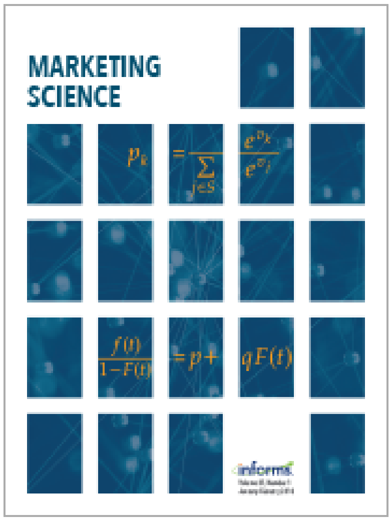

Marketing Science

Technical Note—Aggregating Individual Purchases to the Household Level

Barbara E. Kahn, Donald G. Morrison, Gordon P. Wright,

MARKETING SCIENCE
Vol. 5, No. 3, Summer 1986
Printed in U.S.A.

TECHNICAL Note

AGGREGATING INDIVIDUAL PURCHASES TO THE HOUSEHOLD LEVEL

BARBARA E. KAHN, DONALD G. MORRISON AND GORDON P. WRIGHT
University of California at Los Angeles
Columbia University
Purdue University

Household level panel data are the input for many types of marketing studies. An interesting, but until now unaddressed, question is what is the effect of aggregating individual members' purchases to the household level. Under "standard" assumptions the answer is unambiguous: the household brand switching pattern looks more zero order than the typical individual family member's behavior. When the standard assumptions are relaxed the overall spirit of the results do not change. These conclusions give comfort to those who use brand switching data to partition product categories. Those looking for variety-seeking behavior from household data are given some cause for concern—as well as reasons for reinterpreting previous studies.
(Superposition; Panel Data; Markov Transition Matrices; Zero-Order Behavior; Variety-Seeking Behavior)

Introduction

Aggregating Households to Populations

Panel data in marketing are almost always collected for households (e.g., Lipstein 1959; Massy, Montgomery and Morrison 1970; Jeuland 1979; Givon 1984). These data are sometimes analyzed at the individual household level, but more often they are aggregated across households. This raises an interesting question:

Q1: What can we say about household level behavior from aggregated brand switching data?

This question has a 25-year history in marketing. The Frank (1962) article favoring zero order household brand switching was challenged by Kuehn (1962) who presented data showing higher order brand switching. Morrison's dissertation (Chapters 3 and 4 of Massy, Montgomery and Morrison 1970) showed how to disentangle the spurious effects of heterogeneous preferences across households from the true effect of recent purchases

0732-2399/86/0503/0260$01.25

on current purchases. In a recent article Kahn, Kalwani and Morrison (1986) use these techniques to assess the variety seeking and reinforcement behavior of consumers. Thus we know how to answer Q1. Qualitatively we can say that

A1: Aggregate switching data make the brand choice process look higher order reinforcement than it really is at the household level.

Brand Switching and Partitions

Another use of aggregate switching data is to get "Hendry type" partitions of the market. (See Kalwani and Morrison 1977.) The spirit of these partitioning techniques is to form subsets of the brands within the total product category such that within subsets: switching is proportional to share. This is not the place to comment on the relative merits of this type of market structure analysis. However, we should stress that a primary building block of these brand switching approaches is that *at the household level:*

brand switching is a zero order process.

Is this a good assumption? Bass *et al.* (1984) emphatically say "Yes." Givon (1984) gives a much more qualified "Yes." In any event, we know a lot about aggregate household level switching data.

Aggregating Individuals to Households

Surprisingly, there has been no work on the more fundamental aggregation. Namely,

Q2: What happens when individual family members' purchasing needs are aggregated to the household level?

The answer to this question has two parts.

A2 (Part I): Under "standard" assumptions aggregating individuals to households *always* makes the household behavior look more zero order than the *typical* individual in the household.

A2 (Part II): Under reasonable deviations from the "standard" assumptions, the spirit of the Part I answer still holds.

The remainder of this short note will present the theorems for A2 (Part I) and the numerical analysis that justify A2 (Part II). However, before presenting the probability models, some additional motivation and caveats are needed.

Why Is the Answer to Q2 Relevant?

First, almost all of the recent marketing literature on variety seeking and reinforcement behavior uses household level data, yet makes individual consumer level inferences (Givon 1984; Lattin and McAllister 1985). Thus most of these researchers are loading the dice against themselves. The potentially interesting (i.e., nonzero order) individual consumer level behavior that they are seeking to find is getting masked by the household level data they are using.

Second, regardless of the overall merits of the brand partitions generated from population level switching data, the basic assumption of zero order behavior at the household level is a good assumption. The masking (to zero order) effect that hurts the variety seeking researcher helps the Hendry type partitioner who requires zero order households.

What We Are Not Doing

The key assumption in our analysis is that individuals within a household have independent, identically distributed interpurchase times. Is this realistic for most households? No. Is it a good assumption? Yes. We are assessing the effect of aggregation *per se.* If one member purchases 10 times more frequently than the other member then the household level behavior is almost identical to the heavy consuming individual's behavior. By assuming equal purchasing rates across members of the household we isolate the pure effect of aggregation.

The elegant theorems that we derive assume exponentially distributed interpurchase times. Clearly this is a good starting point since so much of the existing literature makes the same assumption. Therefore, at the very least it is interesting to see the aggregation effect under the "standard" assumption. More importantly, the exponential based result gives us a benchmark. Namely, do intuitively appealing deviations from the exponential retain the spirit of the exponentially based results?

Finally, our numerical results assume that all possible household configurations are equally likely. Again, is this a behaviorally justified assumption? No. Is it an appropriate assumption? Yes—and in the same spirit of simulations that are used to analyze heuristic optimization algorithms. An "equally likely" simulation may show that an algorithm "works" 99 percent of the time. For real world problems we cannot say that the algorithm gets the optimal answer 99 percent of the time—but we can say that for real problems it will work the vast majority of the time. In a similar spirit we can say that our exponential based results are valid "most of the time" even when the exponential assumption is violated.

Thus, our assumptions and simulations are chosen with care. They are not meant to capture the real world precisely. They are made so that we can answer a very real world question on the effect of aggregating individual level consumer purchases to the household level.

Modelling Aggregation: Theorems

Specific Example

Consider the following 2-person household. Each member purchases Brand A or Brand B according to the same first order Markov process with transition matrix:

$$\text{past purchase} \quad \begin{array}{c} \\ A \\ B \end{array} \overset{\text{current purchase}}{\begin{array}{cc} A & B \\ \end{array}} \begin{bmatrix} 0.8 & 0.2 \\ 0.2 & 0.8 \end{bmatrix}$$

where Brand A is the brand of interest and Brand B is "all other brands." This assumption of a two-brand market follows in the tradition of many first-order models, e.g., Lipstein 1959; Massy, Montgomery and Morrison 1970; Jeuland 1979.

Assume that each member's interpurchase times are exponentially distributed with identical means. However, we only observe the superposition of their purchases. That is, we only know when someone made a purchase, but we do not know which one. Given this aggregation (superposition) to the household level, what household transition matrix will we observe?

Let's say that the last household purchase is an "A." Given the lack of memory property of the exponential, the person who made that purchase is no more or less likely to make the next purchase than is the other family member. If the same person makes the next purchase, then the probability is 0.8 that Brand A will be bought. If the other person makes the next purchase, the probability of Brand A is only 0.5 (the second person's steady state, or long run, probability). Thus the household level transition matrix has

$$P(A_{t+1}|A_t) = \tfrac{1}{2}(0.8) + \tfrac{1}{2}(0.5) = 0.65.$$

By similar logic, the complete household transition matrix is

$$\text{past purchase}\quad \begin{matrix} & \text{current purchase} \\ & \begin{matrix} A & \quad B \end{matrix} \\ \begin{matrix} A \\ B \end{matrix} & \begin{bmatrix} 0.65 & 0.35 \\ 0.35 & 0.65 \end{bmatrix} \end{matrix}$$

Note that this household level matrix is "more zero order" than each individual. Specifically, two summary measures are:

D = Deviation (from zero order)

Individual: $|P(A|A) - P(A|B)| = |0.8 - 0.2| = 0.6$,

Household: $|P(A|A) - P(A|B) = |0.65 - 0.35| = 0.3$,

$$RD = \text{Relative Deviation} = \frac{\text{Household Deviation}}{\text{Ave. Individual Deviation}} = \frac{0.3}{\frac{1}{2}(0.6) + \frac{1}{2}(0.6)} = 0.5.$$

That is, the relative deviation from zero order is only half as large for the household as for each member of the household. This is *not* an isolated example. In fact, when the individuals in the household have different first order brand switching matrices with independent, identically distributed exponential interpurchase times, two very powerful general theorems emerge.

General Result

Let family member i have a first order brand switching matrix

$$\begin{matrix} & A & B \\ A & p_i & 1 - p_i \\ B & q_i & 1 - q_i \end{matrix} \quad \begin{bmatrix} p_i & 1-p_i \\ q_i & 1-q_i \end{bmatrix}, \qquad i = 1, 2, \ldots, N.$$

Let each of the N family members purchase with independent, identically distributed exponential interpurchase times.

The N member household level switching matrix is

$$\begin{matrix} & A & B \\ A & V(N) & 1 - V(N) \\ B & W(N) & 1 - W(N) \end{matrix} \quad \begin{bmatrix} V(N) & 1 - V(N) \\ W(N) & 1 - W(N) \end{bmatrix}.$$

Define

$$D_N = |V(N) - W(N)|,$$

$$RD_N = \frac{|V(N) - W(N)|}{\frac{1}{N} \sum_{i=1}^{n} | p_i - q_i|}.$$

Given the above assumptions and definitions, the following two properties hold:

THEOREM 1.
Property 1: $D_N \leq 1/N$.

THEOREM 2.
Property 2: $RD_N \leq 1$
for all possible family size N and all possible individual transition matrices defined by $[p_i, q_i]$, $i = 1, 2, \ldots, N$.

Remarks

Property 1 shows that the household level transition matrix cannot have more than a $1/N$ deviation from zero order. That is, a two person household can have at most a 0.5 difference between the two elements of the first column of the household transition matrix no matter how nonzero-order each member is. A three member household transition matrix has $|V(3) - W(3)| \leq \frac{1}{3}$, etc.

Property 2 says that no household level matrix can be less zero-order than the *average* zero-orderness of the individual family members.

PROOF. Each of these properties is proved in the Appendix.

Modelling Aggregation: Numerical Results

Exponential Assumption Retained

At this point the reader should have a good feel for the worst case scenario. That is, given our i.i.d. exponential interpurchase time assumption, the household level deviation from zero order is *always* less than $1/N$ and the relative deviation is *always* less than 1. But what are more realistic "practical" bounds? We answer this by running 10,000 cases each for family sizes of $N = 2$, 3 and 4. That is, for a family size $N = 2$ we drew 10,000 vectors (p_1, q_1, p_2, q_2) from a uniform distribution over the four dimensional hypercube. When this was done we found that 99 percent of the D_2's were less than 0.33 (the theoretical upper bound is 0.50) and 99 percent of the RD_2's were less than 0.67 (the theoretical upper bound being 1.0). Letting $D(0.99)$ be the value at which 99 percent of the D's were smaller and letting $D(\text{Max})$ be the theoretical upper bound (with analogous definitions for $RD(0.99)$ and $RD(\text{Max})$) we obtained

N	$D(0.99)$	$D(\text{Max})$	$RD(0.99)$	$RD(\text{Max})$
2	0.33	0.50	0.67	1.0
3	0.24	0.33	0.42	1.0
4	0.14	0.25	0.30	1.0

We see that a "practical" bound on D is about $\frac{2}{3}$ of the theoretical bound for typical family sizes of 2, 3 and 4. The theoretical bound on RD is independent of family size but the "practical" bound drops rapidly with increasing family size.

Relaxing the Exponential Assumption

When the interpurchase times are no longer exponentially distributed, then there are no theoretical upper bounds for the constructs D and RD. A simple example will illustrate the problem. Let one family member be a zero order purchaser with a very high probability of purchasing Brand A, while the second family member is also zero order with a high probability of purchasing Brand B. But now each purchases in a deterministic manner with the first member purchasing in weeks 1, 3, 5. . . and the second in weeks 2, 4, 6. . . . Clearly the observed household pattern of purchases will typically be $A\ B\ A\ B\ A\ B$. . . . Thus the observable household switching matrix will have a D value close to 1 while each individual has a $|p - q|$ value of 0.

The above example is contrived, but it does raise an important issue. Namely, how sensitive are our Property 1 and Property 2 conclusions to deviations from the exponential?

To answer this question we allowed the interpurchase times to be independent, identically distributed Erlang (r) random variables. Several researchers have proposed that the Erlang r distribution is reasonable. Chatfield and Goodhart (1975), Lawrence (1980) and Herniter (1980) have found Erlang 2 distributions to fit well on empirical data. When $r = 1$ we have the exponential back again and as r increases, the interpurchase times become more regular. Numerical analyses analogous to the ones discussed above were run. The result of interest relates to *RD*: how often was $RD > 1$ implying that the household matrix was *less* zero order than the average member? (Remember under the i.i.d. exponential scenario the household level matrix is *always more* zero order, i.e., *RD* is always <1.) The median value of *RD* also gives insights into what usually happens over our simulated population of households. The results are:

N	r	*RD*(Median)	Percent $RD > 1$
	1	0.38	0
2	2	0.32	3
	3	0.30	5
	∞	0.22	17

These numerical results confirm that our exponential based theorems for *RD* no longer hold. *RD* can indeed be greater than one. On the other hand, the *RD* (median) results show that typically the spirit of Property 2 holds even *better* when the purchase times are *less* exponential (i.e., more regular).

Discussion

In our view, important efforts in *quantitative* modelling yield *qualitative* insights. In this paper we have, to the best of our knowledge, been the first researchers to investigate the effects of aggregating individual consumer purchases to the household level. Our quantitative model yields the following conclusion:

> Under standard assumptions, aggregating to the household level always makes the household look more zero order than the typical family member (Property 2). The household level matrix gets more zero order as the family size increases (Property 1). Deviations from the standard assumptions destroy the theoretical upper bounds on *D* and *RD*, but the spirit of Properties 1 and 2 remain.

These results have implications for three different sets of researchers.

Probabilists. Those interested in the superposition of renewal processes will gain insights from the proofs of our two properties.

Variety Seekers. Those using household level panel data to detect variety seeking behavior can be comforted. When they find some variety seeking at the household level, chances are there is even more going on at the individual consumer level.

Market Partitioners. Brand switching approaches to obtain market structures have their advocates and critics. However, the key assumption of zero order household level brand switching is given added support by our results.

But what does all this mean to the practitioner who looks at switching matrices? First of all, we need to reinforce the Question 1 answer. The aggregated across households population switching matrix has little, if any, relation to the household level switching matrices. That is, $P(A|A)$ can be high because of heterogeneity of preferences across consumers and/or because of strong reinforcement at the household level. That's the bad news. There is compensating good news.

If—and it is a big if—the practitioner wishes to partition a product category on the basis of observable buying (i.e., switching) data, then the population level switching data are appropriate input to the various "switching proportional to share" methodologies. This is because the crucial zero order household level assumption is almost always a good one.

The population level switching matrix also has another useful—and underutilized—property. The current market shares are a point in time snapshot. But what is the momentum in the market? The steady state (or long-run) shares associated with the switching matrix should tell which way the current shares are being pushed by the current switching behavior. The issue is how good is the steady state which is calculated on the obviously wrong assumption of a homogeneous first order population of consumers. Somewhat surprisingly the answer is that the erroneously calculated solution is very close to the correct answer. The details of this result can be found in Morrison, Massy and Silverman (1971). Thus the population level switching matrix is a good indicator of the momentum in the market.

Finally, there is the managerially relevant issue of whether Brand B is a direct competitor with Brand A or more of a variety seeking alternative. The Kahn, Kalwani and Morrison (1986) approach is a simple, statistically powerful method for answering this question. But any KKM type analysis typically has household level switching data as input. The results in this note show that if variety seeking is detected at the household level then you can be virtually certain that even stronger variety seeking is occurring at the individual level.

In summary, we have added one more item to be considered when a practitioner analyzes switching data. The careful analyst already thinks about the effects of product category definition, population heterogeneity, nonstationarity, sample size, etc. For some uses of switching data the decision maker should now also be concerned about adjusting household level statistics when making individual level conclusions.[1]

[1] This paper was received March 1986 and has been with the authors for 1 revision.

Appendix. *N*-Person Household

Notation

Let N = the number of persons in the household. Let:

$$\text{Last Choice}\ \begin{matrix} \\ A \\ B \end{matrix} \overset{\text{Current Choice}}{\begin{pmatrix} A & B \\ p_i & 1-p_i \\ q_i & 1-q_i \end{pmatrix}} \tag{1}$$

be the switching matrix between brands A and B for Person $i = (1, 2, \cdots)$. Let p_i be the conditional probability that Person i chooses Brand A on his/her choice occasion $t + 1$ given that Brand A was chosen on his/her last choice occasion t. Similarly, q_i is the conditional probability that Person i chooses Brand A on his/her choice occasion $t + 1$ given that Brand B was chosen on his/her last choice occasion t.

Let $[\Pi_{Ai}, \Pi_{Bi}]$ be the steady state probabilities for Person i $(=1, 2, \cdots)$. From (1) we get:

$$\Pi_{Ai} = \frac{q_i}{1 - p_i + q_i} \quad \text{and} \quad \Pi_{Bi} = \frac{1 - p_i}{1 - p_i + q_i}. \tag{2}$$

Household Model (Switching Matrix)

$$\text{Last Purchase}\ \begin{matrix} \\ A \\ B \end{matrix} \overset{\text{Current Purchase}}{\begin{pmatrix} A & B \\ \nu_{(N)} & 1-\nu_{(N)} \\ w_{(N)} & 1-w_{(N)} \end{pmatrix}}, \tag{3}$$

where $\nu_{(N)}$ is the conditional probability that the household purchases Brand A on purchase occasion $t + 1$ given that the household purchases Brand A at t; and $\omega_{(N)}$ is the conditional probability that the household purchases Brand A at $t + 1$ given that the household purchased Brand B at t.

By the use of a similar argument to the one used to derive the probabilities for the two-person household example, given in the section on Modeling Aggregation, we get the following expressions for $\nu_{(N)}$ and $\omega_{(N)}$:

$$\nu_{(N)} = \frac{\frac{1}{N}(\sum_{i=1}^{N} \sum_{\substack{j=1 \\ j\neq i}}^{N} \Pi_{Aj}\Pi_{Ai} + \sum_{i=1}^{N} p_i\Pi_{Ai})}{\sum_{i=1}^{N} \Pi_{Ai}}, \tag{4}$$

$$w_{(N)} = \frac{\frac{1}{N}(\sum_{i=1}^{N} \sum_{\substack{j=1 \\ j\neq i}}^{N} \Pi_{Aj}\Pi_{Bi} + \sum_{i=1}^{N} q_i\Pi_{Bi})}{\sum_{i=1}^{N} \Pi_{Bi}}. \tag{5}$$

Computational Formula

By combining and rearranging terms in (3) and (4) and by using the identity $p_i\Pi_{Ai} - \Pi_{Ai}^2 = (p_i - q_i)\Pi_{Ai}\Pi_{Bi}$ ($i = 1, 2, \ldots, N$), we have the following formula for $\nu_{(N)} - w_{(N)}$:

$$\nu_{(N)} - w_{(N)} = \frac{\sum_{i=1}^{N} (p_i - q_i)\Pi_{Ai}\Pi_{Bi}}{(\sum_{i=1}^{N} \Pi_{Ai})(\sum_{i=1}^{N} \Pi_{Bi})}. \tag{6}$$

Formula (6) is useful for computing $D_{(N)}$ and $RD_{(N)}$ given in the text.

PROOF OF PROPERTY 1: By using formula (6) we have:

$$D_{(N)} = |\nu_{(N)} - w_{(N)}| = \left| \frac{\sum_{i=1}^{N} (p_i - q_i)\Pi_{Ai}\Pi_{Bi}}{(\sum_{i=1}^{N} \Pi_{Ai})(\sum_{i=1}^{N} \Pi_{Bi})} \right|.$$

It follows from the triangle inequality and the inequalities $|p_i - q_i| \leq 1$ ($i = 1, 2, \ldots, N$) that:

$$\left| \frac{\sum_{i=1}^{N} (p_i - q_i)\Pi_{Ai}\Pi_{Bi}}{(\sum_{i=1}^{N} \Pi_{Ai})(\sum_{i=1}^{N} \Pi_{Bi})} \right| \leq \frac{\sum_{i=1}^{N} |p_i - q_i|\Pi_{Ai}\Pi_{Bi}}{(\sum_{i=1}^{N} \Pi_{Ai})(\sum_{i=1}^{N} \Pi_{Bi})} \leq \frac{\sum_{i=1}^{N} \Pi_{Ai}\Pi_{Bi}}{(\sum_{i=1}^{N} \Pi_{Ai})(\sum_{i=1}^{N} \Pi_{Bi})}. \tag{7}$$

Since $N \sum_{i=1}^{N} \Pi_{A}\Pi_{Bi} \leq (\sum_{i=1}^{N} \Pi_{Ai})(\sum_{i=1}^{N} \Pi_{Bi})$ we obtain the desired result. That is that the term on the right side of (7) is equal to or less than $1/N$.

Note:

$$N \sum_{i=1}^{N} \Pi_{Ai}\Pi_{Bi} \leq (\sum_{i=1}^{N} \Pi_{Ai})(\sum_{i=1}^{N} \Pi_{Bi})$$

since

$$(\sum_{i=1}^{N} \Pi_{Ai})(\sum_{i=1}^{N} \Pi_{Bi}) - N \sum_{i=1}^{N} \Pi_{Ai}\Pi_{Bi} = \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} (\Pi_{Ai} - \Pi_{Aj})^2 \geq 0.$$

PROOF OF PROPERTY 2: By use of formula (6) we have:

$$RD_{(N)} = \frac{|\nu_{(N)} - \omega_{(N)}|}{\left(\frac{1}{N}\right) \sum_{i=1}^{N} |p_i - q_i|} = \frac{(N)|\sum_{i=1}^{N} (p_i - q_i)\Pi_{Ai}\Pi_{Bi}}{(\sum_{i=1}^{N} |p_i - q_i|)(\sum_{i=1}^{N} \Pi_{Ai})(\sum_{i=1}^{N} \Pi_{Bi})}.$$

It follows from the triange inequality that

$$\frac{(N)|\sum_{i=1}^{N} (p_i - q_i) \Pi_{Ai}\Pi_{Bi}|}{(\sum_{i=1}^{N} |p_i - q_i|)(\sum_{i=1}^{N} \Pi_{Ai})(\sum_{i=1}^{N} \Pi_{Bi})} \leq \frac{\sum_{i=1}^{N} |p_i - q_i|(N)\Pi_{Ai}\Pi_{Bi}}{(\sum_{i=1}^{N} |p_i - q_i|)(\sum_{i=1}^{N} \Pi_{Ai})(\sum_{i=1}^{N} \Pi_{Bi})}. \tag{8}$$

Since $(N)\Pi_{Ai}\Pi_{Bi} \leq (\sum_{i=1}^{N} \Pi_{Ai})(\sum_{i=1}^{N} \Pi_{Bi})$ for all i (= 1, 2, . . . N) we obtain the desired result which is that the term on the right side of the equation (8) is equal to or less than 1.

Note:

$$(N)\Pi_{Ai}\Pi_{Bi} \leq (\sum_{i=1}^{N} \Pi_{Ai})(\sum_{i=1}^{N} \Pi_{Bi})$$

for all i (= 1, 2, . . . N), since

$$(\sum_{i=1}^{N} \Pi_{Ai})(\sum_{i=1}^{N} \Pi_{Bi}) - (N)\Pi_{Ai}\Pi Bi = (N) \sum_{\substack{l=1 \\ l\neq i}}^{N} \Pi_{Al}(1 - \Pi_{Al}) + \sum_{k=1}^{N-1} \sum_{j=k+1}^{N} (\Pi_{Ak} - \Pi_{Aj})^2 \geq 0.$$

References

Bass, Frank M., Moshe M. Givon, Manohar U. Kalwani, David Reibstein and Gordon P. Wright (1984), "An Investigation into the Order of the Brand Choice Process," *Marketing Science*, 3 (Fall), 267–287.

Chatfield, C. and G. J. Goodhardt (1973), "A Consumer Purchasing Model with Erlang Inter-Purchase Time," *Journal of the American Statistical Association*, 68 (December), 828–835.

Frank, Ronald E. (1962), "Brand Choice as a Probability Process," *Journal of Business*, 35, 43–56.

Givon, Moshe U. (1984), "Variety Seeking Through Brand Switching," *Marketing Science*, 3 (Winter), 1–22.

Herniter, J. (1971), "A Probabilistic Market Model of Purchase Timing and Brand Selection," *Management Science*, 18, 102–113.

Jeuland, Abel P. (1979), "Brand Choice Inertia as One Aspect of the Notion of Brand Loyalty," *Management Science*, 25 (7), 671–682.

Kahn, Barbara E., Manohar U. Kalwani and Donald G. Morrison, "Measuring Variety Seeking and Reinforcement Behavior Using Panel Data," *Journal of Marketing Research*, 23 (May), 89–100.

Kalwani, Manohar U. and Donald G. Morrison (1977), "A Parsimonious Description of the Hendry System," *Management Science*, 23 (January), 467–477.

Kuehn, A. A. (1962) "Consumer Brand Choice—A Learning Process," *Journal of Advertising Research*, 2 (December), 10–17.

Lattin, James and Leigh McAlister (1985), "Using a Variety-Seeking Model to Identify Substitute and Complementary Relationships Among Competing Products," *Journal of Marketing Research*, 22 (August), 330–9.

Lawrence, Raymond J. (1980), "The Lognormal Distribution of Buying Frequency Rates," *Journal of Marketing Research*, 17 (4), 122–128.

Lipstein, Benjamin (1959), "The Dynamics of Brand Loyalty and Brand Switching," *Proceedings* Fifth Annual Conference of the Advertising Research Foundation, New York.

Massy, William F., David B. Montgomery and Donald G. Morrison (1970), *Stochastic Models of Buyer Behavior*, Cambridge, Mass.: The M.I.T. Press.

Morrison, Donald G., William F. Massy and Fred N. Silverman (1971), "The Effect of Nonhomogeneous Populations as Markov Steady-State Probabilities," *Journal of the American Statistical Association*, 66 (June), 268–274.