = 0$ ($\Gamma^b_i = 0$), we can restrict the uncertainty only to the denominator (numerator) of the ratio. Therefore, set $\mathcal{U}^{ab}$ includes subcases in which some ratios are subject to uncertainty only in either their denominators or numerators.

2.2. Joint Uncertainty Sets

We now describe four joint uncertainty sets. In contrast with the preceding disjoint uncertainty set, there is some dependence between the uncertainties related to different numerators and denominators.

- *Shared ratio budget.* Given $\Gamma_i \in \{0, 1, \ldots, 2n\}$, for each $i \in I$, let

$$\mathcal{U}_i = \left\{\left(\widetilde{a}_i, \widetilde{b}_i\right) \in \mathbb{R}^n \times \mathbb{R}^n \,\middle|\, \widetilde{a}_{ij} \in \left[a_{ij} - d^a_{ij}, a_{ij} + d^a_{ij}\right],\ \widetilde{b}_{ij} \in \left[b_{ij} - d^b_{ij}, b_{ij} + d^b_{ij}\right],\ |S_i(\widetilde{a}_i)| + \left|S_i\left(\widetilde{b}_i\right)\right| \le \Gamma_i\right\}.$$

The *shared ratio budget uncertainty set* is

$$\mathcal{U}^{\overline{ab}} = \left\{\left(\widetilde{a}, \widetilde{b}\right) \in \mathbb{R}^{m\times n} \times \mathbb{R}^{m\times n} \,\middle|\, \left(\widetilde{a}_i, \widetilde{b}_i\right) \in \mathcal{U}_i, \text{ for all } i \in I\right\}.$$

Under the shared ratio budget uncertainty set, uncertainty for the ith ratio is independent of other ratios, but the uncertainties of its numerator and denominator are connected by a common budget Γ_i. Specifically, at most, Γ_i of coefficients in the ith ratio's numerator and denominator can change.

The uncertainty sets $\mathcal{U}^{ab}$ and $\mathcal{U}^{\overline{ab}}$ arise naturally when there is uncertainty concerning individual coefficients of FP. In some applications, however, the uncertainty of the original problem may have a specific structure that requires a specialized uncertainty set. We now describe three such sets.

- *Matched sets.* Consider the problem of maximizing return on investment or productivity, where a corresponds to the return of executing a given project (for example, dollar amount) and b corresponds to the investment costs for the project (for example, time). Additionally, suppose that undesirable events may occur (for example, strikes or natural disasters), resulting in a *simultaneous* decrease in the returns and increase in the costs of a given project. Such uncertainty is modeled by the *matched sets uncertainty set*

$$\mathcal{U}^{\overline{ab}}_{=} = \left\{\left(\widetilde{a}, \widetilde{b}\right) \in \mathcal{U}^{\overline{ab}} \,\middle|\, S_i(\widetilde{a}_i) = S_i\left(\widetilde{b}_i\right), \text{ for all } i \in I\right\}.$$

- *Matched effects.* Consider the assortment optimization problem under the mixed multinomial logit model (Boyd and Mellman 1980, Méndez-Díaz et al. 2014)

$$\max_{x\in X} \sum_{i\in I} \frac{\sum_{j\in J} r_{ij}\rho_{ij}x_j}{1 + \sum_{j\in J} \rho_{ij}x_j}, \tag{3}$$

where r_{ij} and ρ_{ij} are the revenue and customer preference weight associated with selling product j to customer class i, respectively. Note that if the revenues are known but the preferences are uncertain, then changes with respect to the nominal values of numerator/denominator coefficients that correspond to the same variable are proportional and of the same sign. The *matched effects uncertainty set*

$$\mathcal{U}^{\overline{ab}}_{\propto} = \left\{\left(\widetilde{a}, \widetilde{b}\right) \in \mathcal{U}^{\overline{ab}}_{=} \,\middle|\, \frac{a_{ij} - \widetilde{a}_{ij}}{d^a_{ij}} = \frac{b_{ij} - \widetilde{b}_{ij}}{d^b_{ij}},\ \text{for all } i \in I,\ j \in J\right\}$$

captures this effect.

- *Single budget.* In all the uncertainty sets defined earlier, we assume that each ratio has its own budget(s) of uncertainty. However, one may consider an uncertainty set in which a single budget controls the degree of conservatism over all ratios. Specifically, the *single-budget uncertainty set* for numerators also arises in the assortment problem (3) when the preferences are known but the revenues are unknown, and it is given by

$$\mathcal{U}^{\bar{a}} = \left\{\widetilde{a} \in \mathbb{R}^{m\times n} \,\middle|\, \widetilde{a}_{ij} \in \left[a_{ij} - d^a_{ij}, a_{ij} + d^a_{ij}\right], \text{for all } i \in I, j \in J, \sum_{i\in I} |S_i(\widetilde{a}_i)| \le \Gamma\right\},$$

where the budget $\Gamma \in \{0, 1, \ldots, m \cdot n\}$ is shared by all ratios. In words, only numerators are subject to uncertainty, and at most, Γ of the numerators coefficients is different from their nominal values.

The five uncertainty sets just defined (i.e., $\mathcal{U}^{ab}, \mathcal{U}^{\overline{ab}}, \mathcal{U}^{\overline{ab}}_{=}, \mathcal{U}^{\overline{ab}}_{\propto}$, and $\mathcal{U}^{\bar{a}}$) aim at modeling a broad range of situations arising in practice; moreover, none are a special case of another. Furthermore, it can be verified that RFP[$\mathcal{U}$], in general, is neither quasi-convex nor quasi-concave.

We show in Section 3 that for a polynomial-time-solvable FP, the considered uncertainty sets lead to polynomial-time-solvable robust counterparts RFP[$\mathcal{U}$]. In contrast, note that the robust counterparts corresponding to general polyhedral uncertainty are NP-hard.

2.3. RFP[$\mathcal{U}$] for General Polyhedral Uncertainty Is NP-Hard

Consider an unconstrained ($X = \mathbb{B}^n$) single-ratio problem with uncertainty limited to the numerator

$$\max_{x\in\mathbb{B}^n} \frac{a_0 + a^T x - \max_{\gamma\in\mathcal{U}}\left\{(A\gamma)^T x\right\}}{b_0 + b^T x}, \tag{4}$$

where $\mathcal{U} = \{\gamma : D\gamma \le d, \gamma \ge 0\}$ is a general polyhedral uncertainty set and Assumption 1 holds. Note that without uncertainty, the deterministic unconstrained single-ratio problem can be solved in polynomial time via a linear time median-finding algorithm (Hansen et al. 1991). However, this property does not follow through to the robust counterpart.

Proposition 1. *Problem (4) is NP-hard.*

Proof. Let $b_0 = 1$ and $b_j = 0$ for $j \in J$; then we have a linear objective with a polyhedral uncertainty set. By theorem 4 of Buchheim and Kurtz (2018), the resulting problem is NP-hard. □

Similarly, consider the problem with uncertainty restricted to the denominator

$$\max_{x\in\mathbb{B}^n} \frac{a_0 + a^T x}{b_0 + b^T x + \max_{\gamma\in\mathcal{U}}\left\{(A\gamma)^T x\right\}}. \tag{5}$$

Proposition 2. *Problem (5) is NP-hard.*

Proof. The proof follows directly from noting that (5) is equivalent to

$$\min_{x\in\mathbb{B}^n} \frac{b_0 + b^T x + \max_{\gamma\in\mathcal{U}}\left\{(A\gamma)^T x\right\}}{a_0 + a^T x}$$

and using an argument similar to the one in Proposition 1. □

In light of these results, in the remainder of this paper, we restrict $\mathcal{U}$ to any disjoint or joint uncertainty sets defined in this section: that is, $\mathcal{U} \in \{\mathcal{U}^{ab}, \mathcal{U}^{\overline{ab}}, \mathcal{U}^{\overline{ab}}_{=}, \mathcal{U}^{\overline{ab}}_{\propto}, \mathcal{U}^{\bar{a}}\}$ and RFP[$\mathcal{U}$] as the corresponding representation of the robust problem.

3. Single-ratio RFP[$\mathcal{U}$]

When the uncertain coefficients of the objective function are in the form of a budgeted uncertainty set, Bertsimas and Sim (2003) prove that the solution of the robust counterpart of the nominal binary linear problem

$$\min_{x\in X} c_0 + \sum_{j\in J} c_j x_j \tag{6}$$

can be found by solving n instances of (6). Therefore, if (6) is polynomially solvable, so is its robust counterpart. Similarly, parametrization algorithms, such as Newton's method (Dinkelbach 1967) and the binary search algorithm (Lawler 1976, Ahuja et al. 1993, Radzik 2013), can find an optimal solution for the constrained single-ratio FPs by solving a sequence of problems in the form of (6).

In this section, we combine and extend the ideas from robust linear programming and deterministic fractional optimization to propose a solution method for single-ratio RFP[$\mathcal{U}$]. In particular, we show that if there exists a polynomial-time algorithm for linear optimization over X, then RFP[$\mathcal{U}$] is polynomial-time solvable when $\mathcal{U}$ is one of the uncertainty sets described in Section 2. We first consider the disjoint uncertainty set $\mathcal{U}^{ab}$ in Section 3.1, and then we tackle the joint uncertainty sets in Section 3.2.

3.1. Disjoint Single-Ratio Case

Herein we demonstrate how to solve single-ratio RFP[$\mathcal{U}^{ab}$] by solving at most $(n+1)^2$ nominal FPs.

Proposition 3. *Problem* RFP[$\mathcal{U}^{ab}$] *is equivalent to*

$$Z^\star_{\mathcal{U}^{ab}} = \max_{\substack{x\in X,\\ \alpha\in\{0,d_1^a,d_2^a,\ldots,d_n^a\},\\ \beta\in\{0,d_1^b,d_2^b,\ldots,d_n^b\}}} \frac{a_0 - \Gamma^a\alpha + \sum_{j\in J}\left(a_j - \left(d_j^a - \alpha\right)^+\right)x_j}{b_0 + \Gamma^b\beta + \sum_{j\in J}\left(b_j + \left(d_j^b - \beta\right)^+\right)x_j}. \tag{7}$$

Proof. Observe that single-ratio RFP[$\mathcal{U}^{ab}$] is equivalent to $\max_{x\in X} \frac{a_0+\min_{\tilde{a}\in\mathcal{U}^a} \tilde{a}^T x}{b_0+\max_{\tilde{b}\in\mathcal{U}^b} \tilde{b}^T x}$, where $\mathcal{U}^a$ and $\mathcal{U}^b$ are the sets given in (1) and (2). Letting u and v be the indicator vectors of sets $S(\widetilde{a})$ and $S(\widetilde{b})$, respectively, we reformulate RFP[$\mathcal{U}^{ab}$] as

$$\max_{x\in X} \quad \frac{a_0 + \sum_{j\in J} a_j x_j - \max_u\left\{\sum_{j\in J} d_j^a x_j u_j\right\}}{b_0 + \sum_{j\in J} b_j x_j + \max_v\left\{\sum_{j\in J} d_j^b x_j v_j\right\}} \tag{8}$$

$$\text{s.t.} \quad \sum_{j\in J} u_j \le \Gamma^a, \quad \sum_{j\in J} v_j \le \Gamma^b \qquad (\alpha,\beta)$$

$$0 \le u_j \le 1, \quad 0 \le v_j \le 1 \qquad \forall j \in J. \quad (p_j, q_j)$$

Note that there exist integral optimal solutions u^* and v^* to the inner optimization problems in (8) because the polytope defined by cardinality and bounding constraints is integral—thus, the preceding formulation is indeed correct. By taking the dual of (independent) inner optimization problems in the numerator and the denominator of (8) with respect to dual variables α, β and p, q, we obtain

$$\max_{\substack{x\in X,\\ \alpha,\beta,p,q\ge 0}} \quad \frac{a_0 + \sum_{j\in J} a_j x_j - \left(\Gamma^a\alpha + \sum_{j\in J} p_j\right)}{b_0 + \sum_{j\in J} b_j x_j + \left(\Gamma^b\beta + \sum_{j\in J} q_j\right)} \tag{9}$$

$$\text{s.t.} \quad p_j + \alpha \ge d_j^a x_j, \quad q_j + \beta \ge d_j^b x_j \qquad \forall j \in J.$$

Clearly, in an optimal solution, we have $p_j^* = (d_j^a x_j^* - \alpha^*)^+ = (d_j^a - \alpha^*)^+ x_j^*$ and $q_j = (d_j^b x_j^* - \beta^*)^+ = (d_j^a - \alpha^*)^+ x_j^*$. Otherwise, we can decrease p_j or q_j and find a solution with a better objective function value.

Additionally, let $E = \{j \in J \,|\, (d_j^a - \alpha^*)^+ x_j^* > 0\}$, and observe that if $\alpha^* > 0$ and $\alpha^* \ne d_j^a$ for all $j \in J$, then

$$\Gamma^a(\alpha^* \pm \epsilon) + \sum_{j\in J}\left(d_j^a - (\alpha^* \pm \epsilon)\right)^+ x_j^* = \Gamma^a(\alpha^*) + \sum_{j\in J}\left(d_j^a - \alpha^*\right)^+ x_j^* \pm \epsilon(\Gamma^a - |E|)$$

for sufficiently small $\epsilon > 0$. In particular, depending on the sign of $\Gamma^a - |E|$, we can increase or decrease α^* and find solutions with greater or equal objective function values. Thus, we conclude that there exists an optimal solution where $\alpha^* \in \{0, d_1^a, \ldots, d_n^a\}$, and similarly, we can conclude that there exists an optimal solution where $\beta^* \in \{0, d_1^b, \ldots, d_n^b\}$. Replacing α, β, p, q in (9) by their corresponding optimal values, we find formulation (7). □

Hence, RFP[$\mathcal{U}^{ab}$] can be tackled by solving (7) for each candidate pair $(\alpha, \beta) \in \{0, d_1^a, d_2^a, \ldots, d_n^a\} \times \{0, d_1^b, d_2^b, \ldots, d_n^b\}$ independently.

Theorem 1. *Single-ratio* RFP[$\mathcal{U}^{ab}$] *can be solved with* $(k^a+1)(k^b+1)$ *calls to an oracle for* (FP), *where* k^a *and* k^b *are the numbers of distinct values of* d_j^a *and* d_j^b, $j \in J$, *respectively.*

Theorem 1 implies that if single-ratio FP over X is solvable in strongly polynomial time, then so is its robust counterpart RFP[$\mathcal{U}^{ab}$]. Note that in the worst case, $(k^a+1)(k^b+1) = (n+1)^2$, and (FP) is polynomial-time solvable when linear optimization over X is polynomial-time solvable.

3.2. Joint Single-Ratio Case

It can be observed that the method of Proposition 3 cannot handle single-ratio RFP under joint uncertainty sets because of the interaction between uncertainties in the numerator and the denominator of each ratio. To solve single-ratio RFP under joint uncertainty sets, we first show that RFP[$\mathcal{U}^{\overline{ab}}$], RFP[$\mathcal{U}_{=}^{\overline{ab}}$], and RFP[$\mathcal{U}_{\propto}^{\overline{ab}}$] can be formulated as mixed-integer nonlinear programs (MINLPs) with a similar structure (Propositions 4–6). Then, by exploring some properties of the resulting reformulations (Propositions 7 and 8), we propose a specialized algorithm for solving them (Proposition 9).

Proposition 4. *Problem* RFP[$\mathcal{U}^{\overline{ab}}$] *is equivalent to*

$$Z^\star_{\mathcal{U}^{\overline{ab}}} = \max_{\substack{x\in X,\\ \mu,\alpha,\beta,\gamma\ge 0}} \quad \mu \tag{10}$$

$$\text{s.t.} \quad \left(b_0 + \sum_{j\in J} b_j x_j\right)\mu + \Gamma\alpha + \sum_{j\in J}\beta_j + \sum_{j\in J}\gamma_j \le a_0 + \sum_{j\in J} a_j x_j$$

$$\alpha + \beta_j \ge d_j^a x_j, \quad \alpha + \gamma_j \ge d_j^b x_j \mu \qquad \forall j \in J.$$

Proof. Let u and v be the indicator variables of the sets $S(\widetilde{a})$ and $S(\widetilde{b})$, respectively. Note that RFP[$\mathcal{U}^{\overline{ab}}$] can be written as

$$Z^{\star}_{\mathcal{U}^{\overline{ab}}} = \max_{x\in X} \min_{u,v\in\mathbb{R}^n} \frac{a_0 + \sum_{j\in J} a_j x_j - \sum_{j\in J} d^a_j x_j u_j}{b_0 + \sum_{j\in J} b_j x_j + \sum_{j\in J} d^b_j x_j v_j} \tag{11}$$

$$\text{s.t.} \sum_{j\in J} u_j + \sum_{j\in J} v_j \le \Gamma \tag{12}$$

$$0 \le u_j \le 1, \quad 0 \le v_j \le 1 \qquad \forall j \in J. \tag{13}$$

Observe that we relaxed the binary constraints $u_j \in \mathbb{B}$ and $v_j \in \mathbb{B}$ to convex bound constraints. Because the inner minimization problem is quasi-concave for any $x \in X$ (Dinkelbach 1967), the nonlinear problem has an optimal solution that is an extreme point of the polytope induced by (12) and (13); in particular, there exists an optimal binary solution.

We now reformulate the inner minimization problem using the transformation proposed in Charnes and Cooper (1962): letting $y = 1/(b_0 + \sum_{j\in J} b_j x_j + \sum_{j\in J} d^b_j x_j v_j)$, $z^u_j = u_j y$, and $z^v_j = v_j y$ for all $j \in J$, we can write (11)–(13) as

$$Z^{\star}_{\mathcal{U}^{\overline{ab}}} = \max_{x\in X} \min_{z^u, z^v, y} \left(a_0 + \sum_{j\in J} a_j x_j\right) y - \sum_{j\in J} d^a_j x_j z^u_j \tag{14}$$

$$\text{s.t.} \left(b_0 + \sum_{j\in J} b_j x_j\right) y + \sum_{j\in J} d^b_j x_j z^v_j = 1 \qquad (\mu) \tag{15}$$

$$\sum_{j\in J} z^u_j + \sum_{j\in J} z^v_j \le \Gamma y \qquad (\alpha) \tag{16}$$

$$0 \le z^u_j \le y \qquad \forall j \in J \quad (\beta_j) \tag{17}$$

$$0 \le z^v_j \le y \qquad \forall j \in J. \quad (\gamma_j) \tag{18}$$

It is seen that for any fixed $x \in X$, the inner minimization problem is a linear program (LP). Thus, using standard LP duality, we obtain formulation (10), where μ, α, β_j, and γ_j are corresponding dual variables to constraints (15)–(18). □

Furthermore, in Propositions 5 and 6, we show that RFP[$\mathcal{U}^{\overline{ab}}_{=}$] and RFP[$\mathcal{U}^{\overline{ab}}_{\propto}$], respectively, can also be formulated as equivalent MINLPs. The proofs are relegated to Appendix A.

Proposition 5. *Problem* RFP[$\mathcal{U}^{\overline{ab}}_{=}$] *is equivalent to*

$$Z^{\star}_{\mathcal{U}^{\overline{ab}}_{=}} = \max_{\substack{x\in X,\\ \mu,\alpha,\beta\ge 0}} \mu \tag{19}$$

$$\text{s.t.} \left(b_0 + \sum_{j\in J} b_j x_j\right)\mu + \Gamma\alpha + \sum_{j\in J} \beta_j \le a_0 + \sum_{j\in J} a_j x_j$$

$$\alpha + \beta_j \ge d^a_j x_j + d^b_j x_j \mu \qquad \forall j \in J.$$

Proposition 6. *Problem* RFP[$\mathcal{U}^{\overline{ab}}_{\propto}$] *is equivalent to*

$$Z^{\star}_{\mathcal{U}^{\overline{ab}}_{\propto}} = \max_{\substack{x\in X,\\ \mu,\alpha,\beta,\gamma\ge 0}} \mu \tag{20}$$

$$\text{s.t.} \left(b_0 + \sum_{j\in J} b_j x_j\right)\mu + \Gamma\alpha + \sum_{j\in J} \beta_j + \sum_{j\in J} \gamma_j \le a_0 + \sum_{j\in J} a_j x_j$$

$$\alpha + \beta_j \ge -d^a_j x_j + d^b_j x_j \mu, \quad \alpha + \gamma_j \ge d^a_j x_j - d^b_j x_j \mu \qquad \forall j \in J.$$

Example 1. Consider a trivariate ($n = 3$) single-ratio RFP[$\mathcal{U}^{\overline{ab}}_{\propto}$], that is, $Z^{\star}_{\mathcal{U}^{\overline{ab}}_{\propto}} = \frac{a_0 + \tilde{a}_1 x_1 + \tilde{a}_2 x_2 + \tilde{a}_3 x_3}{b_0 + \tilde{b}_1 x_1 + \tilde{b}_2 x_2 + \tilde{b}_3 x_3}$, wherein $a_0 = 6$, $\widetilde{a}_1 \in [-3, 13]$, $\widetilde{a}_2 \in [1, 31]$, and $\widetilde{a}_3 \in [1, 5]$ and $b_0 = 3$, $\widetilde{b}_1 \in [0, 4]$, $\widetilde{b}_2 \in [0, 16]$, and $\widetilde{b}_3 \in [1, 3]$ for $\Gamma = 2$. Thus, the nominal values are $a_1 = 5, a_2 = 16, a_3 = 3, b_1 = 2, b_2 = 8, b_3 = 2$, and the deviation values are $d^a_1 = 8, d^a_2 = 15, d^a_3 = 2, d^b_1 = 2$, $d^b_2 = 8, d^b_3 = 1$.

Then, by Proposition 6, the equivalent reformulation of this RFP[$\mathcal{U}^{\overline{ab}}_{\propto}$] is given by

$$
\begin{aligned}
Z^{\star}_{\mathcal{U}^{\overline{ab}}_{\propto}} = \max_{\substack{x\in X,\\ \mu,\alpha,\beta,\gamma\geq 0}} \quad & \mu \\
\text{s.t.} \quad & (3+2x_1+8x_2+2x_3)\mu + 2\alpha + \sum_{j\in\{1,2,3\}}\beta_j + \sum_{j\in\{1,2,3\}}\gamma_j \leq 6+5x_1+16x_2+3x_3\\
& \alpha+\beta_1 \geq -8x_1+2x_1\mu, \qquad \alpha+\gamma_1\geq 8x_1-2x_1\mu\\
& \alpha+\beta_2 \geq -15x_2+8x_2\mu, \qquad \alpha+\gamma_2\geq 15x_2-8x_2\mu\\
& \alpha+\beta_3 \geq -2x_3+x_3\mu, \qquad \alpha+\gamma_3\geq 2x_3-x_3\mu. \quad \square
\end{aligned}
$$

Based on Propositions 4–6, we see that in all cases, single-ratio RFP under the joint uncertainty sets $\mathcal{U}^{\overline{ab}}$, $\mathcal{U}^{\overline{ab}}_{=}$, and $\mathcal{U}^{\overline{ab}}_{\propto}$ can be formulated as

$$\max_{\substack{x\in X,\\ \mu,\alpha,\beta,\gamma\geq 0}} \quad \mu \tag{21}$$

$$\text{s.t.} \quad \left(b_0+\sum_{j\in J}b_jx_j\right)\mu + \Gamma\alpha + \sum_{j\in J}\beta_j + \sum_{j\in J}\gamma_j \leq a_0 + \sum_{j\in J}a_jx_j \tag{22}$$

$$\alpha+\beta_j \geq \left(d_j^1+d_j^2\mu\right)x_j, \quad \alpha+\gamma_j \geq \left(d_j^3+d_j^4\mu\right)x_j \qquad \forall j\in J, \tag{23}$$

for some $d^1, d^2, d^3, d^4 \in \mathbb{Z}^n$, where $d_j^1\cdot d_j^3 \leq 0$ and $d_j^2\cdot d_j^4\leq 0$ for all $j\in J$. In particular, if $d_j^1 = d_j^a, d_j^2 = d_j^3 = 0$ and $d_j^4 = d_j^b$ for all $j\in J$, then problem (21)–(23) is equivalent to the reformulation of RFP[$\mathcal{U}^{\overline{ab}}$] given by (10). Similarly, letting $d_j^1 = d_j^a, d_j^2 = d_j^b, d_j^3 = d_j^4 = 0$ and $d_j^1 = -d_j^3 = -d_j^a, d_j^2 = -d_j^4 = d_j^b$ for all $j\in J$ in (21)–(23) leads to an equivalent reformulation of RFP[$\mathcal{U}^{\overline{ab}}_{=}$] and RFP[$\mathcal{U}^{\overline{ab}}_{\propto}$], respectively, provided in (19) and (20).

Problem (21)–(23) is a mixed-integer nonlinear program. Note that for a fixed value of μ, problem (21)–(23) reduces to a MILP feasibility problem or equivalently checking whether the MILP

$$\psi(\mu) = \min_{x\in X,\alpha,\beta,\gamma}\left\{\left(b_0+\sum_{j\in J}b_jx_j\right)\mu + \Gamma\alpha + \sum_{j\in J}\beta_j + \sum_{j\in J}\gamma_j - \left(a_0+\sum_{j\in J}a_jx_j\right) \;\middle|\; (15c)\right\} \tag{24}$$

has a nonpositive optimal objective function value (i.e., $\psi(\mu)\leq 0$). Proposition 7 shows that $\psi(\mu)$ is a monotone function of μ. Thus, we can solve (21)–(23) by applying the binary search algorithm on μ, where at each iteration of the algorithm, we solve (24) for a fixed value of μ. That is, if $\psi(\mu)>0$, we must decrease μ; otherwise, we can increase μ.

Proposition 7. *For given vectors d^1, d^2, d^3, and d^4 such that $d_j^2\cdot d_j^4\leq 0$ and $|d_j^2|, |d_j^4| \leq d_j^b$ for all $j\in J$, if $\psi(\mu)\leq 0$ for a fixed $\mu\geq 0$, then $\psi(\mu')\leq 0$ for any $0\leq \mu' < \mu$.*

Proof. For fixed $\mu \geq 0$, let (α,β,γ,x) denote a feasible solution of (24) for which the objective function value of (24) is nonpositive. Then we show that for $\mu' = \mu-\epsilon, \epsilon>0$, there exist $\beta', \gamma'\geq 0$ such that $(\alpha,\beta',\gamma',x)$ is a feasible solution of (24) with nonpositive objective function value. Toward this goal, define $J^2 = \{j\in J\,|\, d_j^2<0\}$ and $J^4 = \{j\in J\,|\,d_j^4<0\}$; note that $J^2\cap J^4 = \emptyset$ because $d_j^2\cdot d_j^4\leq 0$ for all $j\in J$. Then let $\beta_j' = \beta_j$ for $j\in J\setminus J^2$ and $\beta_j' = \beta_j - \epsilon d_j^2x_j \geq 0$ for $j\in J^2$. Similarly, let $\gamma_j' = \gamma_j$ for $j\in J\setminus J^4$ and $\gamma_j' = \gamma_j - \epsilon d_j^4 x_j\geq 0$ for $j\in J^4$. Hence, $(\mu',\alpha,\beta',\gamma',x)$ satisfies the constraints of (23).

Next, we show that for $(\mu',\alpha,\beta',\gamma',x)$, the objective function value of (24) is nonpositive:

$$
\begin{aligned}
&\left(b_0+\sum_{j\in J}b_jx_j\right)\mu' + \Gamma\alpha + \sum_{j\in J}\beta_j' + \sum_{j\in J}\gamma_j' - \left(a_0+\sum_{j\in J}a_jx_j\right)\\
&= \left(b_0+\sum_{j\in J}b_jx_j\right)(\mu-\epsilon) + \Gamma\alpha + \sum_{j\in J\setminus J^2}\beta_j + \sum_{j\in J^2}\left(\beta_j - \epsilon d_j^2x_j\right)\\
&\qquad + \sum_{j\in J\setminus J^4}\gamma_j + \sum_{j\in J^4}\left(\gamma_j-\epsilon d_j^4x_j\right) - \left(a_0+\sum_{j\in J}a_jx_j\right)\\
&= \left(b_0+\sum_{j\in J}b_jx_j\right)\mu + \Gamma\alpha + \sum_{j\in J}\beta_j + \sum_{j\in J}\gamma_j - \left(a_0+\sum_{j\in J}a_jx_j\right) - \epsilon\left(b_0+\sum_{j\in J}b_jx_j + \sum_{j\in J^2}d_j^2x_j + \sum_{j\in J^4}d_j^4x_j\right) \leq 0.
\end{aligned}
$$

The last inequality holds because the objective function value of (24) is nonpositive for $(\mu, \alpha, \beta, \gamma, x)$; moreover, because $J^2 \cap J^4 = \emptyset$ and $|d_j^2|, |d_j^4| \leq d_j^b$, for all $j \in J$, by Assumption 1, we have $(b_0 + \sum_{j\in J} b_j x_j + \sum_{j\in J^2} d_j^2 x_j + \sum_{j\in J^4} d_j^4 x_j) > 0$. □

In order to solve (24) efficiently at each iteration of the binary search algorithm, we further simplify it by using an argument similar to the one used for proving Proposition 3.

Proposition 8. *Problem* (24) *can be reformulated as*

$$\begin{aligned}\psi(\mu) = \min_{x\in X, \alpha\in\mathcal{F}} \Bigg\{ \Bigg(b_0 + \sum_{j\in J} b_j x_j\Bigg)\mu + \Gamma\alpha + \sum_{j\in J}\Big(d_j^1 + d_j^2\mu - \alpha\Big)^+ x_j \\ + \sum_{j\in J}\Big(d_j^3 + d_j^4\mu - \alpha\Big)^+ x_j - \Bigg(a_0 + \sum_{j\in J} a_j x_j\Bigg)\Bigg\},\end{aligned} \tag{25}$$

where

$$\mathcal{F} = \{0, (d_1^1 + d_1^2\mu)^+, (d_2^1 + d_2^2\mu)^+, \ldots, (d_n^1 + d_n^2\mu)^+, (d_1^3 + d_1^4\mu)^+, (d_2^3 + d_2^4\mu)^+, \ldots, (d_n^3 + d_n^4\mu)^+\}.$$

Proof. In an optimal solution of (24), we have that for all $j \in J$, $\beta_j^\star = ((d_j^1 + d_j^2\mu)x_j^\star - \alpha^\star)^+ = (d_j^1 + d_j^2\mu - \alpha^\star)^+ x_j^\star$ and $\gamma_j^\star = ((d_j^3 + d_j^4\mu)x_j^\star - \alpha^\star)^+ = (d_j^3 + d_j^4\mu - \alpha^\star)^+ x_j^\star$. Thus, (24) reduces to

$$\psi(\mu) = \min_{x\in X, \alpha\geq 0} \Bigg(b_0 + \sum_{j\in J} b_j x_j\Bigg)\mu + \Gamma\alpha + \sum_{j\in J}\Big(d_j^1 + d_j^2\mu - \alpha\Big)^+ x_j + \sum_{j\in J}\Big(d_j^3 + d_j^4\mu - \alpha\Big)^+ x_j - \Bigg(a_0 + \sum_{j\in J} a_j x_j\Bigg).$$

Additionally, similar to the Proof of Proposition 3, observe that if $\alpha^\star > 0$, $\alpha^\star \neq d_j^1 + d_j^2\mu$, and $\alpha^\star \neq d_j^3 + d_j^4\mu$ for all $j \in J$, then it can be verified that either $\alpha^\star + \epsilon$ or $\alpha^\star - \epsilon$ is also feasible for sufficiently small $\epsilon > 0$. Thus, we may assume without loss of generality that $\alpha^\star \in \{0\} \cup \{(d_j^1 + d_j^2\mu)^+\}_{j\in J} \cup \{(d_j^3 + d_j^4\mu)^+\}_{j\in J}$, which completes the proof. □

Example 2. According to Proposition 8, the corresponding formulation (25) for RFP[$\mathcal{U}_\alpha^{\overline{ab}}$] in Example 1 is

$$\begin{aligned}\psi(\mu) = \min_{x\in X, \alpha\in\mathcal{F}} \{&(3 + 2x_1 + 8x_2 + 2x_3)\mu + 2\alpha \\ &+ (-8 + 2\mu - \alpha)^+ x_1 + (-15 + 8\mu - \alpha)^+ x_2 + (-2 + \mu - \alpha)^+ x_3 \\ &+ (8 - 2\mu - \alpha)^+ x_1 + (15 - 8\mu - \alpha)^+ x_2 + (2 - \mu - \alpha)^+ x_3 - (6 + 5x_1 + 16x_2 + 3x_3)\},\end{aligned}$$

where $\mathcal{F} = \{0, (-8 + 2\mu)^+, (-15 + 8\mu)^+, (-2 + \mu)^+, (8 - 2\mu)^+, (15 - 8\mu)^+, (2 - \mu)^+\}$. □

In the following, we focus our efforts on obtaining the optimal objective function value of (25). To this end, define $T = \{1, 2, \ldots, |\mathcal{F}|\}$ and $|T| \leq 2n + 1$, and for each $t \in T$, define binary linear problem

$$\psi_t(\mu) = \min_{x\in X} g_t(x, \mu), \tag{26}$$

where

$$\begin{aligned}g_t(x, \mu) = \Bigg(b_0 + \sum_{j\in J} b_j x_j\Bigg)\mu + \Gamma(\bar{c}_t + \bar{d}_t\mu)^+ + \sum_{j\in J}\Big(d_j^1 + d_j^2\mu - (\bar{c}_t + \bar{d}_t\mu)^+\Big)^+ x_j \\ + \sum_{j\in J}\Big(d_j^3 + d_j^4\mu - (\bar{c}_t + \bar{d}_t\mu)^+\Big)^+ x_j - a_0 - \sum_{j\in J} a_j x_j,\end{aligned}$$

and $(\bar{c}_t, \bar{d}_t) \in \{(0, 0)\} \cup \{(d_j^1, d_j^2)\}_{j\in J} \cup \{(d_j^3, d_j^4)\}_{j\in J}$.

Evidently, $\psi(\mu) = \min_{t\in T} \psi_t(\mu)$. Thus, for μ fixed, check whether $\psi(\mu) \leq 0$ can be done by verifying whether there exists $t \in T$ such that $\psi_t(\mu) \leq 0$. Thereby, in the following result, we conclude that problem (21)–(23) can be solved efficiently using the binary search method.

Proposition 9. *Problem* (21)–(23) *can be solved with* $O(n \log(U/\epsilon))$ *calls to an oracle for* (26), *where* $U = |a_0| + \sum_{j\in J} |a_j|$, *and* $\epsilon > 0$ *is a precision parameter.*

Proof. The binary search requires $O(\log(\frac{U}{\epsilon}))$ iterations, and each iteration requires solving at most $|\mathcal{F}| = |T| = 2n+1$ problems of the form (26). Moreover, let $\tau(n)$ denote the complexity of solving binary linear problem (26). Then the binary search algorithm to solve problem (21)–(23) has the worst-case complexity $O(n\log(U/\epsilon)\tau(n))$. □

As a direct consequence of Propositions 7–9, we get the main result of this subsection.

Theorem 2. *Single-ratio* RFP$[\mathcal{U}^{\overline{ab}}]$, RFP$[\mathcal{U}^{\overline{ab}}_{=}]$, *and* RFP$[\mathcal{U}^{\overline{ab}}_{\propto}]$ *can be solved in* $O(n\log(U/\epsilon)\tau(n))$, *where* $\tau(n)$ *is the complexity of solving problem* (26). *In particular, if linear optimization over X is polynomial-time solvable, then so is single-ratio* RFP *under the joint uncertainty sets.*

Notably, when $X = \mathbb{B}^n$, the complexity of solving problem (26) is $O(n)$, that is, $\tau(n) = n$, resulting in the overall complexity $O(n^2\log(U/\epsilon))$ to solve RFP$[\mathcal{U}^{\overline{ab}}]$, RFP$[\mathcal{U}^{\overline{ab}}_{=}]$, and RFP$[\mathcal{U}^{\overline{ab}}_{\propto}]$. Additionally, if $X = \{x \in \mathbb{B}^n \mid \sum_{j\in J} x_j \le k\}$ or $X = \{x \in \mathbb{B}^n \mid \sum_{j\in J} x_j = k\}$, we have $\tau(n) = n\log(n)$, resulting in the overall complexity $O(n^2\log(n)\log(U/\epsilon))$. Therefore, we have the following corollary.

Corollary 1. *The unconstrained and cardinality-constrained single-ratio* RFP$[\mathcal{U}]$ *under joint uncertainty sets* $\mathcal{U}^{\overline{ab}}$, $\mathcal{U}^{\overline{ab}}_{=}$, *and* $\mathcal{U}^{\overline{ab}}_{\propto}$ *can be solved in polynomial time.*

It is worth mentioning that the cardinality-constrained ($X = \{x \in \mathbb{B}^n \mid \sum_{j\in J} x_j \le k\}$) single-ratio assortment problem (3) when customer preferences (ρ_j) are subject to rectangular uncertainty $\mathcal{U} = \prod_{j=0}^{n}[l_j, u_j] \subset \mathbb{R}^{n+1}_{++}$, where ℓ_j and u_j are lower and upper bounds on ρ_j, respectively, can be solved in $O(n^2)$ (Rusmevichientong and Topaloglu 2012). This problem is a special case of RFP$[\mathcal{U}^{\overline{ab}}_{\propto}]$ when $\Gamma = n$ and $\widetilde{a}_j, \widetilde{b}_j > 0$. However, the aforementioned result cannot be extended, for example, when revenues (r_j) are uncertain or, more importantly, for generally structured single-ratio RFP$[\mathcal{U}]$ (such as other choice models) under other types of the budgeted uncertainty sets or (weaker) Assumption 1.

We conclude the discussion on single-ratio RFP$[\mathcal{U}]$ with the following remarks.

Remark 1. The solutions methods outlined in this section are particularly efficient for unconstrained problems. Additionally, they are useful when there exist specialized algorithms to solve the corresponding constrained linear binary problem: for example, those that exploit the constraint structure of the underlying combinatorial optimization problem. If these algorithms are polynomial time (e.g., such as those for the linear assignment, the shortest path, and the minimum spanning tree problems; Ahuja et al. 1993), then the single-ratio RFP$[\mathcal{U}]$ is also polynomial-time solvable.

Remark 2. In the case of single-ratio RFPs under the disjoint uncertainty set, the approach of Theorem 1 is superior to the binary search approach developed in Section 3.2 because the former is strongly polynomial $O(n^2)$, whereas the latter involves the binary search algorithm with the number of iterations $O(\log(\frac{U}{\epsilon}))$.

4. Multiple-Ratio RFP[$\mathcal{U}$]

In this section, we present MILP formulations for multiple-ratio RFP$[\mathcal{U}]$. First, for the disjoint uncertainty set, we reformulate RFP$[\mathcal{U}^{ab}]$ as robust linear problems. Second, with these reformulations in hand, we adapt the methods of Bertsimas and Sim (2004) to transform them into MILPs (see Section 4.1). For the joint uncertainty sets (except for $\mathcal{U}^{\bar{a}}$), we use the results from Section 3.2 (see Section 4.2.1); for $\mathcal{U}^{\bar{a}}$, we use a same approach provided in Section 4.1 (see Section 4.2.2). Third, in Section 4.3, we discuss the sizes (numbers of variables and constraints) of the MILP reformulations obtained. Fourth, in Section 4.4, we show that the optimal value of the robust formulations provided in this paper with high probability is not an overestimation of the true value of the fractional problems with symmetric and bounded random coefficients.

4.1. Disjoint Uncertainty Set

For this discussion, we consider the uncertainty set $\mathcal{U}^{ab}$ and present three MILP reformulations of RFP$[\mathcal{U}^{ab}]$. For the first two formulations presented in Sections 4.1.1 and 4.1.2, we exploit the ideas from fractional programming literature (Williams 1974, Li 1994). The third formulation, presented in Section 4.1.3, corresponds to a binary expansion reformulation proposed by Borrero et al. (2016).

4.1.1. Reformulation 1 (MILP$_1[\mathcal{U}^{ab}]$). Note that RFP$[\mathcal{U}^{ab}]$ can be written as

$$\max_{x\in X} \min_{(\tilde{a},\tilde{b})\in\mathcal{U}^{ab}} \sum_{i\in I}\left(a_{i0} + \sum_{j\in J}\widetilde{a}_{ij}x_j\right)\left(\frac{1}{b_{i0} + \sum_{j\in J}\widetilde{b}_{ij}x_j}\right).$$

Using the substitutions

$$\omega_i \le \frac{1}{b_{i0} + \sum_{j\in J} \widetilde{b}_{ij} x_j},$$

for all $\widetilde{b}_i \in \mathcal{U}_i^b$ and $i \in I$, and exploiting the fact that $\mathcal{U}^{ab}$ is disjoint, we find the equivalent formulation

$$\begin{aligned} &\max_{\substack{x\in X,\\ \omega\ge 0}} \min_{\widetilde{a}\in\mathcal{U}^a} \sum_{i\in I} \left(a_{i0} + \sum_{j\in J} \widetilde{a}_{ij} x_j\right)\omega_i \\ &\text{s.t. } \left(b_{i0} + \sum_{j\in J} \widetilde{b}_{ij} x_j\right)\omega_i \le 1 && \forall \widetilde{b}_i \in \mathcal{U}_i^b,\ \forall i \in I, \end{aligned}$$

where $\mathcal{U}^a := \{\widetilde{a} \in \mathbb{R}^{m\times n} \,|\, \widetilde{a}_i \in \mathcal{U}_i^a \text{ for all } i \in I\}$. Similarly, defining new variables μ_i such that $\mu_i \le (a_{i0} + \sum_{j\in J} \widetilde{a}_{ij} x_j)\omega_i$ for all $\widetilde{a}_i \in \mathcal{U}_i^a$ and $i \in I$ yields the robust optimization problem

$$\begin{aligned} &\max_{\substack{x\in X,\\ \mu,\omega\ge 0}} \sum_{i\in I} \mu_i && \left(\text{RFP}_1\left[\mathcal{U}^{ab}\right]\right) \\ &\text{s.t. } \mu_i \le \left(a_{i0} + \sum_{j\in J} \widetilde{a}_{ij} x_j\right)\omega_i && \forall \widetilde{a}_i \in \mathcal{U}_i^a,\ \forall i \in I \\ &\left(b_{i0} + \sum_{j\in J} \widetilde{b}_{ij} x_j\right)\omega_i \le 1 && \forall \widetilde{b}_i \in \mathcal{U}_i^b,\ \forall i \in I. \end{aligned}$$

Note that the directions of the inequalities ($\le$) rely on the sense of the objective function and Assumption 1. Because $x \in X \subseteq \mathbb{B}^n$, we linearize the bilinear terms $x_j\omega_i$ using standard techniques (Wu 1997, Adams and Forrester 2005) as follows:

$$\Delta_{ij} := \left\{(x_j, \omega_i, z_{ij}) \in \mathbb{B} \times \mathbb{R}_+^2 \,|\, z_{ij} \le \omega_i^U x_j, z_{ij} \ge \omega_i^L x_j, z_{ij} \le \omega_i + \omega_i^L(x_j - 1), z_{ij} \ge \omega_i + \omega_i^U(x_j - 1)\right\},$$

where ω_i^U and ω_i^L are an upper bound and a lower bound on ω_i, respectively; note that $(x_j, w_i, z_{ij}) \in \Delta_{ij} \Leftrightarrow z_{ij} = w_i x_j$. Hence, $\text{RFP}_1[\mathcal{U}^{ab}]$ is equivalent to the robust linear problem

$$\begin{aligned} &\max_{\substack{x\in X\\ \omega,\mu,z\ge 0}} \sum_{i\in I} \mu_i && (27) \\ &\text{s.t. } \mu_i \le a_{i0}\omega_i + \sum_{j\in J} \widetilde{a}_{ij} z_{ij} && \forall \widetilde{a}_i \in \mathcal{U}_i^a,\ \forall i \in I \\ &b_{i0}\omega_i + \sum_{j\in J} \widetilde{b}_{ij} z_{ij} \le 1 && \forall \widetilde{b}_i \in \mathcal{U}_i^b,\ \forall i \in I \\ &\left(x_j, \omega_i, z_{ij}\right) \in \Delta_{ij} && \forall i \in I, j \in J. \end{aligned}$$

Following the approach of Bertsimas and Sim (2004), the robust linear problem (27) can be transformed into an MILP reformulation of $\text{RFP}[\mathcal{U}^{ab}]$ as follows:

$$\begin{aligned} &\max \sum_{i\in I} \mu_i && \left(\text{MILP}_1\left[\mathcal{U}^{ab}\right]\right) \\ &\text{s.t. } \mu_i - \sum_{j\in J} a_{ij} z_{ij} + \sum_{j\in J} \beta_{ij} + \Gamma_i^a \alpha_i \le a_{i0}\omega_i && \forall i \in I \\ &b_{i0}\omega_i + \sum_{j\in J} b_{ij} z_{ij} + \sum_{j\in J} \gamma_{ij} + \Gamma_i^b \lambda_i \le 1 && \forall i \in I \\ &\alpha_i + \beta_{ij} \ge d_{ij}^a z_{ij} && \forall i \in I, \forall j \in J \\ &\lambda_i + \gamma_{ij} \ge d_{ij}^b z_{ij} && \forall i \in I, \forall j \in J \\ &x \in X,\ \left(x_j, \omega_i, z_{ij}\right) \in \Delta_{ij},\ \beta_{ij}, \gamma_{ij}, \alpha_i, \lambda_i, \mu_i \ge 0 && \forall i \in I, \forall j \in J. \end{aligned}$$

4.1.2. Reformulation 2 ($\text{MILP}_2[\mathcal{U}^{ab}]$). As an alternative to the approach of Section 4.1.1, one could instead replace each ratio with an auxiliary variable. Let

$$\mu_i \le \frac{a_{i0} + \sum_{j\in J} \widetilde{a}_{ij} x_j}{b_{i0} + \sum_{j\in J} \widetilde{b}_{ij} x_j} \qquad \forall i \in I, \left(\widetilde{a}_i, \widetilde{b}_i\right) \in \mathcal{U}_i^a \times \mathcal{U}_i^b.$$

Then we can write $\text{RFP}[\mathcal{U}^{ab}]$ as

$$\max_{\substack{x\in X,\\ \mu\ge 0}} \sum_{i\in I} \mu_i \qquad \left(\text{RFP}_2\left[\mathcal{U}^{ab}\right]\right)$$

$$\text{s.t.} \left(b_{i0} + \sum_{j\in J} \widetilde{b}_{ij} x_j\right) \mu_i \le a_{i0} + \sum_{j\in J} \widetilde{a}_{ij} x_j \qquad \forall \widetilde{a}_i \in \mathcal{U}_i^a,\ \forall \widetilde{b}_i \in \mathcal{U}_i^b, \forall i \in I.$$

Finally, after linearization of $x_j\mu_i$ using a variant of the set Δ_{ij} and applying the transformation of a robust linear problem to a MILP similar to the one used in Section 4.1.1, we find the MILP reformulation of $\text{RFP}_2[\mathcal{U}^{ab}]$:

$$\begin{aligned} \max \ & \sum_{i\in I} \mu_i && \left(\text{MILP}_2\left[\mathcal{U}^{ab}\right]\right) \\ \text{s.t. } & b_{i0}\mu_i - \sum_{j\in J} a_{ij} x_j + \sum_{j\in J} b_{ij} z_{ij} + \Gamma_i^a \alpha_i + \Gamma_i^b \lambda_i + \sum_{j\in J} \beta_{ij} + \sum_{j\in J} \gamma_{ij} \le a_{i0} && \forall i \in I \\ & \alpha_i + \beta_{ij} \ge d_{ij}^a x_j && \forall i \in I, \forall j \in J \\ & \lambda_i + \gamma_{ij} \ge d_{ij}^b z_{ij} && \forall i \in I, \forall j \in J \\ & x \in X, \left(x_j, \mu_i, z_{ij}\right) \in \Delta_{ij}, \beta_{ij}, \gamma_{ij}, \alpha_i, \lambda_i, \mu_i \ge 0 && \forall i \in I, \forall j \in J. \end{aligned}$$

4.1.3. Binary Expansion Reformulation ($\text{MILP}_2^{\log}[\mathcal{U}^{ab}]$). The third formulation considered uses a base 2 expansion (Borrero et al. 2016) to reduce the number of bilinear terms that require linearization. In the context of RFP, we use this idea to reformulate $\text{RFP}_2[\mathcal{U}^{ab}]$.

Observe that for any $x \in X$ and worst-case realization $\widetilde{b}_i \in \mathcal{U}_i^b$, the term $\sum_{j\in J} \widetilde{b}_{ij} x_j$ is integer because the data are integral (Assumption 2). To ascertain the (logarithmic) number of additional variables needed, let $\max^r(H_i)$ return the rth largest element in the set $H_i = \{d_{ij}^b \,|\, j \in J\}$. Then, for all $i \in I$, we define π_i as follows:

$$\pi_i := \left\lfloor \log_2\left(\sum_{j\in J} |b_{ij}| + \sum_{r \le \Gamma_i^b} \overset{r}{\max}(H_i)\right)\right\rfloor + 1. \tag{28}$$

We then define the binarization variables $w_{ik} \in \mathbb{B}$ for all $k \in K_i := \{1, 2, \ldots, \pi_i\}, i \in I$. We also define $\bar{B}_i := \sum_{j\in J, b_{ij}<0} |b_{ij}|$. Observe that $\sum_{j\in J} \widetilde{b}_{ij} x_j + \bar{B}_i \ge 0$ for any $x \in X$ and $\widetilde{b}_i \in \mathcal{U}_i^b$. Replacing the terms $\sum_{j\in J} \widetilde{b}_{ij} x_j$ with $-\bar{B}_i + \sum_{k=1}^{\pi_i} 2^{k-1} w_{ik}$ for all $i \in I$ in $\text{RFP}_2[\mathcal{U}^{ab}]$ yields

$$\begin{aligned} \max \ & \sum_{i\in I} \mu_i && \left(\text{RFP}_2^{\log}\left[\mathcal{U}^{ab}\right]\right) \\ \text{s.t. } & \left(b_{i0} - \bar{B}_i + \sum_{k\in K_i} 2^{k-1} w_{ik}\right) \mu_i \le a_{i0} + \sum_{j\in J} \widetilde{a}_{ij} x_j && \forall \widetilde{a}_i \in \mathcal{U}_i^a,\ \forall i \in I \\ & \sum_{j\in J} \widetilde{b}_{ij} x_j + \bar{B}_i \le \sum_{k\in K_i} 2^{k-1} w_{ik} && \forall \widetilde{b}_i \in \mathcal{U}_i^b,\ \forall i \in I \\ & x \in X, w_{ik} \in \mathbb{B}, \mu_i \ge 0 && \forall k \in K_i, \forall i \in I. \end{aligned}$$

Let $z_{ik} = w_{ik}\mu_i$. By using a variant of the set Δ_{ij} to linearize bilinear terms of model $\text{RFP}_2^{\log}[\mathcal{U}^{ab}]$ (i.e., $w_{ik}\mu_i$) and applying the transformation of a robust linear problem to a MILP similar to the one used in Section 4.1.1, $\text{RFP}_2^{\log}[\mathcal{U}^{ab}]$ can be reformulated as the following MILP:

$$
\begin{aligned}
&\max \sum_{i\in I} \mu_i && \left(\text{MILP}_2^{\log}[\mathcal{U}^{ab}]\right)\\
&\text{s.t. } (b_{i0} - \bar{B}_i)\mu_i + \sum_{k\in K_i} 2^{k-1} z_{ik} - \sum_{j\in J} a_{ij}x_j + \sum_{j\in J} \beta_{ij} + \Gamma_i^a \alpha_i \le a_{i0} && \forall i \in I\\
&\quad - \sum_{k\in K_i} 2^{k-1} w_{ik} + \sum_{j\in J} b_{ij}x_j + \bar{B}_i + \sum_{j\in J} \gamma_{ij} + \Gamma_i^b \lambda_i \le 0 && \forall i \in I\\
&\quad \alpha_i + \beta_{ij} \ge d_{ij}^a x_j && \forall i \in I, \forall j \in J\\
&\quad \lambda_i + \gamma_{ij} \ge d_{ij} x_j && \forall i \in I, \forall j \in J\\
&\quad x \in X,\ w_{ik} \in \mathbb{B},\ (w_{ik}, \mu_i, z_{ik}) \in \Delta_{ik},\ \beta_{ij}, \gamma_{ij}, \alpha_i, \lambda_i, \mu_i \ge 0 && \forall i \in I, \forall j \in J, \forall k \in K_i.
\end{aligned}
$$

Remark 3. It is also possible to develop a binary expansion reformulation of $\text{RFP}_1[\mathcal{U}^{ab}]$. However, based on our experiments, such a formulation performs poorly in computations; also, refer to Borrero et al. (2016) and Mehmanchi et al. (2019) for an analogous comparison regarding deterministic FP. Hence, we omit this formulation for brevity.

4.2. Joint Uncertainty Sets

We now present MILP formulations of $\text{RFP}[\mathcal{U}]$ for the joint uncertainty sets $\mathcal{U} \in \{\mathcal{U}^{\overline{ab}}, \mathcal{U}_{=}^{\overline{ab}}, \mathcal{U}_{\propto}^{\overline{ab}}, \mathcal{U}^{\overline{a}}\}$. Toward this goal, we use the results of Section 3.2 to develop MILPs for multiple-ratio $\text{RFP}[\mathcal{U}^{\overline{ab}}]$, $\text{RFP}[\mathcal{U}_{=}^{\overline{ab}}]$, and $\text{RFP}[\mathcal{U}_{\propto}^{\overline{ab}}]$ (see Section 4.2.1). For $\text{RFP}[\mathcal{U}^{\overline{a}}]$, we use a similar approach to the one used in Section 4.1.1 (see Section 4.2.2). Note that for the joint uncertainty sets, we cannot take advantage of the binary expansion technique either because of dependencies in the uncertainty sets or because it does not reduce the number of bilinear terms for the joint cases.

4.2.1. Reformulation for RFP[$\mathcal{U}$] When $\mathcal{U} \in \{\mathcal{U}^{\overline{ab}}, \mathcal{U}_{=}^{\overline{ab}}, \mathcal{U}_{\propto}^{\overline{ab}}\}$ (MILP[$\mathcal{U}^{\overline{ab}}$], MILP[$\mathcal{U}_{=}^{\overline{ab}}$], and MILP[$\mathcal{U}_{\propto}^{\overline{ab}}$]).

By Propositions 4–6, it can be verified that multiple-ratio $\text{RFP}[\mathcal{U}]$ under joint uncertainties $\mathcal{U}^{\overline{ab}}$, $\mathcal{U}_{=}^{\overline{ab}}$, and $\mathcal{U}_{\propto}^{\overline{ab}}$ can be represented as the following problem:

$$
\begin{aligned}
&\max_{\substack{x\in X,\\ \mu,\alpha,\beta,\gamma\ge 0}} \sum_{i\in I} \mu_i && (29)\\
&\text{s.t. } \left(b_{i0} + \sum_{j\in J} b_{ij}x_j\right)\mu_i + \Gamma_i\alpha_i + \sum_{j\in J}\beta_{ij} + \sum_{j\in J}\gamma_{ij} \le a_{i0} + \sum_{j\in J} a_{ij}x_j && \forall i \in I\\
&\quad \alpha_i + \beta_{ij} \ \ge\ \left(d_{ij}^1 + d_{ij}^2\mu_i\right)x_j, \quad \alpha_i + \gamma_{ij} \ \ge\ \left(d_{ij}^3 + d_{ij}^4\mu_i\right)x_j && \forall i \in I, \forall j \in J,
\end{aligned}
$$

for some $d^1, d^2, d^3, d^4 \in \mathbb{Z}^{m\times n}$. By linearizing the bilinear terms $x_j\mu_i$, problem (29) can be reformulated as an equivalent MILP:

$$
\begin{aligned}
&\max_{\substack{x\in X,\\ \mu,\alpha,\beta,\gamma\ge 0}} \sum_{i\in I} \mu_i && (30)\\
&\text{s.t. } b_{i0}\mu_i - \sum_{j\in J} a_{ij}x_j + \sum_{j\in J} b_{ij}z_{ij} + \Gamma_i\alpha_i + \sum_{j\in J}\beta_{ij} + \sum_{j\in J}\gamma_{ij} \le a_{i0} && \forall i \in I\\
&\quad \alpha_i + \beta_{ij} \ge d_{ij}^1 x_j + d_{ij}^2 z_{ij}, \quad \alpha_i + \gamma_{ij} \ge d_{ij}^3 x_j + d_{ij}^4 z_{ij} && \forall i \in I, \forall j \in J\\
&\quad x \in X, (x_j, \mu_i, z_{ij}) \in \Delta_{ij}, \beta_{ij}, \gamma_{ij}, \alpha_i, \mu_i \ge 0 && \forall i \in I, \forall j \in J.
\end{aligned}
$$

Specifically, if we let $d_j^1 = d_j^a, d_j^2 = d_j^3 = 0$, and $d_j^4 = d_j^b$ for all $j \in J$, then problem (30) is an equivalent MILP reformulation of $\text{RFP}[\mathcal{U}^{\overline{ab}}]$ denoted by $\text{MILP}[\mathcal{U}^{\overline{ab}}]$. Similarly, letting $d_j^1 = d_j^a, d_j^2 = d_j^b, d_j^3 = d_j^4 = 0$, and $d_j^1 = -d_j^3 = -d_j^a, d_j^2 = -d_j^4 = d_j^b$ for all $j \in J$ in (30) leads to equivalent MILP reformulations of $\text{RFP}[\mathcal{U}_{=}^{\overline{ab}}]$ and $\text{RFP}[\mathcal{U}_{\propto}^{\overline{ab}}]$ indicated by $\text{MILP}[\mathcal{U}_{=}^{\overline{ab}}]$ and $\text{MILP}[\mathcal{U}_{\propto}^{\overline{ab}}]$, respectively. Finally, note that in $\text{MILP}[\mathcal{U}_{=}^{\overline{ab}}]$, because $d_j^3 = d_j^4 = 0$, variable γ_{ij} and constraint $\alpha_i + \gamma_{ij} \ge d_{ij}^3 x_j + d_{ij}^4 z_{ij}$ can be removed for all $i \in I, j \in J$ (see Table 1 for the size of formulations).

4.2.2. Reformulation for RFP[$\mathcal{U}^{\bar{a}}$] (MILP[$\mathcal{U}^{\bar{a}}$]). Let ω as in Section 4.1.1 define a new variable $\mu \le \sum_{i\in I}(a_{i0} + \sum_{j\in J}\widetilde{a}_{ij}x_j)\omega_i$ for all $\widetilde{a} \in \mathcal{U}^{\bar{a}}$, and write RFP[$\mathcal{U}^{\bar{a}}$] as

$$\begin{aligned} \max_{x\in X,\omega,\mu\ge 0} \quad & \mu \\ \text{s.t.} \quad & \mu \le \sum_{i\in I} a_{i0}\omega_i + \sum_{i\in I}\sum_{j\in J}\widetilde{a}_{ij}x_j\omega_i && \forall \widetilde{a} \in \mathcal{U}^{\bar{a}} \\ & b_{i0}\omega_i + \sum_{j\in J} b_{ij}x_j\omega_i \le 1 && \forall i \in I. \end{aligned}$$

Letting $z_{ij} = x_j\omega_i$ and u be the indicator variables of set $S_i(\widetilde{a})$, we obtain

$$\begin{aligned} \max_{\substack{x\in X,\\ \mu,\omega\ge 0}} \quad & \mu \\ \text{s.t.} \quad & \mu - \sum_{i\in I} a_{i0}\omega_i - \sum_{i\in I}\sum_{j\in J} a_{ij}z_{ij} + \max_{u\in V}\left\{\sum_{j\in J} d^a_{ij}z_{ij}u_{ij}\right\} \le 0 && \forall i \in I \\ & b_{i0}\omega_i + \sum_{j\in J} b_{ij}z_{ij} \le 1 && \forall i \in I \\ & (x_j, \omega_i, z_{ij}) \in \Delta_{ij} && \forall i \in I,\ j \in J, \end{aligned}$$

where V is the polytope defined by the constraints

$$\begin{aligned} & \sum_{i\in I}\sum_{j\in J} u_{ij} \le \Gamma && (\alpha) \\ & 0 \le u_{ij} \le 1 && \forall i \in I, j \in J. \quad (\beta_{ij}) \end{aligned}$$

Using LP duality for the inner maximization problem, we obtain the MILP formulation

$$\begin{aligned} \max \quad & \mu && \left(\text{MILP}\left[\mathcal{U}^{\bar{a}}\right]\right) \\ \text{s.t.} \quad & \mu - \sum_{i\in I} a_{i0}\omega_i - \sum_{i\in I}\sum_{j\in J} a_{ij}z_{ij} + \Gamma\alpha + \sum_{i\in I}\sum_{j\in J}\beta_{ij} \le 0 \\ & b_{i0}\omega_i + \sum_{j\in J} b_{ij}z_{ij} \le 1 && \forall i \in I \\ & \alpha + \beta_{ij} \ge d^a_{ij}z_{ij} && \forall i \in I, \forall j \in J \\ & x \in X,\ (x_j, \omega_i, z_{ij}) \in \Delta_{ij},\ \beta_{ij}, \alpha, \mu, \omega_i \ge 0 && \forall i \in I, \forall j \in J. \end{aligned}$$

4.3. Problems Sizes and MILP Enhancement (MILP$^{\log}_{2'}$[$\mathcal{U}^{ab}$])

Table 1 shows the number of variables and constraints for all MILP reformulations provided in Sections 4.1 and 4.2. This table also includes the sizes of the well-known MILPs for FP, denoted by FP_1 (Li 1994, Wu 1997, Tawarmalani et al. 2002) and FP_2 (Tawarmalani et al. 2002), as well as their respective binary expansion versions (Borrero et al. 2016), denoted by FP_3 and FP_4.

Later, in Section 5.2.3, we observe that among the MILPs for the disjoint uncertainty, $\text{MILP}_1[\mathcal{U}^{ab}]$ typically has the best LP relaxation, and $\text{MILP}^{\log}_2[\mathcal{U}^{ab}]$ often has the best performance because of a reduced number of variables and constraints (see Table 1). Hence, we enhance $\text{MILP}^{\log}_2[\mathcal{U}^{ab}]$ by adding the valid inequality $\sum_{i\in I}\mu_i \le z_{LP}^{MILP_1[\mathcal{U}^{ab}]}$ to $\text{MILP}^{\log}_2[\mathcal{U}^{ab}]$, where $z_{LP}^{MILP_1[\mathcal{U}^{ab}]}$ is the objective function value of the LP relaxation of $\text{MILP}_1[\mathcal{U}^{ab}]$, and we call the new formulation $\text{MILP}^{\log}_{2'}[\mathcal{U}^{ab}]$. In the deterministic fractional programming, Borrero et al. (2016) made a similar observation regarding FP_1 and FP_4. They call the new formulation $\text{FP}_{4'}$, and we compare its performance with the performances of the developed MILPs for the disjoint uncertainty in the next section.

4.4. Insights on the Price of Robustness

In robust linear optimization when uncertain coefficients are symmetric, bounded, and independent random variables, Bertsimas and Sim (2004) provide a probabilistic guarantee for each constraint violation. Next, we exploit their approach to establish somewhat similar results for RFPs under disjoint uncertainty sets.

Table 1. Sizes of the MILPs for Nominal Problems FP_1 to FP_4 and the Robust Problems

MILP reformulation	No. of continuous variables	No. of binary variables	No. of linear constraints
Nominal reformulations			
FP_1	$m(n+1)$	n	$m(4n+1)+c$
FP_2	$m(n+1)$	n	$m(4n+1)+c$
FP_3	$m+\sum_{i\in I}(\theta_i^a+\theta_i^b)$	$n+\sum_{i\in I}(\theta_i^a+\theta_i^b)$	$3m+4\sum_{i\in I}(\theta_i^a+\theta_i^b)+c$
FP_4	$m+\sum_{i\in I}\theta_i^b$	$n+\sum_{i\in I}\theta_i^b$	$2m+4\sum_{i\in I}\theta_i^b+c$
Robust reformulations (disjoint)			
$\mathrm{MILP}_1[\mathcal{U}^{ab}]$	$m(3n+4)$	n	$m(6n+2)+c$
$\mathrm{MILP}_2[\mathcal{U}^{ab}]$	$m(3n+3)$	n	$m(6n+1)+c$
$\mathrm{MILP}_2^{\log}[\mathcal{U}^{ab}]$	$m(2n+3)+\sum_{i\in I}\pi_i$	$n+\sum_{i\in I}\pi_i$	$m(2n+2)+4\sum_{i\in I}\pi_i+c$
Robust reformulations (joint)			
$\mathrm{MILP}_2[\mathcal{U}^{\overline{ab}}]$ and $\mathrm{MILP}_2[\mathcal{U}_{\propto}^{\overline{ab}}]$	$m(3n+2)$	n	$m(6n+1)+c$
$\mathrm{MILP}_2[\mathcal{U}_{=}^{\overline{ab}}]$	$m(2n+2)$	n	$m(5n+1)+c$
$\mathrm{MILP}[\mathcal{U}^{\bar{a}}]$	$m(2n+1)+2$	n	$m(5n+1)+c+1$

Notes. n and m are defined as in FP, c is the number of constraints defining X, and π_i is defined as in (28). Moreover, $\theta_i^a := \lfloor \log_2(\sum_{j\in J}|a_{ij}|)\rfloor + 1$ and $\theta_i^b := \lfloor \log_2(\sum_{j\in J}|b_{ij}|)\rfloor + 1$.

Let $x^\star$ and $\mu_i^\star$ denote a robust optimal solution and the robust value of the ith ratio in RFP[$\mathcal{U}$], respectively. By using the binomial distribution

$$B(r,P) = \frac{1}{2^r}\left\{(1-\nu+\lfloor\nu\rfloor)\binom{r}{\lfloor\nu\rfloor} + \sum_{j=\lfloor\nu\rfloor+1}^{r}\binom{r}{j}\right\},$$

for $\nu = (P+r)/2$, and $r, P \in \mathbb{Z}_+$, we show that the probability that $\mu_i^\star$ overestimates the true value of the ith ratio for random variables $\tilde{a}$ and $\tilde{b}$ is bounded above.

Proposition 10. *Let $\tilde{a}$ and $\tilde{b}$ be symmetric, bounded, and independent random variables; that is, $\tilde{a}_{ij} = a_{ij} + \eta_{ij}d_{ij}^a$ and $\tilde{b}_{ij} = b_{ij} + \eta_{i,j+n}d_{ij}^b$, where $\eta_{ij}, \eta_{i,j+n} \in [-1,1]$, for all $i \in I, j \in J$, are independently distributed random variables. For each $i \in I$, in* RFP[$\mathcal{U}$],

$$\text{(a) if } \mathcal{U} = \mathcal{U}^{ab}, \text{ then } \Pr\left(\mu_i^\star > \frac{a_{i0} + \sum_{j\in J}\tilde{a}_{ij}x_j^\star}{b_{i0} + \sum_{j\in J}\tilde{b}_{ij}x_j^\star}\right) \le B(2n, \Gamma_i^a + \Gamma_i^b), \qquad \Gamma_i^a, \Gamma_i^b \in \{0,\ldots,n\};$$

$$\text{(b) if } \mathcal{U} = \mathcal{U}^{\overline{ab}}, \text{ then } \Pr\left(\mu_i^\star > \frac{a_{i0} + \sum_{j\in J}\tilde{a}_{ij}x_j^\star}{b_{i0} + \sum_{j\in J}\tilde{b}_{ij}x_j^\star}\right) \le B(2n, \Gamma_i), \qquad \Gamma_i \in \{0,\ldots,2n\};$$

additionally,

$$\text{(c) if } \mathcal{U} = \mathcal{U}^{\bar{a}}, \text{ then } \Pr\left(\sum_{i\in I}\mu_i^\star > \sum_{i\in I}\frac{a_{i0} + \sum_{j\in J}\tilde{a}_{ij}x_j^\star}{b_{i0} + \sum_{j\in J}b_{ij}x_j^\star}\right) \le B(m\cdot n, \Gamma), \qquad \Gamma \in \{0,\ldots,m\cdot n\}.$$

Proposition 11. *Let $\tilde{a}$ and $\tilde{b}$ be symmetric and bounded random variables; that is, $\tilde{a}_{ij} = a_{ij} + \eta_{ij}d_{ij}^a$ and $\tilde{b}_{ij} = b_{ij} + \eta_{ij}d_{ij}^b$, where $\eta_{ij} \in [-1,1]$, for all $i \in I, j \in J$, are independently distributed random variables. For each $i \in I$, in* RFP[$\mathcal{U}$],

$$\text{if } \mathcal{U} = \mathcal{U}_{\propto}^{\overline{ab}}, \text{ then } \Pr\left(\mu_i^\star > \frac{a_{i0} + \sum_{j\in J}\tilde{a}_{ij}x_j^\star}{b_{i0} + \sum_{j\in J}\tilde{b}_{ij}x_j^\star}\right) \le B(n, \Gamma_i), \qquad \Gamma_i \in \{0,\ldots,n\}.$$

See Appendix A for proofs of Propositions 10 and 11. Evidently, because the decision maker is more conservative and selects larger levels of uncertainty (Γ), the probability that $\mu_i^\star$ is larger than the value of the ith ratio for $x^\star$ and random variables $\tilde{a}$ and $\tilde{b}$ is smaller. Note that we do not derive a similar upper bound when $\mathcal{U} = \mathcal{U}_{=}^{\overline{ab}}$ because we cannot satisfy the key assumption that random variables η are independently distributed.

5. Computational Results

The computational experiments in this section encompass a case study of a particular assortment problem (see Section 5.1) as well as experiments on instances with synthetic data to evaluate the performance of our MILP reformulations (see Section 5.2). In both of the following subsections, we describe the relevant test instances, compare the robust and nominal solutions, and discuss relevant aspects of the solutions. Our experiments were performed using CPLEX 12.7.1 (IBM 2017) on an eight-core central processing unit (3.7 GHz) with 32 GB of random-access memory.

5.1. Case Study: Robust Assortment Optimization for Frozen Pizza

Assortment optimization problems arise in many applications, such as retailing, revenue management, and online advertising. Assortment optimization with uncertainty considerations is a growing area of research; in addition to Rusmevichientong and Topaloglu (2012), discussed in Sections 1.1 and 3.2, Bertsimas and Mišić (2016) and Désir et al. (2019) have proposed robust optimization approaches for different classes of assortment optimization problems.

Our case study, outlined next, optimizes an assortment problem for a real retailer of frozen pizza studied by Keane and Wada (2012); the data are available at http://cblib.zib.de. The objective of the assortment problem is to maximize revenue for a company given a large number of potential product offerings, associated revenues for those offerings, and estimations of customer preferences between those offerings. Additionally, the customers are divided into several different classes; thus, the mixed multinomial logit choice model is a natural fit for the problem.

5.1.1. Test Instances. The test instances comprise customer preference data on frozen pizzas from Keane and Wada (2012). In particular, there are 130 potential product offerings divided into five tiers of revenue ($1.49, $1.75, $1.79, $1.89, and $2.75), and there are three classes of customers. Thus, the problem is an instance of (3) with $m = 3$ ratios and $n = 130$ variables. The same data were used for each test, with variations in the type of uncertainty set as well as the level of uncertainty Γ. We fixed $d^a_{ij} = 0.5a_{ij}$ and $d^b_{ij} = 0.5b_{ij}$ (where relevant) for all uncertainty sets.

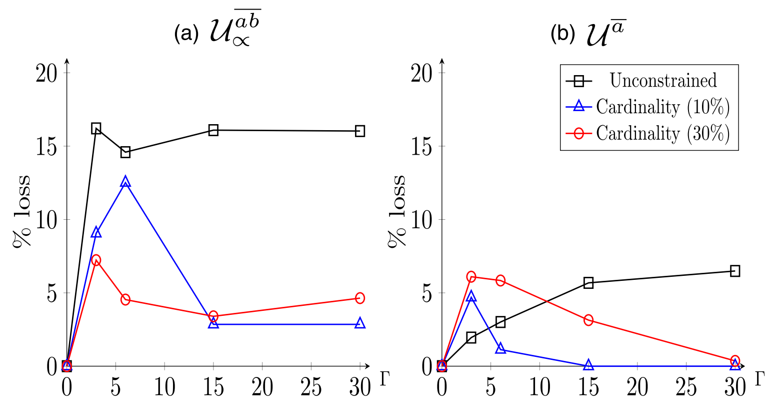
For the case study, we consider four robust problems; specifically, we consider unconstrained ($X = \mathbb{B}^n$) and cardinality-constrained ($X = \{x \in \mathbb{B}^n \mid \sum_{j\in J} x_j \le k\}$) versions of RFP[$\mathcal{U}^{\overline{ab}}_{\propto}$] and RFP[$\mathcal{U}^{\bar{a}}$], which are a natural fit for this application. Uncertainty in customer preferences (ρ_{ij}) and revenues (r_{ij}) can be captured by the matched effects $\mathcal{U}^{\overline{ab}}_{\propto}$ and the single-budget $\mathcal{U}^{\bar{a}}$, uncertainty sets, respectively (see Section 2). With respect to the feasible region, we test the unconstrained case—for an online retailer with the ability to market many options—as well as two sizes of cardinality constraint: $k = 13$ and $k = 39$, corresponding to 10% and 30% of the 130 variables, respectively. The latter problem classes correspond to a small and a large retailer, respectively, where there is a physical limitation on the number of products that can be offered to customers.

5.1.2. The Price of Robustness. The value in the robust approach is demonstrated by checking the performance of the nominal (optimal) solution in the uncertain environment and vice versa. These results are shown in Figures 1 and 2. Figure 1 shows the relative decrease in the robust objective function value when the optimal nominal solution is used in the uncertain setting instead of the optimal robust solution (at the given uncertainty level) as "% loss." Figure 2 depicts the opposite case—the loss of using the robust optimal solution when the unknown coefficients take their nominal values. Thus, higher percentage loss in these two figures implies worse results.

The results for the unconstrained case show that the nominal optimal solution performs worse in the robust setting than the robust solution does in the deterministic environment. Additionally, we observe that as the level of uncertainty increases for both uncertainty sets, the percentage loss ("% loss") of using both nominal and robust solutions in the opposite setting increases.

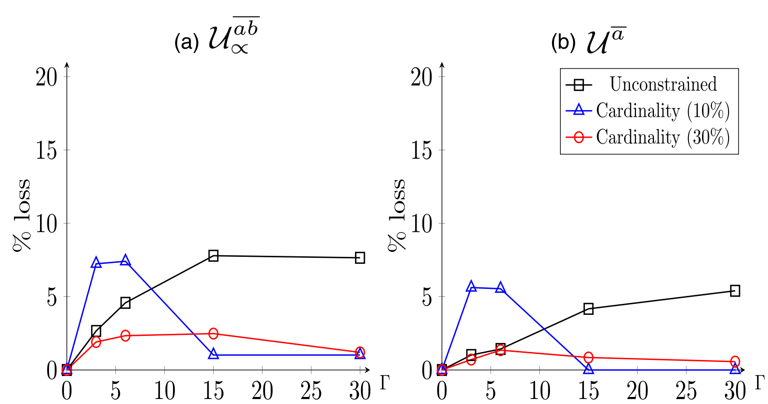
The cardinality results exhibit a somewhat different pattern of behavior, although we continue to see that the robust solution performs better in the nominal setting than vice versa. For the cardinality-feasible regions, in both uncertainty sets, the nominal and robust solutions are different for small to moderate values of Γ_i, but for the larger values of Γ_i, the nominal and robust solutions become similar again. The reason for this behavior is that as Γ_i grows, all (or almost all) of the variable coefficients in the optimal robust solution are reduced by uncertainty; that is, Γ_i is close to or larger than the size of the cardinality k. Because each uncertain coefficient is

Figure 1. (Color online) Decrease in the Robust Optimal Objective Function Value by Plugging a Nominal Optimal Solution into the Robust Problem for Frozen Pizza



Notes. Let $Z^\star_{\mathcal{U}}$ denote the optimal objective function value of RFP[$\mathcal{U}$]. Additionally, let $\hat{Z}_{\mathcal{U}} = \min_{(\tilde{a},\tilde{b})\in\mathcal{U}} \sum_{i\in I} \frac{a_{i0}+\tilde{a}_i^T x^\star}{b_{i0}+\tilde{b}_i^T x^\star}$, where $x^\star$ is a nominal optimal solution. Then the percentage loss for each Γ is $\frac{Z^\star_{\mathcal{U}}-\hat{Z}_{\mathcal{U}}}{Z^\star_{\mathcal{U}}}\times 100\%$.

Figure 2. (Color online) Decrease in the Nominal Optimal Objective Function Value by Plugging a Robust Optimal Solution into the Nominal Problem for Frozen Pizza



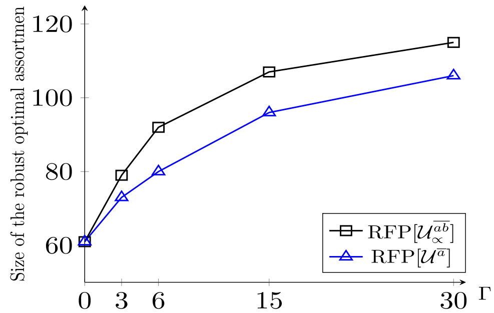
Notes. Let $Z^\star$ denote the optimal objective function value of FP. Additionally, let $\hat{Z} = \sum_{i\in I} \frac{a_{i0}+a_i^T x^\star_{\mathcal{U}}}{b_{i0}+b_i^T x^\star_{\mathcal{U}}}$, where $x^\star_{\mathcal{U}}$ is an optimal solution of RFP[$\mathcal{U}$]. Then the percentage loss for each Γ is $\frac{Z^\star-\hat{Z}}{Z^\star}\times 100\%$.

reduced by 50% (see above), the most favorable products without uncertainty reduction remain the most favorable products when everything (within the limited cardinality size k) is reduced 50% by uncertainty.

5.1.3. Solution Analysis. A salient feature of the unconstrained robust solutions in our case study is that under both uncertainty sets $\mathcal{U}^{\overline{a}}$ and $\mathcal{U}_{\propto}^{\overline{ab}}$, the robust optimal solution contains more variables with $x_j = 1$ as Γ_i increases (see Figure 3). For example, under $\mathcal{U}^{\overline{a}}$, each increase in Γ_i results in roughly 10 more variables included in the optimal solution. With $\Gamma_i = 0$, the optimal solution contains more variables from the highest two revenue classes, and as uncertainty increases, more choices from lower revenue classes become part of the solution. This can be explained by observing that with increasing uncertainty, the Γ_i most favorable products are the ones with their coefficients changed by uncertainty. Hence, the reduction in preference and/or revenue brings these products more in line with the lesser revenue products, which then become part of the optimal solution.

However, somewhat counterintuitively, given a cardinality size of 13, the optimal solutions (both nominal and robust) consist of variables mostly from the second-highest revenue tier, \$1.89. When the cardinality size is expanded to 39, more variables from both the first- and second-highest revenue tiers become part of the optimal solutions. An examination of the data shows that the highest revenue tier items are generally (significantly) less preferred (they have smaller values of preference ρ) than the more reasonably priced second-tier items; hence, the second-tier items show themselves to be superior generators of revenue.

Figure 3. (Color online) Size of the Unconstrained Robust Optimal Assortment vs. the Level of Uncertainty (Γ)



The outlined observations for (either constrained or unconstrained) multiclass deterministic and robust assortment optimizations can be compared with the previous results in the literature for unconstrained single-class deterministic and robust assortment optimizations. For example, assuming (without loss of generality) that the revenues are ordered such that $r_1 \geq r_2 \geq \ldots r_n$, Talluri and Van Ryzin (2004) show that the unconstrained single-class nominal assortment optimization problems under the multinominal logit choice model are *revenue-ordered assortments* (i.e., there exists a set of optimal solutions of the form $\{1, 2, \ldots, j\}$ for some index j). Rusmevichientong and Topaloglu (2012) derive a similar result for the robust case, where uncertainty is limited to customer preferences.

5.2. Synthetic Instances

We now conduct extensive computational experiments on randomly generated instances to gain insights into the performance of the disjoint and joint MILP reformulations provided in Section 4. Additionally, we evaluate the nominal solution in a robust setting and vice versa to determine the *price of robustness*. In Section 5.2.1, we outline the structure and parameters of the computational experiments. The price of robustness is studied in Section 5.2.2. We describe the results for the disjoint and joint uncertainty sets in Sections 5.2.3 and 5.2.4, respectively.

5.2.1. Test Instances. We chose combinations of $m \in \{1, 3, 5\}$ and $n \in \{50, 100, 150\}$. The uncertainty parameters Γ_i^a, Γ_i^b were chosen based on m, n and the relevant uncertainty set $\mathcal{U}$, and these choices are given in the appropriate section below. For each choice of m, n, Γ, and a particular constraint type (detailed below), five instances were sampled, and the results were averaged. The instances were each given a time limit of 1 hour (3,600 seconds).

The LP relaxation quality, denoted by "R" in the following tables, is computed by $\frac{Z_{LP}^*}{Z^*}$, where Z_{LP}^* is the optimal solution of the LP continuous relaxation, and Z^* is the optimal integer solution (if Z^* cannot be found within the time limit by any solution approach, then the best-known integer solution is used in place of Z^*). Moreover, the optimality gap is denoted by "G," and it is computed by $\frac{UB-LB}{LB}$, where UB and LB are the upper and lower bounds on the optimal objective function value, respectively.

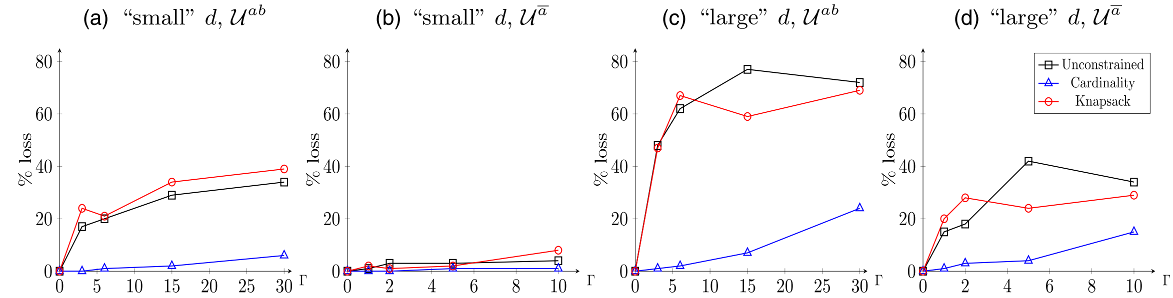
5.2.1.1. Coefficients Sampling. The coefficients a_{ij} and b_{ij} were each sampled from a (discrete) $U[0, 20]$ distribution, except for b_{i0}, which was sampled from a $U[1, 20]$ distribution. Subsequently, d_{ij}^a and d_{ij}^b were sampled from $U[0, \lfloor \frac{1}{2} a_{ij} \rfloor]$ or $U[0, \lfloor \frac{1}{2} b_{ij} \rfloor]$, respectively. Note that these parameter choices satisfy Assumptions 1 and 2.

5.2.1.2. Constraints. Three different constraint types were used: unconstrained (denoted by "U" in the following tables), cardinality constrained ("C"), and knapsack constrained ("K"). The cardinality constraint is of the *equality* type so that $\sum_{j \in J} x_j = k$, where $k = \frac{2}{5} n$. The knapsack constraint is of the *inequality* type, structured so that $\sum_{j \in J} k_j x_j \leq k$, where k_j was sampled from a $U[1, 10]$ distribution and $k = 2n$.

5.2.1.3. Linearization Bounds. For $\text{MILP}_1[\mathcal{U}^{ab}]$, note that $\omega_i^L = 0$ and $\omega_i^U = 1$ are valid bounds. Similar (not necessarily tight) lower- and upper-bound computations were performed for the other linearization procedures.

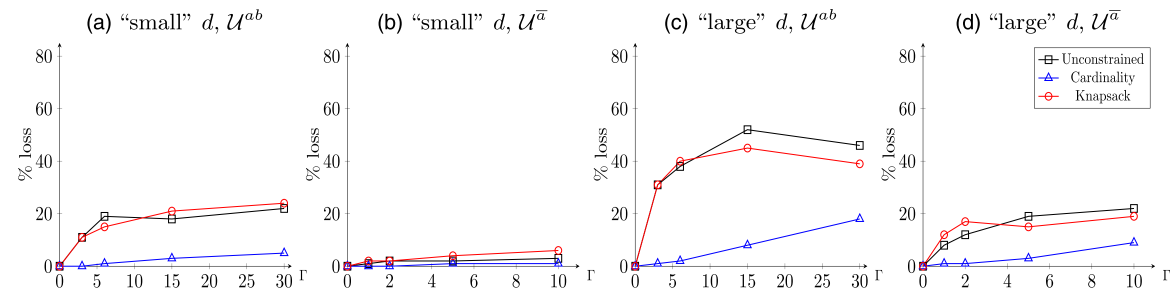
5.2.2. The Price of Robustness. Herein we demonstrate the value of the robust approach; that is, we show that ignoring the possibility of uncertain data can be more costly than being conservative. In Figures 4 and 5, the "small" d_{ij}^a and d_{ij}^b were sampled using the procedure described in Section 5.2.1. The "large" d_{ij}^a and d_{ij}^b in

Figure 4. (Color online) Average Decrease in the Robust Optimal Objective Function Value by Plugging a Nominal Optimal Solution into the Robust Problem for the Synthetic Data and $n = 150$



Notes. Let $Z^\star_{\mathcal{U}}$ denote the optimal objective function value of RFP[$\mathcal{U}$]. Additionally, let $\hat{Z}_{\mathcal{U}} = \min_{(\tilde{a},\tilde{b})\in\mathcal{U}} \sum_{i\in I} \frac{a_{i0}+\tilde{a}_i^T x^\star}{b_{i0}+\tilde{b}_i^T x^\star}$, where $x^\star$ is a nominal optimal solution. Then the percentage loss for each Γ is the average of $\frac{Z^\star_{\mathcal{U}}-\hat{Z}_{\mathcal{U}}}{Z^\star_{\mathcal{U}}} \cdot 100$ over five test instances and three ratio sizes $m \in \{1,3,5\}$.

Figure 5. (Color online) Average Decrease in the Nominal Optimal Objective Function Value by Plugging a Robust Optimal Solution into the Nominal Problem for the Synthetic Data and $n = 150$



Notes. Let $Z^\star$ denote the optimal objective function value of FP. Additionally, let $\hat{Z} = \sum_{i\in I} \frac{a_{i0}+a_i^T x^\star_{\mathcal{U}}}{b_{i0}+b_i^T x^\star_{\mathcal{U}}}$, where $x^\star_{\mathcal{U}}$ is an optimal solution of RFP[$\mathcal{U}$]. Then the percentage loss for each Γ is the average of $\frac{Z^\star-\hat{Z}}{Z^\star} \cdot 100$ over five test instances and three ratio sizes $m \in \{1,3,5\}$.

these two figures were sampled by instead letting d^a_{ij} and d^b_{ij} be distributed as $U[\lfloor\frac{1}{2}a_{ij}\rfloor, a_{ij}]$ and $U[\lfloor\frac{1}{2}b_{ij}\rfloor, b_{ij}]$, respectively (i.e., a higher level of uncertainty). Each subfigure is comparable with the one directly above/below it.

Figure 4 exhibits the benefit from applying the robust approach. It shows that under the worst-case scenario in the robust setting, the objective function value attained by an optimal nominal solution can be rather poor and thus illustrates how much the decision maker can gain by taking into account the data uncertainty. More precisely, Figure 4 depicts the average decrease in the robust objective function value for $m \in \{1,3,5\}$ by inserting optimal x from the associated nominal problem into the robust problem. We observe that in the case of large d, especially for the unconstrained and knapsack-constrained cases, inserting the nominal solution into the robust problem can cause a loss of up to 80%. This observation holds, albeit with scaled-down percentages, for the smaller d values as well.

Therefore, we conclude that the decision maker has more to lose by failing to account for uncertainty than he or she does by being overconservative. Simply speaking, if the decision maker is overly conservative (chooses the Γ_i for all $i \in I$ too large), then the loss on the objective function is outweighed by the amount that he or she would lose by incorrectly ignoring the uncertainty (i.e., assuming $\Gamma_i = 0$ for all $i \in I$). These results are similar to those of robust linear problems (Bertsimas and Sim 2003).

Figure 5 illustrates the opposite situation. That is, it shows how much the decision maker can gain by having precise information about the problem data parameters. Specifically, Figure 5 depicts the average decrease in the nominal objective function value for $m \in \{1,3,5\}$ by inserting robust optimal solution x into the nominal problem. This insertion causes a loss of up to 50% in the objective function value of the nominal problem for large d in case of unconstrained and knapsack-constrained problems.

5.2.3. Disjoint Reformulations. The results for the disjoint uncertainty set $\mathcal{U}^{ab}$ for single-ratio ($m = 1$) and multiple-ratio ($m \in \{3,5\}$) problems, as well as small- ($n = 50$) and large-sized ($n = 150$) problems, are presented in Tables 2 and 3; see also Table B.1 in Appendix B for medium-sized problems ($n = 100$). The uncertainty parameters were chosen so that $\Gamma_i^a = \Gamma_i^b$ for all $i \in I$, as stated in the tables. Observe that, in general, the single-ratio problem is easy to solve for any of the constraint types. In particular, the binary reformulation $\text{MILP}_{2'}^{\log}[\mathcal{U}^{ab}]$ (recall Section 4.3) can handle the single-ratio setting in that its average solution times for $m = 1$ in Tables 2 and 3 are the same as those for the nominal problem $\text{FP}_{4'}$.

As one would expect, increasing either m or n increases the difficulty of the fractional problem under disjoint uncertainty. In the nominal case (Tawarmalani et al. 2002), FP_1 generally outperforms FP_2 across all constraint types for the multiple-ratio problem, and we find that this result carries over into the robust case. Specifically, for $m = 3$ and $m = 5$ in Table 3, $\text{MILP}_1[\mathcal{U}^{ab}]$ solves more than half the unconstrained and knapsack-constrained instances to optimality, whereas $\text{MILP}_2[\mathcal{U}^{ab}]$ solves almost none.

However, the *binarized* $\text{MILP}_2^{\log}[\mathcal{U}^{ab}]$ outperforms both $\text{MILP}_1[\mathcal{U}^{ab}]$ and $\text{MILP}_2[\mathcal{U}^{ab}]$. In Table 3, note that when $m = 5$, $\text{MILP}_{2'}^{\log}[\mathcal{U}^{ab}]$ solves all except one of the unconstrained and knapsack-constrained instances to optimality, whereas $\text{MILP}_2^{\log}[\mathcal{U}^{ab}]$ all solves the cardinality-constrained instances to optimality.

For the multiple-ratio problem, the cardinality-constrained problems seem to be the most computationally difficult (when the best solution approach is chosen for each constraint type), although this observation holds for the nominal case as well—see, for example, the $m = 5$ case under constraint C in Table 3. However, the unconstrained problem is sometimes more difficult than the knapsack-constrained problem (e.g., when $\Gamma_i = 1, m = 5$ in Table 2), although not universally so (e.g., $\Gamma_i = 2, m = 5$ in Table 2). Finally, we note that there seems to be no particular pattern or relationship between the level of uncertainty Γ_i^a, Γ_i^b and the computational difficulty for any of the parameter settings.

To summarize these results, we observe that $\text{MILP}_1[\mathcal{U}^{ab}]$ tends to have the best continuous relaxation bound. This observation is consistent with the earlier observations in the literature that the corresponding nominal reformulation FP_1 typically has the best relaxation quality (Borrero et al. 2016, Mehmanchi et al. 2019). Nonetheless, this does not always (or even often) lead to superior solution times mainly because of the large size of the reformulation. In particular, for a small number of variables (Table 2), it seems that $\text{MILP}_2^{\log}[\mathcal{U}^{ab}]$ is the best choice for disjoint cardinality-constrained problems, whereas $\text{MILP}_1[\mathcal{U}^{ab}]$ is usually better for unconstrained or knapsack-constrained models. However, as the number of variables increases (Table 3), the logarithmic reformulation $\text{MILP}_{2'}^{\log}[\mathcal{U}^{ab}]$ is a better choice for unconstrained and knapsack-constrained problems, although it seems that the binarized reformulations have weaker relaxation qualities than the corresponding original MILPs.

5.2.4. Joint Reformulations. Results for joint uncertainty sets $\mathcal{U}^{\overline{ab}}$, $\mathcal{U}_{=}^{\overline{ab}}$, and $\mathcal{U}_{\propto}^{\overline{ab}}$ are given in Tables 4 and 5 for $n \in \{50, 150\}$ (see also Table B.2 in Appendix B for $n = 100$). These tables also include the respective results of the most efficient reformulation for the disjoint uncertainty (i.e., $\text{MILP}_{2'}^{\log}[\mathcal{U}^{ab}]$ provided in Tables 2 and 3) to compare the difficulty of solving $\text{RFP}[\mathcal{U}]$ under disjoint versus joint uncertainty sets.

The uncertainty parameters were chosen based on those chosen for the disjoint case. With Γ_i^a, Γ_i^b as the relevant disjoint uncertainty parameters, we have, for $\mathcal{U}^{\overline{ab}}$, that $\Gamma_i = 2\,\Gamma_i^a$; for $\mathcal{U}_{=}^{\overline{ab}}$ and $\mathcal{U}_{\propto}^{\overline{ab}}$, that $\Gamma_i = \Gamma_i^a$; and for $\mathcal{U}^{\overline{a}}$, that $\Gamma = m\,\Gamma_i^a$ for problems with similar m, n.

Observe that $\text{MILP}_2[\mathcal{U}]$ performs similarly (with respect to solution times/optimality gap) on both the disjoint and joint uncertainty sets by comparing the $\text{MILP}_2[\mathcal{U}^{ab}]$ of Table 2 with the relevant columns of Table 4 and conducting similar comparisons for columns of the 100- and 150-variable tables. However, for the disjoint uncertainty case, we were able to use a binary reformulation ($\text{MILP}_{2'}^{\log}[\mathcal{U}^{ab}]$) to obtain superior solution times. Thus, the joint problems are generally more computationally difficult than the disjoint problems because of the absence of such a binary reformulation for them, which can be seen by comparing the first column of Tables 4 and 5 with the other columns.

Though the multiple-ratio problem used the entire hour of solution time allowed for most joint uncertainty sets, the single-ratio problem was solved quickly in most cases. Additionally, for the multiple-ratio problem, $\mathcal{U}^{\overline{a}}$ remains tractable for unconstrained and knapsack-constrained problems. In these two special cases, $\text{MILP}[\mathcal{U}^{\overline{a}}]$ typically solved the joint problem to optimality in a similar time as $\text{MILP}_2^{\log}[\mathcal{U}^{ab}]$ solved the disjoint instance. Finally, we observe that the cardinality constraint is universally difficult (as in the disjoint case) for all multiple-ratio instances with the joint uncertainty sets.

Table 2. Results for Disjoint Reformulations

		$FP_{4'}$				$MILP_1[\mathcal{U}^{ab}]$				$MILP_2[\mathcal{U}^{ab}]$				$MILP_2^{\log}[\mathcal{U}^{ab}]$				$MILP_{2'}^{\log}[\mathcal{U}^{ab}]$			
$n = 50$	Constraint type	T	#	G	R	T	#	G	R	T	#	G	R	T	#	G	R	T	#	G	R
$m = 1$																					
	U	0.1	5	0.00	1.0	**0.0**	5	0.00	1.0	0.3	5	0.00	10.8	0.1	5	0.00	12.0	0.1	5	0.00	1.0
$\Gamma_i^a = \Gamma_i^b = 0$	C	0.2	5	0.00	1.0	1.6	5	0.00	1.9	0.3	5	0.00	16.8	**0.1**	5	0.00	17.1	**0.1**	5	0.00	1.9
	K	0.1	5	0.00	1.0	**0.0**	5	0.00	1.0	0.3	5	0.00	9.3	0.1	5	0.00	9.9	0.0	5	0.00	1.0
	U	0.2	5	0.00	1.2	**0.1**	5	0.00	1.0	0.9	5	0.00	16.0	0.2	5	0.00	17.2	0.2	5	0.00	1.0
$\Gamma_i^a = \Gamma_i^b = 1$	C	0.2	5	0.00	1.2	5.1	5	0.00	1.7	0.7	5	0.00	26.0	0.2	5	0.00	27.6	**0.1**	5	0.00	1.7
	K	0.0	5	0.00	1.4	**0.1**	5	0.00	1.0	0.4	5	0.00	23.0	**0.1**	5	0.00	27.2	**0.1**	5	0.00	1.0
	U	0.1	5	0.00	1.5	**0.1**	5	0.00	1.0	0.8	5	0.00	19.4	0.2	5	0.00	21.7	**0.1**	5	0.00	1.0
$\Gamma_i^a = \Gamma_i^b = 2$	C	0.2	5	0.00	1.2	0.8	5	0.00	1.4	0.7	5	0.00	16.4	0.2	5	0.00	16.8	**0.1**	5	0.00	1.4
	K	0.1	5	0.00	1.4	**0.1**	5	0.00	1.0	0.4	5	0.00	13.8	0.2	5	0.00	14.8	**0.1**	5	0.00	1.0
	U	0.1	5	0.00	1.6	**0.1**	5	0.00	1.0	0.7	5	0.00	21.2	0.2	5	0.00	24.7	**0.1**	5	0.00	1.0
$\Gamma_i^a = \Gamma_i^b = 5$	C	0.2	5	0.00	1.5	2.1	5	0.00	1.4	0.6	5	0.00	19.6	0.2	5	0.00	20.0	**0.1**	5	0.00	1.4
	K	0.1	5	0.00	1.9	**0.1**	5	0.00	1.0	0.4	5	0.00	14.8	**0.1**	5	0.00	15.5	**0.1**	5	0.00	1.0
	U	0.1	5	0.00	1.6	**0.1**	5	0.00	1.0	0.5	5	0.00	23.3	0.2	5	0.00	28.0	**0.1**	5	0.00	1.0
$\Gamma_i^a = \Gamma_i^b = 10$	C	0.2	5	0.00	1.7	4.9	5	0.00	2.2	0.7	5	0.00	32.6	0.2	5	0.00	34.6	**0.1**	5	0.00	2.2
	K	0.0	5	0.00	1.5	**0.0**	5	0.00	1.0	0.5	5	0.00	10.4	0.2	5	0.00	10.6	**0.0**	5	0.00	1.0
Average		0.1	5.0	0.00	1.4	1.0	5.0	0.00	1.3	0.5	5.0	0.00	18.2	0.2	5.0	0.00	19.8	**0.1**	5.0	0.00	1.3
$m = 3$																					
	U	0.6	5	0.00	1.5	**0.3**	5	0.00	1.5	2,223.4	3	0.23	18.7	0.5	5	0.00	23.4	0.4	5	0.00	1.5
$\Gamma_i^a = \Gamma_i^b = 0$	C	1.0	5	0.00	1.2	1,798.0	4	0.02	3.1	3,600.0	0	1.07	26.4	1.0	5	0.00	27.8	**0.9**	5	0.00	3.1
	K	0.4	5	0.00	1.5	**0.2**	5	0.00	1.5	1,324.4	4	0.07	16.6	**0.2**	5	0.00	19.9	0.3	5	0.00	1.5
	U	0.4	5	0.00	2.2	**1.1**	5	0.00	1.8	3,600.0	0	0.52	28.5	2.0	5	0.00	34.4	1.2	5	0.00	1.8
$\Gamma_i^a = \Gamma_i^b = 1$	C	0.4	5	0.00	1.2	143.6	5	0.00	2.6	3,600.0	0	0.77	41.4	0.7	5	0.00	48.7	**0.9**	5	0.00	2.6
	K	0.4	5	0.00	1.9	2.0	5	0.00	1.6	2,171.2	2	0.41	19.6	2.0	5	0.00	22.7	**1.3**	5	0.00	1.6
	U	0.3	5	0.00	2.3	**0.6**	5	0.00	1.6	2,972.0	1	0.56	34.7	2.4	5	0.00	44.5	1.4	5	0.00	1.6
$\Gamma_i^a = \Gamma_i^b = 2$	C	0.8	5	0.00	1.3	529.6	5	0.00	2.2	3,600.0	0	0.84	19.4	**1.8**	5	0.00	19.6	2.0	5	0.00	2.2
	K	0.3	5	0.00	2.1	2.7	5	0.00	1.5	1,170.7	4	0.15	19.6	2.9	5	0.00	21.2	**2.3**	5	0.00	1.5
	U	0.4	5	0.00	2.2	7.7	5	0.00	1.5	2,218.2	2	0.43	27.3	2.2	5	0.00	33.5	**1.1**	5	0.00	1.5
$\Gamma_i^a = \Gamma_i^b = 5$	C	0.7	5	0.00	1.6	848.0	4	0.01	2.6	3,600.0	0	0.91	44.1	**3.1**	5	0.00	48.3	12.6	5	0.00	2.6
	K	0.6	5	0.00	2.6	13.7	5	0.00	1.6	2,980.0	2	0.24	26.7	3.9	5	0.00	31.4	**2.3**	5	0.00	1.6
	U	0.4	5	0.00	2.7	**0.5**	5	0.00	1.7	2,920.0	1	0.47	32.8	2.8	5	0.00	40.8	1.3	5	0.00	1.7
$\Gamma_i^a = \Gamma_i^b = 10$	C	0.8	5	0.00	1.8	632.0	5	0.00	2.4	3,600.0	0	1.00	28.2	**6.7**	5	0.00	28.7	29.6	5	0.00	2.4
	K	0.6	5	0.00	2.3	**0.7**	5	0.00	1.5	2,340.3	3	0.18	16.0	2.2	5	0.00	17.2	0.8	5	0.00	1.5
Average		0.5	5.0	0.00	1.9	265.4	4.9	0.00	1.9	2,794.7	1.5	0.52	26.7	**2.3**	5.0	0.00	30.8	3.9	5.0	0.00	1.9
$m = 5$																					
	U	3.3	5	0.00	1.9	**1.0**	5	0.00	1.9	2,883.6	1	0.67	24.0	7.7	5	0.00	30.5	8.0	5	0.00	1.9
$\Gamma_i^a = \Gamma_i^b = 0$	C	57.8	5	0.00	1.2	3,600.0	0	0.16	3.9	3,600.0	0	1.56	46.5	**12.5**	5	0.00	51.8	20.5	5	0.00	3.9
	K	4.8	5	0.00	1.8	**1.2**	5	0.00	1.8	2,884.2	1	0.55	18.7	10.7	5	0.00	20.9	10.9	5	0.00	1.8
	U	8.4	5	0.00	2.4	**76.3**	5	0.00	1.9	2,360.0	2	0.77	34.7	816.3	4	0.00	47.2	307.0	5	0.00	1.9
$\Gamma_i^a = \Gamma_i^b = 1$	C	26.1	5	0.00	1.4	3,080.0	1	0.09	2.6	3,600.0	0	1.11	26.5	**14.4**	5	0.00	27.2	24.0	5	0.00	2.6
	K	9.2	5	0.00	2.5	342.2	5	0.00	1.9	2,948.0	1	0.55	22.3	216.8	5	0.00	25.4	**132.2**	5	0.00	1.9

Table 2. (Continued)

$n = 50$	Constraint type	$FP_{4'}$				$MILP_1[\mathcal{U}^{ab}]$				$MILP_2[\mathcal{U}^{ab}]$				$MILP_2^{\log}[\mathcal{U}^{ab}]$				$MILP_{2'}^{\log}[\mathcal{U}^{ab}]$			
		T	#	G	R	T	#	G	R	T	#	G	R	T	#	G	R	T	#	G	R
	U	7.8	5	0.00	2.4	**74.4**	5	0.00	1.8	2,922.0	1	0.86	29.2	645.0	5	0.00	37.6	111.6	5	0.00	1.8
$\Gamma_i^a = \Gamma_i^b = 2$	C	18.7	5	0.00	1.4	3,600.0	0	0.08	3.1	3,600.0	0	1.20	33.6	**22.4**	5	0.00	36.8	67.3	5	0.00	3.1
	K	16.7	5	0.00	2.7	906.8	4	0.01	1.9	3,600.0	0	0.98	26.0	1,629.0	3	0.01	27.9	**297.0**	5	0.00	1.9
	U	4.7	5	0.00	2.9	**9.3**	5	0.00	1.8	3,600.0	0	0.91	30.4	273.6	5	0.00	37.8	52.2	5	0.00	1.8
$\Gamma_i^a = \Gamma_i^b = 5$	C	25.5	5	0.00	1.6	3,600.0	0	0.08	2.6	3,600.0	0	1.22	34.3	**74.8**	5	0.00	37.3	513.0	5	0.00	2.6
	K	2.9	5	0.00	2.1	**0.7**	5	0.00	1.5	1,521.6	3	0.21	23.6	42.9	5	0.00	28.0	25.4	5	0.00	1.5
	U	4.2	5	0.00	2.6	**22.4**	5	0.00	1.7	2,244.0	2	0.46	28.5	264.6	5	0.00	36.6	59.6	5	0.00	1.7
$\Gamma_i^a = \Gamma_i^b = 10$	C	17.7	5	0.00	1.8	3,600.0	0	0.11	3.5	3,600.0	0	1.36	40.3	**94.4**	5	0.00	43.5	751.6	5	0.00	3.5
	K	3.3	5	0.00	2.8	**0.6**	5	0.00	1.8	3,000.0	2	0.30	22.6	176.2	5	0.00	24.1	51.0	5	0.00	1.8
Average		14.1	5.0	0.00	2.1	1,261.0	3.3	0.03	2.2	3,064.2	0.9	0.85	29.4	286.8	4.8	0.00	34.2	**162.1**	5.0	0.00	2.2

Notes. Average time (T) in seconds with the number (#) of instances solved within default optimality gap 0.01 and the average remaining optimality gap (G) along with the average relaxation quality (R) across instances for $n = 50$. In each row, among the solution methods that solve the most number of instances to optimality, the best average time and the best average gap (if # < 5) are in bold.

Table 3. Results for Disjoint Reformulations

$n = 150$	Constraint type	$FP_{4'}$ T	$FP_{4'}$ #	$FP_{4'}$ G	$FP_{4'}$ R	$MILP_1[\mathcal{U}^{ab}]$ T	$MILP_1[\mathcal{U}^{ab}]$ #	$MILP_1[\mathcal{U}^{ab}]$ G	$MILP_1[\mathcal{U}^{ab}]$ R	$MILP_2[\mathcal{U}^{ab}]$ T	$MILP_2[\mathcal{U}^{ab}]$ #	$MILP_2[\mathcal{U}^{ab}]$ G	$MILP_2[\mathcal{U}^{ab}]$ R	$MILP_2^{\log}[\mathcal{U}^{ab}]$ T	$MILP_2^{\log}[\mathcal{U}^{ab}]$ #	$MILP_2^{\log}[\mathcal{U}^{ab}]$ G	$MILP_2^{\log}[\mathcal{U}^{ab}]$ R	$MILP_{2'}^{\log}[\mathcal{U}^{ab}]$ T	$MILP_{2'}^{\log}[\mathcal{U}^{ab}]$ #	$MILP_{2'}^{\log}[\mathcal{U}^{ab}]$ G	$MILP_{2'}^{\log}[\mathcal{U}^{ab}]$ R
$m = 1$																					
	U	0.1	5	0.00	1.0	**0.1**	5	0.00	1.0	0.7	5	0.00	29.6	0.2	5	0.00	37.5	**0.1**	5	0.00	1.0
$\Gamma_i^a = \Gamma_i^b = 0$	C	0.2	5	0.00	1.0	3,600.0	0	0.31	2.4	32.5	5	0.00	45.6	0.3	5	0.00	46.7	**0.1**	5	0.00	2.4
	K	0.1	5	0.00	1.0	**0.1**	5	0.00	1.0	1.0	5	0.00	20.7	0.2	5	0.00	22.5	**0.1**	5	0.00	1.0
	U	0.1	5	0.00	1.3	**0.2**	5	0.00	1.0	2.3	5	0.00	34.7	**0.2**	5	0.00	42.3	**0.2**	5	0.00	1.0
$\Gamma_i^a = \Gamma_i^b = 3$	C	0.2	5	0.00	1.1	3,600.0	0	0.30	2.0	80.0	5	0.00	31.3	0.3	5	0.00	31.8	**0.2**	5	0.00	2.0
	K	0.1	5	0.00	1.3	**0.1**	5	0.00	1.0	1.3	5	0.00	26.0	0.3	5	0.00	27.4	**0.1**	5	0.00	1.0
	U	0.2	5	0.00	1.4	**0.1**	5	0.00	1.1	6.0	5	0.00	38.8	0.3	5	0.00	47.2	0.2	5	0.00	1.1
$\Gamma_i^a = \Gamma_i^b = 6$	C	0.2	5	0.00	1.1	3,600.0	0	0.31	2.4	1,112.9	4	0.48	88.8	0.3	5	0.00	109.5	**0.2**	5	0.00	2.4
	K	0.1	5	0.00	1.4	**0.2**	5	0.00	1.1	1.5	5	0.00	18.6	0.3	5	0.00	18.9	**0.2**	5	0.00	1.1
	U	0.2	5	0.00	1.6	**0.2**	5	0.00	1.0	2.5	5	0.00	47.3	0.3	5	0.00	57.3	**0.2**	5	0.00	1.0
$\Gamma_i^a = \Gamma_i^b = 15$	C	0.2	5	0.00	1.2	3,600.0	0	0.25	1.8	473.3	5	0.00	46.6	**0.3**	5	0.00	47.3	**0.3**	5	0.00	1.8
	K	0.1	5	0.00	1.9	**0.1**	5	0.00	1.0	1.9	5	0.00	43.8	0.4	5	0.00	46.9	0.2	5	0.00	1.0
	U	0.2	5	0.00	1.9	**0.1**	5	0.00	1.0	1.9	5	0.00	39.8	0.5	5	0.00	43.0	0.2	5	0.00	1.0
$\Gamma_i^a = \Gamma_i^b = 30$	C	0.2	5	0.00	1.4	3,600.0	0	0.28	2.1	931.9	4	0.12	72.2	**0.4**	5	0.00	74.5	**0.4**	5	0.00	2.1
	K	0.1	5	0.00	1.6	**0.1**	5	0.00	1.0	1.6	5	0.00	26.3	0.3	5	0.00	27.4	0.2	5	0.00	1.0
Average		0.2	5.0	0.00	1.4	1,200.1	3.3	0.10	1.4	176.7	4.9	0.04	40.7	0.3	5.0	0.00	45.3	**0.2**	5.0	0.00	1.4
$m = 3$																					
	U	0.7	5	0.00	1.8	721.0	4	0.01	1.8	3,600.0	0	3.62	46.0	0.8	5	0.00	62.3	**0.7**	5	0.00	1.8
$\Gamma_i^a = \Gamma_i^b = 0$	C	4.9	5	0.00	1.2	3,600.0	0	0.99	5.1	3,600.0	0	9.66	118.6	4.2	5	0.00	141.2	**3.2**	5	0.00	5.1
	K	0.5	5	0.00	1.7	3.1	5	0.00	1.7	3,600.0	0	2.86	38.9	0.9	5	0.00	46.4	**0.8**	5	0.00	1.7
	U	0.5	5	0.00	2.3	450.2	5	0.00	1.8	3,600.0	0	5.54	69.9	7.0	5	0.00	92.7	**4.1**	5	0.00	1.8
$\Gamma_i^a = \Gamma_i^b = 3$	C	3.1	5	0.00	1.2	3,600.0	0	0.81	3.9	3,600.0	0	9.36	109.8	**7.4**	5	0.00	122.7	30.8	5	0.00	3.9
	K	0.5	5	0.00	2.4	929.3	4	0.02	1.9	3,600.0	0	4.94	48.0	7.4	5	0.00	52.2	**5.2**	5	0.00	1.9
	U	0.7	5	0.00	2.8	1,660.5	3	0.04	2.0	3,600.0	0	6.62	70.6	11.5	5	0.00	89.2	**5.4**	5	0.00	2.0
$\Gamma_i^a = \Gamma_i^b = 6$	C	8.8	5	0.00	1.3	3,600.0	0	0.68	3.1	3,600.0	0	8.40	56.2	**6.1**	5	0.00	57.4	29.6	5	0.00	3.1
	K	0.9	5	0.00	2.4	1,472.3	3	0.04	1.8	3,600.0	0	4.28	38.5	12.5	5	0.00	43.0	**6.9**	5	0.00	1.8
	U	0.5	5	0.00	2.9	2,164.4	2	0.04	1.8	3,600.0	0	7.84	91.2	20.0	5	0.00	122.4	**13.7**	5	0.00	1.8
$\Gamma_i^a = \Gamma_i^b = 15$	C	6.3	5	0.00	1.4	3,600.0	0	0.59	2.9	3,600.0	0	9.58	62.3	**13.0**	5	0.00	64.0	49.9	5	0.00	2.9
	K	0.8	5	0.00	2.8	2,160.5	2	0.10	1.9	3,600.0	0	5.98	45.1	30.0	5	0.00	47.7	**16.5**	5	0.00	1.9
	U	0.9	5	0.00	2.7	721.5	4	0.05	1.7	3,600.0	0	6.72	58.7	119.8	5	0.00	69.0	**30.0**	5	0.00	1.7
$\Gamma_i^a = \Gamma_i^b = 30$	C	3.7	5	0.00	1.5	3,600.0	0	0.58	3.1	3,600.0	0	11.48	65.1	**22.8**	5	0.00	66.2	1,448.6	5	0.00	3.1
	K	0.8	5	0.00	2.7	730.0	4	0.06	1.6	3,600.0	0	4.68	47.2	**55.4**	5	0.00	53.8	204.1	5	0.00	1.6
Average		2.2	5.0	0.00	2.1	1,934.2	2.4	0.27	2.4	3,600.0	0.0	6.77	64.4	**21.3**	5.0	0.00	75.3	123.3	5.0	0.00	2.4
$m = 5$																					
	U	16.9	5	0.00	2.4	2,004.4	3	0.06	2.4	3,600.0	0	7.18	57.3	46.0	5	0.00	73.3	**32.6**	5	0.00	2.4
$\Gamma_i^a = \Gamma_i^b = 0$	C	2,210.0	4	0.00	1.3	3,600.0	0	1.18	4.5	3,600.0	0	11.58	63.6	**234.0**	5	0.00	65.5	666.6	5	0.00	4.5
	K	20.2	5	0.00	2.3	2,164.8	2	0.09	2.3	3,600.0	0	5.06	40.6	39.1	5	0.00	46.3	**34.7**	5	0.00	2.3
	U	30.6	5	0.00	3.2	2,888.8	1	0.23	2.5	3,600.0	0	10.22	91.4	3,012.0	1	0.11	113.7	**960.0**	5	0.00	2.5
$\Gamma_i^a = \Gamma_i^b = 3$	C	2,250.0	5	0.00	1.3	3,600.0	0	1.03	4.0	3,600.0	0	12.40	105.5	**370.0**	5	0.00	114.4	2,302.0	5	0.00	4.0
	K	15.3	5	0.00	3.0	2,884.2	1	0.25	2.4	3,600.0	0	7.28	58.2	1,726.0	4	0.02	67.5	**734.0**	5	0.00	2.4

Table 3. (Continued)

$n = 150$	Constraint type	$FP_{4'}$				$MILP_1[\mathcal{U}^{ab}]$				$MILP_2[\mathcal{U}^{ab}]$				$MILP_2^{\log}[\mathcal{U}^{ab}]$				$MILP_{2'}^{\log}[\mathcal{U}^{ab}]$			
		T	#	G	R	T	#	G	R	T	#	G	R	T	#	G	R	T	#	G	R
$\Gamma_i^a = \Gamma_i^b = 6$	U	24.2	5	0.00	3.0	2,160.7	2	0.25	2.2	3,600.0	0	8.84	84.6	1,574.6	3	0.05	110.6	**1,578.6**	4	**0.00**	2.2
	C	1,067.8	5	0.00	1.4	3,600.0	0	1.04	4.3	3,600.0	0	13.80	159.8	**1,047.2**	5	0.00	185.9	1,608.8	5	0.00	4.3
	K	7.8	5	0.00	2.6	1,442.0	3	0.16	2.0	3,600.0	0	5.48	55.5	1,505.0	3	0.03	64.1	**417.0**	5	0.00	2.0
$\Gamma_i^a = \Gamma_i^b = 15$	U	17.9	5	0.00	3.0	2,161.8	2	0.16	1.9	3,600.0	0	9.22	82.3	1,478.0	4	0.03	105.7	**1,018.0**	4	**0.02**	1.9
	C	1,356.6	5	0.00	1.5	3,600.0	0	0.81	3.5	3,600.0	0	13.80	102.7	**1,568.0**	5	0.00	111.1	3,600.0	4	0.01	3.5
	K	55.0	5	0.00	3.4	2,166.2	2	0.31	2.0	3,600.0	0	9.84	76.5	2,278.0	2	0.14	85.9	**1,806.0**	3	0.03	2.0
$\Gamma_i^a = \Gamma_i^b = 30$	U	9.8	5	0.00	3.1	741.4	4	0.05	1.9	3,600.0	0	8.46	81.9	1,790.0	3	0.26	100.9	**306.0**	5	0.00	1.9
	C	3,160.0	5	0.00	1.6	3,600.0	0	0.71	3.5	3,600.0	0	14.60	107.0	**3,440.0**	5	0.00	111.5	3,600.0	1	0.01	3.5
	K	20.3	5	0.00	3.3	**782.4**	4	0.13	1.9	3,600.0	0	7.74	75.8	1,308.0	4	0.05	94.9	1,056.0	4	**0.02**	1.9
Average		684.1	4.9	0.00	2.4	2,493.1	1.6	0.43	2.8	3,600.0	0.0	9.70	82.8	1,427.7	3.9	0.05	96.8	**1,314.7**	4.3	**0.01**	2.8

Notes. Average time (T) in seconds with the number (#) of instances solved within default optimality gap 0.01 and the average remaining optimality gap (G) along with the average relaxation quality (R) across instances for $n = 150$. In each row, among the solution methods that solve the most numbers of instances to optimality, the best average time and the best average gap (if # < 5) are in bold.

Table 4. Comparison of Results for the Best Disjoint Reformulation $\mathrm{MILP}^{\log}_{2'}[\mathcal{U}^{ab}]$ vs. Joint Reformulations $\mathrm{MILP}_2[\mathcal{U}^{\overline{ab}}]$, $\mathrm{MILP}_2[\mathcal{U}^{\overline{ab}}_{=}]$, $\mathrm{MILP}_2[\mathcal{U}^{\overline{ab}}_{\propto}]$, and $\mathrm{MILP}[\mathcal{U}^{\bar{a}}]$

		$\mathrm{MILP}^{\log}_{2'}[\mathcal{U}^{ab}]$				$\mathrm{MILP}_2[\mathcal{U}^{\overline{ab}}]$				$\mathrm{MILP}_2[\mathcal{U}^{\overline{ab}}_{=}]$				$\mathrm{MILP}_2[\mathcal{U}^{\overline{ab}}_{\propto}]$				$\mathrm{MILP}[\mathcal{U}^{\bar{a}}]$			
$n = 50$	Constraint type	T	#	G	R	T	#	G	R	T	#	G	R	T	#	G	R	T	#	G	R
$m = 1$																					
	U	0.1	5	0.00	1.0	0.2	5	0.00	10.8	0.1	5	0.00	10.8	0.1	5	0.00	10.8	0.0	5	0.00	1.0
$\Gamma_i^a = 0$	C	0.1	5	0.00	1.9	0.3	5	0.00	16.8	0.3	5	0.00	16.8	0.3	5	0.00	16.8	1.6	5	0.00	1.9
	K	0.0	5	0.00	1.0	0.3	5	0.00	9.3	0.2	5	0.00	9.3	0.3	5	0.00	9.3	0.0	5	0.00	1.0
	U	0.2	5	0.00	1.0	0.7	5	0.00	16.2	0.6	5	0.00	15.7	0.7	5	0.00	14.9	0.1	5	0.00	1.0
$\Gamma_i^a = 1$	C	0.1	5	0.00	1.7	0.6	5	0.00	25.9	0.7	5	0.00	25.7	0.8	5	0.00	24.8	2.3	5	0.00	2.0
	K	0.1	5	0.00	1.0	0.3	5	0.00	22.9	0.4	5	0.00	22.3	0.3	5	0.00	21.1	0.1	5	0.00	1.0
	U	0.1	5	0.00	1.0	0.7	5	0.00	19.2	0.6	5	0.00	19.0	0.7	5	0.00	16.2	0.1	5	0.00	1.0
$\Gamma_i^a = 2$	C	0.1	5	0.00	1.4	0.7	5	0.00	16.2	0.6	5	0.00	15.7	0.7	5	0.00	15.1	0.5	5	0.00	1.6
	K	0.1	5	0.00	1.0	0.4	5	0.00	13.7	0.4	5	0.00	13.6	0.4	5	0.00	12.6	0.0	5	0.00	1.0
	U	0.1	5	0.00	1.0	0.7	5	0.00	20.4	0.7	5	0.00	21.2	0.7	5	0.00	18.5	0.1	5	0.00	1.0
$\Gamma_i^a = 5$	C	0.1	5	0.00	1.4	0.8	5	0.00	19.0	0.7	5	0.00	18.6	0.5	5	0.00	16.7	1.7	5	0.00	1.5
	K	0.1	5	0.00	1.0	0.3	5	0.00	14.2	0.4	5	0.00	14.6	0.4	5	0.00	11.7	0.1	5	0.00	1.0
	U	0.1	5	0.00	1.0	0.5	5	0.00	22.0	0.5	5	0.00	23.3	0.6	5	0.00	20.8	0.1	5	0.00	1.0
$\Gamma_i^a = 10$	C	0.1	5	0.00	2.2	0.7	5	0.00	31.6	0.7	5	0.00	30.3	0.9	5	0.00	26.8	2.3	5	0.00	2.2
	K	0.0	5	0.00	1.0	0.4	5	0.00	10.0	0.4	5	0.00	10.4	0.4	5	0.00	8.7	0.0	5	0.00	1.0
Average		0.1	5.0	0.00	1.3	0.5	5.0	0.00	17.9	0.5	5.0	0.00	17.8	0.5	5.0	0.00	16.3	0.6	5.0	0.00	1.3
$m = 3$																					
	U	0.4	5	0.00	1.5	2,229.4	3	0.23	18.7	2,249.4	3	0.23	18.7	2,225.4	3	0.23	18.7	0.4	5	0.00	1.5
$\Gamma_i^a = 0$	C	0.9	5	0.00	3.1	3,600.0	0	1.07	26.4	3,600.0	0	1.07	26.4	3,600.0	0	1.07	26.4	1,898.0	4	0.02	3.1
	K	0.3	5	0.00	1.5	1,324.4	4	0.07	16.6	1,324.4	4	0.07	16.6	1,324.3	4	0.07	16.6	0.3	5	0.00	1.5
	U	1.2	5	0.00	1.8	3,600.0	0	0.53	28.6	3,600.0	0	0.77	27.7	3,600.0	0	0.43	26.0	0.4	5	0.00	1.7
$\Gamma_i^a = 1$	C	0.9	5	0.00	2.6	3,600.0	0	0.77	41.8	3,600.0	0	0.88	41.1	3,600.0	0	0.78	40.1	833.2	4	0.01	2.8
	K	1.3	5	0.00	1.6	2,170.8	2	0.38	19.7	2,198.6	2	0.39	19.3	2,169.6	2	0.34	17.8	0.9	5	0.00	1.6
	U	1.4	5	0.00	1.6	2,982.0	1	0.56	34.4	3,600.0	0	0.62	34.4	3,064.0	1	0.47	31.5	0.3	5	0.00	1.6
$\Gamma_i^a = 2$	C	2.0	5	0.00	2.2	3,600.0	0	0.85	18.9	3,600.0	0	0.97	18.8	3,600.0	0	0.85	18.1	1,420.0	5	0.00	2.4
	K	2.3	5	0.00	1.5	1,202.7	4	0.14	19.4	2,194.6	3	0.24	18.9	1,204.8	4	0.11	17.5	0.4	5	0.00	1.4
	U	1.1	5	0.00	1.5	2,214.3	2	0.40	26.1	2,420.3	2	0.52	27.3	2,340.3	2	0.40	24.7	0.7	5	0.00	1.6
$\Gamma_i^a = 5$	C	12.6	5	0.00	2.6	3,600.0	0	0.89	42.3	3,600.0	0	0.86	41.8	3,600.0	0	0.85	38.3	2,190.4	2	0.03	3.0
	K	2.3	5	0.00	1.6	2,800.0	2	0.21	25.5	3,580.0	1	0.48	26.7	2,960.0	1	0.23	21.7	5.9	5	0.00	1.6
	U	1.3	5	0.00	1.7	2,900.0	1	0.41	30.6	3,260.0	1	0.56	32.8	2,948.0	1	0.42	28.4	0.4	5	0.00	1.7
$\Gamma_i^a = 10$	C	29.6	5	0.00	2.4	3,600.0	0	0.97	26.9	3,600.0	0	1.17	27.1	3,600.0	0	0.81	22.7	1,242.0	5	0.00	2.6
	K	0.8	5	0.00	1.5	2,040.2	3	0.16	15.2	2,880.3	1	0.37	16.0	2,360.2	3	0.16	14.3	0.5	5	0.00	1.6
Average		3.9	5.0	0.00	1.9	2,764.3	1.5	0.51	26.1	3,020.5	1.1	0.61	26.2	2,813.1	1.4	0.48	24.2	506.2	4.7	0.00	2.0
$m = 5$																					
	U	8.0	5	0.00	1.9	2,883.6	1	0.67	24.0	2,883.6	1	0.67	24.0	2,883.6	1	0.67	24.0	1.0	5	0.00	1.9
$\Gamma_i^a = 0$	C	20.5	5	0.00	3.9	3,600.0	0	1.56	46.7	3,600.0	0	1.56	46.7	3,600.0	0	1.56	46.7	3,600.0	0	0.16	3.9
	K	10.9	5	0.00	1.8	2,884.2	1	0.55	18.7	2,884.2	1	0.55	18.7	2,884.2	1	0.55	18.7	1.2	5	0.00	1.8
	U	307.0	5	0.00	1.9	2,390.0	2	0.75	34.7	2,374.0	2	0.74	34.4	2,390.0	2	0.69	32.2	10.6	5	0.00	2.0
$\Gamma_i^a = 1$	C	24.0	5	0.00	2.6	3,600.0	0	1.07	26.3	3,600.0	0	1.10	26.1	3,600.0	0	1.12	25.5	3,180.0	1	0.09	2.9
	K	132.2	5	0.00	1.9	2,916.0	1	0.51	22.1	2,926.0	1	0.56	22.0	2,968.0	1	0.54	21.7	78.6	5	0.00	1.9

Table 4. (Continued)

		$\text{MILP}_{2'}^{\log}[\mathcal{U}^{ab}]$				$\text{MILP}_2[\mathcal{U}^{a\bar{b}}]$				$\text{MILP}_2[\mathcal{U}_{=}^{a\bar{b}}]$				$\text{MILP}_2[\mathcal{U}_{\propto}^{a\bar{b}}]$				$\text{MILP}[\mathcal{U}^{\bar{a}}]$			
$n = 50$	Constraint type	T	#	G	R	T	#	G	R	T	#	G	R	T	#	G	R	T	#	G	R
	U	111.6	5	0.00	1.8	2,892.4	1	0.85	28.8	2,894.0	1	0.88	29.0	2,902.0	1	0.81	27.5	8.4	5	0.00	1.8
$\Gamma_i^a = 2$	C	67.3	5	0.00	3.1	3,600.0	0	1.18	32.9	3,600.0	0	1.14	32.7	3,600.0	0	1.18	31.0	3,600.0	0	0.10	3.4
	K	297.0	5	0.00	1.9	3,600.0	0	0.97	25.3	3,600.0	0	0.94	25.4	3,600.0	0	0.96	24.1	114.2	5	0.00	2.0
	U	52.2	5	0.00	1.8	3,600.0	0	0.89	29.1	3,600.0	0	0.98	30.4	3,600.0	0	0.82	26.4	2.5	5	0.00	2.0
$\Gamma_i^a = 5$	C	513.0	5	0.00	2.6	3,600.0	0	1.20	33.0	3,600.0	0	1.16	32.7	3,600.0	0	1.15	29.7	3,600.0	0	0.07	2.9
	K	25.4	5	0.00	1.5	1,503.8	3	0.20	22.4	1,559.6	3	0.24	23.6	1,770.8	3	0.27	23.2	1.1	5	0.00	1.8
	U	59.6	5	0.00	1.7	2,207.2	2	0.38	26.7	2,244.0	2	0.48	28.5	2,394.0	2	0.40	25.8	20.4	5	0.00	1.8
$\Gamma_i^a = 10$	C	751.6	5	0.00	3.5	3,600.0	0	1.28	38.6	3,600.0	0	1.22	38.4	3,600.0	0	1.14	32.8	3,600.0	0	0.10	3.5
	K	51.0	5	0.00	1.8	2,880.0	2	0.27	21.1	3,020.0	2	0.31	22.6	3,600.0	0	0.36	20.3	0.8	5	0.00	1.9
Average		162.1	5.0	0.00	2.2	3,050.5	0.9	0.82	28.7	3,065.7	0.9	0.84	29.0	3,132.8	0.7	0.81	27.3	1,187.9	3.4	0.03	2.4

Notes. Average time (T) in seconds with the number (#) of instances solved within default optimality gap 0.01 and the average remaining optimality gap (G) along with the average relaxation quality (R) across instances for $n = 50$. We have, for $\mathcal{U}^{ab}$, that $\Gamma_i = 2\,\Gamma_i^a$; for $\mathcal{U}_{=}^{a\bar{b}}$ and $\mathcal{U}_{\propto}^{a\bar{b}}$, that $\Gamma_i = \Gamma_i^a$; and for $\mathcal{U}^{\bar{a}}$, that $\Gamma = m\,\Gamma_i^a$.

Table 5. Comparison of Results for the Best Disjoint Reformulation $\text{MILP}^{\log}_{2'}[\mathcal{U}^{ab}]$ vs. Joint Reformulations $\text{MILP}_2[\mathcal{U}^{\overline{ab}}]$, $\text{MILP}_2[\mathcal{U}^{\overline{ab}}_{=}]$, $\text{MILP}_2[\mathcal{U}^{\overline{ab}}_{\propto}]$, and $\text{MILP}[\mathcal{U}^{\bar{a}}]$

	Constraint	$\text{MILP}^{\log}_{2'}[\mathcal{U}^{ab}]$				$\text{MILP}_2[\mathcal{U}^{\overline{ab}}]$				$\text{MILP}_2[\mathcal{U}^{\overline{ab}}_{=}]$				$\text{MILP}_2[\mathcal{U}^{\overline{ab}}_{\propto}]$				$\text{MILP}[\mathcal{U}^{\bar{a}}]$			
$n = 150$	type	T	#	G	R	T	#	G	R	T	#	G	R	T	#	G	R	T	#	G	R
$m = 1$																					
	U	0.1	5	0.00	1.0	0.4	5	0.00	29.6	0.4	5	0.00	29.6	0.4	5	0.00	29.6	0.1	5	0.00	1.0
$\Gamma_i^a = 0$	C	0.1	5	0.00	2.4	30.5	5	0.00	45.6	170.6	5	0.00	45.6	30.5	5	0.00	45.6	3,600.0	0	0.32	2.4
	K	0.1	5	0.00	1.0	1.0	5	0.00	20.7	0.9	5	0.00	20.7	0.9	5	0.00	20.7	0.1	5	0.00	1.0
	U	0.2	5	0.00	1.0	2.8	5	0.00	35.4	3.7	5	0.00	33.5	3.1	5	0.00	32.7	0.2	5	0.00	1.0
$\Gamma_i^a = 3$	C	0.2	5	0.00	2.0	41.4	5	0.00	31.6	68.0	5	0.00	31.1	740.6	4	0.01	30.9	3,600.0	0	0.34	2.3
	K	0.1	5	0.00	1.0	1.3	5	0.00	27.9	2.9	5	0.00	25.5	1.1	5	0.00	24.2	0.1	5	0.00	1.0
	U	0.2	5	0.00	1.1	2.5	5	0.00	39.6	3.7	5	0.00	37.8	2.7	5	0.00	36.6	0.1	5	0.00	1.0
$\Gamma_i^a = 6$	C	0.2	5	0.00	2.4	790.4	4	0.70	90.2	1,451.6	3	0.07	87.7	751.8	4	0.30	86.1	3,600.0	0	0.34	3.1
	K	0.2	5	0.00	1.1	1.7	5	0.00	19.4	2.3	5	0.00	18.1	1.3	5	0.00	16.9	0.1	5	0.00	1.0
	U	0.2	5	0.00	1.0	2.1	5	0.00	46.6	3.0	5	0.00	46.8	1.7	5	0.00	42.0	0.1	5	0.00	1.0
$\Gamma_i^a = 15$	C	0.3	5	0.00	1.8	1,451.7	5	0.00	46.8	24.4	5	0.00	45.5	1,843.6	4	0.00	43.7	3,600.0	0	0.33	2.3
	K	0.2	5	0.00	1.0	2.0	5	0.00	42.7	3.1	5	0.00	43.0	1.9	5	0.00	35.7	0.1	5	0.00	1.0
	U	0.2	5	0.00	1.0	2.4	5	0.00	38.3	3.5	5	0.00	39.3	2.3	5	0.00	31.2	0.1	5	0.00	1.0
$\Gamma_i^a = 30$	C	0.4	5	0.00	2.1	690.5	5	0.00	71.2	823.7	4	0.46	68.7	1,539.2	3	0.70	63.9	3,600.0	0	0.40	2.4
	K	0.2	5	0.00	1.0	2.1	5	0.00	25.3	3.4	5	0.00	26.2	1.5	5	0.00	23.7	0.1	5	0.00	1.0
Average		0.2	5.0	0.00	1.4	201.5	4.9	0.05	40.7	171.0	4.8	0.04	39.9	328.2	4.7	0.07	37.6	1,200.1	3.3	0.12	1.5
$m = 3$																					
	U	0.7	5	0.00	1.8	3,600.0	0	3.64	46.0	3,600.0	0	3.64	46.0	3,600.0	0	3.64	46.0	721.0	4	0.01	1.8
$\Gamma_i^a = 0$	C	3.2	5	0.00	5.1	3,600.0	0	9.66	118.7	3,600.0	0	9.74	118.7	3,600.0	0	9.66	118.7	3,600.0	0	1.00	5.1
	K	0.8	5	0.00	1.7	3,600.0	0	2.86	38.9	3,600.0	0	2.86	38.9	3,600.0	0	2.86	38.9	2.9	5	0.00	1.7
	U	4.1	5	0.00	1.8	3,600.0	0	5.26	71.9	3,600.0	0	5.18	67.4	3,600.0	0	4.94	64.8	46.5	5	0.00	1.8
$\Gamma_i^a = 3$	C	30.8	5	0.00	3.9	3,600.0	0	9.42	109.8	3,600.0	0	9.40	109.3	3,600.0	0	10.04	108.1	3,600.0	0	0.91	4.5
	K	5.2	5	0.00	1.9	3,600.0	0	4.88	48.9	3,600.0	0	4.98	48.2	3,600.0	0	4.58	45.2	57.2	5	0.00	1.9
	U	5.4	5	0.00	2.0	3,600.0	0	7.04	71.5	3,600.0	0	6.48	70.5	3,600.0	0	6.18	65.8	322.0	5	0.00	1.9
$\Gamma_i^a = 6$	C	29.6	5	0.00	3.1	3,600.0	0	8.92	56.1	3,600.0	0	8.04	55.5	3,600.0	0	8.44	55.3	3,600.0	0	0.83	3.6
	K	6.9	5	0.00	1.8	3,600.0	0	4.67	38.8	3,600.0	0	4.26	37.7	3,600.0	0	3.90	35.0	375.8	5	0.00	1.8
	U	13.7	5	0.00	1.8	3,600.0	0	7.52	89.6	3,600.0	0	7.40	91.2	3,600.0	0	6.28	73.3	10.9	5	0.00	1.8
$\Gamma_i^a = 15$	C	49.9	5	0.00	2.9	3,600.0	0	9.04	61.4	3,600.0	0	8.56	60.6	3,600.0	0	8.72	58.5	3,600.0	0	0.83	3.7
	K	16.5	5	0.00	1.9	3,600.0	0	5.94	44.0	3,600.0	0	5.72	44.4	3,600.0	0	5.20	40.8	830.6	4	0.01	1.9
	U	30.0	5	0.00	1.7	3,600.0	0	6.54	56.7	3,600.0	0	6.58	59.6	3,600.0	0	6.04	53.2	722.6	4	0.01	1.8
$\Gamma_i^a = 30$	C	1,448.6	5	0.00	3.1	3,600.0	0	9.40	62.6	3,600.0	0	9.50	62.0	3,600.0	0	9.48	58.2	3,600.0	0	0.80	3.8
	K	204.1	5	0.00	1.6	3,600.0	0	5.32	45.3	3,600.0	0	5.02	47.2	3,600.0	0	4.52	41.7	761.3	4	0.00	1.8
Average		123.3	5.0	0.00	2.4	3,600.0	0.0	6.67	64.0	3,600.0	0.0	6.49	63.8	3,600.0	0.0	6.30	60.2	1,456.7	3.1	0.29	2.6
$m = 5$																					
	U	32.6	5	0.00	2.4	3,600.0	0	7.18	58.0	3,600.0	0	7.20	58.0	3,600.0	0	7.18	58.0	1,983.8	3	0.06	2.4
$\Gamma_i^a = 0$	C	666.6	5	0.00	4.5	3,600.0	0	11.58	63.6	3,600.0	0	11.58	63.6	3,600.0	0	11.58	63.7	3,600.0	0	1.18	4.5
	K	34.7	5	0.00	2.3	3,600.0	0	5.08	40.6	3,600.0	0	5.32	40.6	3,600.0	0	5.08	40.6	2,164.6	2	0.09	2.3
	U	960.0	5	0.00	2.5	3,600.0	0	10.08	95.2	3,600.0	0	9.84	91.5	3,600.0	0	9.24	83.4	2,172.4	2	0.14	2.5
$\Gamma_i^a = 3$	C	2,302.0	5	0.00	4.0	3,600.0	0	12.00	105.3	3,600.0	0	12.20	105.1	3,600.0	0	11.80	104.1	3,600.0	0	1.10	4.7
	K	734.0	5	0.00	2.4	3,600.0	0	7.70	61.1	3,600.0	0	6.94	57.5	3,600.0	0	7.06	55.3	2,882.6	1	0.12	2.5

Table 5. (Continued)

$n = 150$	Constraint type	$\text{MILP}_{2'}^{\log}[\mathcal{U}^{ab}]$				$\text{MILP}_2[\mathcal{U}^{a\bar{b}}]$				$\text{MILP}_2[\mathcal{U}_{=}^{a\bar{b}}]$				$\text{MILP}_2[\mathcal{U}_{\propto}^{a\bar{b}}]$				$\text{MILP}[\mathcal{U}^{\bar{a}}]$			
		T	#	G	R	T	#	G	R	T	#	G	R	T	#	G	R	T	#	G	R
	U	1,578.6	4	0.00	2.2	3,600.0	0	8.94	81.3	3,600.0	0	9.12	79.5	3,600.0	0	7.90	73.4	1,447.0	3	0.13	2.3
$\Gamma_i^a = 6$	C	1,608.8	5	0.00	4.3	3,600.0	0	13.40	159.5	3,600.0	0	12.40	159.0	3,600.0	0	13.60	156.1	3,600.0	0	1.22	5.3
	K	417.0	5	0.00	2.0	3,600.0	0	6.10	55.7	3,600.0	0	5.44	54.8	3,600.0	0	5.16	53.3	943.3	4	0.06	2.1
	U	1,018.0	4	0.02	1.9	3,600.0	0	8.44	77.1	3,600.0	0	8.76	77.6	3,600.0	0	8.44	70.9	809.2	4	0.09	2.1
$\Gamma_i^a = 15$	C	3,600.0	4	0.01	3.5	3,600.0	0	13.80	100.5	3,600.0	0	13.40	100.0	3,600.0	0	13.20	96.1	3,600.0	0	1.08	4.4
	K	1,806.0	3	0.03	2.0	3,600.0	0	9.60	71.1	3,600.0	0	9.84	73.1	3,600.0	0	8.24	62.9	2,129.0	3	0.19	2.6
	U	306.0	5	0.00	1.9	3,600.0	0	8.00	79.1	3,600.0	0	8.74	81.9	3,600.0	0	7.54	74.8	291.4	5	0.00	2.2
$\Gamma_i^a = 30$	C	3,600.0	1	0.01	3.5	3,600.0	0	13.20	102.9	3,600.0	0	13.40	102.8	3,600.0	0	13.00	95.5	3,600.0	0	1.03	4.5
	K	1,056.0	4	0.02	1.9	3,600.0	0	7.36	68.0	3,600.0	0	7.44	70.3	3,600.0	0	7.30	64.1	752.3	4	0.08	2.1
Average		1,314.7	4.3	0.01	2.8	3,600.0	0.0	9.50	81.3	3,600.0	0.0	9.44	81.0	3,600.0	0.0	9.09	76.8	2,238.4	2.1	0.44	3.1

Notes. Average time (T) in seconds with the number (#) of instances solved within default optimality gap 0.01 and the average remaining optimality gap (G) along with the average relaxation quality (R) across instances for $n = 150$. We have, for $\mathcal{U}^{a\bar{b}}$, that $\Gamma_i = 2\,\Gamma_i^a$; for $\mathcal{U}_{=}^{a\bar{b}}$ and $\mathcal{U}_{\propto}^{a\bar{b}}$, that $\Gamma_i = \Gamma_i^a$; and for $\mathcal{U}^{\bar{a}}$, that $\Gamma = m\,\Gamma_i^a$.

6. Conclusion

This paper addresses single- and multiple-ratio RFPs defined as the robust counterparts of the FPs under various disjoint and joint uncertainty sets. We demonstrate that single-ratio RFP, contrary to its deterministic counterpart, is NP-hard for a general polyhedral uncertainty set. However, if the uncertainties are in the form of the budgeted uncertainty sets, then we develop polynomial-time-solution methods for single-ratio RFP provided that the nominal problem is polynomial-time solvable.

In particular, for the disjoint uncertainty set, we propose an approach to solve single-ratio RFP by calling at most $(n+1)^2$ instances of FP. Moreover, in the case of joint uncertainty sets, we show that single-ratio RFP can be solved by solving a polynomial number of instances of a linear binary problem. Therefore, if the latter admits a polynomial-time-solution algorithm, then single-ratio RFP under disjoint uncertainty sets is polynomial-time solvable as well.

In the case of multiple-ratio RFPs, we exploit the structure of the budgeted disjoint uncertainty sets in order to propose various MILPs to solve them. In particular, based on our extensive computational experiments, it is noted that RFPs are more challenging to solve under the joint sets than the disjoint sets because the former cannot take advantage of the binary expansion technique. Indeed, it seems that as the size of the problem increases, the binarized formulations are often a better choice for the robust problem under the disjoint uncertainty set.

We also explore the value of the robust optimal solution for instances with both real and synthetic data and find that ignoring the data uncertainty can lead to poor decisions. These results coupled with the insights on the selection of budget(s) of uncertainties can provide guidance to consider the suitable solution method and level of uncertainty in practice.

Finally, it is worth mentioning that conic quadratic programming–based reformulations can be used as an alternative solution approach to tackle deterministic fractional 0-1 programs (see, e.g., Şen et al. 2018, Mehmanchi et al. 2019, Atamtürk and Gómez 2020) because they lead to strong convex relaxations. However, solving large-scale mixed-integer conic quadratic programs is still challenging in practice despite the recent advances in off-the-shelf commercial optimization software. Nevertheless, such approaches can be pursued as a promising future research direction for solving robust fractional 0-1 programs.

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Appendix A. Proofs

Proof of Proposition 5. Let u be the indicator variables of the sets $S(\widetilde{a}) = S(\widetilde{b})$. Note that RFP$[\mathcal{U}^{\overline{ab}}_{=}]$ can be written as

$$\begin{aligned} Z^\star_{\mathcal{U}^{\overline{ab}}_{=}} = \max_{x\in X} \min_{u\in\mathbb{R}^n} & \frac{a_0 + \sum_{j\in J} a_j x_j - \sum_{j\in J} d_j^a x_j u_j}{b_0 + \sum_{j\in J} b_j x_j + \sum_{j\in J} d_j^b x_j u_j} \\ \text{s.t.} & \sum_{j\in J} u_j \le \Gamma \\ & 0 \le u_j \le 1 \qquad \forall j \in J. \end{aligned}$$

Using the Charnes and Cooper (1962) transformation as in the Proof of Proposition 4, we find the equivalent formulation

$$\begin{aligned} Z^\star_{\mathcal{U}^{\overline{ab}}_{=}} = \max_{x\in X} \min_{z,y} & \left(a_0 + \sum_{j\in J} a_j x_j\right) y - \sum_{j\in J} d_j^a x_j z_j \\ \text{s.t.} & \left(b_0 + \sum_{j\in J} b_j x_j\right) y + \sum_{j\in J} d_j^b x_j z_j = 1 && (\mu) \\ & \sum_{j\in J} z_j \le \Gamma y && (\alpha) \\ & 0 \le z_j \le y \qquad \forall j \in J. && (\beta_j) \end{aligned}$$

Using the standard LP duality for the inner minimization problem, we obtain formulation (19). □

Proof of Proposition 6. Let w be the indicator variables of the sets $S(\widetilde{a}) = S(\widetilde{b})$. To model the proportionality conditions (i.e., $\frac{a_j - \widetilde{a}_j}{d_j^a} = \frac{b_j - \widetilde{b}_j}{d_j^b} \in [-1, 1]$ for all $j \in J$), we introduce additional continuous variables $\eta \in [-1, 1]^n$ and write RFP$[\mathcal{U}^{\overline{ab}}_{\propto}]$ as

$$\begin{aligned} Z^\star_{\mathcal{U}^{\overline{ab}}_{\propto}} = \max_{x\in X} \min_{w,\eta} & \frac{a_0 + \sum_{j\in J} a_j x_j + \sum_{j\in J} d_j^a x_j \eta_j}{b_0 + \sum_{j\in J} b_j x_j + \sum_{j\in J} d_j^b x_j \eta_j} \\ \text{s.t.} & \sum_{j\in J} w_j \le \Gamma \\ & -w_j \le \eta_j \le w_j, \quad w_j \in \{0, 1\} \qquad \forall j \in J. \end{aligned}$$

Because the inner minimization problem is quasi-concave, it follows that $\eta_j \in \{-w_j, w_j\}$ in an optimal solution. Letting $u_j = 1$ if $\eta_j = w_j > 0$ and 0 otherwise and letting $v_j = 1$ if $\eta_j = w_j < 0$ and 0 otherwise, we can rewrite RFP$[\mathcal{U}^{\overline{ab}}_{\propto}]$ as

$$\begin{aligned} Z^\star_{\mathcal{U}^{\overline{ab}}_{\propto}} = \max_{x\in X} \min_{u,v\in[0,1]^n} & \frac{a_0 + \sum_{j\in J} a_j x_j + \sum_{j\in J} d_j^a x_j u_j - \sum_{j\in J} d_j^a x_j v_j}{b_0 + \sum_{j\in J} b_j x_j + \sum_{j\in J} d_j^b x_j u_j - \sum_{j\in J} d_j^b x_j v_j} \\ \text{s.t.} & \sum_{j\in J} u_j + \sum_{j\in J} v_j \le \Gamma. \end{aligned}$$

Then, using the Charnes and Cooper (1962) transformation and linear programming duality as in the Proofs of Propositions 4 and 5, we obtain formulation (20). □

Proof of Proposition 10. We prove part (a); parts (b) and (c) can be proved in a similar manner. Note that the fractional binary problems subject to uncertain (random variable) coefficients can be represented as

$$\max_{\substack{x\in X,\\ \mu\ge 0}} \sum_{i\in I} \mu_i \tag{A.1}$$

$$\text{s.t.} \sum_{j\in J} \widetilde{b}_{ij} x_j \mu_i - \sum_{j\in J} \widetilde{a}_{ij} x_j \le a_{i0} - b_{i0}\mu_i \tag{A.2}$$

when $b_{i0} + \sum_{j\in J} \widetilde{b}_{ij} x_j^\star > 0$. For given $(x^\star, \mu^\star)$, random variables $\widetilde{a}$ and $\widetilde{b}$, and for each $i \in I$, we aim to compute an upper bound for the probability that the ith constraint in (A.2) is violated: that is,

$$\Pr\left(\sum_{j\in J} \widetilde{b}_{ij} \mu_i^\star x_j^\star - \sum_{j\in J} \widetilde{a}_{ij} x_j^\star > a_{i0} - \mu_i^\star b_{i0}\right) = \Pr\left(\mu_i^\star > \frac{a_{i0} + \sum_{j\in J} \widetilde{a}_{ij} x_j^\star}{b_{i0} + \sum_{j\in J} \widetilde{b}_{ij} x_j^\star}\right).$$

Then, for each $i \in I$,

$$\Pr\left(\sum_{j\in J}\widetilde{b}_{ij}\mu_i^\star x_j^\star - \sum_{j\in J}\widetilde{a}_{ij}x_j^\star > a_{i0} - \mu_i^\star b_{i0}\right)$$

$$= \Pr\left(\sum_{j\in J} b_{ij}\mu_i^\star x_j^\star + \sum_{j\in J}\eta_{ij}d_{ij}^b\mu_i^\star x_j^\star - \sum_{j\in J}a_{ij}x_j^\star - \sum_{j\in J}\eta_{i,j+n}d_{ij}^a x_j^\star > a_{i0} - \mu_i^\star b_{i0}\right) \tag{A.3}$$

$$= \Pr\left(\sum_{j\in J}\eta_{ij}d_{ij}^b\mu_i^\star x_j^\star + \sum_{j\in J}\eta_{i,j+n}d_{ij}^a x_j^\star > a_{i0} - \mu_i^\star b_{i0} - \sum_{j\in J}b_{ij}\mu_i^\star x_j^\star + \sum_{j\in J}a_{ij}x_j^\star\right) \tag{A.4}$$

$$\le \Pr\left(\sum_{j\in J}\eta_{ij}d_{ij}^b\mu_i^\star x_j^\star + \sum_{j\in J}\eta_{i,j+n}d_{ij}^a x_j^\star > \sum_{j\in S_{i,b}^\star}d_{ij}^b\mu_i^\star x_j^\star + \sum_{j\in S_{i,a}^\star}d_{ij}^a x_j^\star\right) \tag{A.5}$$

$$= \Pr\left(\sum_{j\in J\setminus S_{i,b}^\star}\eta_{ij}d_{ij}^b\mu_i^\star x_j^\star + \sum_{j\in J\setminus S_{i,a}^\star}\eta_{i,j+n}d_{ij}^a x_j^\star > \sum_{j\in S_{i,b}^\star}d_{ij}^b\mu_i^\star x_j^\star(1-\eta_{ij}) + \sum_{j\in S_{i,a}^\star}d_{ij}^a x_j^\star(1-\eta_{i,j+n})\right)$$

$$\le \Pr\left(\sum_{j\in J\setminus S_{i,b}^\star}\eta_{ij}d_{ij}^b\mu_i^\star x_j^\star + \sum_{j\in J\setminus S_{i,a}^\star}\eta_{i,j+n}d_{ij}^a x_j^\star > c_i\sum_{j\in S_{i,b}^\star}(1-\eta_{ij}) + c_i\sum_{j\in S_{i,a}^\star}(1-\eta_{i,j+n})\right) \tag{A.6}$$

$$= \Pr\left(\sum_{j\in S_{i,b}^\star}\eta_{ij} + \sum_{j\in S_{i,a}^\star}\eta_{i,j+n} + \sum_{j\in J\setminus S_{i,b}^\star}\eta_{ij}\frac{d_{ij}^b\mu_i^\star x_j^\star}{c_i} + \sum_{j\in J\setminus S_{i,a}^\star}\eta_{i,j+n}\frac{d_{ij}^a x_j^\star}{c_i} > \Gamma_i^a + \Gamma_i^b\right). \tag{A.7}$$

Probability (A.3) is correct for independently and symmetrically distributed random variables $\eta_{ij} \in [-1,1]$ for all $j \in \{1,\dots,2n\}$. Probability (A.4) is correct becaues $\eta_{i,j+n} \in [-1,1]$. Let $S_{i,a}^\star$ and $S_{i,b}^\star$ be the sets of indices of parameters that take the robust value in the numerator and the denominator of the ith ratio, respectively, in a robust optimal solution. Then note that

$$\sum_{j\in J}b_{ij}\mu_i^\star x_j^\star + \sum_{j\in S_{i,b}^\star}d_{ij}^b\mu_i^\star x_j^\star - \sum_{j\in J}a_{ij}x_j^\star + \sum_{j\in S_{i,a}^\star}d_{ij}^a x_j^\star \le a_{i0} - \mu_i^\star b_{i0}$$

is a valid inequality for problems (A.1) and (A.2) under uncertainty set $\mathcal{U}^{ab}$. Thus, probability (A.5) is correct. Additionally, probability (A.6) is correct for $c_i = \min\{\{d_{ij}^b\mu_i^\star x_j^\star\}_{j\in S_{i,b}^\star}, \{d_{ij}^a x_j^\star\}_{j\in S_{i,a}^\star}\}$. Next, for $j \in \{1,2,\dots,2n\}$, define

$$\gamma_{ij} = \begin{cases} 1, & \text{if } j \in S_{i,b}^\star \text{ or } j-n \in S_{i,a}^\star \\ \dfrac{d_{ij}^b\mu_i^\star x_j^\star}{c_i}, & \text{if } j \in J\setminus S_{i,b}^\star \\ \dfrac{d_{ij}^a x_j^\star}{c_i}, & \text{if } j-n \in J\setminus S_{i,a}^\star, \end{cases}$$

and note that $\gamma_{ij} \le 1$ for all $j \in \{1,\dots,2n\}$; otherwise, $S_{i,a}^\star$ or $S_{i,b}^\star$ are not the robust optimal set of indices. Hence, probability (A.7) is equivalent to

$$\Pr\left(\sum_{j\in\{1,\dots,2n\}}\gamma_{ij}\eta_{ij} > \Gamma_i^a + \Gamma_i^b\right) \le \Pr\left(\sum_{j\in\{1,\dots,2n\}}\gamma_{ij}\eta_{ij} \ge \Gamma_i^a + \Gamma_i^b\right) \le B(2n, \Gamma_i^a + \Gamma_i^b).$$

This last inequality follows from theorem 3, part (a), in Bertsimas and Sim (2004) for independent and symmetrically distributed random variables $\eta_j \in [-1,1]$ and $\gamma_{ij} \le 1$ for $j \in J$. □

Proof of Proposition 11. Following the Proof of Proposition 10, for uncertainty set $\mathcal{U}^{ab}_{\propto}$ and each $i \in I$, we have

$$
\begin{aligned}
&\Pr\left(\mu_i^\star > \frac{a_{i0} + \sum_{j\in J} \widetilde{a}_{ij} x_j^\star}{b_{i0} + \sum_{j\in J} \widetilde{b}_{ij} x_j^\star}\right)\\
&\quad = \Pr\left(\sum_{j\in J} \widetilde{b}_{ij}\mu_i^\star x_j^\star - \sum_{j\in J} \widetilde{a}_{ij} x_j^\star > a_{i0} - \mu_i^\star b_{i0}\right)\\
&\quad = \Pr\left(\sum_{j\in J} b_{ij}\mu_i^\star x_j^\star + \sum_{j\in J} \eta_{ij} d^b_{ij}\mu_i^\star x_j^\star - \sum_{j\in J} a_{ij} x_j^\star - \sum_{j\in J} \eta_{ij} d^a_{ij} x_j^\star > a_{i0} - \mu_i^\star b_{i0}\right)\\
&\quad = \Pr\left(\sum_{j\in J} \eta_{ij}\left(d^b_{ij}\mu_i^\star x_j^\star - d^a_{ij} x_j^\star\right) > a_{i0} - \mu_i^\star b_{i0} - \sum_{j\in J} b_{ij}\mu_i^\star x_j^\star + \sum_{j\in J} a_{ij} x_j^\star\right)\\
&\quad = \Pr\left(\sum_{j\in J} \eta_{ij}\left|d^b_{ij}\mu_i^\star x_j^\star - d^a_{ij} x_j^\star\right| > a_{i0} - \mu_i^\star b_{i0} - \sum_{j\in J} b_{ij}\mu_i^\star x_j^\star + \sum_{j\in J} a_{ij} x_j^\star\right) && \text{(A.8)}\\
&\quad \le \Pr\left(\sum_{j\in J} \eta_{ij}\left|d^b_{ij}\mu_i^\star x_j^\star - d^a_{ij} x_j^\star\right| > \sum_{j\in S_i^\star}\left|d^b_{ij}\mu_i^\star x_j^\star - d^a_{ij} x_j^\star\right|\right) && \text{(A.9)}\\
&\quad = \Pr\left(\sum_{j\in J\setminus S_i^\star} \eta_{ij}\left|d^b_{ij}\mu_i^\star x_j^\star - d^a_{ij} x_j^\star\right| > \sum_{j\in S_i^\star}\left|d^b_{ij}\mu_i^\star x_j^\star - d^a_{ij} x_j^\star\right|(1-\eta_{ij})\right)\\
&\quad \le \Pr\left(\sum_{j\in J\setminus S_i^\star} \eta_{ij}\left|d^b_{ij}\mu_i^\star x_j^\star - d^a_{ij} x_j^\star\right| > \sum_{j\in S_i^\star} c_i(1-\eta_{ij})\right) && \text{(A.10)}\\
&\quad = \Pr\left(\sum_{j\in S_i^\star} \eta_{ij} + \sum_{j\in J\setminus S_i^\star} \eta_{ij}\frac{\left|d^b_{ij}\mu_i^\star x_j^\star - d^a_{ij} x_j^\star\right|}{c_i} > \Gamma_i\right). && \text{(A.11)}
\end{aligned}
$$

Probability (A.8) is correct for $\eta_{ij} \in [-1, 1]$. Let $S_i^\star$ be the set of indices of parameters that take the robust value in a robust optimal solution of the ith ratio. Then note that

$$\sum_{j\in J} b_{ij}\mu_i^\star x_j^\star - \sum_{j\in J} a_{ij} x_j^\star + \sum_{j\in S_i^\star}\left|d^b_{ij}\mu_i^\star x_j^\star - d^a_{ij} x_j^\star\right| \le a_{i0} - \mu_i^\star b_{i0}$$

is a valid inequality for problems (A.1) and (A.2) under uncertainty set $\mathcal{U}^{ab}_{\propto}$. Thus, probability (A.9) is correct. Additionally, probability (A.10) is correct for $c_i = \min\{|d^b_{ij}\mu_i^\star x_j^\star - d^a_{ij} x_j^\star|\}_{j\in S_i^\star}$. Next, for $j \in \{1, 2, \ldots, n\}$, define

$$\gamma_{ij} = \begin{cases} 1, & \text{if } j \in S_i^\star\\ \dfrac{\left|d^b_{ij}\mu_i^\star x_j^\star - d^a_{ij} x_j^\star\right|}{c_i}, & \text{if } j \in J\setminus S_i^\star, \end{cases}$$

and note that $\gamma_{ij} \le 1$ for all $j \in J$; otherwise, $S_i^\star$ is not the robust optimal set of indices. Hence, probability (A.11) is equivalent to

$$\Pr\left(\sum_{j\in J} \gamma_{ij}\eta_{ij} > \Gamma_i\right) \le \Pr\left(\sum_{j\in J} \gamma_{ij}\eta_{ij} \ge \Gamma_i\right) \le B(n, \Gamma_i).$$

The last inequality follows from theorem 3, part (a), in Bertsimas and Sim (2004) for independent and symmetrically distributed random variables $\eta_j \in [-1, 1]$ and $\gamma_{ij} \le 1$ for $j \in J$. □

Appendix B. Additional Computational Results

Table B.1. Results for Disjoint Reformulations

		$FP_{4'}$				$MILP_1[\mathcal{U}^{ab}]$				$MILP_2[\mathcal{U}^{ab}]$				$MILP_2^{log}[\mathcal{U}^{ab}]$				$MILP_{2'}^{log}[\mathcal{U}^{ab}]$			
$n = 100$	Constraint type	T	#	G	R	T	#	G	R	T	#	G	R	T	#	G	R	T	#	G	R
$m = 1$																					
	U	0.3	5	0.00	1.0	**0.1**	5	0.00	1.0	0.5	5	0.00	20.9	0.2	5	0.00	24.0	**0.1**	5	0.00	1.0
$\Gamma_i^a = \Gamma_i^b = 0$	C	0.2	5	0.00	1.0	3,048.0	1	0.08	2.8	1.0	5	0.00	64.9	0.2	5	0.00	74.4	**0.1**	5	0.00	2.8
	K	0.0	5	0.00	1.0	**0.0**	5	0.00	1.0	0.6	5	0.00	14.4	0.2	5	0.00	15.0	**0.0**	5	0.00	1.0
	U	0.1	5	0.00	1.2	**0.1**	5	0.00	1.0	1.4	5	0.00	25.4	0.3	5	0.00	31.8	0.2	5	0.00	1.0
$\Gamma_i^a = \Gamma_i^b = 2$	C	0.2	5	0.00	1.1	2,942.0	1	0.18	2.7	13.1	5	0.00	64.8	0.3	5	0.00	69.5	**0.2**	5	0.00	2.7
	K	0.1	5	0.00	1.2	**0.1**	5	0.00	1.0	1.2	5	0.00	16.1	**0.1**	5	0.00	16.6	**0.1**	5	0.00	1.0
	U	0.1	5	0.00	1.5	**0.1**	5	0.00	1.0	1.2	5	0.00	34.0	0.3	5	0.00	42.5	**0.1**	5	0.00	1.0
$\Gamma_i^a = \Gamma_i^b = 4$	C	0.2	5	0.00	1.2	2,891.2	1	0.13	2.3	70.9	5	0.00	80.1	0.4	5	0.00	87.5	**0.1**	5	0.00	2.3
	K	0.1	5	0.00	1.5	**0.1**	5	0.00	1.1	1.2	5	0.00	23.5	0.2	5	0.00	25.3	0.2	5	0.00	1.1
	U	0.1	5	0.00	1.6	**0.1**	5	0.00	1.0	1.0	5	0.00	39.8	0.3	5	0.00	46.4	0.2	5	0.00	1.0
$\Gamma_i^a = \Gamma_i^b = 10$	C	0.2	5	0.00	1.3	3,600.0	0	0.14	2.4	125.6	5	0.00	96.5	**0.3**	5	0.00	111.1	**0.3**	5	0.00	2.4
	K	0.0	5	0.00	1.4	**0.1**	5	0.00	1.0	1.2	5	0.00	19.2	0.2	5	0.00	19.9	**0.1**	5	0.00	1.0
	U	0.1	5	0.00	1.4	**0.1**	5	0.00	1.0	1.2	5	0.00	31.3	0.3	5	0.00	40.4	**0.1**	5	0.00	1.0
$\Gamma_i^a = \Gamma_i^b = 20$	C	0.2	5	0.00	1.5	3,600.0	0	0.13	2.9	14.4	5	0.00	74.8	0.4	5	0.00	80.8	**0.3**	5	0.00	2.9
	K	0.1	5	0.00	1.6	**0.0**	5	0.00	1.0	1.1	5	0.00	35.2	0.2	5	0.00	39.0	0.1	5	0.00	1.0
Average		0.1	5.0	0.00	1.3	1,072.1	3.5	0.04	1.5	15.7	5.0	0.00	42.7	0.3	5.0	0.00	48.3	**0.1**	5.0	0.00	1.5
$m = 3$																					
	U	1.0	5	0.00	1.9	6.4	5	0.00	1.9	3,600.0	0	2.48	31.5	1.5	5	0.00	35.9	**0.9**	5	0.00	1.9
$\Gamma_i^a = \Gamma_i^b = 0$	C	2.2	5	0.00	1.2	3,600.0	0	0.49	3.3	3,600.0	0	4.22	35.0	1.8	5	0.00	35.7	**1.5**	5	0.00	3.3
	K	0.4	5	0.00	1.7	2.0	5	0.00	1.7	3,160.0	1	1.30	31.5	1.0	5	0.00	38.4	**0.6**	5	0.00	1.7
	U	0.4	5	0.00	2.1	4.6	5	0.00	1.7	3,600.0	0	2.28	47.1	4.1	5	0.00	65.4	**1.3**	5	0.00	1.7
$\Gamma_i^a = \Gamma_i^b = 2$	C	3.4	5	0.00	1.2	3,600.0	0	0.46	3.5	3,600.0	0	4.84	66.3	**4.2**	5	0.00	69.4	5.0	5	0.00	3.5
	K	0.4	5	0.00	2.0	17.5	5	0.00	1.6	3,600.0	0	1.99	32.6	3.3	5	0.00	39.2	**2.3**	5	0.00	1.6
	U	0.7	5	0.00	2.5	981.4	5	0.00	1.8	3,600.0	0	3.62	48.5	9.0	5	0.00	61.3	**4.8**	5	0.00	1.8
$\Gamma_i^a = \Gamma_i^b = 4$	C	1.9	5	0.00	1.3	3,600.0	0	0.36	2.7	3,600.0	0	4.40	60.1	5.3	5	0.00	67.1	**5.1**	5	0.00	2.7
	K	1.0	5	0.00	2.7	1,656.0	5	0.00	2.0	3,600.0	0	3.46	31.6	17.3	5	0.00	32.8	**7.8**	5	0.00	2.0
	U	0.5	5	0.00	2.4	47.4	5	0.00	1.6	3,600.0	0	3.35	54.2	6.7	5	0.00	73.9	**2.4**	5	0.00	1.6
$\Gamma_i^a = \Gamma_i^b = 10$	C	5.2	5	0.00	1.4	3,600.0	0	0.33	2.9	3,600.0	0	4.92	86.6	**11.6**	5	0.00	100.2	407.9	5	0.00	2.9
	K	0.3	5	0.00	2.1	**0.5**	5	0.00	1.4	3,600.0	0	1.42	41.1	2.4	5	0.00	50.4	0.8	5	0.00	1.4
	U	0.9	5	0.00	2.8	11.2	5	0.00	1.7	3,600.0	0	4.42	55.7	7.3	5	0.00	68.3	**3.3**	5	0.00	1.7
$\Gamma_i^a = \Gamma_i^b = 20$	C	1.4	5	0.00	1.6	3,600.0	0	0.32	2.8	3,600.0	0	5.44	63.6	**30.4**	5	0.00	65.8	1,273.2	5	0.00	2.8
	K	0.5	5	0.00	2.3	724.9	4	0.00	1.8	3,600.0	0	2.84	28.9	5.9	5	0.00	31.0	**2.2**	5	0.00	1.8
Average		1.4	5.0	0.00	1.9	1,430.1	3.3	0.13	2.2	3,570.7	0.1	3.40	47.6	**7.4**	5.0	0.00	55.7	114.6	5.0	0.00	2.2
$m = 5$																					
	U	10.4	5	0.00	2.1	282.6	5	0.00	2.1	3,600.0	0	2.98	41.5	58.4	5	0.00	56.4	**24.0**	5	0.00	2.1
$\Gamma_i^a = \Gamma_i^b = 0$	C	1,676.0	5	0.00	1.3	3,600.0	0	0.67	4.3	3,600.0	0	5.92	70.5	**93.4**	5	0.00	79.9	393.8	5	0.00	4.3
	K	8.3	5	0.00	2.0	169.6	5	0.00	2.0	3,600.0	0	1.21	23.7	**21.6**	5	0.00	26.8	21.8	5	0.00	2.0

Table B.1. (Continued)

$n = 100$	Constraint type	FP$_{4'}$ T	FP$_{4'}$ #	FP$_{4'}$ G	FP$_{4'}$ R	MILP$_1[\mathcal{U}^{ab}]$ T	#	G	R	MILP$_2[\mathcal{U}^{ab}]$ T	#	G	R	MILP$_2^{\log}[\mathcal{U}^{ab}]$ T	#	G	R	MILP$_{2'}^{\log}[\mathcal{U}^{ab}]$ T	#	G	R
$\Gamma_i^a = \Gamma_i^b = 2$	U	7.1	5	0.00	2.6	1,567.5	3	0.07	2.1	3,600.0	0	4.84	50.5	1,312.4	4	0.01	64.8	**206.4**	5	0.00	2.1
	C	1,560.6	5	0.00	1.4	3,600.0	0	0.62	3.4	3,600.0	0	5.94	53.7	**313.6**	5	0.00	55.6	1,299.6	5	0.00	3.4
	K	9.7	5	0.00	2.7	2,198.4	2	0.07	2.2	3,600.0	0	2.28	25.0	400.0	5	0.00	26.3	**167.2**	5	0.00	2.2
$\Gamma_i^a = \Gamma_i^b = 4$	U	4.8	5	0.00	3.3	1,692.8	3	0.07	2.2	3,600.0	0	5.40	66.9	1,139.8	5	0.00	89.8	**480.0**	5	0.00	2.2
	C	1,146.2	5	0.00	1.4	3,600.0	0	0.57	3.3	3,600.0	0	6.16	79.1	**98.0**	5	0.00	86.0	431.4	5	0.00	3.3
	K	5.9	5	0.00	3.1	2,161.2	2	0.12	2.1	3,600.0	0	1.98	33.7	1,023.8	4	0.03	37.2	**378.2**	5	0.00	2.1
$\Gamma_i^a = \Gamma_i^b = 10$	U	13.4	5	0.00	3.1	2,166.1	2	0.20	2.1	3,600.0	0	5.78	70.3	1,870.0	3	0.06	99.3	**1,524.8**	3	**0.02**	2.1
	C	1,764.0	4	0.00	1.5	3,600.0	0	0.50	3.1	3,600.0	0	6.86	70.5	**804.0**	5	0.00	77.1	3,600.0	4	0.01	3.1
	K	11.3	5	0.00	2.7	**725.6**	4	0.07	1.7	3,600.0	0	1.61	30.3	897.4	4	**0.01**	32.6	964.0	4	**0.01**	1.7
$\Gamma_i^a = \Gamma_i^b = 20$	U	14.3	5	0.00	3.2	904.0	4	0.01	1.9	3,600.0	0	5.12	56.2	1,304.0	4	0.02	66.6	**182.0**	5	0.00	1.9
	C	839.0	5	0.00	1.7	3,600.0	0	0.50	3.5	3,600.0	0	7.44	64.7	**1,614.0**	5	0.00	66.2	3,600.0	1	0.01	3.5
	K	16.2	5	0.00	3.3	751.6	4	0.04	2.0	3,600.0	0	2.06	25.1	762.0	5	0.00	25.8	**238.2**	5	0.00	2.0
Average		472.5	4.9	0.00	2.4	2,041.3	2.3	0.23	2.5	3,600.0	0.0	4.37	50.8	**780.8**	4.6	**0.01**	59.4	900.8	4.5	0.00	2.5

Notes. Average time (T) in seconds with the number (#) of instances solved within default optimality gap 0.01 and the average remaining optimality gap (G) along with the average relaxation quality (R) across instances for $n = 100$. In each row, among the solution methods that solve the most number of instances to optimality, the best average time and the best average gap (if # < 5) are in bold.

Table B.2. Comparison of Results for the Best Disjoint Reformulation $\text{MILP}^{\log}_{2'}[\mathcal{U}^{ab}]$ vs. Joint Reformulations $\text{MILP}_2[\mathcal{U}^{\overline{ab}}]$, $\text{MILP}_2[\mathcal{U}^{\overline{ab}}_{=}]$, $\text{MILP}_2[\mathcal{U}^{\overline{ab}}_{\propto}]$, and $\text{MILP}[\mathcal{U}^{\bar{a}}]$

		$\text{MILP}^{\log}_{2'}[\mathcal{U}^{ab}]$				$\text{MILP}_2[\mathcal{U}^{\overline{ab}}]$				$\text{MILP}_2[\mathcal{U}^{\overline{ab}}_{=}]$				$\text{MILP}_2[\mathcal{U}^{\overline{ab}}_{\propto}]$				$\text{MILP}[\mathcal{U}^{\bar{a}}]$			
$n = 100$	Constraint type	T	#	G	R	T	#	G	R	T	#	G	R	T	#	G	R	T	#	G	R
$m = 1$																					
	U	0.1	5	0.00	1.0	0.4	5	0.00	20.9	0.5	5	0.00	20.9	0.4	5	0.00	20.9	0.1	5	0.00	1.0
$\Gamma^a_i = 0$	C	0.1	5	0.00	2.8	1.0	5	0.00	64.9	1.1	5	0.00	64.9	1.1	5	0.00	64.9	3,046.0	1	0.10	2.8
	K	0.0	5	0.00	1.0	0.5	5	0.00	14.4	0.6	5	0.00	14.4	0.6	5	0.00	14.4	0.0	5	0.00	1.0
	U	0.2	5	0.00	1.0	1.3	5	0.00	25.7	1.0	5	0.00	25.3	1.1	5	0.00	24.0	0.1	5	0.00	1.0
$\Gamma^a_i = 2$	C	0.2	5	0.00	2.7	27.8	5	0.00	65.0	8.3	5	0.00	63.8	21.7	5	0.00	63.4	3,600.0	1	0.16	3.0
	K	0.1	5	0.00	1.0	1.1	5	0.00	16.9	1.0	5	0.00	15.9	1.0	5	0.00	15.1	0.1	5	0.00	1.0
	U	0.1	5	0.00	1.0	1.1	5	0.00	34.3	1.3	5	0.00	33.6	1.1	4	0.28	30.9	0.1	5	0.00	1.0
$\Gamma^a_i = 4$	C	0.1	5	0.00	2.3	247.1	5	0.00	80.5	108.3	5	0.00	79.0	56.0	5	0.00	76.8	2,340.0	2	0.14	2.8
	K	0.2	5	0.00	1.1	1.2	5	0.00	23.4	1.0	5	0.00	23.2	1.0	5	0.00	21.8	0.1	5	0.00	1.0
	U	0.2	5	0.00	1.0	1.0	5	0.00	38.9	1.0	5	0.00	39.6	0.9	5	0.00	34.3	0.1	5	0.00	1.0
$\Gamma^a_i = 10$	C	0.3	5	0.00	2.4	128.3	5	0.00	95.7	42.2	5	0.00	93.1	81.9	5	0.00	86.7	3,600.0	0	0.21	3.1
	K	0.1	5	0.00	1.0	1.2	5	0.00	18.6	1.1	5	0.00	18.9	1.1	5	0.00	17.2	0.0	5	0.00	1.0
	U	0.1	5	0.00	1.0	1.3	5	0.00	30.1	1.3	5	0.00	31.3	1.2	5	0.00	29.6	0.1	5	0.00	1.0
$\Gamma^a_i = 20$	C	0.3	5	0.00	2.9	11.8	5	0.00	73.2	12.3	5	0.00	69.4	74.0	5	0.00	64.4	3,440.0	1	0.14	3.0
	K	0.1	5	0.00	1.0	1.0	5	0.00	33.5	0.9	5	0.00	35.2	1.1	5	0.00	30.6	0.0	5	0.00	1.0
Average		0.1	5.0	0.00	1.5	28.4	5.0	0.00	42.4	12.1	5.0	0.00	41.9	16.3	4.9	0.02	39.7	1,068.5	3.7	0.05	1.6
$m = 3$																					
	U	0.9	5	0.00	1.9	3,600.0	0	2.48	31.5	3,600.0	0	2.50	31.5	3,600.0	0	2.48	31.5	6.4	5	0.00	1.9
$\Gamma^a_i = 0$	C	1.5	5	0.00	3.3	3,600.0	0	4.22	35.4	3,600.0	0	4.22	35.4	3,600.0	0	4.22	35.4	3,600.0	0	0.49	3.3
	K	0.6	5	0.00	1.7	3,160.0	1	1.30	31.5	3,160.0	1	1.30	31.5	3,160.0	1	1.30	31.5	1.7	5	0.00	1.7
	U	1.3	5	0.00	1.7	3,600.0	0	2.27	47.6	3,600.0	0	2.24	46.8	3,600.0	0	2.22	44.4	1.1	5	0.00	1.7
$\Gamma^a_i = 2$	C	5.0	5	0.00	3.5	3,600.0	0	4.76	66.2	3,600.0	0	4.84	65.8	3,600.0	0	4.94	64.9	3,600.0	0	0.54	4.1
	K	2.3	5	0.00	1.6	3,600.0	0	1.96	33.0	3,600.0	0	1.90	32.1	3,600.0	0	1.89	31.8	1.8	5	0.00	1.6
	U	4.8	5	0.00	1.8	3,600.0	0	3.54	48.3	3,600.0	0	3.60	48.0	3,600.0	0	3.14	44.5	8.9	5	0.00	1.9
$\Gamma^a_i = 4$	C	5.1	5	0.00	2.7	3,600.0	0	4.62	60.3	3,600.0	0	4.40	59.2	3,600.0	0	4.40	57.9	3,600.0	0	0.46	3.5
	K	7.8	5	0.00	2.0	3,600.0	0	3.68	32.2	3,600.0	0	3.44	30.6	3,600.0	0	3.24	28.9	26.3	5	0.00	1.9
	U	2.4	5	0.00	1.6	3,600.0	0	3.38	52.9	3,600.0	0	3.18	54.0	3,600.0	0	2.90	49.9	2.1	5	0.00	1.8
$\Gamma^a_i = 10$	C	407.9	5	0.00	2.9	3,600.0	0	5.02	85.1	3,600.0	0	4.80	83.0	3,600.0	0	4.84	79.7	3,600.0	0	0.46	3.4
	K	0.8	5	0.00	1.4	3,600.0	0	1.43	39.8	3,600.0	0	1.48	41.1	3,600.0	0	1.56	40.3	0.6	5	0.00	1.6
	U	3.3	5	0.00	1.7	3,600.0	0	4.26	53.1	3,600.0	0	4.40	55.7	3,600.0	0	4.00	50.5	3.3	5	0.00	1.9
$\Gamma^a_i = 20$	C	1,273.2	5	0.00	2.8	3,600.0	0	5.10	61.3	3,600.0	0	5.10	60.4	3,600.0	0	4.86	54.4	3,600.0	0	0.47	3.5
	K	2.2	5	0.00	1.8	3,600.0	0	2.90	28.0	3,600.0	0	2.94	28.9	3,600.0	0	2.62	26.5	20.0	5	0.00	1.8
Average		114.6	5.0	0.00	2.2	3,570.7	0.1	3.39	47.1	3,570.7	0.1	3.36	46.9	3,570.7	0.1	3.24	44.8	1,204.8	3.3	0.16	2.4
$m = 5$		T																			
	U	24.0	5	0.00	2.1	3,600.0	0	2.98	41.5	3,600.0	0	2.98	41.6	3,600.0	0	2.98	41.6	262.8	5	0.00	2.1
$\Gamma^a_i = 0$	C	393.8	5	0.00	4.3	3,600.0	0	5.92	70.6	3,600.0	0	5.94	70.6	3,600.0	0	5.92	70.7	3,600.0	0	0.67	4.3
	K	21.8	5	0.00	2.0	3,600.0	0	1.21	23.7	3,600.0	0	1.21	23.7	3,600.0	0	1.21	23.7	197.8	5	0.00	2.0
	U	206.4	5	0.00	2.1	3,600.0	0	4.96	52.8	3,600.0	0	4.62	49.9	3,600.0	0	4.38	47.6	159.3	5	0.00	2.1
$\Gamma^a_i = 2$	C	1,299.6	5	0.00	3.4	3,600.0	0	5.64	53.3	3,600.0	0	5.80	53.3	3,600.0	0	5.66	52.6	3,600.0	0	0.68	3.9
	K	167.2	5	0.00	2.2	3,600.0	0	2.12	25.1	3,600.0	0	2.14	24.5	3,600.0	0	2.04	23.5	773.6	4	0.01	2.2

Table B.2. (Continued)

$n = 100$	Constraint type	$\text{MILP}_{2'}^{\log}[\mathcal{U}^{ab}]$				$\text{MILP}_2[\mathcal{U}^{\overline{ab}}]$				$\text{MILP}_2[\mathcal{U}_{=}^{\overline{ab}}]$				$\text{MILP}_2[\mathcal{U}_{\propto}^{\overline{ab}}]$				$\text{MILP}[\mathcal{U}^{\bar{a}}]$			
		T	#	G	R	T	#	G	R	T	#	G	R	T	#	G	R	T	#	G	R
$\Gamma_i^a = 4$	U	480.0	5	0.00	2.2	3,600.0	0	5.40	67.2	3,600.0	0	5.66	67.7	3,600.0	0	4.84	58.4	10.3	5	0.00	2.1
	C	431.4	5	0.00	3.3	3,600.0	0	6.66	78.6	3,600.0	0	5.90	77.8	3,600.0	0	6.60	76.0	3,600.0	0	0.68	4.1
	K	378.2	5	0.00	2.1	3,600.0	0	1.98	33.3	3,600.0	0	2.06	33.4	3,600.0	0	1.94	31.8	786.6	4	0.01	2.2
$\Gamma_i^a = 10$	U	1,524.8	3	0.02	2.1	3,600.0	0	5.72	67.4	3,600.0	0	5.52	69.3	3,600.0	0	5.16	62.0	779.0	4	0.03	2.3
	C	3,600.0	4	0.01	3.1	3,600.0	0	6.16	68.4	3,600.0	0	6.14	68.0	3,600.0	0	6.66	64.6	3,600.0	0	0.65	4.0
	K	964.0	4	0.01	1.7	3,600.0	0	1.56	27.5	3,600.0	0	1.66	28.8	3,600.0	0	1.58	26.7	723.7	4	0.01	1.9
$\Gamma_i^a = 20$	U	182.0	5	0.00	1.9	3,600.0	0	4.70	53.4	3,600.0	0	4.86	56.2	3,600.0	0	4.36	50.5	180.9	5	0.00	2.2
	C	3,600.0	1	0.01	3.5	3,600.0	0	6.88	61.7	3,600.0	0	6.36	61.3	3,600.0	0	6.38	55.6	3,600.0	0	0.67	4.0
	K	238.2	5	0.00	2.0	3,600.0	0	2.20	23.3	3,600.0	0	2.24	25.1	3,600.0	0	1.86	21.3	969.5	5	0.00	2.2
Average		900.8	4.5	0.00	2.5	3,600.0	0.0	4.27	49.9	3,600.0	0.0	4.21	50.1	3,600.0	0.0	4.10	47.1	1,522.9	3.1	0.23	2.8

Notes. Average time (T) in seconds with the number (#) of instances solved within default optimality gap 0.01 and the average remaining optimality gap (G) along with the average relaxation quality (R) across instances for $n = 100$. We have, for $\mathcal{U}^{\overline{ab}}$, that $\Gamma_i = 2\,\Gamma_i^a$; for $\mathcal{U}_{=}^{\overline{ab}}$ and $\mathcal{U}_{\propto}^{\overline{ab}}$, that $\Gamma_i = \Gamma_i^a$; and for $\mathcal{U}^{\bar{a}}$, that $\Gamma = m\,\Gamma_i^a$.

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