Management Science

Publication details, including instructions for authors and subscription information:
http://pubsonline.informs.org

Note—Single and Double Vertex Substitution in Heuristic Procedures for the p-Median Problem

Samuel Eilon, Roberto D. Galvão,

MANAGEMENT SCIENCE
Vol. 24, No. 16, December 1978
Printed in U.S.A.

Notes*
IV

SINGLE AND DOUBLE VERTEX SUBSTITUTION IN HEURISTIC PROCEDURES FOR THE *p*-MEDIAN PROBLEM†

SAMUEL EILON‡ AND ROBERTO D. GALVÃO§

The p-median problem is an uncapacitated minisum network location problem where it is required to site p facilities in a network, so that the sum of the shortest distances from each of the nodes of the network to its nearest facility is minimized. An existing heuristic procedure for this problem is extended, and computational experience is provided for several cases. A simple vertex addition heuristic and its use as a "pre-processor" to the extended procedure is described and tested for a number of problems.
(p-MEDIAN; HEURISTIC)

We are concerned in this paper with the p-median problem, where no constraints are placed on the capacities of the facilities and where it is assumed that the fixed costs of the facilities are independent of their location (and hence have no effect on the objective cost function). The p-median problem has attracted some attention because of its applications in many practical situations, for example in the location of switching centres in telephone networks, in siting supply depots in road networks, and in determining desirable locations for various facilities of public services.

Heuristic methods for the p-median problem first appeared in papers by Maranzana [10] and Teitz and Bart [12], the latter describing a heuristic method based on single vertex substitution. Their vertex substitution method is in fact one of a family based on local optimization and the idea of λ-optimality, which was first introduced by Lin [9] for the travelling salesman problem and subsequently extended by others [1], [2], [8] for a variety of combinatorial problems.

This note extends the method proposed by Teitz and Bart [12] by examining a double vertex substitution heuristic procedure for the p-median problem; details are given on our computational experience for various examples. A simple vertex addition ("greedy") heuristic, and its use as a "pre-processor" to the vertex substitution methods, is also described.

λ-Optimal Substitution Methods for the p-Median Problem

The λ substitution procedure involves examining a given solution S of a set of p points by exchanging λ of its points (where $\lambda \leqslant p$) with λ points taken from the total set X of the n points in the network. The replacement set of λ vertices chosen from X must obviously satisfy the condition that at least one of the λ points does not belong to S. In the p-median problem a set $\bar{S}$ of p vertices is called λ-optimal if the substitution of any λ vertices in $\bar{S}$ does not improve the solution corresponding to $\bar{S}$. In this context, the answer produced by the single vertex substitution algorithm of Teitz and Bart relates to $\lambda = 1$ and may be called 1-optimal.

* All Notes are refereed.

† Accepted by Michael Held; received March 14, 1978. This paper has been with the authors 1 month for 1 revision.

‡ Imperial College of Science and Technology, London.
§ Universidade Federal do Rio de Janiero.

0025-1909/78/2416/1763$01.25

From the definition of λ-optimality given above, it is not difficult to see that in order to ensure that a given set is λ-optimal a total of

$$T(\lambda) = \sum_{\lambda_i = 1}^{\lambda} \binom{p}{\lambda_i}\binom{n-p}{\lambda_i}$$

potential substitutions must be performed and the corresponding objective functions computed in each cycle of the algorithm. This number $T(\lambda)$ increases rapidly with λ, and hence practical algorithms cannot use values of λ much above 2.

Another heuristic for the p-median problem is the vertex addition procedure, which in fact is not an original idea (see [3], [6], [11]). Even though this procedure does not perform badly on its own, especially for large values of p, the main interest in this heuristic in the present context is in its use as a "pre-processor" to vertex substitution methods. The idea behind this "combined approach" is that since λ-optimal substitution algorithms must start from a set S of p vertices, some advantage might be gained by starting from a "good" initial set.

Used on its own, the vertex addition heuristic procedure starts from an available solution to the $(p-1)$-median problem and adds to this solution the vertex that produces the maximum possible reduction in the objective function as the number of centres is increased from $(p-1)$ to p. The combined approach is initialized with the 1-median solution. It then proceeds in a stepwise fashion, with the appropriate λ-optimal substitution method being applied to the set of cardinality p generated by the vertex addition heuristic. The procedure terminates after solutions are produced for the desired range of values of p.

Computational Results

The algorithms described above were coded and tested for λ = 1 and λ = 2, and a sample of the corresponding computational results is given in Table 1. Due to the high

TABLE 1

Computational Results for λ = 1 and λ = 2

Problem Size		%Deviation of Heuristic Solution from Optimal Solution				TIME IN SECONDS*			
n	p	1-Optimal Substitution	Combined App., λ = 1	2-Optimal Substitution	Combined App., λ = 2	1-Optimal Substitution	Combined App., λ = 1	2-Optimal Substitution	Combined App., λ = 2
10⁺	3	0	0	0	0	0.01	0.01	0.04	0.02
10⁺	5	5.55	0	0	0	0.02	0.01	0.06	0.04
10	3	0	0	0	0	0.01	0.01	0.04	0.02
10	5	0	0	0	0	0.02	0.01	0.05	0.02
15	5	0	0	0	0	0.08	0.05	0.48	0.33
15	7	0	0	0	0	0.07	0.03	0.50	0.17
20	3	0.96	0.96	0	0	0.15	0.13	0.66	1.00
20	5	0	0.59	0	0	0.19	0.07	1.38	1.36
20	7	6.17	1.32	0	0	0.20	0.08	4.62	1.83
25	4	0	0	0	0	0.37	0.12	4.37	1.49
25	6	0	0	0	0	0.45	0.14	7.77	2.65
25	8	2.35	0	0	0	0.84	0.14	13.72	3.48
30	5	2.71	3.29	0	0	1.35	0.25	19.16	14.34
30	7	0	0.78	0	0	1.59	0.28	30.94	15.31
30	9	0	0	—	0	1.91	0.29	—	29.36
33⁺⁺	4	2.12	0	—	—	1.09	0.70	—	—
33⁺⁺	6	0	0	—	—	3.10	0.89	—	—
40	6	—	—	—	—	3.87	2.31	—	—
40	8	—	—	—	—	6.55	0.74	—	—
50	7	—	—	—	—	6.47	1.43	—	—
50	10	—	—	—	—	16.27	1.74	—	—

* CPU Time, in CDC 7600 Seconds

⁺ Garfinkel et al. Example [5, p. 231]

⁺⁺ Karg and Thompson 33 City Example [7, p. 244] computing costs involved when $\lambda = 2$, results for $n > 30$ were only obtained for $\lambda = 1$. Except for the 10-vertex network of Garfinkel et al. [5] and the 33-city example of Karg and Thompson [7], the networks used in the examples were randomly generated. The optimal solution in each case, up to $n = 33$, was obtained by the branch-and-bound method described in [4]. The quality of the solutions obtained is summarized (for $\lambda = 1$) in Table 2.

TABLE 2

Frequency Distribution of % Deviation from Optimal Solution for $\lambda = 1$

	1-Optimal Substitution	Combined Approach, $\lambda = 1$
0%	51	53
0–2%	5	9
2–4%	7	6
4–6%	4	1
6–8%	2	1
> 8%	1	0
All points considered		
No. of points	70	70
Mean	0.98%	0.56%
St. deviation	2.01%	1.32%
Zero deviation points excluded		
No. of points	19	17
Mean	3.61%	2.29%
St. deviation	2.34%	1.82%

In relation to the methods tested in this note, it can be observed that 2-optimal substitution solutions are only slightly better than the corresponding 1-optimal solutions. From a cost-effectiveness point of view, the combination of the vertex addition heuristic procedure with the 1-optimal substitution method appears to be the best of the methods studied. It produces solutions that are on average of better quality than the solutions produced by its "pure" 1-optimal counterpart, and are not much worse than the solutions produced by the 2-optimal methods. The method has a good worst-case performance (see Cornhuejols et al. [3]), and the corresponding computing times are the lowest of the methods tested.

References

1. Christofides, N. and Eilon, S., "An Algorithm for the Vehicle Dispatching Problem," *Operational Res. Quart.*, Vol. 20 (1969), pp. 309–318.
   See Also: Eilon, S., Watson-Gandy, C. D. T. and Christofides, N., *Distribution Management, Mathematical Modelling and Practical Analysis* (Chapter 7), Griffin, London, 1971.
2. ——— and ———, "Algorithms for Large Scale Travelling Salesman Problems," *Operational Res. Quart.*, Vol. 23 (1972), pp. 511–518.
3. Cornuejols, G., Fisher, M. L. and Nemhauser, G. L., "Location of Bank Accounts to Optimize Float: An Analytic Study of Exact and Approximate Algorithms," *Management Sci.*, Vol. 23 (1977), pp. 789–810.
4. Galvão, R. D., "The Optimal Location of Facilities on a Network," Ph.D. Thesis, Department of Management Science, Imperial College of Science and Technology, University of London.
5. Garfinkel, R. S., Neebe, A. W. and Rao, M. R. "An Algorithm for the m-Median Plant Location Problem," *Trans. Sci*, Vol. 8 (1974), pp. 217–236.
6. Järvinen, P., Rajala, J. and Sinervo, H., "A Branch-and-Bound Algorithm for Seeking the p-Median," *Operations Res.*, Vol. 20 (1972), pp. 173–178.

7. KARG, R. L. AND THOMPSON, G. L., "A Heuristic Approach to Solving Travelling Salesman Problems," *Management Sci.*, Vol. 10 (1964), pp. 225–248.
8. KERNINGAN, B. W. AND LIN, S., "An Efficient Heuristic Procedure for Partitioning Graphs," *Bell System Tech. J.*, Vol. 49 (1970), pp. 291–307.
9. LIN, S., "Computer Solutions of the Travelling Salesman Problem," *Bell System Tech. J.*, Vol. 44 (1965), pp. 2245–2269.
10. MARANZANA, F. E., "On the Location of Supply Points to Minimize Transport Costs," *Operational Res. Quart.*, Vol. 15 (1964), pp. 261–270.
11. SÁ, G., "Branch-and-Bound and Approximate Solutions to the Capacitated Plant Location Problem," *Operations Res.*, Vol. 17 (1969), pp. 1005–1016.
12. TEITZ, M. B. AND BART, P., "Heuristic Methods for Estimating the Generalized Vertex Median of a Weighted Graph," *Operations Res.*, Vol. 16 (1968), pp. 955–961.

Notes*
V

MANAGEMENT SCIENCE
Vol. 24, No. 16, December 1978
Printed in U.S.A.

COMMENTS ON "SINGLE CYCLE CONTINUOUS REVIEW POLICIES FOR ARBORESCENT PRODUCTION/INVENTORY SYSTEMS"†

JOHN A. MUCKSTADT‡ AND HOWARD M. SINGER§

(INVENTORY/PRODUCTION; INVENTORY/PRODUCTION–DETERMINISTIC MODELS)

In the paper by Graves and Schwarz [1], we observed an error in their discussion of optimal policies for the one warehouse-*m* retailer system. Specifically, Theorem 1, which states that the optimal stationary policy is a single-cycle policy, is incorrect.

The proofs for property P4, the Lemma for Theorem 1, and Theorem 1 are in error. Property P4 states that the time interval between successive simultaneous production points for all stages is finite for an optimal policy. The proof presented implicitly assumes that the infimum of an infinite set of positive echelon inventory levels is a positive quantity. This need not be the case. The Lemma for Theorem 1 states that given zero initial inventories, the optimal stationary policy is a renewal policy. The proofs of the Lemma and Theorem 1 rely on the application of the properties P4 and P1 respectively. Both of these results, however, deal with optimal *stationary* policies and, unfortunately, it is not necessarily the case that the same properties which hold for overall optimal policies are valid for optimal stationary policies as well.

The following example demonstrates that a single-cycle policy need not be the best stationary policy. Consider the following one warehouse-two retailer system with costs

* All Notes are refereed.

† Accepted by Warren H. Hausman; received March 10, 1977. This paper was with the authors 6 months for 2 revisions.

‡ Cornell University.

§ Bell Telephone Laboratories, Holmdel, N.J.

0025-1909/78/2416/1766$01.25