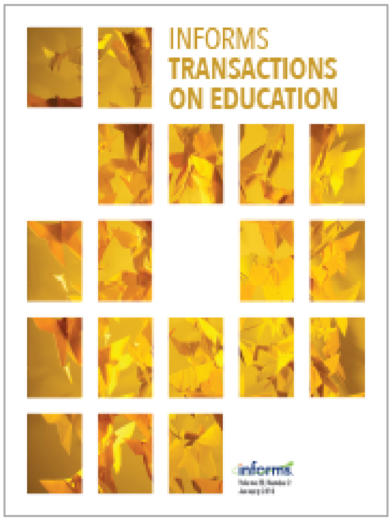

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Puzzle

Verbal Arithmetic and Mastermind

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Verbal arithmetic (also known as cryptarithmetic) problems and a numerical version of the game Mastermind are both solvable as integer programs. This article describes these puzzles and develops an integer programming model and an Excel based implementation for each of them.

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Introduction

As noted in Chlond and Toase (2002), games and puzzles are an excellent way to introduce integer programming and "making learning fun." Several puzzles such as logic grid problems and Sudoku have been written about in this journal (see, for example, Chlond 2014 or Weiss and Rasmussen 2007). Also most students in business schools are familiar with Microsoft Excel. In view of these, here we present a few additional puzzles with their solutions implemented using Excel Solver.

Verbal Arithmetic

In this puzzle words are used to represent an arithmetical result, with each letter standing for a unique digit (Dudeney 1924). A common example is

		S	E	N	D
+		M	O	R	E
=	M	O	N	E	Y

Decision Variables. We use 0-1 variables which assign a digit (0 to 9) to each of the letters that appear in the puzzle. In the given example, the letters are S, E, N, D, M, O, R, and Y (eight letters). Hence we need 0-1 variables X_{ij}, $i = 1$ to 8 (for the letters), $j = 0$ to 9 (for the digits). We also need additional 0-1 variables that represent the "carry over" digit in each addition of the positional (units, tenths, ...) digits in the given sum. Let these be C_k, $k = 1$ to 4.

Constraints. Each letter corresponds to one digit:

$$\sum_{j=0}^{9} X_{ij} = 1, \quad \forall\, i$$

Each digit is to be assigned to only one letter (or none):

$$\sum_{i=1}^{8} X_{ij} \leq 1, \quad \forall\, j$$

Once the assignments are made, we can obtain the digit for each letter. For example, the digit that corresponds to the letter S ($i = 1$) would be $\sum_{j=0}^{9} jX_{1j}$.

Now the sum D + E = Y or Y + 10 can be written as (note that $i = 4$ for D, $i = 2$ for E, and $i = 8$ for Y)

$$\sum_{j=0}^{9} jX_{4j} + \sum_{j=0}^{9} jX_{2j} = \sum_{j=0}^{9} jX_{8j} + 10C_1$$

The next sum would be $C_1 + N + R = E + 10\, C_2$, and so on.

Objective. Though this is more about finding a feasible solution, we could maximize the value of M (in our setup, this is $\sum_{j=0}^{9} jX_{5j}$) so that it would be 1 rather than 0.

The Excel implementation is in the file "SendMore Money.xlsx" (available as supplemental material at http://dx.doi.org/10.1287/ited.2016.0163). The solution is

		9	5	6	7
+		1	0	8	5
=	1	0	6	5	2

Mastermind

There are many versions of this game[1]. In one version (see "Ask Dr. Math," 2007, for a closely similar version), a person selects a five digit number (where all digits are different), and an opponent has to figure out this number through a series of guesses and their feedback. Each guess should also be a five-digit number (where all digits are different), and the feedback would be (1) the number of right digits in the right position and (2) the number of right digits regardless of position. So for example, if the original selection is 47135, and a guess was 12345, the feedback would be that one digit is in the right position, and four digits are right regardless of position.

After each guess and its feedback (and incorporating all previous guesses and their feedback), we run an integer program (IP) to obtain a feasible solution. One could start with a generic guess of 12345, and then use the result from the IP solution as the next guess. The IP implementation in Excel is given in the file "MasterMind.xlsm" (which has a macro to initialize the spreadsheet at the start of a new problem; if the macro is disabled, it would be sufficient to reopen the spreadsheet for each new problem).

Decision Variables. We use 0-1 variables X_{ij} ($i = 1$ to 5 for each position, and $j = 0$ to 9 for each digit) for assigning each position to a digit.

Constraints. We start with the assignment of digits to each position, so

$$\sum_{j=0}^{9} X_{ij} = 1 \quad \forall\, i$$

and

$$\sum_{i=1}^{5} X_{ij} \le 1 \quad \forall\, j$$

Let G_{ik} be the guess at the kth turn ($i = 1$ to 5 for each position, $k = 1, 2, \ldots,$ for successive guesses). For this guess, let the number of right digits in the right position be P_k, and the number of right digits regardless of position be D_k (clearly, $P_k \le D_k$). It helps to transform G_{ik} (whose values are 0 to 9) into G_{ijk} (whose values are 0 or 1) according to this rule: if $G_{ik} = j$, then $G_{ijk} = 1$; if $G_{ik} \ne j$, then $G_{ijk} = 0$. For example, if the second guess is 36290, then $G_{12} = 3$, $G_{22} = 6$, $G_{32} = 2$, $G_{42} = 9$, and $G_{52} = 0$. In terms of G_{ijk}, we have

	j									
G_{ij2}	0	1	2	3	4	5	6	7	8	9
i										
1	0	0	0	1	0	0	0	0	0	0
2	0	0	0	0	0	0	1	0	0	0
3	0	0	1	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0	0	1
5	1	0	0	0	0	0	0	0	0	0

Now we need

$$\sum_{i=1}^{5}\sum_{j=0}^{9} X_{ij} G_{ijk} = P_k$$

$\forall\, k$ (for the number of right digits in the right position)

and

$$\sum_{j=0}^{9}\left(\sum_{i=1}^{5} X_{ij} \sum_{i=1}^{5} G_{ijk}\right) = D_k$$

$\forall\, k$ (for the number of right digits regardless of position)

The Excel file "MasterMindFindingPandD.xls" is helpful in explaining the logic of the constraints given above.

Objective. Again, we only have a feasibility problem, and any formula containing the variables can be set as the objective.

The Excel formulation for solving this is given in the file "MasterMind.xlsm"—it has provision for up to 15 guesses, though typically five to six is sufficient.

Mastermind (Variation)

We could drop the restriction that all digits be distinct. Now the constraint regarding D_k is more difficult. Again, the Excel file "MasterMindFinding PandD.xls" is helpful in explaining the logic of what follows. For each k, we introduce variables D_{jk} ($j =$ 0 to 9) and conditions $D_{jk} \le \sum_{i=1}^{5} G_{ijk}$ and $D_{jk} \le \sum_{i=1}^{5} X_{ij}$ for every j. Finally we maximize $\sum_k \sum_{j=0}^{9} D_{jk}$ (the underlying logic is that the number of right digits regardless of position for the given guess ($[G_{ijk}]$) and the solution under consideration ($[X_{ij}]$) is $\sum_{j=0}^{9} \min\{\sum_{i=1}^{5} G_{ijk}, \sum_{i=1}^{5} X_{ij}\}$; but the function min{} introduces a nonlinear constraint, and we get around that by finding a value for D_{jk} which is less than or equal to each of the function arguments, and then finding the largest such value). Each guess adds many constraints, and with eight guesses we

[1] For history and additional references, see *Mathworld*, s.v. "Mastermind," by Eric Weisstein, retrieved January 11, 2016, http://mathworld.wolfram.com/Mastermind.html.

reach the limit for the standard Solver in Excel. The implementation with up to eight guesses is provided in the file "MasterMindVariation.xlsm," though most situations need more than that to reach the solution.

To illustrate the viability of this approach, consider a smaller version of this puzzle, where the selected number (and guesses) are four-digit numbers and each digit is restricted to takes values 0–5 (actually this corresponds to a more standard version of Mastermind which uses four choices from six colors—see Kooi (2005), for this and additional references). This is implemented in the Excel spreadsheet "MasterMind SmallVersion.xlsm." The spreadsheet allows for up to ten guesses, though the solution is reached typically within eight.

The additional file "MasterMindFindingPandD.xls" is included for two reasons: as mentioned earlier, it makes it easier to follow the logic of the integer programming formulation; additionally, it automates the feedback of P and D, preventing errors in it. One could have this file and a solution file (Master Mind.xlsm or MasterMindVariation.xlsm) open side by side, so that the feedback of P and D can be entered into the solution file.

Supplemental Material

Supplemental material to this paper is available at http://dx.doi.org/10.1287/ited.2016.0163.

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