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Note on Level-Debt-Service Municipal Bidding

idea not often fully understood, an example may help to give us a firmer grasp of it.

Consider a proposed investment with the following sequence of cash flows:

Year-end	0	1	2
Cash Flow	−\$10	\$40	−\$40

The large cash outflow at the end of the project's life may be interpreted as a negative salvage value. The project balances of this investment are all zero or negative if the assumed rate of return is at least 300 percent, which is $r_{\min}$ for the project. However, although a rate of 300 percent would turn this investment into a pure investment, 300 percent is not the IRR of the investment, because $S_2(300\%)$ equals $-\$40$, not zero. Since $S_n(r_{\min}) < 0$, there is some rate $r^* < r_{\min}$ which will equate $S_n(r^*)$ to zero. Solving the equation $-\$10 + \$40/(1 + r^*) - \$40/(1 + r^*)^2 = 0$, one obtains a unique rate of 100 percent for r^*, the IRR of the investment. If project balances are compounded at r^* of 100 percent, $S_0(100\%) = -\$10$, but $S_1(100\%) = \$20$. Hence, the project is a mixed investment. This example illustrates that the existence of a unique IRR does not imply that the project is necessarily a pure investment.

References

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Note on Level-Debt-Service Municipal Bidding

Cohen and Hammer [1] have shown that under certain conditions the problem of selecting a coupon-principal schedule, when bidding on a new issue of municipal bonds with a level-debt-service constraint, is a linear programming problem. A similar result was independently obtained by the present author, and is presented here, using the same definitions and notations as [1].

The principal difference between [1] and the present paper is that here we have prejudged, to a certain extent, the form of the optimal solution. The judgments are in accord with the experience of market professionals and lead to a linear programming problem with $2n + 3$ constraints as opposed to the $n^2 + 2$ in [1] (both problems involve $2n$ variables, n being the number of yearly maturities).

In addition to the notation in [1], we define d_t as the difference in debt service (principal and interest) between years t and $t + 1$. By this definition

$$d_t = A_t + \sum_{j=t}^{L} c_j A_j - A_{t+1} - \sum_{j=t+1}^{L} c_j A_j$$
$$= A_t + c_t A_t - A_{t+1} \qquad t = F, F+1, \cdots, L-1.$$

The form of the solution is prejudged in that it is assumed

$$\sum_{t=F}^{L} (\alpha_t + \beta_t c_t) A_t = A + \pi,$$

i.e., the optimal coupon-amount schedule will involve no payment of premium. In addition, we assume

$$d_t \geqq 0 \qquad t = F, F+1, \cdots, L-1.$$

The objective function then becomes

$$\text{(1)} \qquad \text{Minimize} \quad NIC = \sum_{t=F}^{L} t A_t c_t ,$$

as in [1], the coupon,[1] production, and amount constraints are written

$$\text{(2)} \qquad c_t^{\,l} \leqq c_t \leqq c_t^{\,u} \qquad t = F, \cdots, L$$

$$\text{(3)} \qquad \sum_{t=F}^{L} (\alpha_t + \beta_t c_t) A_t = A + \pi$$

$$\text{(4)} \qquad \sum_{t=F}^{L} A_t = A.$$

The level debt restriction, however, takes on the form

$$\text{(5)} \qquad \sum_{t=F}^{L-1} d_t \leqq D.$$

From the definition of d_t, one has

$$\text{(6)} \qquad c_t A_t = d_t - A_t + A_{t+1} \qquad t = F, \cdots, L-1,$$

and if we adopt the convention that $A_{L+1} = 0$, the above is true for $t = L$. Substituting (6), the problem becomes linear, i.e., choose A_t, d_t ($t = F, \cdots, L$) to

$$\text{(1')} \qquad \text{Minimize} \quad \sum_{t=F}^{L} t\, d_t - F A_F$$

subject to

$$\text{(2')} \qquad A_t(1 + c_t^{\,l} \leqq d_t + A_{t+1} \leqq A_t(1 + c_t^{\,u}), \qquad t = F, \cdots, L$$

$$\text{(3')} \qquad \sum_{t=F}^{L} (\gamma_t A_t + \beta_t d_t) = A + \pi$$

$$\text{(4')} \qquad \sum_{t=F}^{L} A_t = A$$

$$\text{(5')} \qquad \sum_{t=F}^{L-1} dt \leqq D \qquad d_t \geqq 0, A_t \geqq 0, t = F, \cdots, L,$$

where

$$\gamma_t = \alpha_t - \beta_t + \beta_{t+1} \qquad t = F, \cdots, L-1$$
$$= \alpha_t - \beta_t \qquad t = L.$$

As in [1], F, L, π, D, and the $c_t^{\,l}$, $c_t^{\,u}$, γ_t, β_t are given. The A_t of the optimal

[1] This note uses the notation $c_t^{\,l}$ and $c_t^{\,u}$ in place of the under- and overbars, respectively, used in [1].

solution give the principal amount schedule, and the coupons are determined by (6). A completely general formulation of the problem of using the above conventions would have

$$\sum_{t=F}^{L} (\alpha_t + \beta_t c_t) A_t \geqq A + \pi$$

with the objective function modified as follows:

$$\text{Minimize} \quad \sum_{t=F}^{L} t A_t c_t + \{A + \pi - \sum_{t=F}^{L} (\alpha_t + \beta_t c_t) A_t\}.$$

In addition, the d_t would be unrestricted in sign and the level debt constraint would be replaced by the following inequalities:

$$-D \leqq \sum_{t=i}^{j} dt \leqq D \quad \text{for} \quad i, j = F, F + 1, \cdots, L, \quad \text{and} \quad i \leqq j.$$

The optimal solution to the simplified formulation is obviously a feasible solution (satisfies constraints) to the more general problem. It remains to state a sufficiency test to insure that the solution is also optimal for the general problem.

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Reference

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