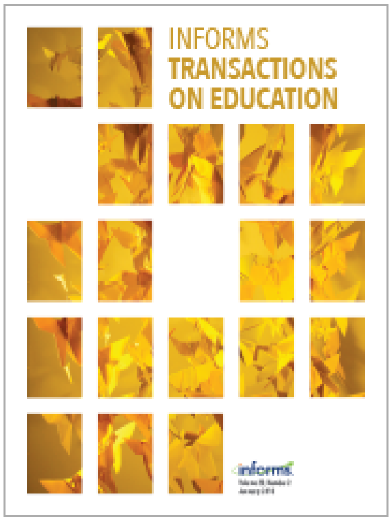

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Abstract

We consider the problem of planning a mix of products with a limited budget constraint. This problem is an extension of the well known product mix problem. The problem is easily formulated as a linear programming problem. In this paper we find the optimal solution by an explicit formula without applying any linear programming solution method. Therefore, very large problems can be easily solved. The problem is illustrated by an example. It can be used to demonstrate the usefulness of the dual problem in linear programming and the utility of using problem structure insight as a means to provide a more meaningful tool for management decision making. An Excel spreadsheet is constructed for illustrating the procedure in class.

Introduction

Consider a common production problem of the selection of a mix of products subject to limited resources. We have a choice of n products x_i for $i = 1, ..., n$ The profit per unit is c_i 0 for $i = 1, ..., n$. We have a set of m resources that are needed for the production. The available amount of each resource is b_j for $j = 1, ..., m$. The amount of each resource that is required for the production of each unit is $a_{ij} \geq 0$ for $i=1, ..., n; j = 1, ..., m$. This leads to the well known product mix problem in introductory linear programming (for example, Lawrence and Pasternack, 2002):

$$\max\left\{\sum_{i=1}^{n} c_i x_i\right\} \tag{1}$$

subject to :

$$\sum_{i=1}^{n} a_{ij} x_i \leq b_j \text{ for } j = 1, \ldots, m$$

$$x_i \geq 0 \text{ for } i = 1, \ldots, n$$

We now modify the problem to model a realistic situation. The availability of resources is not a constraint, but one has a limited budget to be used for purchasing of resources. This is similar to the concept of soft constraints (Byrne et al., 1998). Resource j costs $p_j > 0$ per unit, and there is a budget B for purchases. This is also a linear programming problem. We now have $m + n$ variables x_i for i = 1, ..., n and $b_j = 1, ..., m$, and $m + 1$ constraints.

$$\max\left\{\sum_{i=1}^{n} c_i x_i\right\} \tag{2}$$

subject to :

$$\sum_{i=1}^{n} a_{ij} x_i - b_j \leq 0 \text{ for } j = 1, \ldots, m$$

$$\sum_{j=1}^{m} p_j b_j \leq B$$

$$x_i, b_j \geq 0$$

Note that in the budget constraint we make the standard assumption of no economies of scale.

We show that this problem has a special structure that allows us to solve it without a need for a linear programming procedure. This can be very helpful, especially if m and/or n are very large.

We show in the appendix that the optimal solution to problem (2) is to select the production of only one product (and set the other product quantities to zero). The selected product is the one for which the maximum of the potential profit i is obtained; that is,

$$\theta_i = \frac{c_i}{\sum_{j=1}^{m} a_{ij} p_j} \tag{3}$$

and

$$\mu = \max_i \{\theta_i\}$$

Let k be the i which yields μ in (3), then

$$x_k = \frac{B}{\sum_{j=1}^{m} a_{kj} p_j} \tag{4}$$

$$b_j = a_{kj} x_k \text{ for } j = 1, \ldots, m.$$

and all other $x_i = 0$ for $i \neq k$.

Two proofs for the explicit formula are provided in the appendix. One is based on duality in linear programming and the other is a direct algebraic proof. Such proofs can be used in graduate classes where the students can appreciate them.

Practical Considerations

Even though the optimal solution suggests the production of only one product, managers and decision makers do not like to put all the eggs in one basket and would prefer to give up some of the profit in order to reduce the risk and balance the production process. We suggest to prepare a summary of all potential products (or a partial list of the best ones) sorted by the potential profit i defined in equation (3). The potential profit is divided by so that the best product has a relative profitability of 1.00. This idea is similar to the Data Envelopment Analysis (DEA) concept (Soteriou and Zenios, 1998). A list of the maximum number of units that can be produced for each item is constructed using equation (4). The decision maker can blend some of the top ranked products and get a percentage of the total projected profit as compared with the optimal solution. This concept is illustrated by an example.

An Example

A paint factory can produce various shades of colors, and can sell any quantity it produces. In Table 1 we depict the list of colors that are produced of four possible basic colors: white, red, blue, and yellow. The cost is calculated by adding the cost of each fraction of the basic colors for each paint using the costs given in the last row.

		Parts of				Calculated Cost
Color	Profit/gal	White	Red	Blue	Yellow	
Pink	1.24	0.7	0.3	--	--	0.893
Orange	1.37	--	0.5	--	0.5	1.000
Purple	1.43	--	0.3	0.7	--	1.026
Green	1.09	--	--	0.5	0.5	1.040
Peach	1.35	0.2	0.3	--	0.5	0.978
Lt. Blue	1.29	0.6	--	0.4	--	0.936
Brown	0.87	--	0.4	0.3	0.3	1.012
Lt. Green	1.27	0.6	--	0.2	0.2	0.932
Cost/gal		0.86	0.97	1.05	1.03	

Table 1: Parameters for the Example problem

The available budget is $10,000. How much to produce of each color? First we calculate the values of the potential profit i for each color.

Color	θ_i
Pink	1.3886
Orange	1.3700
Purple	1.3938
Green	1.0481
Peach	1.3804
Lt. Blue	1.3782
Brown	0.8597
Lt. Green	1.3627

Table 2: The Potential Profit of Each Color

From Table 2 we find that =1.3938 and the optimal solution is to produce only purple paint. The total profit is 1.3938*$10,000=$13,938. The relative profitability is obtained by dividing each i by . The sorted colors by relative profitability are depicted in Table 3. In the table we also present the maximum quantity that can be produced for each color calculated by equation (4).

The attached Excel spreadsheet illustrates the procedure. In the first sheet the relative profitability is calculated. In the second sheet the linear programming formulation (2) is set up for Solver to just click and solve. Of course, the same solution is obtained: Producing just 9,747 gallons of the color purple and nothing else. Sensitivity analysis of the results can further illustrate the procedure. For example, changing the profit of a gallon of purple to $1.42 will result in a different color to be produced, a result that is confirmed by the linear programming formulation.

Color	Relative Profitability	Quantity
Purple	1.000	9,747
Pink	0.996	11,198
Peach	0.990	10,225
Lt. Blue	0.989	10,684
Orange	0.983	10,000
Lt. Green	0.978	10,730
Green	0.752	9,615
Brown	0.617	9,881

Table 3: Relative Profitability and Quantity of Colors

Product Mix

The first six colors are comparable in profitability. Consider producing 30% of maximum quantity of purple in Table 3 (i.e. 30% of 9747 = 2924 gallons), 20% of pink, 25% of peach, 15% of Lt. Blue, and 10% of Lt. Green. The relative profitability is: 0.3*1.000 + 0.2*0.996 + 0.25*0.990 + 0.15*0.983 + 0.1*0.978 = 0.99295 which yields a profit of $13,938*0.99295=$13,839. The calculations are detailed in Table 4 and are also included in the attached Excel file. We also verify the value of the profit by multiplying each quantity by the profit per gallon given in Table 1. In conclusion, by lowering the profit by about $100 (0.7%), a variety of colors can be produced.

Color	Percentage	Quantity
Pink	20%	2240
Orange		0
Purple	30%	2924
Green		0
Peach	25%	2556
Lt. Blue	15%	1603
Brown		0
Lt. Green	10%	1073
	Profitability:	0.99295
	Profit:	$ 13,839.33
	Verify:	$ 13,839.33

Table 4: Production Mix

If there were minimum production requirements for some colors, the optimal solution would be to produce the minimum requirements for all products except the most profitable one, and allocate the slack to the most profitable color.

If there were production limitations on some products, the optimal solution is to produce the maximum possible quantity (considering the budget constraint) of the most profitable product, then allocate the maximum possible for the second most profitable, and so on, until the budget is exhausted. For example, if there is a maximum production limit of 4000 gallons for each color, then the production mix should be 4000 gallons of purple paint at a cost of \$4,104, leaving \$5,896 for other paints. The next profitable one is pink. 4000 gallons of pink cost \$3,572, leaving \$2,324 for peach that allows for 2376 gallons of peach, and the budget is exhausted. This result can be verified by adding the constraints that all variables are bounded by 4000 gallons and re-solving the linear programming problem.

Appendix: An Explicit Formula

A Dual Derivation

Let the dual variables for problem (2) be λ_j for j =1, ..., m for the first m constraints and μ for the last constraint. The dual problem to problem (2) is:

$$\min \{B\mu\} \tag{5}$$

Subject to :

$$\sum_{j=1}^{m} a_{ij}\lambda_j \geq c_i \text{ for } i = 1, \ldots, n$$
$$-\lambda_j + p_j\mu \geq 0 \text{ for } j = 1, \ldots, m$$

The j values are the shadow prices for the modified resources constraints obtained when solving formulation (2) above and is the shadow price of the single resource budget constraint again obtained from solving (2) above. If a feasible solution of λ_j's exists for a given , then since a_{ij}, ≥ 0 greater λ_j's are also feasible for the first n constraints of (5). Therefore, in the optimum, we must have $\lambda_j = p_j\mu$.

Substituting this into the first n constraints of (5) yields:

$$\mu\left\{\sum_{j=1}^{m} a_{ij} p_j\right\} \geq c_i \quad \text{for } i = 1, \ldots, n.$$

Therefore, the solution to the dual problem is:

$$\mu = \max_i \{\theta_i\} \text{ where } \theta_i = \frac{c_i}{\sum_{j=1}^{m} a_{ij} p_j} \tag{6}$$
$$\lambda_j = \mu p_j$$

with the optimal cost of B. Note that is a logical optimal solution as it represents the highest profit per resource expense.

Let k be the i for which the maximum of i is obtained (and if there is a tie for the maximum in (6), any of the tied variables is selected). The solution to (2) is:

$$x_k = \frac{B}{\sum_{j=1}^{m} a_{kj} p_j} \tag{7}$$
$$b_j = a_{kj} x_k \quad \text{for } j = 1, \ldots, m.$$

and all other $x_i = 0$ for $i \neq k$.

An Algebraic Derivation [1]

The problem under consideration is as follows:

$$\max_{x_i, b_i} \left\{ \sum_{i=1}^{n} c_i x_i \right\}$$

subject to :

$$\sum_{i=1}^{n} a_{ij} x_i - b_j \le 0 \text{ for } j = 1, \ldots, m$$

$$\sum_{j=1}^{m} p_j b_j \le B$$

$$x_i \ge 0; b_j \ge 0 \text{ for all } i = 1, \ldots, n; j = 1, \ldots, m.$$

There is an optimal solution such that the first m constraints are binding, namely

$$\sum_{i=1}^{n} a_{ij} x_i = b_j \text{ for } j = 1, \ldots, m$$

This is because for any non-binding constraint, the value of b_j can be reduced to its binding value. No constraints are violated, and the value of the objective function is unchanged. Therefore, the problem can be rewritten as follows:

$$\max_{x_i} \left\{ \sum_{i=1}^{n} c_i x_i \right\}$$

subject to :

$$\sum_{j=1}^{m} p_j \sum_{i=1}^{n} a_{ij} x_i \le B$$

$$x_i \ge 0 \text{ for all } i = 1, \ldots, n$$

This, in turn is equivalent to

$$\max_{x_i} \left\{ \sum_{i=1}^{n} c_i x_i \right\} \qquad (8)$$

subject to :

$$\sum_{i=1}^{n} d_i x_i \le B$$

$$x_i \ge 0 \text{ for all } i = 1, \ldots, n$$

where $d_i = \sum_{j=1}^{m} a_{ij} p_j \; i=1,2, \ldots, n$

This is the knapsack problem with continuous variables. According to the Fundamental Theorem of Linear Programming, if (8) has an optimal solution it must also have a basic feasible optimal solution. Since there is only one functional constraint in (8), a basic feasible solution for (8) has only one nonzero component. It follows that the optimal basic feasible solution is such that

$$x_k = \frac{B}{d_k} \text{ for some } k=1,2,\ldots,n$$
$$\text{and } x_i = 0 \text{ for } i \ne k.$$

Therefore, the optimal value of k should then be such that

$$\frac{c_k}{d_k} = \max_i \left\{ \frac{c_i}{d_i} \right\}$$

[1] We are thankful to an anonymous referee for suggesting this proof

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