Management Science

Publication details, including instructions for authors and subscription information:
http://pubsonline.informs.org

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MANAGEMENT SCIENCE
Vol. 50, No. 12 Supplement, December 2004, pp. 1872–1874
ISSN 0025-1909 | EISSN 1526-5501 | 04 | 5012S | 1872

DOI 10.1287/mnsc.1040.0307

Comments on "The Lagrangian Relaxation Method for Solving Integer Programming Problems"

Marshall L. Fisher
The Wharton School, University of Pennsylvania, Philadelphia, Pennsylvania 19104, fisher@wharton.upenn.edu

Congratulations to Wally Hopp and our society for conceiving this idea. What a wonderful way to take stock of what we have accomplished in the last 50 years and to hopefully distill from those accomplishments lessons for the future.

I was of course pleased that the membership of INFORMS judged my paper "The Lagrangian Relaxation Method for Solving Integer Programming Problems" to have been influential, but I must hasten to add that this is surely an endorsement of the impact of the concept of Lagranigan relaxation as much as of one paper, and that many people contributed to that concept. There isn't space here to name all of them, but I'd like to reiterate a few individuals mentioned at the start of the 1981 article who played especially pivotal roles in the early formation of Lagrangian relaxation: Michael Held and Richard Karp, whose novel algorithm for the traveling salesman problem (Held and Karp 1970) marked the birth of Lagrangian relaxation; Arthur Geoffrion, who coined the term Lagrangian Relaxation in an early exposition of the concept (Geoffrion 1974); and three people with whom I had the pleasure of working—Jeremy Shapiro, on general concepts (Fisher and Shapiro 1974), and Gerard Cornuejols and George Nemhauser, on an early application (Cornuejols, Fisher, and Nemhauser 1977).

In this note, I'll offer comments on events during three time periods: (1) prior to publication of the paper in 1981, (2) 1981 to the present, and (3) the future.

Background on the Paper

After getting my SB from MIT in 1965, I took a job as a systems engineer in the IBM Boston Manufacturing and Distribution sales office, where I was assigned to about a dozen companies in industries ranging from shoes to consumer electronics. This was a time when using computers in industry was still quite new and there was great excitement around finding new applications. One inescapable observation was that creating a schedule that specified what each person and piece of equipment in a plant would do each day had a huge impact on the plant's productivity, was tailor-made for application of a computer, and was a problem that had been barely dented at the time.

When I returned to MIT for graduate study, I encountered the jobshop scheduling problem, which asks for a sequence in which to perform n jobs that require a known amount of processing on m machines in order to minimize the time to complete all tasks on all jobs. I was struck that this simple, elegant model captured the essential features of the manufacturing application to which I had been exposed at IBM, so I decided to focus my dissertation research on the jobshop scheduling problem.

While at MIT I learned about the Dantzig-Wolfe Decomposition (Dantzig and Wolfe 1961) and was intrigued by the concept of mimicking in a mathematical algorithm the way prices matched supply with demand in a market. I conceived that this idea could be exploited in solving the jobshop scheduling problem by relaxing the machine constraints with prices charged for using the machines during each time period. This work was published in Fisher (1973a, b).

My research was "turbo-charged" when Michael Held spoke at MIT in the late 1960s on his work with Richard Karp on the traveling salesman problem. I was able to immediately employ many of the components they had developed in this research, such as the subgradient method for setting Lagrange multipliers.

Seeing another application of Lagrange multipliers to relax complicating constraints in a discrete optimization problem helped me to appreciate the potential generality of this approach and to think about general concepts (Fisher and Shaprio 1974) and an additional application (Cornuejols et al. 1977).

In 1978 I was invited to give some lectures on Lagrangian relaxation at a summer school on combinatorial optimization in Urbino, Italy. This was a wonderful chance to "organize my thinking" on the topic. The paper I wrote for the proceedings of that conference (Fisher 1978) was later extended to create the paper that appeared in *Management Science* in 1981.

What's Happened Since 1981

To gain some insight into the level and type of research on Lagrangian relaxation since 1981, I used the ISI Web of Knowledge to identify all papers that had referenced Fisher (1981). For 1981 and prior I used the references in Fisher (1981), supplemented by JSTOR, to determine the number of papers written about Lagrangian relaxation each year. The results are compiled in Table 1, which shows that Lagrangian relaxation has continued to be an area of active research to the present.

What are the topics of these papers? Mostly, they are applications of Lagrangian relaxation to particular problems. Of course our favorite classes of discrete optimization problems are heavily represented, including facility location and capacity planning, scheduling, vehicle routing and network flow, set covering, generalized assignment, and other variations on the knapsack problem. My own research on Lagrangian relaxation after 1981 focused on applications to vehicle routing, both theoretical (Fisher 1994) and applied (Bell et al. 1983) and an expository piece intended to fuel further applications (Fisher 1985).

I was also intrigued by the large number of applications of Lagrangian relaxation I found beyond these traditional resource allocation problems, including applications to biology (Caprara et al. 2004); electric power generation (Ongsakul and Petcharaks 2004); the chemical and petroleum industries (van den Heever and Grossmann 2004, van den Heever et al. 2001); electronics (Feo and Hochbaum 1986, Hwang and Chang 2003); forest management (Hoganson and Borges 2000); computer science (Batra 2000); and marketing (Green and Kreiger 1985, Menon and Amiri 2004).

I concluded from this scan of papers about Lagrangian relaxation that the concept has become popular because it is applicable to many problems in a way that allows room for researcher creativity on issues like which constraints to relax and how to solve the resultant Lagrangian problem.

The Future

In preparing these comments I "traded notes" with several authors on the top 10 list, and had a chance to read an early draft of Frank Bass's comments regarding his paper on new product diffusion (Bass 1969). This paper postulates that first-time buyers of new products fall into two group: innovators, who discover the product on their own, and imitators, who learn about the product from other users. The rate of adoption by imitators is thus proportional to cumulative prior sales, and this may be used to derive a formula for purchases over time that has been shown to well fit many real examples. A key feature of this formula is that sales to first-time buyers peak and eventually fall to 0, although replacement sales will continue after the peak.

It occurred to me that the assumptions of this process should also fit the diffusion of ideas, like Lagrangian relaxation, so I sent the data in the table above to Frank Bass who found that a version of his model with $m = 667$, $p = 0.0022$ and $q = 0.1625$ fit the data well (See Bass 1969 for a definition of these parameters.) He also commented that "As you see from the estimates of the Bass model on your references data the Lagrangian relaxation method references do indeed follow a Bass curve. The estimated value of m is 667 and the predicted year of peak is 1996." A graph of the model fit to the data is shown in Figure 1.

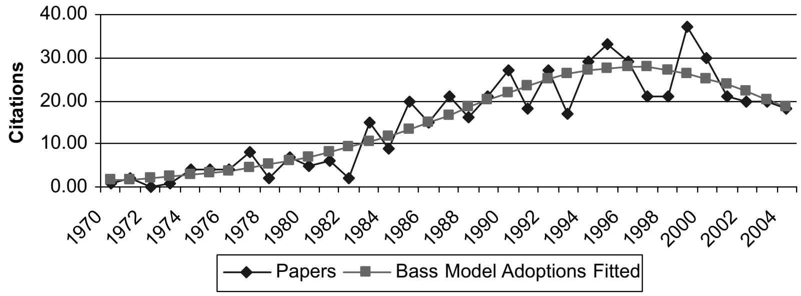
Note that an implication of this model is that research on Langrangian relaxation is trending down and will eventually end. I don't know if this is true. The Bass model, as applied to first-time purchases of new products, assumes a finite market, and once this market is fully saturated, first-time sales end. This doesn't quite fit academic research, because new researchers are entering the field every year.

Nonetheless, the notion that ideas in our field are "born," grow to a peak of interest, and then stabilize, seems quite plausible, and hence it is interesting to ask what drives this life cycle of ideas. I believe one factor is that new ideas, like Lagrangian relaxation in

Table 1 **Number of Published Papers on Lagrangian Relaxation by Year**

Year	Papers on Lagrangian relaxation	Year	Papers on Lagrangian relaxation	Year	Papers on Lagrangian relaxation	Year	Papers on Lagrangian relaxation
1970	1	1980	5	1990	27	2000	30
1971	2	1981	6	1991	18	2001	21
1972	0	1982	2	1992	27	2002	20
1973	1	1983	15	1993	17	2003	20
1974	4	1984	9	1994	29	2004	18
1975	4	1985	20	1995	33		
1976	4	1986	15	1996	29		
1977	8	1987	21	1997	21		
1978	2	1988	16	1998	21		
1979	7	1989	21	1999	37		

Figure 1 **Bass Model Adoptions Count of Lagrangian Relaxation References by Year**



the 1970s, grow out of old ideas (e.g. Dantzig-Wolfe decomposition), but they also spawn other new ideas that eventually replace them. For example, it eventually became apparent that Lagrangian relaxation, as used in the Held-Karp TSP algorithm, was actually a way to efficiently solve a very large linear program, which led researchers to focus on other ways to solve this large LP, as well as on how to enhance the process with cutting planes. As a result, better algorithms have since been developed for the TSP, resting on the shoulders of Held and Karp's seminal work.

It is wonderful that our field is constantly refreshed by succeeding generations of big, new ideas, and this project launched by INFORMS can only enhance that process. I can't wait to see what the next generation of our research will create!

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