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Communications to the Editor—A Counterexample in Game Theory

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A Counterexample in Game Theory†

1. Introduction

This note describes a 5-person von Neumann-Morgenstern game which has a unique solution that is strictly larger than its core. This game provides a counterexample for the following two conjectures stated by L. S. Shapley in [1]:

(a) The intersection of all solutions of a game is the core.

(b) For every game, a solution exists which is contained in the region defined by the inequalities

$$\sum_{i \varepsilon S_j} x_i \geqq v(S_j) \qquad j = 1, 2, \cdots, m \tag{1}$$

where $S_1, S_2, \cdots, S_m$ is an arbitrary partition of the set of players and v is the characteristic function.

A brief review of the essential definitions for a classical n-person game in characteristic function form will be given before the particular example is discussed.

2. Definitions

An *n-person game* is a pair (N, v) where $N = \{1, 2, \cdots, n\}$ and v is a real valued characteristic function on 2^N; i.e., v assigns the real number $v(S)$ to each subset S of N and $v(\emptyset) = 0$. The set of all *imputations* is

$$A = \{x: \textstyle\sum_{i \varepsilon N} x_i = v(N) \quad \text{and} \quad x_i \geqq v(\{i\}) \quad \text{for all} \quad i \varepsilon N\}$$

where $x = (x_1, x_2, \cdots, x_n)$ is a vector with real components. For any $B \subset A$, define Dom B to be the set of all $x \varepsilon A$ such that there exists a $y \varepsilon B$ and a nonempty $S \subset N$ with $y_i > x_i$ for all $i \varepsilon S$ and with $\sum_{i \varepsilon S} y_i \leqq v(S)$. A subset K of A is a *solution* if $K \cap \text{Dom } K = \emptyset$ and $K \cup \text{Dom } K = A$. The *core* of a game is $C = \{x \varepsilon A: \sum_{i \varepsilon S} x_i \geqq v(S) \text{ for all } S \subset N\}$.

3. Example

Consider the game (N, v) where $N = \{1, 2, 3, 4, 5\}$ and v is given by $v(N) = 1$, $v(\{1, 2\}) = v(\{4, 5\}) = v(\{2, 3, 4\}) = \frac{1}{2}$, and $v(S) = 0$ for all other $S \subset N$. For this game

$$A = \{x: \textstyle\sum_{i \varepsilon N} x_i = 1 \quad \text{and} \quad x_i \geqq 0 \quad \text{for all} \quad i \varepsilon N\},$$

$$C = \{x \varepsilon A: x_1 + x_2 = \tfrac{1}{2}, x_4 + x_5 = \tfrac{1}{2}, x_2 + x_3 + x_4 \geqq \tfrac{1}{2}\},$$

and it is not difficult to show that the *unique* solution is

$$K = A - \text{Dom } C = \{x \varepsilon A: x_1 + x_2 = \tfrac{1}{2}, x_4 + x_5 = \tfrac{1}{2}\}.$$

Hence (a) is false, because $K - C$ is nonempty. Also (b) is false, since K is not contained in the region described by (1) when $S_1 = \{2, 3, 4\}$ and $S_2 = \{1, 5\}$.

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This example may be extended to a superadditive game; i.e., $v(S_1 \cup S_2) \geqq v(S_1) + v(S_2)$ whenever $S_1 \cap S_2 = \emptyset$) by changing $v(\{1, 2, 4, 5\})$ from 0 to 1 and $v(S)$ from 0 to $\frac{1}{2}$ for all other S that are proper supersets of $\{1, 2\}$ and $\{4, 5\}$. The resulting core and unique solution remain the same.

Reference

1. KUHN, H. W. (ed.), *Report of an Informal Conference on the Theory of n-Person Games*, held at Princeton University, March 20–21, 1953 (dittoed).

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ERRATA†

In the paper, "The Sequential Selection of Approaches to a Task," by J. A. Yahav and myself, the following errata should be corrected:

p. 634, third line of *Definition*, for "lower" substitute "not higher"

pp. 641, 642, in the sequence of three equations which begins on p. 641, substitute c for k and substitute h for b.

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† MARSCHAK, T. A. AND J. A. YAHAV, "The Sequential Selection of Approaches to a Task," *Management Science*, Vol. 12, No. 9 (May 1966), pp. 627–647.