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# Note—Solving the Generalized Market Area Problem

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One need only change $G$ to $x^*/t^*$ and delete $w_1$ (or $w_2$ if $x^* > Gt^*$). With these changes the linear program (4) becomes

$$\begin{array}{llll} \min & & & w_2 \\ & x - (x^*/t^*)\,t & & +w_2 = 0 \\ & x & -\mathbf{a}_2 \cdot \mathbf{y} & \phantom{+w_2} = 0 \\ & & t - \mathbf{a}_1 \cdot \mathbf{y} & \phantom{+w_2} = 0 \\ & & \mathbf{A}\mathbf{y} & \phantom{+w_2} \leqslant \mathbf{b} \\ & \mathbf{y},\ w_2 \geqslant 0, & & \end{array}$$

which is equivalent to (A) since min $w_2$ is the same as $-\max(-w_2)$, and $-w_2 = x - (x^*/t^*)t$. Hence, one first solves (4), and then reoptimizes (4) when the single parameter $G$ is changed to $x^*/t^*$. If the objective value of this perturbed program is zero, then the solution of (4) also solves (2).

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# Notes* IV

# SOLVING THE GENERALIZED MARKET AREA PROBLEM†

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Lowe and Hurter introduced the "generalized market area problem" of simultaneously determining production levels and distribution patterns at $n$ plants to satisfy demand distributed over a subset of $R^k$. Here we give simpler proofs and extensions of their results. We also show how the problem can be solved using unconstrained optimization techniques. (PROGRAMMING–NONLINEAR ALGORITHMS; INVENTORY/PRODUCTION)

## 1. Introduction

Lowe and Hurter [3] studied the "generalized market area problem" of simultaneously determining production levels and distribution patterns at $n$ plants to satisfy

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demand distributed over a subset of $R^k$. The purpose of this note is to provide simpler proofs and extensions of results of Lowe and Hurter. Our arguments are based on nonlinear programming duality. The dual problem is an unconstrained finite dimensional convex minimization problem, and can thus be solved by known techniques.

Let $S \subseteq R^k$ be a compact Jordan-measurable set. Following Lowe and Hurter [3] we use Jordan measure (see [1]) and Riemann integration, but our results are also valid with Lebesgue measure and integration. Note that Lowe and Hurter also assume $S$ connected. The demand density is given by a function $\Delta : S \to R$ that is bounded, nonnegative and continuous almost everywhere (that is, except on a set of Jordan measure zero). In other words, $\int_A \Delta$ is the demand of the Jordan-measurable subset $A$ of $S$. Without loss of generality we assume that the total demand $\int_S \Delta$ is 1. Costs are described by $n+1$ continuous functions $f_i : S \to R$, $i = 1, \ldots, n$, and $c : Z \to R \cup \{+\infty\}$, where $Z = \{z \in R^n_+ \mid \sum_i z_i = 1\}$. Here $f_i(x)$ is the unit transportation cost from plant $i$ to $x \in S$ and $c(z)$ is the total production cost when production at plant $i$ is $z_i$, $i = 1, \ldots, n$. Lowe and Hurter suppose $c(z)$ is separable, $c(z) = \sum_i c_i(z_i)$, but the general case is no more difficult.

Let $\Omega = \{\sigma = (\sigma_1, \ldots, \sigma_n) \mid$ each $\sigma_i : S \to R_+$ is bounded and continuous almost everywhere, and $\sum_i \sigma_i = \Delta$ almost everywhere$\}$. Each $\sigma \in \Omega$ represents a supply policy with plant $i$ supplying $\int_A \sigma_i$ to the subset $A$ of $S$. Denote by $\int_S \sigma$ the vector $(\int_S \sigma_1, \ldots, \int_S \sigma_n)$ of production levels. Then $\sigma \in \Omega$ and $\int_S \Delta = 1$ imply that $\int_S \sigma$ is in $Z$.

For any $\sigma \in \Omega$ denote by $T(\sigma)$ the associated transportation cost $\sum_i \int_S \sigma_i f_i$. The problem we consider is

$$\min_{\sigma \in \Omega} \left\{ T(\sigma) + c\left( \int_S \sigma \right) \right\}. \tag{P}$$

A related problem fixes the total production $\bar{z} \in Z$, yielding a "semi-infinite" transportation problem

$$\min_{\sigma \in \Omega} \left\{ T(\sigma) \mid \int_S \sigma = \bar{z} \right\}. \qquad \mathrm{P}(\bar{z})$$

When $\Delta \equiv 1$, P($\bar{z}$) is the partition problem of Corley and Roberts [2].

## 2. Preliminary Lemmas and Weak Duality

Problem (P) and P($\bar{z}$) involve the infinite-dimensional supply policy $\sigma$. We use duality to obtain finite-dimensional optimization problems. First write (P) as $\min_{\sigma \in \Omega,\, z \in Z} \{ T(\sigma) + c(z) \mid \int_S \sigma - z = 0 \}$. Then dualizing with respect to the explicit constraints we get

$$\max_{u \in R^n} \left\{ \min_{\sigma \in \Omega} \left\{ T(\sigma) + u \cdot \int_S \sigma \right\} + \min_{z \in Z} \{ c(z) - u \cdot z \} \right\}.$$

Similarly, from P($\bar{z}$) we are led to

$$\max_{u \in R^n} \left\{ \min_{\sigma \in \Omega} \left\{ T(\sigma) + u \cdot \int_S \sigma \right\} - u \cdot \bar{z} \right\}.$$

To simplify these dual problems we put further restrictions on the $f_i$'s.

Assumption 1. For any $1 \leqslant i, j \leqslant n$ and any $u_i, u_j \in R$, the set $\{x \in S \mid f_i(x) + u_i = f_j(x) + u_j\}$ has (Jordan) measure zero.

This is assumption a of Lowe and Hurter [3].

Definitions. For any $u \in R^n$, define

$$\theta(u) = \int_S \min_{1 \leqslant i \leqslant n} \{ f_i + u_i \} \Delta;$$

$$S_i(u) = \{x \in S \mid f_i(x) + u_i \leqslant f_j(x) + u_j,\ 1 \leqslant j \leqslant n\};$$

$$g_i(u) = \int_{S_i(u)} \Delta; \quad \text{and}$$

$$\sigma_i(u) = \Delta \quad \text{on } S_i(u), \quad \text{for } i = 1, 2, \ldots, n,$$
$$0 \quad \text{elsewhere.}$$

By our assumption, the boundary of $S_i(u)$ has measure zero; hence $S_i(u)$ is measurable and $g(u) \in Z$ for any $u \in R^n$. Also $\sigma(u) \in \Omega$ with $\int_S \sigma(u) = g(u)$.

LEMMA 1. *For any $u \in R^n$, $\theta(u) = \min_{\sigma \in \Omega}\{T(\sigma) + u \cdot \int_S \sigma\}$ and the minimum is achieved for $\sigma = \sigma(u)$.*

PROOF. For any $\sigma \in \Omega$ we have

$$\begin{aligned} T(\sigma) + u \cdot \int_S \sigma &= \sum_i \int_S \sigma_i(f_i + u_i) \\ &= \sum_i \sum_j \int_{S_j(u)} \sigma_i(f_i + u_i) \\ &\geqslant \sum_i \sum_j \int_{S_j(u)} \sigma_i(f_j + u_j) \\ &= \sum_j \int_{S_j(u)} \Big(\sum_i \sigma_i\Big)(f_j + u_j) \\ &= \sum_j \int_{S_j(u)} \Delta(f_j + u_j) = \theta(u). \end{aligned} \tag{1}$$

But equality is attained when $\sigma = \sigma(u)$, whence the lemma follows.

Using Lemma 1 we can write our dual problems as

$$\max_{u \in R^n} \{\theta(u) - c^*(u)\} \tag{D}$$

where $c^*(u)$ is defined as $\max_{z \in Z}\{u \cdot z - c(z)\}$, and

$$\max_{u \in R^n} \{\theta(u) - u \cdot \bar{z}\}. \tag{D($\bar{z}$)}$$

THEOREM 1 (WEAK DUALITY). *Let $u \in R^n$. Then for any $\sigma \in \Omega$, $\theta(u) - c^*(u) \leqslant T(\sigma) + c(\int_S \sigma)$, and for any $\sigma \in \Omega$ with $\int_S \sigma = \bar{z}$, $\theta(u) - u \cdot \bar{z} \leqslant T(\sigma)$.*

PROOF. Let $z = \int_S \sigma$. Then by Lemma 1, $\theta(u) \leqslant T(\sigma) + u \cdot \int_S \sigma$. By definition of $c^*$, $-c^*(u) \leqslant c(z) - u \cdot z$. Adding gives the first part. The second part follows directly from Lemma 1.

COROLLARY 1. *If $\sigma^* \in \Omega$ satisfies $T(\sigma^*) + c(\int_S \sigma^*) = \theta(u^*) - c^*(u^*)$ for some $u^* \in R^n$, $\sigma^*$ is an optimal solution to* (P) *and $u^*$ is an optimal solution to* (D). *Similarly if $\sigma^* \in \Omega$ satisfies $\int_S \sigma^* = \bar{z}$ and $T(\sigma^*) = \theta(u^*) - u^* \cdot \bar{z}$ for some $u^* \in R^n$, $\sigma^*$ are optimal solutions to* (P($\bar{z}$)) *and* (D($\bar{z}$)) *respectively.*

In order to use the dual problems, we must investigate the functions appearing in them.

LEMMA 2. *$\theta$ is concave and $c^*$ is convex.*

PROOF. Since $c^*$ is the maximum of a family of affine functions, it is convex. Similarly, for each $x \in S$, $\min_{1 \leqslant i \leqslant n}\{f_i(x) + u_i\}$ is a concave function of $u$. Since $\Delta$ is nonnegative, the integral $\theta(u)$ is also concave.

LEMMA 3. *$\theta$ is continuously differentiable with gradient $g$.*

PROOF. We show first that $g$ is continuous. Suppose $u^k \to u$. Let $Q_i^k = S_i(u^k) \setminus S_i(u)$ and $R_i^k = S_i(u) \setminus S_i(u^k)$. Then

$$g_i(u) - \delta\mu(R_i^k) \leqslant g_i(u^k) \leqslant g_i(u) + \delta\mu(Q_i^k),$$

where $\mu$ denotes (Jordan) measure and $\Delta(x) \leqslant \delta$ for all $x$. We must show that $\mu(Q_i^k)$ and $\mu(R_i^k)$ converge to zero.

Let $Q_{ij}(u, \epsilon) = \{x \in S \mid u_i - u_j - \epsilon \leqslant f_j(x) - f_i(x) \leqslant u_i - u_j\}$. With $Q_{ij}^k = Q_{ij}(u, u_i - u_i^k - u_j + u_j^k)$, we have $Q_i \subseteq \bigcup_j Q_{ij}^k$. It is therefore sufficient to show that $\mu(Q_{ij}(u, \epsilon)) \to 0$ as $\epsilon \to 0$. Note that $\mu(Q_{ij}(u, 0)) = 0$ by Assumption 1. Hence there is a measurable set $T_{ij}$ with $\mu(T_{ij}) \leqslant \eta$ and $Q_{ij}(u, 0) \subseteq \operatorname{int} T_{ij}$ for all $\eta > 0$. For sufficiently small $\epsilon$, $Q_{ij}(u, \epsilon) \subseteq T_{ij}$; otherwise take a sequence $x^r \in Q_{ij}(u, 1/r) \setminus T_{ij}$ and a convergent subsequence $x^r \to x^\infty$. Then $x^\infty \notin \operatorname{int} T_{ij}$, but $x^\infty \in Q_{ij}(u, 0)$ by continuity of $f_i, f_j$. This contradiction shows that for sufficiently small $\epsilon$, $Q_{ij}(u, \epsilon) \subseteq T_{ij}$; hence $\mu(Q_{ij}(u, \epsilon)) \leqslant \eta$. But $\eta$ was arbitrary, so that the continuity of $g$ is established.

We can now prove the lemma. We only need to show that $\partial\theta(u)/\partial u_i = g_i(u)$. But if $e^i$ is the $i$th unit vector in $R^n$, then for any $\epsilon > 0$ we have

$$\theta_i(u, \epsilon) \equiv (\theta(u + \epsilon e^i) - \theta(u))/\epsilon$$

$$= \int_S \epsilon^{-1}\Big[\min\Big\{f_i(x) + u_i + \epsilon,\ \min_{j \neq i}\{f_j(x) + u_j\}\Big\} - \min\{f_j(x) + u_j\}\Big]\Delta.$$

Now the term in brackets is at most $\epsilon$, and is only positive if $x \in S_i(u)$. Hence $\theta_i(u, \epsilon) \leqslant \int_{S_i(u)} \Delta = g_i(u)$. Conversely, the term in brackets is always nonnegative and is equal to $\epsilon$ if $x \in S_i(u)$ but $x \notin Q_{ji}(u, \epsilon)$ for all $j$. Hence $\theta_i(u, \epsilon) \geqslant g_i(u) - \sum_j (\int_{Q_{ji}(u, \epsilon)} \Delta)$. Now as $\epsilon \to 0$, $\mu(Q_{ji}(u, \epsilon))$ tends to zero as proved above. It follows that $\lim_{\epsilon \downarrow 0} \theta_i(u, \epsilon) = g_i(u)$. A similar proof establishes $\lim_{\epsilon \uparrow 0} \theta_i(u, \epsilon) = g_i(u)$, and the lemma is proved.

## 3. Strong Duality and Algorithms

We consider first P($\bar{z}$). Note first that $\theta(u) - u \cdot \bar{z} = \theta(v) - v \cdot \bar{z}$ if $v - u$ is a multiple of $(1, \ldots, 1)$. Hence we may restrict $u$ so that $\sum_i u_i = 0$. Now let $\nu = \max\{f_i(x) - f_j(x) \mid 1 \leqslant i, j \leqslant n,\ x \in S\} < \infty$. Set $U = \{u \in R^n \mid \sum_i u_i = 0,\ u_i \leqslant \nu$ for all $i\}$.

THEOREM 2 (STRONG DUALITY FOR P($\bar{z}$), D($\bar{z}$)). *There is a $u^* \in U$ that is an optimal solution to* D($\bar{z}$). *Further $\theta(u^*) - u^* \cdot \bar{z} = T(\sigma(u^*))$ and $\sigma(u^*)$ is an optimal solution to* P($\bar{z}$).

PROOF. Let $u^*$ maximize $\theta(u) - u \cdot \bar{z}$ over $U$. Then by the Karush-Kuhn-Tucker conditions, $g_i(u^*) - \bar{z}_i = \lambda + \pi_i$ for $1 \leqslant i \leqslant n$, with $\pi_i \geqslant 0$ and $\pi_i(u_i^* - \nu) = 0$ for all $i$.

If all $\pi_i$'s are zero, $n\lambda = \sum_i (g_i(u^*) - \bar{z}_i) = 0$ gives $\lambda = 0$ and $g(u^*) = \bar{z}$. Otherwise, let $\pi_j > 0$ be the maximum of the $\pi_i$'s. Then $u_j^* = \nu$ and some $u_i^* < 0$; hence by definition of $\nu$, $S_j(u^*) = \emptyset$ and $g_j(u^*) = 0$. Thus $g_j(u^*) - \bar{z}_j \leqslant 0$, and since $g_i(u^*) - \bar{z}_i \leqslant g_j(u^*) - \bar{z}_j$ for all $i$ and $\sum_i (g_i(u^*) - \bar{z}_i) = 0$, we again have $g(u^*) = \bar{z}$.

Since $g$ is the gradient of $\theta$ (Lemma 3), it follows that $u^*$ is an optimal solution of D($\bar{z}$). Now we have $\int_S \sigma(u^*) = g(u^*) = \bar{z}$, so from Lemma 1, $\theta(u^*) - u^* \cdot \bar{z} = T(\sigma(u^*))$. Corollary 1 now implies the optimality of $\sigma(u^*)$.

Our Theorem 2 implies Theorem 2 of [3].

The proof above also gives the following result, which is Assumption b in Lowe and Hurter [3]:

COROLLARY 2. *For any $\bar{z} \in Z$, there is some $u \in R^n$ with $g(u) = \bar{z}$.*

Suppose that given any $u \in R^n$ the sets $S_j(u)$ and the function $g(u)$ can easily be found. Then Theorem 2 implies that maximizing the continuously differentiable concave function $\theta(u) - u \cdot \bar{z}$ gives a solution of P($\bar{z}$). Note that $\sum_i (g_i(u) - \bar{z}_i) = 0$; hence a gradient or Quasi-Newton algorithm initiated in $\{u \in R^n \mid \sum_i u_i = 0\}$ will remain in this set. In practice one would delete any plants with $\bar{z}_i = 0$. For the remaining subproblem we have $\bar{z} > 0$; thus any $u$ maximizing $\theta(u) - u \cdot \bar{z}$ has $g(u) > 0$. It follows that any optimal solution to D($\bar{z}$) with $\sum_i u_i = 0$ also lies in the compact set $U$, and that every level set of $\theta(u) - u \cdot \bar{z}$ has a compact intersection with $\{u \in R^n \mid \sum_i u_i = 0\}$.

We now turn to the general problems (P) and (D). We make

ASSUMPTION 2. $c : Z \to R \cup \{+\infty\}$ is a proper closed convex function.

This assumption is satisfied if $c : Z \to R$ is convex, but also allows some production vectors $z$ to be infeasible—then set $c(z) = \infty$. The function $c^*$ is then the convex conjugate of $c$ when $c$ is extended to $R^n$ by setting $c(z) = \infty$ if $z \notin Z$—see Rockafellar [4]. From [4, Theorems 23.5 and 23.8], we have

$$u \in \partial c(z) \Leftrightarrow c(z) + c^*(u) = u \cdot z \Leftrightarrow z \in \partial c^*(u) \text{ and}$$

$$\partial(c^* - \theta)(u) = \partial c^*(u) - \{g(u)\},$$

where $\partial$ is the subgradient mapping for a convex function:

$$\partial f(x) = \{y \mid f(z) \geqslant f(x) + y \cdot (z - x) \text{ for all } z\}.$$

THEOREM 3 (STRONG DUALITY FOR (P), (D)). *There is a $u^* \in U$ that is an optimal solution to* (D). *Further, $\theta(u^*) - c^*(u^*) = T(\sigma(u^*)) + c(\int_S \sigma(u^*))$ and $\sigma(u^*)$ is an optimal solution to* (P).

PROOF. By [4, Theorem 27.3], the convex function $c^* - \theta$ attains its infimum over the compact set $U$ at some $u^*$. By [4, Corollary 28.2.2], there is some $z \in \partial c^*(u^*)$ with $g_i(u^*) - z_i = \lambda + \pi_i$ for $1 \leqslant i \leqslant n$ with $\pi_i \geqslant 0$ and $\pi_i(u_i^* - \nu) = 0$ for all $i$. Now $z \in \partial c^*(u^*)$ implies $c(z) < \infty$ and hence $z \in Z$. Exactly as in Theorem 2 we deduce that $g(u^*) = z$ and thus $u^*$ is an optimal solution to (D); indeed $0 = z - z \in \partial(c^* - \theta)(u^*)$.

Now $\int_S \sigma(u^*) = g(u^*) = z$ and $c(z) + c^*(u^*) = u^* \cdot z$. Hence $\theta(u^*) - c^*(u^*) = T(\sigma(u^*)) + c(\int_S \sigma(u^*))$ from Lemma 1. Corollary 1 implies that $\sigma(u^*)$ is optimal in (P).

Theorem 3 shows that one may solve (P) by maximizing the concave function $\theta - c^*$. (Again we assume that $S_j(u)$ and $g(u)$ can be evaluated for any $u \in R^n$.) If $c$ is essentially strictly convex [3, §26], $c^*$ is continuously differentiable. Otherwise we may have to maximize the nondifferentiable function $\theta - c^*$, by subgradient optimization, for example. Here we assume that for any $u \in R^n$, $c^*(u)$ and some element of $\partial c^*(u)$ can be obtained.

An alternative approach is to find $z \in Z$ to minimize $b(z) + c(z)$, where $b(z)$ is the optimal value of P($z$). By Theorem 2, $b(z)$ is also the optimal value of D($z$), i.e. $\max_{u \in R^n}(\theta(u) - u \cdot z) = \max_{u \in R^n}(u \cdot (-z) - (-\theta)(u))$. But the latter expression is merely the convex conjugate of $-\theta$ evaluated at $-z$ : $b(z) = (-\theta)^*(-z)$. It follows that $b$ is convex. A subgradient of $b$ at $z$ is a byproduct of finding $b(z)$ by solving D($z$). For $u \in \partial b(z) \Leftrightarrow -u \in \partial(-\theta)^*(-z) \Leftrightarrow -z \in \partial(-\theta)(-u) \Leftrightarrow z = g(-u) \Leftrightarrow -u$ solves D($z$). Hence the alternative approach iterates on $z$; for each $z$, $b(z)$ and some $u \in \partial b(z)$ is found by solving D($z$). Of course, in this approach one must deal with the constraints $z \in Z$. Note that if $\theta$ restricted to $\{u \in R^n \mid \sum_i u_i = 0\}$ is strictly convex, then an optimal solution of D($z$) in this set is unique and $b$ restricted to $Z$ is continuously differentiable.

The result $u \in \partial b(z) \Leftrightarrow g(-u) = z$ above is related to theorem 8 of Lowe and Hurter [3], wherein the "partial derivatives" of $b$ at $z$ are determined under an additional assumption. It is claimed that this assumption always holds if $S$ is connected and $\Delta$ is continuous almost everywhere. However, the claim fails even with $\Delta$ constant. The following example shows that partial derivatives may not exist. Let $S$ be the "barbell" in $R^2$:

$$\{x \in R^2 \mid \|x - (-2, 0)\| \leqslant 1\} \cup \{x \in R^2 \mid -1 \leqslant x_1 \leqslant 1, x_2 = 0\} \cup \{x \in R^2 \mid \|x - (2, 0)\| \leqslant 1\}.$$

Let $\Delta \equiv 1/2\pi$ on $S$, and let $f_i(x)$ be the Euclidean distance from $x \in R^2$ to $((-1)^i 2, 0)$, $i = 1, 2$. Then $\partial b/\partial z_1\ (\frac{1}{2}, \frac{1}{2})$ does not exist. For $z = (\frac{1}{2} + \epsilon, \frac{1}{2} - \epsilon)$, the first plant must supply a small region of demand $\epsilon$ near $(1, 0)$ and the cost is $\frac{2}{3} + 2\epsilon$ to first order in $\epsilon$. The cost is also $\frac{2}{3} + 2\epsilon$ to first order in $\epsilon$ when $z = (\frac{1}{2} - \epsilon, \frac{1}{2} + \epsilon)$ for similar reasons.[1]

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