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Quantitative Convergence Analysis of Iterated Expansive, Set-Valued Mappings

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Abstract. We develop a framework for quantitative convergence analysis of Picard iterations of expansive set-valued fixed point mappings. There are two key components of the analysis. The first is a natural generalization of single-valued averaged mappings to expansive set-valued mappings that characterizes a type of strong calmness of the fixed point mapping. The second component to this analysis is an extension of the well-established notion of metric subregularity—or inverse calmness—of the mapping at fixed points. Convergence of expansive fixed point iterations is proved using these two properties, and quantitative estimates are a natural by-product of the framework. To demonstrate the application of the theory, we prove, for the first time, a number of results showing local linear convergence of nonconvex cyclic projections for inconsistent (and consistent) feasibility problems, local linear convergence of the forward-backward algorithm for structured optimization without convexity, strong or otherwise, and local linear convergence of the Douglas-Rachford algorithm for structured nonconvex minimization. This theory includes earlier approaches for known results, convex and nonconvex, as special cases.

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1. Introduction

We present a program of analysis that enables one to quantify the rate of convergence of sequences generated by fixed point iterations of expansive set-valued mappings. The framework presented here subsumes earlier approaches for analyzing fixed point iterations of relaxed nonexpansive mappings and opens up new results for *expansive* mappings. Our approach has its roots in the pioneering work of Mann, Krasnoselski, Edelstein, Gurin,¹ Polyak, and Raik who wrote seminal papers in the analysis of (firmly) nonexpansive and averaged mappings (Mann [54], Krasnoselski [41], Edelstein [31], Gubin et al. [32]) although the terminology “averaged” wasn’t coined until sometime later in Baillon et al. [8]. Our strategy is also indebted to the developers of notions of stability, in particular, metric regularity and its more recent refinements (Penot [67], Azé [7], Dontchev and Rockafellar [29], Ioffe [36, 37]). We follow a pattern of proof used in Hesse and Luke [33] and Aspelmeier et al. [3] for Picard iterations of set-valued mappings, though this approach was actually inspired by the analysis of alternating projections in Gubin et al. [32].

The idea is to isolate two properties of the fixed point mapping. The first property is a generalization of the averaging property, what we call *almost averaging*. When a self-mapping is averaged and fixed points exist, then the Picard iteration converges to a fixed point (weakly in the infinite dimensional setting) without any additional assumptions. (See Opial [65, theorem 3]. See also Schaefer [74, 3. Satz] for the statement under the assumption that the mapping is weakly continuous.) To *quantify* convergence, a second property is needed. In their analysis of Krasnoselski-Mann relaxed cyclic projections for convex feasibility, Gubin et al. [32] assume that the set-intersection has interior (Gubin et al. [32, theorem 1]). Interiority is an assumption about *stability*

of the fixed points of the mapping, and this generalizes considerably. Even if rates of convergence are not the primary interest, if the averaging property is relaxed in any meaningful way, monotonicity of Picard iterations with respect to the set of fixed points is lost. To recover convergence in this case, we appeal to stability of the set of fixed points to overcome the lack of monotonicity of the fixed point mapping. The second property we require of the mapping is a characterization of the needed stability at fixed points. Metric *subregularity* of the mapping at fixed points is one well-established notion that fulfills this stability and provides quantitative estimates for the rate of convergence of the iterates. This is closely related (actually synonymous) to the existence of *error bounds*. The almost averaging and the stability properties are defined and quantified on local neighborhoods, but our approach is not asymptotic. Indeed, when convexity or nonexpansivity is assumed, these local neighborhoods extend to the whole space and the corresponding results are global and recover the classical results.

We take care to introduce the notions of almost averaging, stability, and metric subregularity, and to present the most general abstract results in Section 2. Almost averaged mappings are developed first in Section 2.1, after which abstract convergence results are presented in Section 2.2. In Section 2.3, the notion of metric regularity and its variants is presented and applied to the abstract results of Section 2.2. The rest of the paper, Section 3, is a tutorial on the application of these ideas to quantitative convergence analysis of algorithms for, respectively, nonconvex and inconsistent feasibility (Section 3.1) and structured optimization (Section 3.2). We focus our attention on just a few simple algorithms; namely, cyclic projections, projected gradients and Douglas-Rachford.

Among the new and recent concepts are: almost nonexpansive/averaged mappings (Section 2.1), which are a generalization of averaged mappings (Baillon et al. [8]) and satisfy a type of strong calmness of set-valued mappings; a generalization of hypomonotonicity of set-valued self-mappings (Definition 2.3), which is equivalent to almost firm-nonexpansiveness of their resolvents (Proposition 2.3) generalizing Minty's classical identification of monotone mappings with firmly-nonexpansive resolvents (Minty [55], Reich [71]); elementally subregular sets (Definition 3.1 from Kruger et al. [44, definition 5]); subtransversality of collections of sets at points of non-intersection (Definition 3.2); and gauge metric subregularity (Definition 2.5 from Ioffe [36, 37]). These objects are applied to obtain a number of new results: local linear convergence of nonconvex cyclic projections for inconsistent feasibility problems (Theorem 3.2) with some surprising special cases like two nonintersecting circles (Example 3.5) and practical (inconsistent) phase retrieval (Example 3.6); global R-linear convergence of cyclic projections onto convex sets (Corollary 3.1); local linear convergence of forward-backward-type algorithms without convexity or strong monotonicity (Theorem 3.3); local linear convergence of the Douglas-Rachford algorithm for structured nonconvex optimization (Theorem 3.4) and a specialization to the relaxed averaged alternating reflections (RAAR) algorithm (Luke [47, 48]) for inconsistent phase retrieval (Example 3.8).

The quantitative convergence results presented here focus on linear convergence, but this framework is appropriate for a wider range of behaviors, particularly, sublinear convergence. The emphasis on linear convergence is, in part, due to its simplicity, but also because it is surprisingly prevalent in first-order algorithms for common problem structures (see the discussions of phase retrieval in Examples 3.6 and 3.8). To be sure, there are constants that would, if known, determine the exact rate, and these are either hard or impossible to calculate. But in many instances, the *order* of convergence—linear or sublinear—can be determined a priori. As such, a posteriori *error bounds* can be estimated in some cases, with the usual epistemological caveats, from the observed behavior of the algorithm. For problems where the *solution* to the underlying variational problem, as opposed to its *optimal value*, is the only meaningful result of the numerical algorithm, such error bounds are essential. One important example is image processing with statistical constraints studied in Aspelmeier et al. [3] and Luke and Shefi [51]. Here, the images are physical measurements and solutions to the variational image processing problems have a quantitative statistical interpretation in terms of the experimental data. In contrast, the more common analysis determining that an algorithm for computing these solutions merely converges, or even that the *objective value* converges at a given rate, leads unavoidably to vacuous assurances.

1.1. Basic Definitions and Notation

The setting throughout this work is a finite dimensional Euclidean space $\mathbb{E}$. The norm $\|\cdot\|$ denotes the Euclidean norm. The open unit ball and the unit sphere in a Euclidean space are denoted $\mathbb{B}$ and $\mathbb{S}$, respectively. $\mathbb{B}_\delta(x)$ stands for the open ball with radius $\delta > 0$ and center x . We denote the extended reals by $(-\infty, +\infty] := \mathbb{R} \cup \{+\infty\}$. The domain of a function $f: U \rightarrow (-\infty, +\infty]$ is defined by $\text{dom } f = \{u \in U \mid f(u) < +\infty\}$. The *subdifferential* of f at $\bar{x} \in \text{dom } f$, for our purposes, can be defined by

$$\partial f(\bar{x}) := \left\{ v \mid \exists v^k \rightarrow v \text{ and } x^k \rightarrow \bar{x} \text{ such that } f(x) \geq f(x^k) + \langle v^k, x - x^k \rangle + o(\|x - x^k\|) \right\}. \quad (1)$$

Here, the notation $x^k \xrightarrow{f} \bar{x}$ means that $x^k \rightarrow \bar{x} \in \text{dom } f$ and $f(x^k) \rightarrow f(\bar{x})$. When f is convex, (1) reduces to the usual convex subdifferential given by

$$\partial f(\bar{x}) := \{v \in U \mid \langle v, x - \bar{x} \rangle \leq f(x) - f(\bar{x}), \text{ for all } x \in U\}. \quad (2)$$

When $\bar{x} \notin \text{dom } f$ the subdifferential is defined to be empty. Elements of the subdifferential are called *subgradients*.

A set-valued mapping T from $\mathbb{E}$ to another Euclidean space $\mathbb{Y}$ is denoted $T: \mathbb{E} \rightrightarrows \mathbb{Y}$ and its *inverse* is given by

$$T^{-1}(y) := \{x \in \mathbb{E} \mid y \in T(x)\}. \quad (3)$$

The mapping $T: \mathbb{E} \rightrightarrows \mathbb{E}$ is said to be *monotone* on $\Omega \subset \mathbb{E}$ if

$$\forall x, y \in \Omega \quad \inf_{\substack{x' \in T(x) \\ y' \in T(y)}} \langle x' - y', x - y \rangle \geq 0. \quad (4)$$

T is called *strongly monotone* on Ω if there exists a $\tau > 0$ such that

$$\forall x, y \in \Omega \quad \inf_{\substack{x' \in T(x) \\ y' \in T(y)}} \langle x' - y', x - y \rangle \geq \tau \|x - y\|^2. \quad (5)$$

A *maximally monotone* mapping is a monotone mapping whose graph cannot be augmented by any more points without violating monotonicity. The subdifferential of a proper, l.s.c., convex function, for example, is a maximally monotone set-valued mapping (Rockafellar and Wets [72, theorem 12.17]). We denote the *resolvent* of T by $J_T := (\text{Id} + T)^{-1}$ where Id denotes the identity mapping. The corresponding *reflector* is defined by $R_T := 2J_T - \text{Id}$. A basic and fundamental fact is that the resolvent of a monotone mapping is firmly nonexpansive, and hence single-valued (Minty [55], Bruck and Reich [22]). Of particular interest are *polyhedral* (or *piecewise polyhedral* Rockafellar and Wets [72]) mappings; that is, mappings $T: \mathbb{E} \rightrightarrows \mathbb{Y}$ whose graph is the union of finitely many sets that are polyhedral convex in $\mathbb{E} \times \mathbb{Y}$ (Dontchev and Rockafellar [29]).

Notions of *continuity* of set-valued mappings have been thoroughly developed over the last 40 years. Readers are referred to the monographs (Aubin and Frankowska [6], Rockafellar and Wets [72], Dontchev and Rockafellar [29]) for basic results. A mapping $T: \mathbb{E} \rightrightarrows \mathbb{Y}$ is said to be *Lipschitz continuous* if it is closed-valued and there exists a $\tau \geq 0$ such that, for all $u, u' \in \mathbb{E}$,

$$T(u') \subset T(u) + \tau \|u' - u\| \mathbb{B}. \quad (6)$$

Lipschitz continuity is, however, too strong a notion for set-valued mappings. We will mostly only require *calmness*, which is a pointwise version of Lipschitz continuity. A mapping $T: \mathbb{E} \rightrightarrows \mathbb{Y}$ is said to be *calm* at $\bar{u}$ for $\bar{v}$ if $(\bar{u}, \bar{v}) \in \text{gph } T$, and there is a constant κ together with neighborhoods $U \times V$ of $(\bar{u}, \bar{v})$ such that

$$T(u) \cap V \subset T(\bar{u}) + \kappa \|u - \bar{u}\|, \quad \forall u \in U. \quad (7)$$

When T is single-valued, calmness is just pointwise Lipschitz continuity:

$$\|T(u) - T(\bar{u})\| \leq \kappa \|u - \bar{u}\|, \quad \forall u \in U. \quad (8)$$

Closely related to calmness is *metric subregularity*, which can be understood as the property corresponding to a calmness of the inverse mapping. As the name suggests, it is a weaker property than *metric regularity*, which in the case of an $n \times m$ matrix, for instance ($m \leq n$), is equivalent to surjectivity. Our definition follows the characterization of this property given in Ioffe [36, 37], and appropriates the terminology of Dontchev and Rockafellar [29] with slight but significant variations. The *graphical derivative* of a mapping $T: \mathbb{E} \rightrightarrows \mathbb{Y}$ at a point $(x, y) \in \text{gph } T$ is denoted $DT(x \mid y): \mathbb{E} \rightrightarrows \mathbb{Y}$ and defined as the mapping whose graph is the tangent cone to $\text{gph } T$ at (x, y) (see Aubin and Center [5] where it is called the contingent derivative). That is,

$$v \in DT(x \mid y)(u) \iff (u, v) \in \mathcal{T}_{\text{gph } T}(x, y), \quad (9)$$

where $\mathcal{T}_\Omega$ is the tangent cone mapping associated with the set Ω defined by

$$\mathcal{T}_\Omega(\bar{x}) := \left\{ w \mid \frac{(x^k - \bar{x})}{\tau} \rightarrow w \text{ for some } x^k \xrightarrow{\Omega} \bar{x}, \tau \searrow 0 \right\}. \quad (10)$$

Here, the notation $x^k \xrightarrow{\Omega} \bar{x}$ means that the sequence of points $\{x^k\}$ approaches $\bar{x}$ from within Ω .

The distance to a set $\Omega \subset \mathbb{E}$ with respect to the bivariate function $\text{dist}(\cdot, \cdot)$ is defined by

$$\text{dist}(\cdot, \Omega): \mathbb{E} \rightarrow \mathbb{R}: x \mapsto \inf_{y \in \Omega} \text{dist}(x, y) \quad (11)$$

and the set-valued mapping

$$P_\Omega: \mathbb{E} \rightrightarrows \mathbb{E}: x \mapsto \{y \in \Omega \mid \text{dist}(x, \Omega) = \text{dist}(x, y)\} \quad (12)$$

is the corresponding *projector*. An element $y \in P_\Omega(x)$ is called a *projection*. Closely related to the projector is the *prox* mapping (Moreau [57])

$$\text{prox}_{\lambda, f}(x) := \arg \min_{y \in \mathbb{E}} \left\{ f(y) + \frac{1}{2\lambda} \|y - x\|^2 \right\}.$$

When $f(x) = \iota_\Omega$, then $\text{prox}_{\lambda, \iota_\Omega} = P_\Omega$ for all $\lambda > 0$. The value function corresponding to the prox mapping is known as the *Moreau envelope*, which we denote by $e_{\lambda, f}(x) := \inf_{y \in \mathbb{E}} \{f(y) + (1/(2\lambda))\|y - x\|^2\}$. When $\lambda = 1$ and $f = \iota_\Omega$, the Moreau envelope is just one-half the squared distance to the set Ω : $e_{1, \iota_\Omega}(x) = \frac{1}{2} \text{dist}^2(x, \Omega)$. The *inverse* projector P_Ω^{-1} is defined by

$$P_\Omega^{-1}(y) := \{x \in \mathbb{E} \mid P_\Omega(x) \ni y\}. \quad (13)$$

Throughout this note, we will assume the distance corresponds to the Euclidean norm, though most of the statements are not limited to this. When $\text{dist}(x, y) = \|x - y\|$, then one has the following variational characterization of the projector: $\bar{z} \in P_\Omega^{-1}\bar{x}$ if and only if

$$\langle \bar{z} - \bar{x}, x - \bar{x} \rangle \leq \frac{1}{2} \|x - \bar{x}\|^2 \quad \forall x \in \Omega. \quad (14)$$

Following Bauschke et al. [17], we use this object to define the various normal cone mappings, which, in turn, lead to the *subdifferential* of the indicator function ι_Ω .

The ε -normal cone to Ω at $\bar{x} \in \Omega$ is defined

$$\hat{N}_\Omega^\varepsilon(\bar{x}) := \left\{ v \mid \limsup_{x \rightarrow \bar{x}, x \neq \bar{x}} \frac{\langle v, x - \bar{x} \rangle}{\|x - \bar{x}\|} \leq \varepsilon \right\}. \quad (15)$$

The (limiting) normal cone to Ω at $\bar{x} \in \Omega$, denoted $N_\Omega(\bar{x})$, is defined as the limsup of the ε -normal cones. That is, a vector $v \in N_\Omega(\bar{x})$ if there are sequences $x^k \xrightarrow{\Omega} \bar{x}$, $v^k \rightarrow v$ with $v^k \in \hat{N}_\Omega^{\varepsilon_k}(x^k)$ and $\varepsilon_k \searrow 0$. The *proximal normal cone* to Ω at $\bar{x}$ is the set

$$N_\Omega^{\text{prox}}(\bar{x}) := \text{cone}(P_\Omega^{-1}\bar{x} - \bar{x}). \quad (16)$$

If $\bar{x} \notin \Omega$, then all normal cones are defined to be empty.

The proximal normal cone need not be closed. The limiting normal cone is, of course, closed by definition. See Mordukhovich [56, definition 1.1] or Rockafellar and Wets [72, definition 6.3] (where this is called the regular normal cone) for an in-depth treatment as well as (Mordukhovich [56, p. 141]) for historical notes. When the projection is with respect to the Euclidean norm, the limiting normal cone can be written as the limsup of proximal normals:

$$N_\Omega(\bar{x}) = \overline{\lim_{x \rightarrow \bar{x}} N_\Omega^{\text{prox}}(x)}. \quad (17)$$

2. General Theory: Picard Iterations

2.1. Almost Averaged Mappings

Our ultimate goal is a quantitative statement about convergence to fixed points for set-valued mappings. Preparatory to this, we first must be clear what is meant by a fixed point of a set-valued mapping.

Definition 2.1 (Fixed Points of Set-Valued Mappings). The set of fixed points of a set-valued mapping $T: \mathbb{E} \rightrightarrows \mathbb{E}$ is defined by

$$\text{Fix } T := \{x \in \mathbb{E} \mid x \in T(x)\}.$$

In the set-valued setting, it is important to keep in mind a few things that can happen that cannot happen when the mapping is single-valued.

Example 2.1 (Inhomogeneous Fixed Point Sets). Let $T := P_A P_B$, where

$$A = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 \geq -2x_1 + 3\} \cap \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 \geq 1\}, \quad B = \mathbb{R}^2 \setminus \mathbb{R}_{++}^2.$$

Here, $P_B(1, 1) = \{(0, 1), (1, 0)\}$ and the point $(1, 1)$ is a fixed point of T since $(1, 1) \in P_A\{(0, 1), (1, 0)\}$. However, the point $P_A(0, 1)$ is also in $T(1, 1)$, and this is not a fixed point of T . $\square$

To help rule out inhomogeneous fixed point sets like the one in the previous example, we introduce the following strong calmness of fixed point mappings that is an extension of conventional nonexpansiveness and firm nonexpansiveness. What we call *almost nonexpansive mappings* below were called (S, ϵ) -nonexpansive mappings in Hesse and Luke [33, definition 2.3], and almost averaged mappings are slight generalization of (S, ϵ) -firmly nonexpansive mappings also defined there.

Definition 2.2 (Almost Nonexpansive/Averaged Mappings). Let D be a nonempty subset of $\mathbb{E}$ and let T be a (set-valued) mapping from D to $\mathbb{E}$:

- (i) T is said to be *pointwise almost nonexpansive on D at $y \in D$* if there exists a constant $\epsilon \in [0, 1)$ such that

$$\|x^+ - y^+\| \leq \sqrt{1 + \epsilon} \|x - y\|, \quad \forall y^+ \in Ty \text{ and } \forall x^+ \in Tx \text{ whenever } x \in D. \quad (18)$$

If (18) holds with $\epsilon = 0$, then T is called *pointwise nonexpansive at y on D* .

If T is pointwise (almost) nonexpansive at every point on a neighborhood of y (with the same violation constant ϵ) on D , then T is said to be *(almost) nonexpansive at y (with violation ϵ) on D* .

If T is pointwise (almost) nonexpansive on D at every point $y \in D$ (with the same violation constant ϵ), then T is said to be *pointwise (almost) nonexpansive on D (with violation ϵ)*. If D is open and T is pointwise (almost) nonexpansive on D , then it is (almost) nonexpansive on D .

- (ii) T is called *pointwise almost averaged on D at y* if there is an averaging constant $\alpha \in (0, 1)$ and a violation constant $\epsilon \in [0, 1)$ such that the mapping $\tilde{T}$ defined by

$$T = (1 - \alpha)\text{Id} + \alpha\tilde{T}$$

is pointwise almost nonexpansive at y with violation ϵ/α on D .

Likewise, if $\tilde{T}$ is (pointwise) (almost) nonexpansive on D (at y) (with violation ϵ), then T is said to be *(pointwise) (almost) averaged on D (at y) (with averaging constant α and violation $\alpha\epsilon$)*.

If the averaging constant $\alpha = 1/2$, then T is said to be *(pointwise) (almost) firmly nonexpansive on D (with violation ϵ) (at y)*.

Note that the mapping T need not be a self-mapping from D to itself. In the special case where T is (firmly) nonexpansive at all points $y \in \text{Fix } T$, mappings satisfying (18) are also called *quasi-(firmly)nonexpansive* (Bauschke and Combettes [10]).

The term “almost nonexpansive” has been used for different purposes by Nussbaum [64] and Rouhani [73]. Rouhani uses the term to indicate sequences in the Hilbert space setting that are asymptotically nonexpansive. Nussbaum’s definition is the closest in spirit and definition to ours, except that he defines f to be locally almost nonexpansive when $\|f(y) - f(x)\| \leq \|y - x\| + \epsilon$. In this context, see also Reich [70]. At the risk of some confusion, we re-purpose the term here. Our definition of pointwise almost nonexpansiveness of T at $\bar{x}$ is stronger than *calmness* Rockafellar and Wets [72, chapter 8.F] with constant $\lambda = \sqrt{1 + \epsilon}$ since the inequality must hold for all pairs $x^+ \in Tx$ and $y^+ \in Ty$, while for calmness, the inequality would hold only for points $x^+ \in Tx$ and their *projections* onto Ty . We have avoided the temptation to call this property “strong calmness” to make clearer the connection to the classical notions of (firm) nonexpansiveness. A theory based only on calm mappings, what one might call “weakly almost averaged/nonexpansive” operators is possible and would yield statements about the existence of convergent *selections* from sequences of iterated set-valued mappings. In light of the other requirement of the mapping T that we will explore in Section 2.3, namely, metric subregularity, this would illuminate an aesthetically pleasing and fundamental symmetry between requirements on T and its inverse. We leave this avenue of investigation open. Our development of the properties of almost averaged operators parallels the treatment of averaged operators in Bauschke and Combettes [10].

Proposition 2.1 (Characterizations of Almost Averaged Operators). Let $T: \mathbb{E} \rightrightarrows \mathbb{E}$, $U \subset \mathbb{E}$, and $\alpha \in (0, 1)$. The following are equivalent;

- (i) T is pointwise almost averaged at y on U with violation ϵ and averaging constant α .
(ii) $(1 - 1/\alpha)\text{Id} + (1/\alpha)T$ is pointwise almost nonexpansive at y on $U \subset \mathbb{E}$ with violation ϵ/α .

(iii) For all $x \in U$, $x^+ \in T(x)$, and $y^+ \in T(y)$, it holds that

$$\|x^+ - y^+\|^2 \leq (1 + \varepsilon)\|x - y\|^2 - \frac{1 - \alpha}{\alpha} \|(x - x^+) - (y - y^+)\|^2. \quad (19)$$

Consequently, if T is pointwise almost averaged at y on U with violation ε and averaging constant α , then T is pointwise almost nonexpansive at y on U with violation at most ε .

Proof. This is a slight extension of Bauschke and Combettes [10, proposition 4.25]. $\square$

Example 2.2 (Alternating Projections). Let $T := P_A P_B$ for the closed sets A and B defined below:

- (i) If A and B are convex, then T is nonexpansive and averaged (i.e., pointwise everywhere, no violation).
- (ii) Packman eating a piece of pizza:

$$A = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \leq 1, -1/2x_1 \leq x_2 \leq x_1, x_1 \geq 0\} \subset \mathbb{R}^2, \\ B = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \leq 1, x_1 \leq |x_2|\} \subset \mathbb{R}^2, \quad \bar{x} = (0, 0).$$

The mapping T is not almost nonexpansive on any neighborhood for any finite violation at $y = (0, 0) \in \text{Fix } T$, but it is pointwise nonexpansive (no violation) at $y = (0, 0)$ and nonexpansive at all $y \in (A \cap B) \setminus \{(0, 0)\}$ on small enough neighborhoods of these points.

(iii) T is pointwise averaged at $(1, 1)$ when

$$A = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 \leq 2x_1 - 1\} \cap \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 \geq 1/2x_1 + 1/2\}, \quad B = \mathbb{R}^2 \setminus \mathbb{R}_{++}^2.$$

This illustrates that whether or not A and B have points in common is not relevant to the property.

(iv) T is not pointwise almost averaged at $(1, 1)$ for any $\varepsilon > 0$ when

$$A = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 \geq -2x_1 + 3\} \cap \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 \geq 1\}, \quad B = \mathbb{R}^2 \setminus \mathbb{R}_{++}^2.$$

In light of Example 2.1, this shows that the pointwise almost averaged property is incompatible with inhomogeneous fixed points (see Proposition 2.2). $\square$

Proposition 2.2 (Pointwise Single-Valuedness). If $T: \mathbb{E} \rightrightarrows \mathbb{E}$ is pointwise almost nonexpansive on $D \subseteq \mathbb{E}$ at $\bar{x} \in D$ with violation $\varepsilon \geq 0$, then T is single-valued at $\bar{x}$. In particular, if $\bar{x} \in \text{Fix } T$ (that is, $\bar{x} \in T\bar{x}$), then $T\bar{x} = \{\bar{x}\}$.

Proof. By the definition of pointwise nonexpansive on D at $\bar{x}$, it holds that

$$\|x^+ - \bar{x}^+\| \leq \sqrt{1 + \varepsilon} \|x - \bar{x}\|$$

for all $x \in D$, $x^+ \in T(x)$ and $\bar{x}^+ \in T(\bar{x})$. In particular, setting $x = \bar{x}$ yields

$$\|x^+ - \bar{x}^+\| \leq \sqrt{1 + \varepsilon} \|\bar{x} - \bar{x}\| = 0.$$

That is, $x^+ = \bar{x}^+$ and hence we conclude that T is single-valued at $\bar{x}$. $\square$

Example 2.3 (Pointwise Almost Nonexpansive Mappings Not Single-Valued). Although a pointwise almost nonexpansive mapping is single-valued at the reference point, it need not be single-valued on neighborhoods of the reference points. Consider, for example, the coordinate axes in $\mathbb{R}^2$,

$$A = \mathbb{R} \times \{0\} \cup \{0\} \times \mathbb{R}.$$

The metric projector P_A is single-valued and even pointwise nonexpansive (no “almost”) at every point in A , but multivalued on $L := \{(x, y) \in \mathbb{R}^2 \setminus \{0\} \mid |x| = |y|\}$. $\square$

Almost firmly nonexpansive mappings have particularly convenient characterizations. In our development below and thereafter, we use the set S to denote the collection of points at which the property holds. This is useful for distinguishing points where the regularity holds from other points of interest, like fixed points. In Section 2.3, the set S is used to isolate a subset of fixed points. The idea here is that the properties needed to quantify convergence need not hold on the space where a problem is formulated, but may only hold on a subset of this space where the iterates of a particular algorithm may be naturally confined. This is used in Aspelmeier et al. [3] to achieve linear convergence results for the alternating directions method of multipliers algorithm.

Alternatively, S can also include points that are not fixed points of constituent operators in an algorithm, but are closely related to fixed points. One example of this is *local best approximation points*; that is, points in one set that are locally nearest to another. In Section 3.1, we will need to quantify the violation of the averaging property for a projector onto a nonconvex set A at points in another set; say, B that are locally nearest points to A . This will allow us to tackle *inconsistent feasibility* where the alternating projections iteration converges not to the intersection, but to local best approximation points.

Proposition 2.3 (Almost Firmly Nonexpansive Mappings). *Let $S \subset U \subset \mathbb{E}$ be nonempty and $T: U \rightrightarrows \mathbb{E}$. The following are equivalent:*

- (i) T is pointwise almost firmly nonexpansive on U at all $y \in S$ with violation ε .
- (ii) The mapping $\tilde{T}: U \rightrightarrows \mathbb{E}$ given by

$$\tilde{T}x := (2Tx - x), \quad \forall x \in U \quad (20)$$

is pointwise almost nonexpansive on U at all $y \in S$ with violation 2ε ; that is, T can be written as

$$Tx = \frac{1}{2}(x + \tilde{T}x), \quad \forall x \in U. \quad (21)$$

- (iii) $\|x^+ - y^+\|^2 \leq (\varepsilon/2)\|x - y\|^2 + \langle x^+ - y^+, x - y \rangle$ for all $x^+ \in Tx$, and all $y^+ \in Ty$ at each $y \in S$ whenever $x \in U$.
- (iv) Let $F: \mathbb{E} \rightrightarrows \mathbb{E}$ be a mapping whose resolvent is T , i.e., $T = (\text{Id} + F)^{-1}$. At each $x \in U$ for all $u \in Tx$, $y \in S$, and $v \in Ty$, the points (u, z) and (v, w) are in $\text{gph } F$, where $z = x - u$ and $w = y - v$, and satisfy

$$-\frac{\varepsilon}{2}\|(u + z) - (v + w)\|^2 \leq \langle z - w, u - v \rangle. \quad (22)$$

Proof. (i) $\Leftrightarrow$ (ii): Follows from Proposition 2.1 when $\alpha = 1/2$.

(ii) $\Rightarrow$ (iii): Note first that, at each $x \in U$ and $y \in S$,

$$\|(2x^+ - x) - (2y^+ - y)\|^2 = 4\|x^+ - y^+\|^2 - 4\langle x^+ - y^+, x - y \rangle + \|x - y\|^2 \quad (23a)$$

for all $x^+ \in Tx$ and $y^+ \in Ty$. Repeating the definition of pointwise almost nonexpansiveness of $2T - \text{Id}$ at $y \in S$ with violation 2ε on U ,

$$\|(2x^+ - x) - (2y^+ - y)\|^2 \leq (1 + 2\varepsilon)\|x - y\|^2. \quad (23b)$$

Together (23) yields

$$\|x^+ - y^+\|^2 \leq \frac{\varepsilon}{2}\|x - y\|^2 + \langle x^+ - y^+, x - y \rangle,$$

as claimed.

(iii) $\Rightarrow$ (ii): Use (23a) to replace $\langle x^+ - y^+, x - y \rangle$ in (iii) and rearrange the resulting inequality to conclude that $2T - \text{Id}$ is pointwise almost nonexpansive at $y \in S$ with violation 2ε on U .

(iv) $\Leftrightarrow$ (iii): First, note that $(u, z) \in \text{gph } F$ if and only if $(u + z, u) \in \text{gph}(\text{Id} + F)^{-1}$. From this, it follows that, for $u \in Tx$ and $v \in Ty$, the points (u, z) and (v, w) with $z = x - u$ and $w = y - v$, are in $\text{gph } F$. Therefore starting with (iii), at each $x \in U$ and $y \in S$,

$$\|u - v\|^2 \leq \frac{\varepsilon}{2}\|x - y\|^2 + \langle u - v, x - y \rangle \quad (24)$$

$$= \frac{\varepsilon}{2}\|(u + z) - (v + w)\|^2 + \langle u - v, (u + z) - (v + w) \rangle \quad (25)$$

for all $u \in Tx$ and $v \in Ty$. Separating out $\|u - v\|^2$ from the inner product on the left-hand side of (25) yields the result. $\square$

Property (iv) of Proposition 2.3 characterizes a type of nonmonotonicity of the mapping F on D with respect to S ; for lack of a better terminology, we call this *Type-I nonmonotonicity*. It can be shown that, for small enough parameter values, this is a generalization of another well-established property known as *hypomonotonicity* (Poliquin et al. [69]). In Daniilidis and Georgiev [27], the notion of *submonotonicity* proposed by Spingarn [75] in relation to *approximate convexity* Ngai et al. [62] was studied. Their relation to the definition below is the topic of future research.

Definition 2.3 (Nonmonotone Mappings). (a) A mapping $F: \mathbb{E} \rightrightarrows \mathbb{E}$ is *pointwise Type-I nonmonotone* at $\bar{v}$ if there is a constant τ together with a neighborhood U of $\bar{v}$ such that

$$-\tau\|(u + z) - (\bar{v} + w)\|^2 \leq \langle z - w, u - \bar{v} \rangle \quad \forall z \in Fu, \quad \forall u \in U, \quad \forall w \in F\bar{v}. \quad (26)$$

The mapping F is said to be *Type-I nonmonotone* on U if (26) holds for all $\bar{v}$ on U .

(b) The mapping $F: \mathbb{E} \rightrightarrows \mathbb{E}$ is said to be *pointwise hypomonotone at $\bar{v}$ with constant τ on U* if

$$-\tau \|u - \bar{v}\|^2 \leq \langle z - w, u - \bar{v} \rangle, \quad \forall z \in Fu, \quad \forall u \in U, \quad \forall w \in F\bar{v}. \quad (27)$$

If (27) holds for all $\bar{v} \in U$ then F is said to be hypomonotone with constant τ on U .

If T is, in fact, firmly nonexpansive (that is, $S = D$ and $\tau = 0$), then Proposition 2.3(iv) just establishes the well-known equivalence between monotonicity of a mapping and firm nonexpansiveness of its resolvent (Minty [55]). Moreover, if a single-valued mapping $f: \mathbb{E} \rightarrow \mathbb{E}$ is calm at $\bar{v}$ with calmness modulus L , then it is pointwise hypomonotone at $\bar{v}$ with violation at most L . Indeed,

$$\langle u - \bar{v}, f(u) - f(\bar{v}) \rangle \geq -\|u - \bar{v}\| \|f(u) - f(\bar{v})\| \geq -L \|u - \bar{v}\|^2. \quad (28)$$

This also points to a relationship to *cohypomonotonicity* developed in Combettes and Pennanen [26]. More recently, the notion of pointwise quadratically supportable functions was introduced (Luke and Shefi [51, definition 2.1]); for smooth functions, this class—which is not limited to convex functions—was shown to include functions whose gradients are pointwise strongly monotone (pointwise hypomonotone with constant $\tau < 0$) Luke and Shefi [51, proposition 2.2]. A deeper investigation of the relationships between these different notions is postponed to future work.

The next result shows the inheritance of the averaging property under compositions and averages of averaged mappings.

Proposition 2.4 (Compositions and Averages of Relatively Averaged Operators). *Let $T_j: \mathbb{E} \rightrightarrows \mathbb{E}$ for $j = 1, 2, \dots, m$ be pointwise almost averaged on U_j at all $y_j \in S_j \subset \mathbb{E}$ with violation ε_j and averaging constant $\alpha_j \in (0, 1)$, where $U_j \supset S_j$ for $j = 1, 2, \dots, m$.*

(i) *If $U := U_1 = U_2 = \dots = U_m$ and $S := S_1 = S_2 = \dots = S_m$, then the weighted mapping $T := \sum_{j=1}^m w_j T_j$ with weights $w_j \in [0, 1]$, $\sum_{j=1}^m w_j = 1$ is pointwise almost averaged at all $y \in S$ with violation $\varepsilon = \sum_{j=1}^m w_j \varepsilon_j$ and averaging constant $\alpha = \max_{j=1,2,\dots,m} \{\alpha_j\}$ on U .*

(ii) *If $T_j U_j \subseteq U_{j-1}$ and $T_j S_j \subseteq S_{j-1}$ for $j = 2, 3, \dots, m$, then the composite mapping $T := T_1 \circ T_2 \circ \dots \circ T_m$ is pointwise almost nonexpansive at all $y \in S_m$ on U_m with violation at most*

$$\varepsilon = \prod_{j=1}^m (1 + \varepsilon_j) - 1. \quad (29)$$

(iii) *If $T_j U_j \subseteq U_{j-1}$ and $T_j S_j \subseteq S_{j-1}$ for $j = 2, 3, \dots, m$, then the composite mapping $T := T_1 \circ T_2 \circ \dots \circ T_m$ is pointwise almost averaged at all $y \in S_m$ on U_m with violation at most ε given by (29) and averaging constant at least*

$$\alpha = \frac{m}{m - 1 + 1/(\max_{j=1,2,\dots,m} \{\alpha_j\})}. \quad (30)$$

Proof. Statement (i) is a formal generalization of Bauschke and Combettes [10, proposition 4.30] and follows directly from convexity of the squared norm and Proposition 2.1(iii).

Statement (ii) follows from applying the definition of almost nonexpansivity to each of the operators T_j inductively from $j = 1$ to $j = m$.

Statement (iii) is formal generalization of Bauschke and Combettes [10, proposition 4.32] and follows from more or less the same pattern of proof. Since it requires a little more care, the proof is given here. Define $\kappa_j := \alpha_j / (1 - \alpha_j)$ and set $\kappa = \max_j \{\kappa_j\}$. Identify y_{j-1} with any $y_j^+ \in T_j y_j \subseteq S_{j-1}$ for $j = 2, 3, \dots, m$ and choose any $y_m \in S_m$. Likewise, identify x_{j-1} with any $x_j^+ \in T_j x_j \subseteq U_{j-1}$ for $j = 2, 3, \dots, m$ and choose any $x_m \in U_m$. Denote $u^+ \in T_1 \circ T_2 \circ \dots \circ T_m u$ for $u := x_m$ and $v^+ \in T_1 \circ T_2 \circ \dots \circ T_m v$ for $v := y_m$. By convexity of the squared norm and Proposition 2.1(iii), one has

$$\begin{aligned} \frac{1}{m} \|(u - u^+) - (v - v^+)\|^2 &\leq \|(x_1 - u^+) - (y_1 - v^+)\|^2 + \|(x_2 - x_1) - (y_2 - y_1)\|^2 + \dots + \|(x_m - x_{m-1}) - (y_m - y_{m-1})\|^2 \\ &\leq \kappa_1((1 + \varepsilon_1)\|x_1 - y_1\|^2 - \|u^+ - v^+\|^2) + \kappa_2((1 + \varepsilon_2)\|x_2 - y_2\|^2 - \|x_1 - y_1\|^2) + \dots \\ &\quad + \kappa_m((1 + \varepsilon_m)\|u - v\|^2 - \|x_{m-1} - y_{m-1}\|^2). \end{aligned}$$

Replacing κ_j by κ yields

$$\frac{1}{m} \|(u - u^+) - (v - v^+)\|^2 \leq \kappa \left((1 + \varepsilon_m) \|u - v\|^2 - \|u^+ - v^+\|^2 + \sum_{i=1}^{m-1} \varepsilon_i \|x_i - y_i\|^2 \right). \quad (31)$$

From part (ii), one has

$$\|x_i - y_i\|^2 = \|x_{i+1}^+ - y_{i+1}^+\|^2 \leq \left(\prod_{j=i+1}^m (1 + \varepsilon_j) \right) \|u - v\|^2, \quad i = 1, 2, \dots, m-1$$

so that

$$\sum_{i=1}^{m-1} \varepsilon_i \|x_i - y_i\|^2 \leq \left(\sum_{i=1}^{m-1} \varepsilon_i \left(\prod_{j=i+1}^m (1 + \varepsilon_j) \right) \right) \|u - v\|^2. \quad (32)$$

Putting (31) and (32) together yields

$$\frac{1}{m} \|(u - u^+) - (v - v^+)\|^2 \leq \kappa \left(\left(1 + \varepsilon_m + \sum_{i=1}^{m-1} \varepsilon_i \left(\prod_{j=i+1}^m (1 + \varepsilon_j) \right) \right) \|u - v\|^2 - \|u^+ - v^+\|^2 \right). \quad (33)$$

The composition T is therefore almost averaged with violation

$$\varepsilon = \varepsilon_m + \sum_{i=1}^{m-1} \varepsilon_i \left(\prod_{j=i+1}^m (1 + \varepsilon_j) \right)$$

and averaging constant $\alpha = m/(m + 1/\kappa)$. Finally, an induction argument shows that

$$\varepsilon_m + \sum_{i=1}^{m-1} \varepsilon_i \left(\prod_{j=i+1}^m (1 + \varepsilon_j) \right) = \prod_{j=1}^m (1 + \varepsilon_j) - 1,$$

which is the claimed violation. $\square$

Remark 2.1. We remark that Proposition 2.4(ii) holds in the case when T_j ($j = 1, 2, \dots, m$) are merely pointwise almost nonexpansive. The counterpart for T_j ($j = 1, \dots, m$) pointwise almost nonexpansive to Proposition 2.4(i) is given by allowing $\alpha = 0$.

Corollary 2.1 (Krasnoselski-Mann Relaxations). *Let $\lambda \in [0, 1]$ and define $T_\lambda := (1 - \lambda)\text{Id} + \lambda T$ for T pointwise almost averaged at y with violation ε and averaging constant α on U . Then, T_λ is pointwise almost averaged at y with violation $\lambda\varepsilon$ and averaging constant α on U . In particular, when $\lambda = 1/2$, the mapping $T_{1/2}$ is pointwise almost firmly nonexpansive at y with violation $\varepsilon/2$ on U .*

Proof. Noting that Id is averaged everywhere on $\mathbb{E}$ with zero violation and all averaging constants $\alpha \in (0, 1)$, the statement is an immediate specialization of Proposition 2.4(i). $\square$

A particularly attractive consequence of Corollary 2.1 is that the violation of almost averaged mappings can be mitigated by taking smaller steps via Krasnoselski-Mann relaxation.

To conclude this section, we prove the following lemma, a special case of which will be required in Section 3.1.3, which relates the fixed point set of the composition of pointwise almost averaged operators to the corresponding difference vector.

Definition 2.4 (Difference Vectors of Composite Mappings). For a collection of operators $T_j: \mathbb{E} \rightrightarrows \mathbb{E}$ ($j = 1, 2, \dots, m$) and $T := T_1 \circ T_2 \circ \dots \circ T_m$, the set of difference vectors of T at u is given by the mapping $\mathcal{Z}: \mathbb{E} \rightrightarrows \mathbb{E}^m$ defined by

$$\mathcal{Z}(u) := \{\zeta := z - \Pi z \mid z \in W_0 \subset \mathbb{E}^m, z_1 = u\}, \quad (34)$$

where $\Pi: z = (z_1, z_2, \dots, z_m) \mapsto (z_2, \dots, z_m, z_1)$ is the permutation mapping on the product space $\mathbb{E}^m$ for $z_j \in \mathbb{E}$ ($j = 1, 2, \dots, m$) and

$$W_0 := \{x = (x_1, \dots, x_m) \in \mathbb{E}^m \mid x_m \in T_m x_1, x_j \in T_j(x_{j+1}), j = 1, 2, \dots, m-1\}.$$

Lemma 2.1 (Difference Vectors of Averaged Compositions). *Given a collection of operators $T_j: \mathbb{E} \rightrightarrows \mathbb{E}$ ($j = 1, 2, \dots, m$), set $T := T_1 \circ T_2 \circ \dots \circ T_m$. Let $S_0 \subset \text{Fix } T$ and U_0 be a neighborhood of S_0 and define $U := \{z = (z_1, z_2, \dots, z_m) \in W_0 \mid z_1 \in U_0\}$. Fix $\bar{u} \in S_0$ and the difference vector $\bar{\zeta} \in \mathcal{Z}(\bar{u})$ with $\bar{\zeta} = \bar{z} - \Pi \bar{z}$ for the point $\bar{z} = (\bar{z}_1, \bar{z}_2, \dots, \bar{z}_m) \in W_0$ having $\bar{z}_1 = \bar{u}$. Let T_j be pointwise almost averaged at $\bar{z}_j$ with violation ε_j and averaging constant α_j on $U_j := p_j(U)$, where $p_j: \mathbb{E}^m \rightarrow \mathbb{E}$*

denotes the j th coordinate projection operator ($j = 1, 2, \dots, m$). Then, for $u \in S_0$ and $\zeta \in \mathcal{X}(u)$ with $\zeta = z - \Pi z$ for $z = (z_1, z_2, \dots, z_m) \in W_0$ having $z_1 = u$,

$$\frac{1-\alpha}{\alpha} \|\bar{\zeta} - \zeta\|^2 \leq \sum_{j=1}^m \varepsilon_j \|\bar{z}_j - z_j\|^2, \quad \text{where } \alpha = \max_{j=1,2,\dots,m} \alpha_j. \quad (35)$$

If the mapping T_j is, in fact, pointwise averaged at $\bar{z}_j$ on U_j ($j = 1, 2, \dots, m$), then the set of difference vectors of T is a singleton and independent of the initial point; that is, there exists $\bar{\zeta} \in \mathbb{E}^m$ such that $\mathcal{X}(u) = \{\bar{\zeta}\}$ for all $u \in S_0$.

Proof. First, observe that, since $\bar{\zeta} \in \mathcal{X}(\bar{u})$, there exists $\bar{z} = (\bar{z}_1, \bar{z}_2, \dots, \bar{z}_m) \in W_0$ with $\bar{z}_1 = \bar{u}$ such that $\bar{\zeta} = \bar{z} - \Pi \bar{z}$, hence U , and thus $U_j = p_j(U)$ is nonempty since it at least contains $\bar{z}$ (and $\bar{z}_j \in U_j$ for $j = 1, 2, \dots, m$). Consider a second point $u \in S_0$ and let $\zeta \in \mathcal{X}(u)$. Similarly, there exists $z = (z_1, z_2, \dots, z_m) \in W_0$ such that $z_1 = u$ and $\zeta = z - \Pi z$ with $z \in U$. For each $j = 1, 2, \dots, m$, we therefore have that

$$\|(\bar{z}_j - \bar{z}_{j-1}) - (z_j - z_{j-1})\| = \|\bar{\zeta}_j - \zeta_j\|, \quad (36)$$

and since T_j is pointwise almost averaged at $\bar{z}_j$ with constant α_j and violation ε_j on U_j ,

$$\|\bar{z}_j - z_j\|^2 + \frac{1-\alpha_j}{\alpha_j} \|\bar{\zeta}_j - \zeta_j\|^2 \leq (1+\varepsilon_j) \|\bar{z}_{j-1} - z_{j-1}\|^2, \quad (37)$$

where $\bar{z}_0 := \bar{z}_m$ and $z_0 = z_m$. Altogether this yields

$$\frac{1-\alpha}{\alpha} \|\bar{\zeta} - \zeta\|^2 \leq \sum_{j=1}^m \frac{1-\alpha_j}{\alpha_j} \|\bar{\zeta}_j - \zeta_j\|^2 \leq \sum_{j=1}^m ((1+\varepsilon_j) \|\bar{z}_{j-1} - z_{j-1}\|^2 - \|\bar{z}_j - z_j\|^2) = \sum_{j=1}^m \varepsilon_j \|\bar{z}_j - z_j\|^2,$$

which proves (35). If in addition, for all $j = 1, 2, \dots, m$, the mappings T_j are pointwise averaged, then $\varepsilon_1 = \varepsilon_2 = \dots = \varepsilon_m = 0$, and the proof is complete. $\square$

2.2. Convergence of Picard Iterations

The next theorem serves as the basic template for the quantitative convergence analysis of fixed point iterations and generalizes (Hesse and Luke [33, lemma 3.1]). By the notation $T: \Lambda \rightrightarrows \Lambda$, where Λ is a subset or an affine subspace of $\mathbb{E}$, we mean that $T: \mathbb{E} \rightrightarrows \mathbb{E}$, and $T(x) \subset \Lambda$ for all $x \in \Lambda$. This simplification of notation should not lead to any confusion if one keeps in mind that there may exist fixed points of T that are not in Λ . For the importance of the use of Λ in isolating the desirable fixed point, we refer the reader to Aspelmeier et al. [3, example 1.8].

Theorem 2.1. Let $T: \Lambda \rightrightarrows \Lambda$ for $\Lambda \subset \mathbb{E}$ and let $S \subset \text{ri } \Lambda$ be closed and nonempty with $Ty \subset \text{Fix } T \cap S$ for all $y \in S$. Let $\mathcal{O}$ be a neighborhood of S such that $\mathcal{O} \cap \Lambda \subset \text{ri } \Lambda$. Suppose

- (a) T is pointwise almost averaged at all points $y \in S$ with violation ε and averaging constant $\alpha \in (0, 1)$ on $\mathcal{O} \cap \Lambda$ and
- (b) there exists a neighborhood $\mathcal{V}$ of $\text{Fix } T \cap S$ and a $\kappa > 0$ such that for all $y^+ \in Ty$, $y \in S$, and $x^+ \in Tx$, the estimate

$$\text{dist}(x, S) \leq \kappa \|(x - x^+) - (y - y^+)\| \quad (38)$$

holds whenever $x \in (\mathcal{O} \cap \Lambda) \setminus (\mathcal{V} \cap \Lambda)$.

Then, for all $x^+ \in Tx$,

$$\text{dist}(x^+, \text{Fix } T \cap S) \leq \sqrt{1 + \varepsilon - \frac{1-\alpha}{\kappa^2 \alpha}} \text{dist}(x, S) \quad (39)$$

whenever $x \in (\mathcal{O} \cap \Lambda) \setminus (\mathcal{V} \cap \Lambda)$.

In particular, if $\kappa < \sqrt{(1-\alpha)/(\varepsilon \alpha)}$, then for all $x^0 \in \mathcal{O} \cap \Lambda$, the iteration $x^{j+1} \in Tx^j$ satisfies

$$\text{dist}(x^{j+1}, \text{Fix } T \cap S) \leq c^j \text{dist}(x^0, S) \quad (40)$$

with $c := (1 + \varepsilon - (1-\alpha)/(\alpha \kappa^2))^{1/2} < 1$ for all j such that $x^i \in (\mathcal{O} \cap \Lambda) \setminus (\mathcal{V} \cap \Lambda)$ for $i = 1, 2, \dots, j$.

Before presenting the proof, some remarks will help clarify the technicalities. The role of assumption (a) is clear in the two-property scheme we have set up. The second assumption (b) is a characterization of the required stability of the fixed points and their preimages. It is helpful to consider a specialization of this assumption, which simplifies things considerably. First, by Proposition 2.2, since T is almost averaged at all points in S , then

it is single-valued there and one can simply write Ty for all $y \in S$ instead of $y^+ \in Ty$. The real simplification comes when one considers the case $S = \text{Fix } T$. In this case, $Ty = y$ for all $y \in S$ and condition (38) simplifies to

$$\text{dist}(x, \text{Fix } T) \leq \kappa \text{dist}(0, x - Tx) \iff \text{dist}(x, \Phi^{-1}(0)) \leq \kappa \text{dist}(0, \Phi(x)) \quad (41)$$

for all $x \in (\mathcal{C} \cap \Lambda) \setminus (\mathcal{V} \cap \Lambda)$, where $\Phi := T - \text{Id}$. The statement on annular regions $(\mathcal{C} \cap \Lambda) \setminus (\mathcal{V} \cap \Lambda)$ can be viewed as an assumption about the existence of an *error bound* on that region. For earlier manifestations of this and connections to previous work on error bounds, see Luo and Tseng [53] and Ngai and Théra [60, 61]. In the present context, this condition will be identified in Section 2.3 with *metric subregularity* of Φ , though, of course, error bounds and metric subregularity are related.

The assumptions lead to the conclusion that the iterates approach the set of fixed points at some rate that can be bounded below by a linear characterization on the region $(\mathcal{C} \cap \Lambda) \setminus (\mathcal{V} \cap \Lambda)$. This will lead to *convergence* in Corollary 2.2 where on all such annular regions, there is some lower linear convergence bound.

The possibility to have $S \subset \text{Fix } T$ and not $S = \text{Fix } T$ allows one to sidestep complications arising from the not-so-exotic occurrence of fixed point mappings that are almost nonexpansive at some points in $\text{Fix } T$ and not at others (see Example 2.2(ii)). It would be too restrictive in the statement of the theorem, however, to have $S \subseteq \text{Fix } T$, since this does not allow one to tackle inconsistent feasibility, studied in depth in Section 3.1. In particular, we have in mind the situation where sets A and B do not intersect, but still the alternating projections mapping $T_{AP} := P_A P_B$ has nice properties at points in B that, while not fixed points, at least locally are nearest to A . The full richness of the structure is used in Theorem 3.2 where we establish, for the first time, sufficient conditions for local linear convergence of the method of cyclic projections for nonconvex inconsistent feasibility.

Proof of Theorem 2.1. If $\mathcal{C} \cap \mathcal{V} = \mathcal{C}$, there is nothing to prove. Assume then that there is some $x \in (\mathcal{C} \cap \Lambda) \setminus (\mathcal{V} \cap \Lambda)$. Choose any $x^+ \in Tx$ and define $\bar{x}^+ \in T\bar{x}$ for $\bar{x} \in P_S x$. Inequality (38) implies

$$\frac{1-\alpha}{\kappa^2 \alpha} \|x - \bar{x}\|^2 \leq \frac{1-\alpha}{\alpha} \|(x - x^+) - (\bar{x} - \bar{x}^+)\|^2. \quad (42)$$

Assumption (a) and Proposition 2.1(iii) together with (42), then yield

$$\|x^+ - \bar{x}^+\|^2 \leq \left(1 + \varepsilon - \frac{1-\alpha}{\alpha \kappa^2}\right) \|x - \bar{x}\|^2. \quad (43)$$

Note, in particular, that $0 \leq 1 + \varepsilon - (1-\alpha)/(\alpha \kappa^2)$. Since $\bar{x}^+ \in T(\bar{x}) \subset \text{Fix } T \cap S$, this proves the first statement.

If, in addition $\kappa < \sqrt{(1-\alpha)/(\varepsilon \alpha)}$, then $c := (1 + \varepsilon - (1-\alpha)/(\alpha \kappa^2))^{1/2} < 1$. Since clearly $S \supset \text{Fix } T \cap S$, (39) yields

$$\text{dist}(x^1, S) \leq \text{dist}(x^1, \text{Fix } T \cap S) \leq c \text{dist}(x^0, S).$$

If $x^1 \in \mathcal{C} \setminus \mathcal{V}$, then the first part of this theorem yields

$$\text{dist}(x^2, S) \leq \text{dist}(x^2, \text{Fix } T \cap S) \leq c \text{dist}(x^1, S) \leq c^2 \text{dist}(x^0, S).$$

Proceeding inductively then, the relation $\text{dist}(x^j, \text{Fix } T \cap S) \leq c^j \text{dist}(x^0, S)$ holds until the first time $x^{j-1} \notin \mathcal{C} \setminus \mathcal{V}$. $\square$

The inequality (39) by itself says nothing about convergence of the iteration $x^{j+1} = Tx^j$, but it does clearly indicate what needs to hold for the iterates to move closer to a fixed point of T . This is stated explicitly in the next corollary.

Corollary 2.2 (Convergence). Let $T: \Lambda \rightrightarrows \Lambda$ for $\Lambda \subset \mathbb{E}$ and let $S \subset \text{ri } \Lambda$ be closed and nonempty with $T\bar{x} \subset \text{Fix } T \cap S$ for all $\bar{x} \in S$. Define $\mathcal{C}_\delta := S + \delta \mathbb{B}$ and $\mathcal{V}_\delta := \text{Fix } T \cap S + \delta \mathbb{B}$. Suppose that for $\gamma \in (0, 1)$ fixed and for all $\delta > 0$ small enough, there is a triplet $(\varepsilon, \delta, \alpha) \in \mathbb{R}_+ \times (0, \gamma \delta] \times (0, 1)$ such that

- (a) T is pointwise almost averaged at all $y \in S$ with violation ε and averaging constant α on $\mathcal{C}_\delta \cap \Lambda$, and
- (b) at each $y^+ \in Ty$ for all $y \in S$ there exists a $\kappa \in [0, \sqrt{(1-\alpha)/(\varepsilon \alpha)})$ such that

$$\text{dist}(x, S) \leq \kappa \|(x - x^+) - (y - y^+)\|$$

at each $x^+ \in Tx$ for all $x \in (\mathcal{C}_\delta \cap \Lambda) \setminus (\mathcal{V}_\delta \cap \Lambda)$.

Then, for any x^0 close enough to S , the iterates $x^{i+1} \in Tx^i$ satisfy $\text{dist}(x^i, \text{Fix } T \cap S) \rightarrow 0$ as $i \rightarrow \infty$.

Proof. Let $\Delta > 0$ be such that for all $\bar{\delta} \in (0, \Delta]$, there is a triplet $(\varepsilon, \delta, \alpha) \in \mathbb{R}_+ \times (0, \gamma\bar{\delta}] \times (0, 1)$ for which (a) and (b) hold. Choose any $x^0 \in \mathcal{C}_{\Delta} \cap \Lambda$ and define $\bar{\delta}_0 := \text{dist}(x^0, S)$ so that (a) and (b) are satisfied for the parameter values $(\varepsilon_0, \delta_0, \alpha_0) \in \mathbb{R}_+ \times (0, \gamma\bar{\delta}_0] \times (0, 1)$. Define $x^{(0,j)} \in Tx^{(0,j-1)}$ for $j = 1, 2, \dots$ with $x^{(0,0)} := x^0$. At $j = 1$, there are two possible cases: either $x^{(0,1)} \in \mathcal{V}_{\delta_0} \cap \mathcal{C}_{\bar{\delta}_0}$ or $x^{(0,1)} \notin \mathcal{V}_{\delta_0} \cap \mathcal{C}_{\bar{\delta}_0}$. In the former case,

$$\text{dist}(x^{(0,1)}, \text{Fix } T \cap S) \leq \delta_0 \leq \gamma\bar{\delta}_0 < \bar{\delta}_0,$$

therefore for $J_0 = 1$ it holds that

$$\text{dist}(x^{(0,J_0)}, \text{Fix } T \cap S) \leq \delta_0 \leq \gamma\bar{\delta}_0 < \bar{\delta}_0.$$

In the latter case, since $x^{(0,0)} \in \mathcal{C}_{\bar{\delta}_0} \cap \Lambda$, Theorem 2.1 shows that

$$\text{dist}(x^{(0,1)}, \text{Fix } T \cap S) \leq c_0 \text{dist}(x^{(0,0)}, S)$$

for $c_0 := \sqrt{1 + \varepsilon_0 - (1 - \alpha_0)/(\kappa_0^2 \alpha_0)} < 1$. Moreover, clearly, $\text{dist}(x^{(0,1)}, S) \leq \text{dist}(x^{(0,1)}, \text{Fix } T \cap S)$, therefore in either case $x^{(0,1)} \in \mathcal{C}_{\bar{\delta}_0}$, and the alternative reduces to either $x^{(0,1)} \in \mathcal{V}_{\delta_0}$ or $x^{(0,1)} \notin \mathcal{V}_{\delta_0}$. Proceeding by induction for some $j \geq 1$, it holds that $x^{(0,\nu)} \in (\mathcal{C}_{\bar{\delta}_0} \cap \Lambda) \setminus (\mathcal{V}_{\delta_0} \cap \Lambda)$ for all $\nu = 0, 1, 2, \dots, j-1$ and $x^{(0,j)} \in \mathcal{C}_{\bar{\delta}_0} \cap \Lambda$ with either $x^{(0,j)} \notin \mathcal{V}_{\delta_0}$ or $x^{(0,j)} \in \mathcal{V}_{\delta_0}$. If $x^{(0,j)} \notin \mathcal{V}_{\delta_0}$, then since $x^{(0,j)} \in \mathcal{C}_{\bar{\delta}_0} \cap \Lambda$ by Theorem 2.1,

$$\text{dist}(x^{(0,j+1)}, \text{Fix } T \cap S) \leq c_0 \text{dist}(x^{(0,j)}, S).$$

Iterating this process, there must eventually be a $J_0 \in \mathbb{N}$ such that

$$\text{dist}(x^{(0,J_0)}, \text{Fix } T \cap S) \leq \delta_0 \leq \gamma\bar{\delta}_0 < \bar{\delta}_0. \quad (44)$$

To see this, suppose that there is no such J_0 . Then, $x^{(0,j)} \in (\mathcal{C}_{\bar{\delta}_0} \cap \Lambda) \setminus (\mathcal{V}_{\delta_0} \cap \Lambda)$ and

$$\text{dist}(x^{(0,j+1)}, \text{Fix } T \cap S) \leq c_0 \text{dist}(x^{(0,j)}, S) \leq c_0^j \text{dist}(x^{(0,0)}, S)$$

for all $j \geq 1$. Since, by assumption $c_0 < 1$, it holds that $\text{dist}(x^{(0,j)}, \text{Fix } T \cap S) \rightarrow 0$ at least linearly with constant c_0 , in contradiction with the assumption that $x^{(0,j)} \notin \mathcal{V}_{\delta_0}$ for all j .

Therefore, with J_0 being the first iteration where (44) occurs, we update the region $\bar{\delta}_1 := \text{dist}(x^{(0,J_0)}, S) \leq \text{dist}(x^{(0,J_0)}, \text{Fix } T \cap S) \leq \delta_0 \leq \gamma\bar{\delta}_0$, and set $x^1 := x^{(0,J_0)}$ and $x^{(1,0)} := x^1$. By assumption, there is a triplet $(\varepsilon_1, \delta_1, \alpha_1) \in \mathbb{R}_+ \times (0, \gamma\bar{\delta}_1] \times (0, 1)$ for which (a) and (b) hold.

Proceeding inductively, this generates the sequence $(x^i)_{i \in \mathbb{N}}$ with

$$x^i := x^{(i-1, J_{i-1})}, \quad \bar{\delta}_i := \text{dist}(x^i, S) \leq \text{dist}(x^i, \text{Fix } T \cap S) \leq \gamma^i \bar{\delta}_0.$$

Therefore $\text{dist}(x^i, \text{Fix } T \cap S) \rightarrow 0$ as $i \rightarrow \infty$. As this is just a reindexing of the Picard iteration, this completes the proof. $\square$

An interesting avenue of investigation would be to see to what extent the proof mining techniques of Kohlenbach et al. [40] could be applied to quantify convergence in the present setting.

2.3. Metric Regularity

The key insight into condition (b) of Theorem 2.1 is the connection to *metric regularity* of set-valued mappings (cf. Rockafellar and Wets [72], Dontchev and Rockafellar [29]). This approach to the study of algorithms has been advanced by several authors (Pennanen [66], Iusem et al. [38], Artacho et al. [2], Artacho and Geoffroy [1], Klatte and Kummer [39]). We modify the concept of *metric regularity with functional modulus on a set* suggested in Ioffe [36, definition 2.1(b)] and Ioffe [37, definition 1(b)] so that the property is relativized to appropriate sets for iterative methods. Recall that $\mu: [0, \infty) \rightarrow [0, \infty)$ is a *gauge function* if μ is continuous strictly increasing with $\mu(0) = 0$ and $\lim_{t \rightarrow \infty} \mu(t) = \infty$.

Definition 2.5 (Metric Regularity on a Set). Let $\Phi: \mathbb{E} \rightrightarrows \mathbb{Y}$, $U \subset \mathbb{E}$, $V \subset \mathbb{Y}$. The mapping Φ is called *metrically regular with gauge μ on $U \times V$ relative to $\Lambda \subset \mathbb{E}$* if

$$\text{dist}(x, \Phi^{-1}(y) \cap \Lambda) \leq \mu(\text{dist}(y, \Phi(x))) \quad (45)$$

holds for all $x \in U \cap \Lambda$, and $y \in V$ with $0 < \mu(\text{dist}(y, \Phi(x)))$. When the set V consists of a single point, $V = \{\bar{y}\}$, then Φ is said to be *metrically subregular for $\bar{y}$ on U with gauge μ relative to $\Lambda \subset \mathbb{E}$* .

When μ is a linear function (that is, $\mu(t) = \kappa t$, $\forall t \in [0, \infty)$), one says “with constant κ ” instead of “with gauge $\mu(t) = \kappa t$.” When $\Lambda = \mathbb{E}$, the quantifier “relative to” is dropped. When μ is linear, the smallest constant κ for which (45) holds is called the *modulus* of metric regularity.

The conventional concept of *metric regularity* (Azé [7], Dontchev and Rockafellar [29], Rockafellar and Wets [72]) (and *metric regularity of order ω* , respectively, Kruger and Thao [43]) at a point $\bar{x} \in \mathbb{E}$ for $\bar{y} \in \Phi(\bar{x})$ corresponds to the setting in Definition 2.5 where $\Lambda = \mathbb{E}$, U , and V are neighborhoods of $\bar{x}$ and $\bar{y}$, respectively, and the gauge function $\mu(t) = \kappa t$ ($\mu(t) = \kappa t^\omega$ for metric regularity of order $\omega < 1$) for all $t \in [0, \infty)$ with $\kappa > 0$.

Relaxing the requirements on the sets U and V from neighborhoods to the more ambiguous sets in Definition 2.5 allows the same definition and terminology to unambiguously cover well-known relaxations of metric regularity such as *metric subregularity* (U is a neighborhood of $\bar{x}$ and $V = \{\bar{y}\}$, Dontchev and Rockafellar [29]) and *metric hemi/semiregularity* ($U = \{\bar{x}\}$ and V is a neighborhood of $\bar{y}$ Mordukhovich [56, definition 1.47]). For our purposes, we will use the flexibility of choosing U and V in Definition 2.5 to *exclude* the reference point $\bar{x}$ and to *isolate* the image point $\bar{y}$. This is reminiscent of the Kurdyka-Łojasiewicz (KL) property (Bolte et al. [20]) for functions, which require that the subdifferential possesses a sharpness property near (but not at) critical points of the function. However, since the restriction of V to a point features prominently in our development, we retain the terminology *metric subregularity* to ease the technicality of the presentation. The reader is cautioned, however, that our use of metric subregularity does not precisely correspond to the usual definition (see Dontchev and Rockafellar [29]) since we do not require the domain U to be a neighborhood.

Theorem 2.2 ((Sub)linear Convergence with Metric Regularity). *Let $T: \Lambda \rightrightarrows \Lambda$ for $\Lambda \subset \mathbb{E}$, $\Phi := T - \text{Id}$ and let $S \subset \text{ri } \Lambda$ be closed and nonempty with $TS \subset \text{Fix } T \cap S$. Denote $(S + \delta\mathbb{B}) \cap \Lambda$ by S_δ for a nonnegative real δ . Suppose that, for all $\bar{\delta} > 0$ small enough, there are $\gamma \in (0, 1)$, a nonnegative sequence of scalars $(\varepsilon_i)_{i \in \mathbb{N}}$ and a sequence of positive constants α_i bounded above by $\bar{\alpha} < 1$ such that, for each $i \in \mathbb{N}$,*

- (a) *T is pointwise almost averaged at all $y \in S$ with averaging constant α_i and violation ε_i on $S_{\gamma^i \bar{\delta}}$ and*
- (b) *for*

$$R_i := S_{\gamma^i \bar{\delta}} \setminus (\text{Fix } T \cap S + \gamma^{i+1} \bar{\delta} \mathbb{B}),$$

- (i) *$\text{dist}(x, S) \leq \text{dist}(x, \Phi^{-1}(\bar{y}) \cap \Lambda)$ for all $x \in R_i$, and $\bar{y} \in \Phi(P_S(x)) \setminus \Phi(x)$,*
- (ii) *Φ is metrically regular with gauge μ_i relative to Λ on $R_i \times \Phi(P_S(R_i))$, where μ_i satisfies*

$$\sup_{x \in R_i, \bar{y} \in \Phi(P_S(R_i)), \bar{y} \notin \Phi(x)} \frac{\mu_i(\text{dist}(\bar{y}, \Phi(x)))}{\text{dist}(\bar{y}, \Phi(x))} \leq \kappa_i < \sqrt{\frac{1 - \alpha_i}{\varepsilon_i \alpha_i}}. \quad (46)$$

Then, for any $x^0 \in \Lambda$ close enough to S , the iterates $x^{j+1} \in Tx^j$ satisfy $\text{dist}(x^j, \text{Fix } T \cap S) \rightarrow 0$, and

$$\text{dist}(x^{j+1}, \text{Fix } T \cap S) \leq c_i \text{dist}(x^j, S), \quad \forall x^j \in R_i, \quad (47)$$

where $c_i := \sqrt{1 + \varepsilon_i - ((1 - \alpha_i)/\kappa_i^2 \alpha_i)} < 1$.

In particular, if ε_i is bounded above by $\bar{\varepsilon}$ and $\kappa_i \leq \bar{\kappa} < \sqrt{(1 - \bar{\alpha})/(\bar{\alpha} \bar{\varepsilon})}$ for all i large enough, then convergence is eventually at least linear with rate at most $\bar{c} := \sqrt{1 + \bar{\varepsilon} - ((1 - \bar{\alpha})/(\bar{\kappa}^2 \bar{\alpha}))} < 1$.

The first inequality in (46) is a condition on the gauge function μ_i and would not be needed if the statement were limited to linearly metrically regular mappings. Essentially, it says that the gauge function characterizing metric regularity of Φ can be bounded above by a linear function. The second inequality states that the constant of metric regularity κ_i is small enough relative to the violation of the averaging property ε_i to guarantee a linear progression of the iterates through the region R_i .

Proof of Theorem 2.2. To begin, note that by assumption (b), for any $x \in R_i$, $\bar{x} \in P_S(x)$, and $\bar{y} \in \Phi(\bar{x})$ with $\bar{y} \notin \Phi(x)$,

$$\text{dist}(x, S) \leq \text{dist}(x, \Phi^{-1}(\bar{y}) \cap \Lambda) \leq \mu_i(\text{dist}(\bar{y}, \Phi(x))) \leq \kappa_i \text{dist}(\bar{y}, \Phi(x)). \quad (48)$$

Let $\bar{y} = \bar{x}^+ - \bar{x}$ for $\bar{x}^+ \in T\bar{x}$. The above statement yields

$$\text{dist}(x, S) \leq \kappa_i \|(x^+ - x) - (\bar{x}^+ - \bar{x})\|, \quad \forall x \in R_i, \quad \forall x^+ \in Tx, \quad \forall \bar{x} \in P_S(x), \quad \forall \bar{x}^+ \in T\bar{x}. \quad (49)$$

The convergence of the sequence $\text{dist}(x^j, \text{Fix } T \cap S) \rightarrow 0$ then follows from Corollary 2.2 with the sequence of triplets $(\varepsilon_i, \gamma^{i+1} \bar{\delta}, \alpha_i)_{i \in \mathbb{N}}$. By Theorem 2.1, the rate of convergence on R_i is characterized by

$$\text{dist}(x^+, \text{Fix } T \cap S) \leq \sqrt{1 + \varepsilon_i - \frac{1 - \alpha_i}{\kappa_i^2 \alpha_i}} \text{dist}(x, S), \quad \forall x^+ \in Tx, \quad (50)$$

whence (47) holds with constant $c_i < 1$ given by (46).

The final claim of the theorem follows immediately. $\square$

When $S = \text{Fix } T \cap \Lambda$ in Theorem 2.2, the condition (b) (i) can be dropped from the assumptions, as the next corollary shows.

Corollary 2.3. Let $T: \Lambda \rightrightarrows \Lambda$ for $\Lambda \subset \mathbb{E}$ with $\text{Fix } T$ nonempty and closed, $\Phi := T - \text{Id}$. Denote $(\text{Fix } T + \delta\mathbb{B}) \cap \Lambda$ by S_δ for a nonnegative real δ . Suppose that, for all $\bar{\delta} > 0$ small enough, there are $\gamma \in (0, 1)$, a nonnegative sequence of scalars $(\varepsilon_i)_{i \in \mathbb{N}}$ and a sequence of positive constants α_i bounded above by $\bar{\alpha} < 1$ such that, for each $i \in \mathbb{N}$,

- (a) T is pointwise almost averaged at all $y \in \text{Fix } T \cap \Lambda$ with averaging constant α_i and violation ε_i on $S_{\gamma^i \bar{\delta}}$ and
- (b) for

$$R_i := S_{\gamma^i \bar{\delta}} \setminus (\text{Fix } T + \gamma^{i+1} \bar{\delta} \mathbb{B}),$$

Φ is metrically subregular for 0 on R_i (metrically regular on $R_i \times \{0\}$) with gauge μ_i relative to Λ , where μ_i satisfies

$$\sup_{x \in R_i} \frac{\mu_i(\text{dist}(0, \Phi(x)))}{\text{dist}(0, \Phi(x))} \leq \kappa_i < \sqrt{\frac{1 - \alpha_i}{\varepsilon_i \alpha_i}}. \quad (51)$$

Then, for any $x^0 \in \Lambda$ close enough to $\text{Fix } T \cap \Lambda$, the iterates $x^{j+1} \in Tx^j$ satisfy $\text{dist}(x^j, \text{Fix } T \cap \Lambda) \rightarrow 0$ and

$$\text{dist}(x^{j+1}, \text{Fix } T \cap \Lambda) \leq c_i \text{dist}(x^j, \text{Fix } T \cap \Lambda), \quad \forall x^j \in R_i, \quad (52)$$

where $c_i := \sqrt{1 + \varepsilon_i - ((1 - \alpha_i)/\kappa_i^2 \alpha_i)} < 1$.

In particular, if ε_i is bounded above by $\bar{\varepsilon}$ and $\kappa_i \leq \bar{\kappa} < \sqrt{(1 - \bar{\alpha})/(\bar{\alpha} \bar{\varepsilon})}$ for all i large enough, then convergence is eventually at least linear with rate at most $\bar{c} := \sqrt{1 + \bar{\varepsilon} - ((1 - \bar{\alpha})/(\bar{\kappa}^2 \bar{\alpha}))} < 1$.

Proof. To deduce Corollary 2.3 from Theorem 2.2, it suffices to check that when $S = \text{Fix } T \cap \Lambda$, condition (46) becomes (51), and condition (i) is always satisfied. This follows immediately from the fact that $\Phi(P_{\text{Fix } T \cap \Lambda}(\mathbb{E})) = \{0\}$ and $\Phi^{-1}(0) = \text{Fix } T$. $\square$

The following example explains why gauge metric regularity on a set (Definition 2.5) fits well in the framework of Theorem 2.2, whereas the conventional metric (sub)regularity does not.

Example 2.4 (A Line Tangent to a Circle). In $\mathbb{R}^2$, consider the two sets:

$$A := \{(u, -1) \in \mathbb{R}^2 \mid u \in \mathbb{R}\}, \quad B := \{(u, v) \in \mathbb{R}^2 \mid u^2 + v^2 = 1\},$$

and the point $\bar{x} = (0, -1)$. It is well known that the alternating projection algorithm $T := P_A P_B$ does not converge linearly to $\bar{x}$ unless with the starting points on $\{(0, v) \in \mathbb{R}^2: v \in \mathbb{R}\}$ (in this special case, the method reaches $\bar{x}$ in one step). Note that T behaves the same if B is replaced by the closed unit ball (the case of two closed-convex sets). In particular, T is averaged with constant $\alpha = 2/3$ by Proposition 2.4(iii). Hence the absence of linear convergence of T here can be explained as the lack of regularity of the fixed point set $A \cap B = \{\bar{x}\}$. In fact, the mapping $\Phi := T - \text{Id}$ is not (linearly) metrically subregular at $\bar{x}$ for 0 on any set $\mathbb{B}_\delta(\bar{x})$ for any $\delta > 0$. However, T does converge sublinearly to $\bar{x}$. This can be characterized in the following two different ways:

- Using Corollary 2.3, we characterize sublinear convergence in this example as linear convergence on annular sets. To proceed, we set

$$R_i := \mathbb{B}_{2^{-i}}(\bar{x}) \setminus \mathbb{B}_{2^{-(i+1)}}(\bar{x}), \quad (i = 0, 1, \dots).$$

This corresponds to setting $\bar{\delta} = 1$ and $\gamma = 1/2$ in Corollary 2.3. The task that remains is to estimate the constant of metric subregularity, κ_i , of Φ on each R_i . Indeed, we have

$$\inf_{x \in R_i \cap A} \frac{\|x - Tx\|}{\|x - \bar{x}\|} = \frac{\|x^* - Tx^*\|}{\|x^* - \bar{x}\|} = 1 - \frac{1}{\sqrt{2^{-2(i+1)} + 1}} := \kappa_i > 0, \quad (i = 0, 1, \dots),$$

where $x^* = (2^{-(i+1)}, -1)$.

Hence, on each ring R_i , T converges linearly to a point in $\mathbb{B}_{2^{-(i+1)}}(\bar{x})$ with rate c_i not worse than $\sqrt{1 - 1/(2\kappa_i^2)} < 1$ by Corollary 2.3.

- The discussion above uses the linear gauge functions $\mu_i(t) := t/\kappa_i$ on annular regions, and hence a piecewise linear gauge function for the characterization of metric subregularity. Alternatively, we can construct a smooth gauge function μ that works on neighborhoods of the fixed point. For analyzing convergence of $P_A P_B$, we must have Φ metrically subregular at 0 with gauge μ on $\mathbb{R}^2$ relative to A . But we have

$$\text{dist}(0, \Phi(x)) = \|x - x^+\| = f(\|x - \bar{x}\|) = f(\text{dist}(x, \Phi^{-1}(0))), \quad \forall x \in A, \quad (53)$$

where $f: [0, \infty) \rightarrow [0, \infty)$ is given by $f(t) := t(1 - 1/\sqrt{t^2 + 1})$. The function f is continuous strictly increasing and satisfies $f(0) = 0$ and $\lim_{t \rightarrow \infty} f(t) = \infty$. Hence f is a gauge function.

We can now characterize sublinear convergence of $P_A P_B$ explicitly without resorting to annular sets. Note first that, since $f(t) < t$ for all $t \in (0, \infty)$, the function $g: [0, \infty) \rightarrow [0, \infty)$ given by

$$g(t) := \sqrt{t^2 - \frac{1}{2}(f(t))^2}$$

is a gauge function and satisfies $g(t) < t$ for all $t \in (0, \infty)$. Note next that, $T := P_A P_B$ is (for all points in A) averaged with constant two-thirds together with (53), we get for any $x \in A$,

$$\|x^+ - \bar{x}\|^2 \leq \|x - \bar{x}\|^2 - (1/2)\|x - x^+\|^2 = \|x - \bar{x}\|^2 - (1/2)(f(\|x - \bar{x}\|))^2.$$

This implies

$$\text{dist}(x^+, S) = \|x^+ - \bar{x}\| \leq \sqrt{\|x - \bar{x}\|^2 - (1/2)(f(\|x - \bar{x}\|))^2} = g(\|x - \bar{x}\|) = g(\text{dist}(x, S)), \quad \forall x \in A. \quad \square$$

Remark 2.2 (Global (Sub)linear Convergence of Pointwise Averaged Mappings). As Example 2.4 illustrates, Theorem 2.2 is not an asymptotic result and does not gainsay the possibility that the required properties hold with neighborhood $U = \mathbb{E}$, which would then lead to a global quantification of convergence. First-order methods for convex problems lead generically to *globally* averaged fixed point mappings T . Convergence for convex problems can be determined from the averaging property of T and existence of fixed points. Hence, to *quantify* convergence, the only thing to be determined is the gauge of metric regularity at the fixed points of T . In this context, see Borwein et al. [21]. Example 2.4 illustrates how this can be done. This instance will be revisited in Example 3.5.

The following proposition, taken from Dontchev and Rockafellar [29], characterizes metric subregularity in terms of the graphical derivative defined by (9).

Proposition 2.5 (Dontchev and Rockafellar [29], theorems 4B.1 and 4C.2). *Let $T: \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ have locally closed graph at $(\bar{x}, \bar{y}) \in \text{gph } T$, $\Phi := T - \text{Id}$, and $\bar{z} := \bar{y} - \bar{x}$. Then, Φ is metrically subregular for 0 on U (metrically regular on $U \times \{\bar{z}\}$) with constant κ for U some neighborhood of $\bar{x}$ satisfying $U \cap \Phi^{-1}(\bar{z}) = \{\bar{x}\}$ if and only if the graphical derivative satisfies*

$$D\Phi(\bar{x} \mid \bar{z})^{-1}(0) = \{0\}. \quad (54)$$

If, in addition, T is single-valued and continuously differentiable on U , then the two conditions hold if and only if $\nabla\Phi$ has rank n at $\bar{x}$ with $\|[\nabla\Phi(x)]^\top\|^{-1} \leq \kappa$ for all x on U .

While the characterization (54) appears daunting, the property comes almost free for polyhedral mappings.

Proposition 2.6 (Polyhedrality Implies Metric Subregularity). *Let $\Lambda \subset \mathbb{E}$ be an affine subspace and $T: \Lambda \rightrightarrows \Lambda$. If T is polyhedral and $\text{Fix } T \cap \Lambda$ is an isolated point, $\{\bar{x}\}$, then $\Phi := T - \text{Id}$ is metrically subregular for 0 on U (metrically regular on $U \times \{0\}$) relative to Λ with some constant κ for some neighborhood U of $\bar{x}$. In particular, $U \cap \Phi^{-1}(0) = \{\bar{x}\}$.*

Proof. If T is polyhedral, so is $\Phi^{-1} := (T - \text{Id})^{-1}$. The statement now follows from Dontchev and Rockafellar [29], propositions 3I.1 and 3I.2] since Φ^{-1} is polyhedral and $\bar{x}$ is an isolated point of $\Phi^{-1}(0) \cap \Lambda$. $\square$

Proposition 2.7 (Local Linear Convergence: Polyhedral Fixed Point Iterations). *Let $\Lambda \subset \mathbb{E}$ be an affine subspace and $T: \Lambda \rightrightarrows \Lambda$ be pointwise almost averaged at $\{\bar{x}\} = \text{Fix } T \cap \Lambda$ on Λ with violation constant ε and averaging constant α . If T is polyhedral, then there is a neighborhood U of $\bar{x}$ such that*

$$\|x^+ - \bar{x}\| \leq c\|x - \bar{x}\| \quad \forall x \in U \cap \Lambda, \quad x^+ \in Tx,$$

where $c = \sqrt{1 + \varepsilon - (1 - \alpha)/(\kappa^2 \alpha)}$ and κ is the modulus of metric subregularity of $\Phi := T - \text{Id}$ for 0 on U relative to Λ . If, in addition $\kappa < \sqrt{(1 - \alpha)/(\alpha \varepsilon)}$, then the fixed point iteration $x^{j+1} \in Tx^j$ converges linearly to $\bar{x}$ with rate $c < 1$ for all $x^0 \in U \cap \Lambda$.

Proof. The result follows immediately from Proposition 2.6 and Corollary 2.3. $\square$

3. Applications

The idea of the previous section is simple. Formulated as Picard iterations of a fixed point mapping T , to establish the quantitative convergence of an algorithm, one must establish two properties of this mapping: first, that T is almost averaged, and second, that $T - \text{Id}$ is metrically subregular at fixed points relative to an appropriate subset. This section serves as a tutorial for how to do this for fundamental first-order algorithms. Each of the problems studied below represents a distinct region on the map of numerical analysis, each with its own dialect. Part of our goal is to show that the phenomena that these different dialects describe sort into one of the two more general properties of fixed point mappings established above. While the technicalities can become quite dense, particularly for feasibility, the two principles above offer a reliable guide through the details.

3.1. Feasibility

The feasibility problem is to find $\bar{x} \in \bigcap_{j=1}^m \Omega_j$. If the intersection is empty, the problem is called *inconsistent*, but a meaningful solution still can be found in the sense of best approximation in the case of just two sets, or in some other appropriate sense when there are three or more sets. The most prevalent algorithms for solving these problems are built on projectors onto the individual sets (indeed, we are aware of no other approach to the problem). The regularity of the fixed point mapping T that encapsulates a particular algorithm (in particular, pointwise almost averaging and coercivity at the fixed point set) stems from the regularity of the underlying projectors and the way the projectors are put together to construct T . Our first task is to show in what way the regularity of the underlying projectors is inherited from the regularity of the sets Ω_j .

3.1.1. Elemental Set Regularity. The following definition of what we call *elemental regularity* was first presented in Kruger et al. [44, definition 5]. This places under one schema the many different kinds of set regularity appearing in Lewis et al. [46], Bauschke et al. [17, 16], Hesse and Luke [33], Bauschke et al. [18], Noll and Rondepierre [63].

Definition 3.1 (Elemental Regularity of Sets). Let $\Omega \subset \mathbb{E}$ be nonempty and let $(\bar{y}, \bar{v}) \in \text{gph}(N_\Omega)$.

(i) Ω is *elementally subregular of order σ relative to Λ at $\bar{x}$ for $(\bar{y}, \bar{v})$ with constant ε* if there exists a neighborhood U of $\bar{x}$ such that

$$\langle \bar{v} - (x - x^+), x^+ - \bar{y} \rangle \leq \varepsilon \|\bar{v} - (x - x^+)\|^{1+\sigma} \|x^+ - \bar{y}\|, \quad \forall x \in \Lambda \cap U, \quad x^+ \in P_\Omega(x). \quad (55)$$

(ii) The set Ω is said to be *uniformly elementally subregular of order σ relative to Λ at $\bar{x}$ for $(\bar{y}, \bar{v})$* if for any $\varepsilon > 0$ there is a neighborhood U (depending on ε) of $\bar{x}$ such that (55) holds.

(iii) The set Ω is said to be *elementally regular of order σ at $\bar{x}$ for $(\bar{y}, \bar{v})$ with constant ε* if it is elementally subregular of order σ relative to $\Lambda = \Omega$ at $\bar{x}$ for all $(\bar{y}, \bar{v})$ with constant ε , where $\bar{v} \in N_\Omega(\bar{y}) \cap V$ for some neighborhood V of $\bar{v}$.

(iv) The set Ω is said to be *uniformly elementally regular of order σ at $\bar{x}$ for $(\bar{y}, \bar{v})$* if it is uniformly elementally subregular of order σ relative to $\Lambda = \Omega$ at $\bar{x}$ for all $(\bar{y}, \bar{v})$, where $\bar{v} \in N_\Omega(\bar{y}) \cap V$ for some neighborhood V of $\bar{v}$. If $\Lambda = \{\bar{x}\}$ in (i) or (ii), then the respective qualifier “relative to” is dropped. If $\sigma = 0$, then the respective qualifier “of order” is dropped in the description of the properties. The *modulus of elemental (sub)regularity* is the infimum over all ε for which (55) holds.

In all properties in Definition 3.1, $\bar{x}$ need not be in Λ and $\bar{y}$ need not be in either U or Λ . In case of order $\sigma = 0$, the properties are trivial for any constant $\varepsilon \geq 1$. When saying a set is not elementally (sub)regular but without specifying a constant, it is meant for any constant $\varepsilon < 1$.

Example 3.1. (a) (cross) Recall the set in Example 2.3,

$$A = \mathbb{R} \times \{0\} \cup \{0\} \times \mathbb{R}.$$

This example is of particular interest for the study of *sparsity constrained optimization*. A is elementally regular at any $\bar{x} \neq 0$; say, $\|\bar{x}\| > \delta > 0$ for all $(a, v) \in \text{gph } N_A$, where $a \in \mathbb{B}_\delta(\bar{x})$ with constant $\varepsilon = 0$ and neighborhood $\mathbb{B}_\delta(\bar{x})$. The set A is not elementally regular at $\bar{x} = 0$ for any $(0, v) \in \text{gph } N_A$ since $N_A(0) = A$. However, A is elementally subregular at $\bar{x} = 0$ for all $(a, v) \in \text{gph } N_A$ with constant $\varepsilon = 0$ and neighborhood $\mathbb{E}$ since all vectors $a \in A$ are orthogonal to $N_A(a)$.

(b) (circle) The humble circle is central to the phase retrieval problem,

$$A = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 = 1\}.$$

The set A is uniformly elementally regular at any $\bar{x} \in A$ for all $(\bar{x}, v) \in \text{gph } N_A$. Indeed, note first that for any $\bar{x} \in A$, $N_A(\bar{x})$ consists of the line passing through the origin and $\bar{x}$. Now, for any $\varepsilon \in (0, 1)$, we choose $\delta = \varepsilon$. Then, for any $x \in A \cap \mathbb{B}_\delta(\bar{x})$, it holds $\cos \angle(-\bar{x}, x - \bar{x}) \leq \delta \leq \varepsilon$. Hence, for all $x \in A \cap \mathbb{B}_\delta(\bar{x})$ and $v \in N_A(\bar{x})$,

$$\langle v, x - \bar{x} \rangle = \cos \angle(v, x - \bar{x}) \|v\| \|x - \bar{x}\| \leq \cos \angle(-\bar{x}, x - \bar{x}) \|v\| \|x - \bar{x}\| \leq \varepsilon \|v\| \|x - \bar{x}\|.$$

(c) (Packman eating a piece of pizza) Consider again the sets

$$\begin{aligned} A &= \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \leq 1, -1/2x_1 \leq x_2 \leq x_1, x_1 \geq 0\} \subset \mathbb{R}^2, \\ B &= \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \leq 1, x_1 \leq |x_2|\} \subset \mathbb{R}^2, \end{aligned}$$

at the point $\bar{x} = (0, 0)$ from Example 2.2(ii). The set B is elementally subregular relative to A at $\bar{x} = 0$ for all $(b, v) \in \text{gph}(N_B \cap A)$ with constant $\varepsilon = 0$ and neighborhood $\mathbb{E}$ since for all $a \in A$, $a_B \in P_B(a)$, and $v \in N_B(b) \cap A$, there holds

$$\langle v - (a - a_B), a_B - b \rangle = \langle v, a_B - b \rangle - \langle a - a_B, a_B - b \rangle = 0.$$

The set B , however, is not elementally regular at $\bar{x} = 0$ for any $(0, v) \in \text{gph } N_B$ because by choosing $x = tv \in B$ (where $0 \neq v \in B \cap N_B(0)$, $t \downarrow 0$), we get $\langle v, x \rangle = \|v\| \|x\| > 0$.

To see how the language of elemental regularity unifies the existing terminology, we list the following equivalences first established in Kruger et al. [44, proposition 4].

Proposition 3.1 (Equivalences of Elemental (Sub)regularity). *Let A , A' , and B be closed nonempty subsets of $\mathbb{E}$.*

(i) *Let $A \cap B \neq \emptyset$ and suppose that there is a neighborhood W of $\bar{x} \in A \cap B$ and a constant $\varepsilon > 0$ such that for each*

$$(a, v) \in V := \{(b_A, u) \in \text{gph } N_A^{\text{prox}} \mid u = b - b_A, \text{ for } b \in B \cap W \text{ and } b_A \in P_A(b) \cap W\}, \quad (56)$$

it holds that

$$\bar{x} \in \text{int } U(a, v), \quad \text{where } U(a, v) := \mathbb{B}_{(1+\varepsilon^2)\|v\|}(a + v). \quad (57)$$

Then, A is σ -Hölder regular relative to B at $\bar{x}$ in the sense of Noll and Rondepierre [63, definition 2] if and only if A is elementally subregular of order σ relative to $A \cap P_B^{-1}(a + v)$ at $\bar{x}$ for each $(a, v) \in V$ with constant $\varepsilon = \sqrt{c}$ and the respective neighborhood $U(a, v)$.

(ii) *Let $B \subset A$. The set A is (ε, δ) -subregular relative to B at $\bar{x} \in A$ in the sense of Hesse and Luke [33, definition 2.9] if and only if A is elementally subregular relative to B at $\bar{x}$ for all $(a, v) \in \text{gph } N_A^{\text{prox}}$, where $a \in \mathbb{B}_\delta(\bar{x})$ with constant ε and neighborhood $\mathbb{B}_\delta(\bar{x})$. Consequently, (ε, δ) -subregularity implies 0-Hölder regularity.*

(iii) *If the set A is $(\mathbb{E}, \varepsilon, \delta)$ -regular at $\bar{x}$ in the sense of Bauschke et al. [17, definition 8.1], then A is elementally regular at $\bar{x}$ for all $(\bar{x}, v)$ with constant ε , where $0 \neq v \in N_A^{\text{prox}}(\bar{x})$. Consequently, $(\mathbb{E}, \varepsilon, \delta)$ -regularity implies (ε, δ) -subregularity.*

(iv) *The set A is Clarke regular at $\bar{x} \in A$ (Rockafellar and Wets [72, definition 6.4]) if and only if A is uniformly elementally regular at $\bar{x}$ for all $(\bar{x}, v)$ with $v \in N_A(\bar{x})$. Consequently, Clarke regularity implies (ε, δ) -regularity.*

(v) *The set A is super-regular at $\bar{x} \in A$ Lewis et al. [46, definition 4.3] if and only if for any $\varepsilon > 0$, there is a $\delta > 0$ such that A is elementally regular at $\bar{x}$ for all $(a, v) \in \text{gph } N_A$, where $a \in \mathbb{B}_\delta(\bar{x})$ with constant ε and neighborhood $\mathbb{B}_\delta(\bar{x})$. Consequently, super-regularity implies Clarke regularity.*

(vi) *If A is prox-regular at $\bar{x}$ Poliquin et al. [69, definition 1.1], then there exist positive constants $\bar{\varepsilon}$ and $\bar{\delta}$ such that, for any $\varepsilon > 0$ and $\delta := \varepsilon \bar{\delta} / \bar{\varepsilon}$ defined correspondingly, A is elementally regular at $\bar{x}$ for all $(a, v) \in \text{gph } N_A$, where $a \in \mathbb{B}_\delta(\bar{x})$ with constant ε and neighborhood $\mathbb{B}_\delta(\bar{x})$. Consequently, prox-regularity implies super-regularity.*

(vii) *If A is convex, then it is elementally regular at all $x \in A$ for all $(a, v) \in \text{gph } N_A$ with constant $\varepsilon = 0$ and the neighborhood $\mathbb{E}$ for x and v .*

The following relations reveal a similarity to almost firm nonexpansiveness of the projector onto elementally subregular sets on the one hand, and almost nonexpansiveness of the same projector on the other.

Proposition 3.2 (Characterizations of Elemental Subregularity). (i) *A nonempty set $\Omega \subset \mathbb{E}$ is elementally subregular at $\bar{x}$ relative to Λ for $(y, v) \in \text{gph}(N_\Omega^{\text{prox}})$, where $y \in P_\Omega(y + v)$ if and only if there is a neighborhood U of $\bar{x}$ together with a constant $\varepsilon \geq 0$ such that*

$$\|x - y\|^2 \leq \varepsilon \|(y' - y) - (x' - x)\| \|x - y\| + \langle x' - y', x - y \rangle \quad (58)$$

holds with $y' = y + v$ whenever $x' \in U \cap \Lambda$ and $x \in P_\Omega x'$.

(ii) Let the nonempty set $\Omega \subset \mathbb{E}$ be elementally subregular at $\bar{x}$ relative to Λ for $(y, v) \in \text{gph}(N_{\Omega}^{\text{prox}})$, where $y \in P_{\Omega}(y+v)$ with the constant $\varepsilon \geq 0$ for the neighborhood U of $\bar{x}$. Then,

$$\|x - y\| \leq \varepsilon \|(y' - y) - (x' - x)\| + \|x' - y'\| \quad (59)$$

holds with $y' = y + v$ whenever $x' \in U \cap \Lambda$ and $x \in P_{\Omega}x'$.

Proof. (i) This is just a rearrangement of the inequality in (55). (ii) Follows by applying the Cauchy-Schwarz inequality to the inner product on the right-hand side of (58). $\square$

The next theorem is an update of Hesse and Luke [33, theorem 2.14] to the current terminology. It establishes the connection between elemental subregularity of a set and almost nonexpansiveness/averaging of the projector onto that set. Since the cyclic projections algorithm applied to inconsistent feasibility problems involves the properties of the projectors at points that are *outside* the sets, we show how the properties depend on whether the reference points are inside or outside of the sets. The theorem uses the symbol Λ to indicate subsets of the sets and the symbol Λ' to indicate points on some neighborhood whose projection lies in Λ . Later, the sets Λ' will be specialized in the context of cyclic projections to sets of points S_j whose projections lie in Ω_j . One thing to note in the theorem below is that the almost nonexpansive/averaging property degrades rapidly as the reference points move away from the sets. Our estimate is severe and could be sharpened somewhat, but it serves our purposes.

Theorem 3.1 (Projectors and Reflectors onto Elementally Subregular Sets). *Let $\Omega \subset \mathbb{E}$ be nonempty closed, and let U be a neighborhood of $\bar{x} \in \Omega$. Let $\Lambda \subset \Omega \cap U$, and $\Lambda' := P_{\Omega}^{-1}(\Lambda) \cap U$. If Ω is elementally subregular at $\bar{x}$ relative to Λ' for each*

$$(x, v) \in V := \{(z, w) \in \text{gph } N_{\Omega}^{\text{prox}} \mid z + w \in U \text{ and } z \in P_{\Omega}(z + w)\}$$

with constant ε on the neighborhood U , then the following hold:

(i) The projector P_{Ω} is pointwise almost nonexpansive at each $y \in \Lambda$ on U with violation $\varepsilon' := 2\varepsilon + \varepsilon^2$. That is, at each $y \in \Lambda$,

$$\|x - y\| \leq \sqrt{1 + \varepsilon'} \|x' - y\|, \quad \forall x' \in U, \quad x \in P_{\Omega}x'.$$

(ii) Let $\varepsilon \in [0, 1)$. The projector P_{Ω} is pointwise almost nonexpansive at each $y' \in \Lambda'$ with violation $\tilde{\varepsilon}$ on U for $\tilde{\varepsilon} := 4\varepsilon/(1 - \varepsilon)^2$. That is, at each $y' \in \Lambda'$,

$$\|x - y\| \leq \frac{1 + \varepsilon}{1 - \varepsilon} \|x' - y'\|, \quad \forall x' \in U, \quad x \in P_{\Omega}x', \quad y \in P_{\Omega}y'.$$

(iii) The projector P_{Ω} is pointwise almost firmly nonexpansive at each $y \in \Lambda$ with violation $\varepsilon'_2 := 2\varepsilon + 2\varepsilon^2$ on U . That is, at each $y \in \Lambda$,

$$\|x - y\|^2 + \|x' - x\|^2 \leq (1 + \varepsilon'_2) \|x' - y\|^2, \quad \forall x' \in U, \quad x \in P_{\Omega}x'.$$

(iv) Let $\varepsilon \in [0, 1)$. The projector P_{Ω} is pointwise almost firmly nonexpansive at each $y' \in \Lambda'$ with violation $\tilde{\varepsilon}_2 := 4\varepsilon(1 + \varepsilon)/(1 - \varepsilon)^2$ on U . That is, at each $y' \in \Lambda'$,

$$\|x - y\|^2 + \|(x' - x) - (y' - y)\|^2 \leq (1 + \tilde{\varepsilon}_2) \|x' - y'\|^2, \quad \forall x' \in U, \quad x \in P_{\Omega}x', \quad y \in P_{\Omega}y'.$$

(v) The reflector R_{Ω} is pointwise almost nonexpansive at each $y \in \Lambda$ (respectively, $y' \in \Lambda'$) with violation $\varepsilon'_3 := 4\varepsilon + 4\varepsilon^2$ (respectively, $\tilde{\varepsilon}_3 := 8\varepsilon(1 + \varepsilon)/(1 - \varepsilon)^2$) on U ; that is, for all $y \in \Lambda$ (respectively, $y' \in \Lambda'$),

$$\begin{aligned} \|x - y\| &\leq \sqrt{1 + \varepsilon'_3} \|x' - y\|, \quad \forall x' \in U, \quad x \in R_{\Omega}x', \\ (\text{respectively, } \|x - y\| &\leq \sqrt{1 + \tilde{\varepsilon}_3} \|x' - y'\|, \quad \forall x' \in U, \quad x \in R_{\Omega}x', \quad y \in R_{\Omega}y'). \end{aligned}$$

Proof. First, some general observations about the assumptions. The projector is nonempty since Ω is closed. Note also that, since $\Lambda \subset \Omega$, $P_{\Omega}y = y$ for all $y \in \Lambda$. Since $\Lambda \subset \Lambda'$, Ω is elementally subregular at $\bar{x}$ relative to Λ for each $(x, v) \in V$ with constant ε on the neighborhood U of $\bar{x}$, though the constant ε may not be optimal for Λ even if it is optimal for Λ' . Finally, for all $x' \in U$, it holds that $(x, x' - x) \in V$ for all $x \in P_{\Omega}(x')$. To see this, take any $x' \in U$ and any $x \in P_{\Omega}(x')$. Then, $v = x' - x \in N_{\Omega}^{\text{prox}}(x)$ and, by definition $(x, v) \in V$.

(i) By the Cauchy-Schwarz inequality,

$$\|x - y\|^2 = \langle x' - y, x - y \rangle + \langle x - x', x - y \rangle \leq \|x' - y\| \|x - y\| + \langle x' - x, y - x \rangle. \quad (60)$$

Now, with $v = x' - x \in N_{\Omega}^{\text{prox}}(x)$ such that $x' = x + v \in U$, one has $(x, v) \in V$ and by the definition of elemental subregularity of Ω at $\bar{x}$ relative to $\Lambda \subset \Omega$ for each $(x, v) \in V$ with constant ε on the neighborhood U of $\bar{x}$, the inequality $\langle x' - x, y - x \rangle \leq \varepsilon \|x' - x\| \|y - x\|$ holds for all $y \in \Lambda = U \cap \Lambda$. But $\|x' - x\| \leq \|x' - y\|$ since $x \in P_{\Omega}(x')$, and $y \in \Omega$, therefore, in fact, the inequality $\langle x' - x, y - x \rangle \leq \varepsilon \|x' - y\| \|y - x\|$ holds whenever $y \in \Lambda$. Combining this with (60) yields, for all $(x, x' - x) \in V$, and $y \in \Lambda$,

$$\|x - y\| \leq (1 + \varepsilon) \|x' - y\| = \sqrt{1 + (2\varepsilon + \varepsilon^2)} \|x' - y\|. \quad (61)$$

Equivalently, since for all $x' \in U$, it holds that $(x, x' - x) \in V$ for all $x \in P_{\Omega}(x')$, (61) holds at each $y \in \Lambda$ for all $x \in P_{\Omega}(x')$ whenever $x' \in U$; that is, P_{Ω} is almost nonexpansive at each $y \in \Lambda \subset U$ with violation $(2\varepsilon + \varepsilon^2)$ on U , as claimed. $\triangle$

(ii) Since any point $(x, x' - x) \in V$ satisfies $x' - x \in N_{\Omega}^{\text{prox}}(x)$, and $x \in P_{\Omega}(x')$, Proposition 3.2(ii) applies with Λ replaced by Λ' ; namely,

$$\|y - x\| \leq \varepsilon \|(y' - y) - (x' - x)\| + \|x' - y'\|$$

for all $y' \in U \cap \Lambda'$ and for every $y \in P_{\Omega}(y')$. The triangle inequality applied to $\|(y' - y) - (x' - x)\|$ then establishes the result. $\triangle$

(iii) Expanding and rearranging the norm yields, for all $y \in U \cap \Lambda$,

$$\begin{aligned} \|x - y\|^2 + \|x' - x\|^2 &= \|x - y\|^2 + \|x' - y + y - x\|^2 = \|x - y\|^2 + \|x' - y\|^2 + 2\langle x' - y, y - x \rangle + \|x - y\|^2 \\ &= 2\|x - y\|^2 + \|x' - y\|^2 + 2\langle x - y, y - x \rangle + 2\langle x' - x, y - x \rangle \leq \|x' - y\|^2 + 2\varepsilon \|x' - x\| \|x - y\| \end{aligned} \quad (62)$$

for each $(x, x' - x) \in V$, where the last inequality follows from the definition of elemental subregularity of Ω at $\bar{x}$ relative to Λ for $(x, x' - x) \in V$. As in Part (i), since $y \in \Omega$, and $x \in P_{\Omega}(x')$, it holds that $\|x' - x\| \leq \|x' - y\|$. Combining (62) and part (i) yields, at each $y \in \Lambda$,

$$\|x - y\|^2 + \|x' - x\|^2 \leq (1 + 2\varepsilon(1 + \varepsilon)) \|x' - y\|^2 \quad (63)$$

for all $(x, x' - x) \in V$. Again, since for all $x' \in U$, it holds that $(x, x' - x) \in V$ for all $x \in P_{\Omega}(x')$, (63) holds at each $y \in \Lambda$ for all $x \in P_{\Omega}(x')$ whenever $x' \in U$. By Proposition 2.1(iii) with $\alpha = 1/2$, and $y^+ = P_{\Omega}(y) = y$, it follows that P_{Ω} is almost firmly nonexpansive at each $y \in \Lambda \subset U$ with violation $(2\varepsilon + 2\varepsilon^2)$ on U , as claimed. $\triangle$

(iv) As in part (ii), Proposition 3.2 applies with Λ replaced by Λ' . Proceeding as in part (iii),

$$\begin{aligned} \|x - y\|^2 + \|(x' - x) - (y' - y)\|^2 &= 2\|x - y\|^2 + \|x' - y'\|^2 + 2\langle x' - y', y - x \rangle \\ &\leq \|x' - y'\|^2 + 2\varepsilon \|(x' - x) - (y' - y)\| \|x - y\|, \end{aligned} \quad (64)$$

where by elemental subregularity of Ω at $\bar{x}$ relative to Λ' for $(x, x' - x) \in V$ and (58) of Proposition 3.2, the last inequality holds for each $(x, x' - x) \in V$ for every $y' \in U \cap \Lambda'$ for all $y \in P_{\Omega}(y')$. This together with the triangle inequality yields

$$\|x - y\|^2 + \|(x' - x) - (y' - y)\|^2 \leq \|x' - y'\|^2 + 2\varepsilon \|x - y\| (\|x' - y'\| + \|x - y\|). \quad (65)$$

Part (ii) and (65) then give

$$\|x - y\|^2 + \|(x' - x) - (y' - y)\|^2 \leq \left(1 + 4\varepsilon \frac{1 + \varepsilon}{(1 - \varepsilon)^2}\right) \|x' - y'\|^2 \quad (66)$$

for all $(x, x' - x) \in V$ and for all $y \in P_{\Omega}(y')$ at each $y' \in U \cap \Lambda'$. Again, since for all $x' \in U$, it holds that $(x, x' - x) \in V$ for all $x \in P_{\Omega}(x')$, (66) holds at each $y' \in \Lambda' = U \cap \Lambda'$ for all $x \in P_{\Omega}(x')$ whenever $x' \in U$. By Proposition 2.1(iii) with $\alpha = 1/2$ and y^+ replaced by $y \in P_{\Omega}(y')$, it follows that P_{Ω} is almost firmly nonexpansive at each $y' \in \Lambda' \subset U$ with violation $4\varepsilon(1 + \varepsilon)/(1 - \varepsilon)^2$ on U as claimed. $\triangle$

(v) By part (iii) (respectively, part (iv)), the projector is pointwise almost firmly nonexpansive at each $y \in \Lambda$ (respectively, $y' \in \Lambda'$) with violation $(2\varepsilon + 2\varepsilon^2)$ (respectively, $4\varepsilon(1 + \varepsilon)/(1 - \varepsilon)^2$) on U , and therefore, by Proposition 2.3(iii), $R_{\Omega} = 2P_{\Omega} - \text{Id}$ is pointwise almost nonexpansive at each $y \in \Lambda$ (respectively, $y' \in \Lambda'$) with violation $(4\varepsilon + 4\varepsilon^2)$ (respectively, $8\varepsilon(1 + \varepsilon)/(1 - \varepsilon)^2$) on U . This completes the proof. $\square$

3.1.2. Subtransversal Collections of Sets. Elemental regularity of sets has been shown to be the source of the almost averaging property of the corresponding projectors. We show in this section that metric subregularity of the composite/averaged fixed point mapping is a consequence of how the individual sets align with each other. This impinges on a literature rich in terminology and competing notions of stability that have been energetically promoted recently in the context of consistent feasibility (see Kruger et al. [44] and references therein). Our placement of metric subregularity as the central organizing principle allows us to extend these notions beyond consistent feasibility to *inconsistent feasibility*. Before we can translate the dialect of set feasibility into the language of metric subregularity, we need to first extend one of the main concepts describing the regularity of collections of sets to collections that don't necessarily intersect. The idea behind the following definition stems from the equivalence between metric subregularity of an appropriate set-valued mapping on the product space and subtransversality of sets at common points (Kruger et al. [44, theorem 3]). The trick to extending this to points that do not belong to all the sets is to define the correct set-valued mapping.

Definition 3.2 (Subtransversal Collections of Sets). Let $\{\Omega_1, \Omega_2, \dots, \Omega_m\}$ be a collection of nonempty closed subsets of $\mathbb{E}$ and define $\Psi: \mathbb{E}^m \rightrightarrows \mathbb{E}^m$ by $\Psi(x) := P_\Omega(\Pi x) - \Pi x$, where $\Omega := \Omega_1 \times \Omega_2 \times \dots \times \Omega_m$, the projection P_Ω is with respect to the Euclidean norm on $\mathbb{E}^m$, and $\Pi: x = (x_1, x_2, \dots, x_m) \mapsto (x_2, \dots, x_m, x_1)$ is the permutation mapping on the product space $\mathbb{E}^m$ for $x_j \in \mathbb{E}$ ($j = 1, 2, \dots, m$). Let $\bar{x} = (\bar{x}_1, \bar{x}_2, \dots, \bar{x}_m) \in \mathbb{E}^m$, and $\bar{y} \in \Psi(\bar{x})$.

(i) The collection of sets is said to be *subtransversal with gauge μ relative to $\Lambda \subset \mathbb{E}^m$ at $\bar{x}$ for $\bar{y}$* if Ψ is metrically subregular at $\bar{x}$ for $\bar{y}$ on some neighborhood U of $\bar{x}$ (metrically regular on $U \times \{\bar{y}\}$) with gauge μ relative to Λ .

(ii) The collection of sets is said to be *transversal with gauge μ relative to $\Lambda \subset \mathbb{E}^m$ at $\bar{x}$ for $\bar{y}$* if Ψ is metrically regular with gauge μ relative to Λ on $U \times V$, for some neighborhoods U of $\bar{x}$ and V of $\bar{y}$.

As in Definition 2.5, when $\mu(t) = \kappa t$, $\forall t \in [0, \infty)$, one says “constant κ ” instead of “gauge $\mu(t) = \kappa t$.” When $\Lambda = \mathbb{E}$, the quantifier “relative to” is dropped.

Consistent with the terminology of metric regularity and subregularity, the prefix “sub” is meant to indicate the pointwise version of the more classical, though restrictive, idea of transversality. When the point $\bar{x} = (\bar{u}, \dots, \bar{u})$ for $\bar{u} \in \bigcap_{j=1}^m \Omega_j$, the following characterization of subtransversality holds.

Proposition 3.3 (Subtransversality at Common Points). Let $\mathbb{E}^m$ be endowed with the 2-norm; that is, $\|(x_1, x_2, \dots, x_m)\|_2 = (\sum_{j=1}^m \|x_j\|_{\mathbb{E}}^2)^{1/2}$. A collection $\{\Omega_1, \Omega_2, \dots, \Omega_m\}$ of nonempty closed subsets of $\mathbb{E}$ is subtransversal relative to $\Lambda := \{x = (u, u, \dots, u) \in \mathbb{E}^m \mid u \in \mathbb{E}\}$ at $\bar{x} = (\bar{u}, \dots, \bar{u})$ with $\bar{u} \in \bigcap_{j=1}^m \Omega_j$ for $\bar{y} = 0$ with gauge μ if there exists a neighborhood U' of $\bar{u}$ together with a gauge μ' satisfying $\sqrt{m}\mu' \leq \mu$ such that

$$\text{dist}\left(u, \bigcap_{j=1}^m \Omega_j\right) \leq \mu' \left(\max_{j=1, \dots, m} \text{dist}(u, \Omega_j) \right), \quad \forall u \in U'. \quad (67)$$

Conversely, if $\{\Omega_1, \Omega_2, \dots, \Omega_m\}$ is subtransversal relative to Λ at $\bar{x}$ for $\bar{y} = 0$ with gauge μ , then (67) is satisfied with any gauge μ' for which $\mu(\sqrt{m}t) \leq \sqrt{m}\mu'(t)$ for all $t \in [0, \infty)$.

Proof. Let $x = (u, u, \dots, u) \in \mathbb{E}^m$ with $u \in U'$, where U' denotes a neighborhood of $\bar{u}$. Note that $\Pi x = x$ for all $x \in \Lambda$. Moreover, for $\Psi(x) := P_\Omega(\Pi x) - \Pi x$ with $\Omega := \Omega_1 \times \Omega_2 \times \dots \times \Omega_m$, it holds that

$$\left(\bigcap_{j=1}^m \Omega_j, \bigcap_{j=1}^m \Omega_j, \dots, \bigcap_{j=1}^m \Omega_j \right) \cap \Lambda = \Psi^{-1}(0) \cap \Lambda. \quad (68)$$

To see this, note that any element $z \in \Psi^{-1}(0) \cap \Lambda$ satisfies $z \in \Lambda$ and $0 \in P_\Omega(\Pi z) - \Pi z$, which means that $z_i = z_j$, and $z_j \in \Omega_j$ for $i, j = 1, 2, \dots, m$. In other words, $z_i \in \bigcap_{j=1}^m \Omega_j$, and $z_i = z_j$ for $i, j = 1, 2, \dots, m$, which is just (68).

Denote $\Omega := \Omega_1 \times \Omega_2 \times \dots \times \Omega_m$. For the first implication, if (67) is satisfied, it holds that

$$\begin{aligned} \text{dist}\left(x, \left(\bigcap_{j=1}^m \Omega_j, \bigcap_{j=1}^m \Omega_j, \dots, \bigcap_{j=1}^m \Omega_j \right) \cap \Lambda\right) &= \sqrt{m} \text{dist}\left(u, \bigcap_{j=1}^m \Omega_j\right) \leq \sqrt{m} \mu' \left(\max_{j=1, \dots, m} \{\text{dist}(u, \Omega_j)\} \right) \\ &\leq \sqrt{m} \mu'(\text{dist}(x, \Omega)) \leq \mu(\text{dist}(x, \Omega)) = \mu(\text{dist}(\Pi x, P_\Omega(\Pi x))) = \mu(\text{dist}(0, \Psi(x))) \end{aligned} \quad (69)$$

whenever $x \in U \cap \Lambda = \{x = (u, u, \dots, u) \mid u \in U'\}$. By (68), the inequality (69) is equivalent to

$$\text{dist}(x, \Psi^{-1}(0) \cap \Lambda) \leq \mu(\text{dist}(0, \Psi(x))), \quad \forall x \in U \cap \Lambda, \quad (70)$$

which is the definition of subtransversality of $\{\Omega_1, \Omega_2, \dots, \Omega_m\}$ relative to Λ at $\bar{x}$ for 0 with gauge μ .

For the reverse implication, if (70) is satisfied, (using (68)), it holds that

$$\begin{aligned} \text{dist}\left(u, \bigcap_{j=1}^m \Omega_j\right) &= \frac{1}{\sqrt{m}} \text{dist}\left(x, \left(\bigcap_{j=1}^m \Omega_j, \bigcap_{j=1}^m \Omega_j, \dots, \bigcap_{j=1}^m \Omega_j\right) \cap \Lambda\right) = \frac{1}{\sqrt{m}} \text{dist}(x, \Psi^{-1}(0) \cap \Lambda) \leq \frac{1}{\sqrt{m}} \mu(\text{dist}(0, \Psi(x))) \\ &= \frac{1}{\sqrt{m}} \mu(\text{dist}(x, \Omega)) \leq \frac{1}{\sqrt{m}} \mu\left(\sqrt{m} \max_{j=1, \dots, m} \{\text{dist}(u, \Omega_j)\}\right) \leq \mu'\left(\max_{j=1, \dots, m} \{\text{dist}(u, \Omega_j)\}\right), \quad \forall u \in U'. \end{aligned}$$

This is (67). $\square$

Note that if one endows $\mathbb{E}^m$ with the maximum norm, $(\|(x_1, x_2, \dots, x_m)\|_{\mathbb{E}^m} := \max_{1 \leq j \leq m} \|x_j\|_{\mathbb{E}})$, it holds that

$$\begin{aligned} \text{dist}\left(x, \left(\bigcap_{j=1}^m \Omega_j, \bigcap_{j=1}^m \Omega_j, \dots, \bigcap_{j=1}^m \Omega_j\right) \cap \Lambda\right) &= \text{dist}\left(u, \bigcap_{j=1}^m \Omega_j\right); \\ \text{dist}(x, \Omega) &= \max_{j=1, \dots, m} \text{dist}(u, \Omega_j) \quad \text{for all } u \text{ and } x \text{ as above.} \end{aligned}$$

Then, the two properties in Proposition 3.3 are equivalent for the same gauge $\mu' = \mu$.

By Kruger et al. [44, theorem 1], Proposition 3.3 shows that Definition 3.2(i) coincides with subtransversality defined in Kruger et al. [44, definition 6] for points of intersection. This notion was developed to bring many other definitions of regularities of collections of sets (Lewis and Malick [45], Lewis et al. [46], Bauschke et al. [17, 16], Drusvyatskiy et al. [30], Kruger and Thao [43], Kruger [42]) under a common framework. The definition given in Kruger et al. [44], however, does not immediately lead to a characterization of the relation between sets at points that are not common to all sets. There is much to be done to align the many different characterizations of (sub)transversality studied in Kruger et al. [44] with Definition 3.2 above, but this is not our main interest here.

3.1.3. Cyclic Projections. Having established the basic geometric language of set feasibility and its connection to the averaging and stability properties of fixed point mappings, we can now pursue our main goal for this section: new convergence results for cyclic projections between sets with possibly empty intersection, Theorem 3.2, and Corollary 3.1. The majority of the work, and the source of technical complications, lies in constructing an appropriate fixed point mapping in the right space to be able to apply Theorem 2.2. As we have already said, establishing the extent of almost averaging is a straightforward application of Theorem 3.1. Thanks to Proposition 2.4, this can be stated in terms of the more primitive property of elemental set regularity. The challenging part is to show that subtransversality as introduced above leads to metric subregularity of an appropriate fixed point surrogate for cyclic projections, Proposition 3.4. In the process, we show in Proposition 3.5 that elemental regularity and subtransversality become entangled and it is not clear whether they can be completely separated when it comes to necessary conditions for convergence of cyclic projections.

Given a collection of closed subsets of $\mathbb{E}$, $\{\Omega_1, \Omega_2, \dots, \Omega_m\}$ ($m \geq 2$), and an initial point u^0 , the cyclic projections algorithm generates the sequence $(u^k)_{k \in \mathbb{N}}$ by

$$u^{k+1} \in P_0 u^k, \quad P_0 := P_{\Omega_1} P_{\Omega_2} \cdots P_{\Omega_m} P_{\Omega_1}. \quad (71)$$

Since projectors are idempotent, the initial P_{Ω_1} at the right end of the cycle has no real effect on the sequence, though we retain it for technical reasons. We will assume throughout this section that $\text{Fix } P_0 \neq \emptyset$.

Our analysis proceeds on an appropriate product space designed for the cycles associated with a given fixed point of P_0 . As above, we will use Ω to denote the sets Ω_j on $\mathbb{E}^m$: $\Omega := \Omega_1 \times \Omega_2 \times \cdots \times \Omega_m$. Let $\bar{u} \in \text{Fix } P_0$ and let $\bar{\zeta} \in \mathcal{Z}(\bar{u})$, where

$$\mathcal{Z}(u) := \{\bar{\zeta} := z - \Pi z \mid z \in W_0 \subset \mathbb{E}^m, z_1 = u\} \quad (72)$$

for

$$W_0 := \{x \in \mathbb{E}^m \mid x_m \in P_{\Omega_m} x_1, x_j \in P_{\Omega_j} x_{j+1}, j = 1, 2, \dots, m-1\}. \quad (73)$$

Note that $\sum_{j=1}^m \bar{\zeta}_j = 0$. The vector $\bar{\zeta}$ is a *difference vector*, which gives information regarding the intrasteps of the cyclic projections operator P_0 at the fixed point $\bar{u}$. In the case of only two sets, a difference vector is frequently called a *gap* vector (Bauschke and Borwein [9], Bauschke et al. [15], Luke [48], Bauschke and Moursi [11]). This is unique in the convex case, but need not be in the nonconvex case (see Lemma 3.2 below). In the more general setting we have here, this corresponds to nonuniqueness of cycles for cyclic projections. This greatly complicates matters since the fixed points associated with P_0 will not, in general, be associated with cycles that are the same

length and orientation. Consequently, the usual trick of looking at the zeros of $P_0 - \text{Id}$ is rather uninformative, and another mapping needs to be constructed, which distinguishes fixed points associated with different cycles. The following development establishes some of the key properties of difference vectors and cycles, which then motivates the mapping that we construct for this purpose.

To analyze the cyclic projections algorithm, we consider the sequence on the product space on $\mathbb{E}^m$, $(x^k)_{k \in \mathbb{N}}$ generated by $x^{k+1} \in T_{\bar{\zeta}} x^k$ with

$$T_{\bar{\zeta}}: \mathbb{E}^m \rightrightarrows \mathbb{E}^m: x \mapsto \left\{ \left(x_1^+, x_1^+ - \bar{\zeta}_1, \dots, x_1^+ - \sum_{j=1}^{m-1} \bar{\zeta}_j \right) \mid x_1^+ \in P_0 x_1 \right\} \quad (74)$$

for $\bar{\zeta} \in \mathcal{Z}(\bar{u})$, where $\bar{u} \in \text{Fix } P_0$. To isolate cycles, we restrict our attention to relevant subsets of $\mathbb{E}^m$. These are

$$W(\bar{\zeta}) := \{x \in \mathbb{E}^m \mid x - \Pi x = \bar{\zeta}\}, \quad (75a)$$

$$L := \text{an affine subspace with } T_{\bar{\zeta}}: L \rightrightarrows L, \quad \text{and} \quad (75b)$$

$$\Lambda := L \cap W(\bar{\zeta}). \quad (75c)$$

The set $W(\bar{\zeta})$ is an affine transformation of the diagonal of the product space and thus an affine subspace: for $x, y \in W(\bar{\zeta})$, $z = \lambda x + (1 - \lambda)y$ satisfies $z - \Pi z = \bar{\zeta}$ for all $\lambda \in \mathbb{R}$. This affine subspace is used to characterize the local geometry of the sets in relation to each other at fixed points of the cyclic projections operator.

Points in $\text{Fix } P_0$ can correspond to cycles of different lengths, hence an element $x \in \text{Fix } T_{\bar{\zeta}}$ need not be in W_0 and vice versa, as the next example demonstrates.

Example 3.2 ($\text{Fix } T_{\bar{\zeta}}$ and W_0). Consider the sets $\Omega_1 = \{0, 1\}$ and $\Omega_2 = \{0, 3/4\}$. The cyclic projections operator P_0 has fixed points $\{0, 1\}$ and two corresponding cycles, $\mathcal{Z}(0) = \{(0, 0)\}$, and $\mathcal{Z}(1) = \{(1/4, -1/4)\}$. Let $\bar{\zeta} = (1/4, -1/4)$. Then, $(0, -1/4) \in \text{Fix } T_{\bar{\zeta}}$ but $(0, -1/4) \notin W_0$. Conversely, the vector $(0, 0) \in W_0$, but $(0, 0) \notin \text{Fix } T_{\bar{\zeta}}$. The point $(1, 3/4)$, however, belongs to W_0 and $\text{Fix } T_{\bar{\zeta}}$.

The example above shows that what distinguishes elements in $\text{Fix } T_{\bar{\zeta}}$ from each other is whether or not they also belong to W_0 . The next lemma establishes that, on appropriate subsets, a fixed point of $T_{\bar{\zeta}}$ can be identified meaningfully with a vector in the image of the mapping Ψ in Definition 3.2, which is used to characterize the alignment of the sets Ω_j to each other at points of interest (in particular, fixed points of the cyclic projections operator).

Lemma 3.1. Let $\bar{u} \in \text{Fix } P_0$ and let $\bar{\zeta} \in \mathcal{Z}(\bar{u})$. Define $\Psi := (P_{\Omega} - \text{Id}) \circ \Pi$ and $\Phi_{\bar{\zeta}} := T_{\bar{\zeta}} - \text{Id}$.

(i) $T_{\bar{\zeta}}$ maps $W(\bar{\zeta})$ to itself. Moreover, $x \in \text{Fix } T_{\bar{\zeta}}$ if and only if $x \in W(\bar{\zeta})$ with $x_1 \in \text{Fix } P_0$. Indeed,

$$\text{Fix } T_{\bar{\zeta}} = \left\{ x = (x_1, x_2, \dots, x_m) \in \mathbb{E}^m \mid x_1 \in \text{Fix } P_0, x_j = x_1 - \sum_{i=1}^{j-1} \bar{\zeta}_i, j = 2, 3, \dots, m \right\}. \quad (76)$$

(ii) A point $\bar{z} \in \text{Fix } T_{\bar{\zeta}} \cap W_0$ if and only if $\bar{\zeta} \in \Psi(\bar{z})$ if and only if $\bar{\zeta} \in (\Phi_{\bar{\zeta}} \circ \Pi)(\bar{z})$.

(iii) $\Psi^{-1}(\bar{\zeta}) \cap W(\bar{\zeta}) \subseteq \Phi_{\bar{\zeta}}^{-1}(0) \cap W(\bar{\zeta})$.

(iv) If the distance is with respect to the Euclidean norm, then $\text{dist}(0, \Phi_{\bar{\zeta}}(x)) = \sqrt{m} \text{dist}(x_1, P_0 x_1)$.

Proof. (i) This is immediate from the definitions of $W(\bar{\zeta})$ and $T_{\bar{\zeta}}$.

(ii) From the definition of W_0 , it follows directly that $\bar{\zeta} \in \Psi(\bar{z})$ if and only if $\bar{z} \in \text{Fix } T_{\bar{\zeta}} \cap W_0$. Moreover, $\bar{\zeta} \in \Phi_{\bar{\zeta}}(\Pi \bar{z}) = T_{\bar{\zeta}} \Pi \bar{z} - \Pi \bar{z}$ if and only if for each $j = 1, 2, \dots, m$, it holds that $\bar{z}_{j+1} + \bar{\zeta}_j \in (T_{\bar{\zeta}} \Pi \bar{z})_j = u - \sum_{i=1}^{j-1} \bar{\zeta}_i$ for some $u \in P_0 \bar{z}_2$, and $\bar{z}_{m+1} := \bar{z}_1$. Equivalently, for some $u \in P_0 \bar{z}_2$, it holds that $\bar{z}_{j+1} + \sum_{i=1}^j \bar{\zeta}_i = u$ for all $j = 1, 2, \dots, m$. Since $\sum_{i=1}^m \bar{\zeta}_i = 0$, then $\bar{z}_1 = u$, therefore $\bar{z}_{j+1} = \bar{z}_1 - \sum_{i=1}^j \bar{\zeta}_i$ for all $j = 1, 2, \dots, m$, and $\bar{z}_1 \in P_0 \bar{z}_2$, which thanks to the redundancy of the first projector in the definition of P_0 (71) and the definition of W_0 , is equivalent to $\bar{z} \in \text{Fix } T_{\bar{\zeta}} \cap W_0$, as claimed.

To establish (iii), let $\bar{z} \in \Psi^{-1}(\bar{\zeta}) \cap W(\bar{\zeta})$. Then, $\bar{\zeta} \in ((P_{\Omega} - \text{Id}) \circ \Pi)(\bar{z})$, and since $\bar{z} \in W(\bar{\zeta})$, also $\bar{\zeta} = \bar{z} - \Pi \bar{z}$. Hence $\bar{\zeta} = \bar{z} - \Pi \bar{z}$ and $\bar{z} \in P_{\Omega}(\Pi \bar{z})$. But this implies that $\bar{\zeta} = \bar{z} - \Pi \bar{z}$, and $\bar{z}_1 \in P_0 \bar{z}_1$, hence $\Phi_{\bar{\zeta}}(\bar{z}) = 0$, and $\bar{z} \in W(\bar{\zeta})$. That is, $\bar{z} \in \Phi_{\bar{\zeta}}^{-1}(0) \cap W(\bar{\zeta})$, which verifies (iii).

Relation (iv) is obvious from the definition of $\Phi_{\bar{\zeta}}$. $\square$

Lemma 3.2 (Difference Vectors: Cyclic Projections). Let $\Omega_j \subseteq \mathbb{E}$ be nonempty and closed ($j = 1, 2, \dots, m$). Let $S_0 \subset \text{Fix } P_0$ and U_0 be a neighborhood of S_0 and define $U := \{z = (z_1, z_2, \dots, z_m) \in W_0 \mid z_1 \in U_0\}$. Fix $\bar{u} \in S_0$ and the difference vector

$\bar{\zeta} \in \mathcal{X}(\bar{u})$ with $\bar{\zeta} = \bar{z} - \Pi\bar{z}$ for the point $\bar{z} = (\bar{z}_1, \bar{z}_2, \dots, \bar{z}_m) \in W_0$ having $\bar{z}_1 = \bar{u}$. If Ω_j is elementally subregular at $\bar{z}_j$ for $(\bar{z}_j, 0) \in \text{gph } N_{\Omega_j}^{\text{prox}}$ with constant $\bar{\varepsilon}_j$ and neighborhood $U_j := p_j(U)$ of $\bar{z}_j$ (where p_j is the j th coordinate projection operator), then

$$\|\bar{\zeta} - \zeta\|^2 \leq \sum_{j=1}^m \varepsilon_j \|\bar{z}_j - z_j\|^2 \quad (\varepsilon_j := 2\bar{\varepsilon}_j + 2\bar{\varepsilon}_j^2) \quad (77)$$

for the difference vector $\zeta \in \mathcal{X}(u)$ with $u \in S_0$, and $\zeta = z - \Pi z$, where $z = (z_1, z_2, \dots, z_m) \in W_0$ with $z_1 = u$. If the sets Ω_j ($j = 1, 2, \dots, m$) are, in fact, convex, then the difference vector is unique and independent of the initial point $\bar{u}$; that is, $\mathcal{X}(u) = \{\bar{\zeta}\}$ for all $u \in S_0$.

Proof. Note that $U_0 \subset \Omega_1$ and $U_j \subset \Omega_j$ ($j = 1, 2, \dots, m$). By Theorem 3.1(iii), the projectors P_{Ω_j} are pointwise almost firmly nonexpansive at $\bar{z}_j$ on U_j with violation $\varepsilon_j := 2\bar{\varepsilon}_j + 2\bar{\varepsilon}_j^2$ (and averaging constant $\alpha_j = 1/2$). If the sets Ω_j are convex, then the violation $\varepsilon_j = 0$ and the projectors are firmly nonexpansive (globally). The result then follows by specializing Lemma 2.1 to pointwise almost firmly nonexpansive (respectively, firmly nonexpansive) projectors. $\square$

Proposition 3.4 (Metric Subregularity of Cyclic Projections). *Let $\bar{u} \in \text{Fix } P_0$ and $\bar{\zeta} \in \mathcal{X}(\bar{u})$ and let $\bar{x} = (\bar{x}_1, \bar{x}_2, \dots, \bar{x}_m) \in W_0$ satisfy $\bar{\zeta} = \bar{x} - \Pi\bar{x}$ with $\bar{x}_1 = \bar{u}$. For L , an affine subspace containing $\bar{x}$, let $T_{\bar{\zeta}}: L \rightrightarrows L$ and define the mapping $\Phi_{\bar{\zeta}} := T_{\bar{\zeta}} - \text{Id}$. Suppose the following hold:*

- (a) *the collection of sets $\{\Omega_1, \Omega_2, \dots, \Omega_m\}$ is subtransversal at $\bar{x}$ for $\bar{\zeta}$ relative to $\Lambda := L \cap W(\bar{\zeta})$ with constant κ and neighborhood U of $\bar{x}$;*
- (b) *there exists a positive constant σ such that*

$$\text{dist}(\bar{\zeta}, \Psi(x)) \leq \sigma \text{dist}(0, \Phi_{\bar{\zeta}}(x)), \quad \forall x \in \Lambda \cap U \text{ with } x_1 \in \Omega_1.$$

Then, Φ is metrically subregular for 0 on U (metrically regular on $U \times \{0\}$) relative to Λ with constant $\bar{\kappa} = \kappa\sigma$.

Proof. A straightforward application of the assumptions and Lemma 3.1(iii) yields

$$(\forall x \in U \cap \Lambda \text{ with } x_1 \in \Omega_1) \quad \text{dist}(x, \Phi_{\bar{\zeta}}^{-1}(0) \cap \Lambda) \leq \text{dist}(x, \Psi^{-1}(\bar{\zeta}) \cap \Lambda) \leq \kappa \text{dist}(\bar{\zeta}, \Psi(x)) \leq \kappa\sigma \text{dist}(0, \Phi_{\bar{\zeta}}(x)).$$

In other words, $\Phi_{\bar{\zeta}}$ is metrically subregular for 0 on U relative to Λ with constant $\bar{\kappa}$, as claimed. $\square$

Example 3.3 (Two Intersecting Sets). To provide some insight into condition (b) of Proposition 3.4, it is instructive to examine the case of two sets with nonempty intersection. Let $\bar{x} = (\bar{u}, \bar{u})$ with $\bar{u} \in \Omega_1 \cap \Omega_2$ and the difference vector $\bar{\zeta} = 0 \in \mathcal{X}(\bar{x})$. To simplify the presentation, let us consider $L = \mathbb{E}^2$ and $U = U' \times U'$, where U' is a neighborhood of $\bar{u}$. Then, one has $\Lambda = W(0) = \{(u, u): u \in \mathbb{E}\}$, and hence $x \in \Lambda \cap U$ with $x_1 \in \Omega_1$ is equivalent to $x = (u, u) \in U$ with $u \in \Omega_1 \cap U'$. For such a point $x = (u, u)$, one has

$$\text{dist}(0, \Psi(x)) = \text{dist}(u, \Omega_2), \quad \text{dist}(0, \Phi_0(x)) = \sqrt{2} \text{dist}(u, P_{\Omega_1} P_{\Omega_2}(u)),$$

where the last equality follows from the representation $\Phi_0(x) = \{(z - u, z - u) \in \mathbb{E}^2: z \in P_{\Omega_1} P_{\Omega_2}(u)\}$.

(b) of Proposition 3.4 becomes

$$\text{dist}(u, \Omega_2) \leq \gamma \text{dist}(u, P_{\Omega_1} P_{\Omega_2}(u)), \quad \forall u \in \Omega_1 \cap U', \quad (78)$$

where $\gamma := \sqrt{2}\sigma > 0$. In Kruger et al. [44, Remark 12], the phenomenon of entanglement of elemental subregularity and regularity of collections of sets is briefly discussed in the context of other notions of regularity in the literature. Inequality (78) serves as a type of conduit for this entanglement of regularities as Proposition 3.5 demonstrates.

Proposition 3.5 (Elemental Subregularity and (78) Imply Subtransversality). *Let $\bar{u} \in \Omega_1 \cap \Omega_2$, and U' be the neighborhood of $\bar{u}$ as in Example 3.3. Suppose that condition (78) holds and that the set Ω_1 is elementally subregular relative to Ω_2 at $\bar{u}$ for all $(\bar{y}, 0)$ with $\bar{y} \in \Omega_1 \cap U'$ with constant $\varepsilon < 1/(1 + \gamma^2)$ and the neighborhood U' . Then, $\{\Omega_1, \Omega_2\}$ is subtransversal at $\bar{u}$.*

Proof. Choose a number $\delta' > 0$ such that $\mathbb{B}_{2\delta'}(\bar{u}) \subset U'$. Take any $u \in \Omega_1 \cap \mathbb{B}_{\delta'}(\bar{u})$, and $u^+ \in P_{\Omega_1} P_{\Omega_2}(u)$. Let $u' \in P_{\Omega_2}(u)$ such that $u^+ \in P_{\Omega_1}(u')$. Note that $u' \in \Omega_2 \cap U'$. Without loss of generality, we can assume $u' \notin \Omega_1$. Then, $\|u - u'\| \geq \|u' - u^+\| > 0$. The elemental regularity of Ω_1 relative to Ω_2 at $\bar{u}$ for $(u, 0)$ with constant ε and neighborhood U' yields

$$\langle u' - u^+, u - u^+ \rangle \leq \varepsilon \|u' - u^+\| \|u - u^+\|.$$

This inequality and condition (78) (note that $\text{dist}(u, \Omega_2) = \|u - u'\|$ and $\text{dist}(u, P_{\Omega_1}P_{\Omega_2}(u)) \leq \|u - u^+\|$) yield

$$\begin{aligned} \|u - u'\|^2 &= \|u - u^+\|^2 + \|u^+ - u'\|^2 + 2\langle u - u^+, u^+ - u' \rangle \geq \|u - u^+\|^2 + \|u^+ - u'\|^2 - 2\varepsilon \|u' - u^+\| \|u - u^+\| \\ &= (1 - \varepsilon)(\|u - u^+\|^2 + \|u^+ - u'\|^2) + \varepsilon(\|u' - u^+\| - \|u - u^+\|)^2 \geq (1 - \varepsilon)(\|u - u^+\|^2 + \|u^+ - u'\|^2) \\ &\geq (1 - \varepsilon)(\gamma^{-2}\|u - u'\|^2 + \|u^+ - u'\|^2). \end{aligned}$$

It is clear that $1/(1 - \varepsilon) \geq 1/\gamma^2$, and hence

$$\|u' - u^+\| \leq c \|u - u'\|, \quad (79)$$

where $c := \sqrt{1/(1 - \varepsilon) - 1/\gamma^2} \in [0, 1)$ as $\varepsilon < 1/(1 + \gamma^2)$.

Choose a number $\delta > 0$ such that $((1 + c)/(1 - c))\delta \leq \delta'$. Employing the basic argument originated in Lewis et al. [46, theorem 5.2], one can derive that for any given point $u \in \mathbb{B}_\delta(\bar{u}) \cap \Omega_1$, there exists a point $\tilde{u} \in \Omega_1 \cap \Omega_2$ such that

$$\|u - \tilde{u}\| \leq \frac{2}{1 - c} \|u - u'\|.$$

In other words,

$$\frac{1 - c}{2} \text{dist}(u, \Omega_1 \cap \Omega_2) \leq \frac{1 - c}{2} \|u - \tilde{u}\| \leq \|u - u'\| = \text{dist}(u, \Omega_2), \quad \forall u \in \mathbb{B}_\delta(\bar{u}) \cap \Omega_1.$$

The subtransversality of $\{\Omega_1, \Omega_2\}$ at $\bar{u}$ now follows from Proposition 3.3 (or alternatively Kruger et al. [44, theorem 1(iii)]). $\square$

The main result of this section can now be presented. This statement uses the full technology of regularities relativized to certain sets of points S_j introduced in Definitions 3.1 and 2.2 and used in Proposition 2.4, as well as the expanded notion of subtransversality of sets at points of nonintersection introduced in Definition 3.2 and applied in Proposition 3.4.

Theorem 3.2 (Convergence of Cyclic Projections). *Let $S_0 \subset \text{Fix } P_0 \neq \emptyset$, and $Z := \bigcup_{u \in S_0} \mathcal{Z}(u)$. Define*

$$S_j := \bigcup_{\zeta \in Z} \left(S_0 - \sum_{i=1}^{j-1} \zeta_i \right) \quad (j = 1, 2, \dots, m). \quad (80)$$

Let $U := U_1 \times U_2 \times \dots \times U_m$ be a neighborhood of $S := S_1 \times S_2 \times \dots \times S_m$ and suppose that

$$P_{\Omega_j} \left(u - \sum_{i=1}^j \zeta_i \right) \subseteq S_0 - \sum_{i=1}^{j-1} \zeta_i, \quad \forall u \in S_0, \quad \forall \zeta \in Z \text{ for each } j = 1, 2, \dots, m, \quad (81a)$$

$$P_{\Omega_j} U_{j+1} \subseteq U_j, \quad \text{for each } j = 1, 2, \dots, m \quad (U_{m+1} := U_1). \quad (81b)$$

For $\bar{\zeta} \in Z$ fixed and $\bar{x} \in S$ with $\bar{\zeta} = \Pi \bar{x} - \bar{x}$, generate the sequence $(x^k)_{k \in \mathbb{N}}$ by $x^{k+1} \in T_{\bar{\zeta}} x^k$ for $T_{\bar{\zeta}}$ defined by (74), seeded by a point $x^0 \in W(\bar{\zeta}) \cap U$ for $W(\bar{\zeta})$ defined by (75a) with $x_1^0 \in \Omega_1 \cap U_1$.

Suppose that, for $\Lambda := L \cap \text{aff}(\bigcup_{\zeta \in Z} W(\zeta)) \supset S$ such that $T_{\bar{\zeta}}: \Lambda \rightrightarrows \Lambda$ for all $\zeta \in Z$ and an affine subspace $L \supset \text{aff}(x^k)_{k \in \mathbb{N}}$, the following hold:

(a) *the set Ω_j is elementally subregular at all $\hat{x}_j \in S_j$ relative to S_j for each*

$$(x_j, v_j) \in V_j := \{(z, w) \in \text{gph } N_{\Omega_j}^{\text{prox}} \mid z + w \in U_j, \text{ and } z \in P_{\Omega_j}(z + w)\}$$

with constant $\varepsilon_j \in (0, 1)$ on the neighborhood U_j for $j = 1, 2, \dots, m$;

(b) *for each $\hat{x} = (\hat{x}_1, \hat{x}_2, \dots, \hat{x}_m) \in S$, the collection of sets $\{\Omega_1, \Omega_2, \dots, \Omega_m\}$ is subtransversal at $\hat{x}$ for $\hat{\zeta} := \hat{x} - \Pi \hat{x}$ relative to Λ with constant κ on the neighborhood U ;*

(c) *for $\Phi_{\hat{\zeta}} := T_{\hat{\zeta}} - \text{Id}$ and $\Psi := (P_{\Omega} - \text{Id}) \circ \Pi$, there exists a positive constant σ such that for all $\hat{\zeta} \in Z$,*

$$\text{dist}(\hat{\zeta}, \Psi(x)) \leq \sigma \text{dist}(0, \Phi_{\hat{\zeta}}(x))$$

holds whenever $x \in \Lambda \cap U$ with $x_1 \in \Omega_1$;

(d) *$\text{dist}(x, S) \leq \text{dist}(x, \Phi_{\hat{\zeta}}^{-1}(0) \cap \Lambda)$ for all $x \in U \cap \Lambda$, for all $\hat{\zeta} \in Z$.*

Then, the sequence $(x^k)_{k \in \mathbb{N}}$ seeded by a point $x^0 \in W(\bar{\zeta}) \cap U$ with $x_1^0 \in \Omega_1 \cap U_1$ satisfies

$$\text{dist}(x^{k+1}, \text{Fix } T_{\bar{\zeta}} \cap S) \leq c \text{dist}(x^k, S)$$

whenever $x^k \in U$ with

$$c := \sqrt{1 + \bar{\varepsilon} - \frac{1 - \alpha}{\alpha \bar{\kappa}^2}} \quad (82)$$

for

$$\bar{\varepsilon} := \prod_{j=1}^m (1 + \varepsilon_j) - 1, \quad \bar{\varepsilon}_j := 4\varepsilon_j \frac{1 + \varepsilon_j}{(1 - \varepsilon_j)^2}, \quad \alpha := \frac{m}{m + 1}, \quad (83)$$

and $\bar{\kappa} = \kappa\sigma$. If, in addition,

$$\bar{\kappa} < \sqrt{\frac{1 - \alpha}{\bar{\varepsilon}\alpha}}, \quad (84)$$

then $\text{dist}(x^k, \text{Fix } T_{\bar{\zeta}} \cap S) \rightarrow 0$, and hence $\text{dist}(x_1^k, \text{Fix } P_0 \cap S_1) \rightarrow 0$, at least linearly with rate $c < 1$.

Proof. The neighborhood U can be replaced by an enlargement of S , hence the result follows from Theorem 2.2 once it can be shown that the assumptions are satisfied for the mapping $T_{\bar{\zeta}}$ on the product space $\mathbb{E}^m$ restricted to Λ . To see that Assumption (a) of Theorem 2.2 is satisfied, note first that, by condition (81a) and definition (80), $P_{\Omega_j} S_{j+1} \subset S_j$. This together with condition (81b) and Assumption (a) allow one to conclude from Theorem 3.1(iv) that the projector P_{Ω_j} is pointwise almost firmly nonexpansive at each $y_j \in S_j$ with violation ε_j on U_j given by (83). Then, by Proposition 2.4(iii), the cyclic projections mapping P_0 is pointwise almost averaged at each $y_1 \in S_1$ with violation $\bar{\varepsilon}$ and averaging constant α given by (83) on U_1 . Since $T_{\bar{\zeta}}$ is just P_0 shifted by $\bar{\zeta}$ on the product space, it follows that $T_{\bar{\zeta}}$ is pointwise almost averaged at each $y \in S := S_1 \times S_2 \times \cdots \times S_m$ with the same violation $\bar{\varepsilon}$ and averaging constant α on U .

Assumption (b) of Theorem 2.2 for $\Phi_{\bar{\zeta}}$ follows from Assumptions (b)–(d) and Proposition 3.4. This completes the proof. $\square$

Corollary 3.1 (Global R -Linear Convergence of Convex Cyclic Projections). *Let the sets Ω_j ($j = 1, 2, \dots, m$) be nonempty, closed, and convex, let $S_0 = \text{Fix } P_0 \neq \emptyset$ and $S = S_1 \times S_2 \times \cdots \times S_m$ for S_j defined by (80). Let $\Lambda := W(\bar{\zeta})$ for $\bar{\zeta} \in \mathcal{X}(u)$ and any $u \in S_0$. Suppose, in addition, that*

(b') for each $\hat{x} = (\hat{x}_1, \hat{x}_2, \dots, \hat{x}_m) \in S$, the collection of sets $\{\Omega_1, \Omega_2, \dots, \Omega_m\}$ is subtransversal at $\hat{x}$ for $\bar{\zeta} = \hat{x} - \Pi\hat{x}$ relative to Λ with neighborhood $U \supset S$;

(c') there exists a positive constant σ such that

$$\text{dist}(\bar{\zeta}, \Psi(x)) \leq \sigma \text{dist}(0, \Phi_{\bar{\zeta}}(x))$$

holds whenever $x \in \Lambda \cap U$ with $x_1 \in \Omega_1$.

Then, the sequence $(x^k)_{k \in \mathbb{N}}$ generated by $x^{k+1} \in T_{\bar{\zeta}} x^k$ seeded by any point $x^0 \in W(\bar{\zeta})$ with $x_1^0 \in \Omega_1$ satisfies

$$\text{dist}(x^{k+1}, \text{Fix } T_{\bar{\zeta}} \cap S) \leq c \text{dist}(x^k, S)$$

for all k large enough where

$$c := \sqrt{1 - \frac{1 - \alpha}{\alpha \bar{\kappa}^2}} < 1 \quad (85)$$

with $\bar{\kappa} = \kappa\sigma$ for κ a constant of metric subregularity of Ψ for $\bar{\zeta}$ on U relative to Λ and α given by (83). In other words, $\text{dist}(x^k, \text{Fix } T_{\bar{\zeta}} \cap S) \rightarrow 0$, and hence $\text{dist}(x_1^k, \text{Fix } P_0 \cap S_0) \rightarrow 0$, at least R -linearly with rate $c < 1$.

Proof. By Lemma 3.2, $Z = \mathcal{X}(u) = \{\bar{\zeta}\}$ for any $u \in S_0$. Moreover, since Ω_j is convex, the projector is single-valued and firmly nonexpansive, and further the conditions (81) are satisfied with $U_j = \mathbb{E}$ ($j = 1, 2, \dots, m$) since $S_0 = S_1$ and

$$P_{\Omega_j} \left(S_1 - \sum_{i=1}^j \bar{\zeta}_i \right) = P_{\Omega_j} (S_{j+1}) = S_j = S_1 - \sum_{i=1}^{j-1} \bar{\zeta}_i, \quad \text{for each } j = 1, 2, \dots, m. \quad (86)$$

Also, by convexity, Ω_j ($j = 1, 2, \dots, m$) is elementally regular with constant $\varepsilon_j = 0$ globally ($U_j = \mathbb{E}$), therefore Assumption (a) of Theorem 3.2 is satisfied. Moreover, $\Phi_{\bar{\zeta}}^{-1}(0) = S_0$ therefore condition (d) holds trivially. The result then follows immediately from Theorem 3.2. $\square$

When the sets Ω_j are affine, then it is easy to see that the sets are subtransversal to each other at collections of nearest points corresponding to the gap between the sets. If the cyclic projections algorithm does not converge in one step (which it will in the case of either parallel or orthogonally arranged sets), the above corollary shows that cyclic projections converge linearly with rate $\sqrt{1 - \kappa}$, where κ is the constant of metric subregularity, reflecting the angle between the affine subspaces. This much for the affine case has already been shown in Bauschke et al. [12, theorem 5.7.8].

Remark 3.1 (Global Convergence for Nonconvex Alternating Projections). Convexity is not necessary for global linear convergence of alternating projections. This has been demonstrated using earlier versions of the theory presented here for sparse affine feasibility in Hesse et al. [34, corollary III.13 and theorem III.15]. A sufficient property for global results in sparse affine feasibility is a common *restricted isometry property* Hesse et al. [34, equation (32)] familiar to experts in signal processing with sparsity constraints. The restricted isometry property was shown in Hesse et al. [34, proposition III.14] to imply *transversality* of the affine subspace with all subspaces of a certain dimension.

Example 3.4 (An Equilateral Triangle—Three Affine Subspaces with a Hole). Consider the problem specified by the following three sets in $\mathbb{R}^2$:

$$\begin{aligned}\Omega_1 &= \mathbb{R}(1, 0) = \{x \in \mathbb{R}^2 \mid \langle (0, 1), x \rangle = 0\}, \\ \Omega_2 &= (0, -1) + \mathbb{R}(-\sqrt{3}, 1) = \{x \in \mathbb{R}^2 \mid \langle (-\sqrt{3}, 1), x \rangle = \sqrt{3}\}, \\ \Omega_3 &= (0, 1) + \mathbb{R}(\sqrt{3}, 1) = \{x \in \mathbb{R}^2 \mid \langle (\sqrt{3}, 1), x \rangle = 1\}.\end{aligned}$$

The following statements regarding the assumptions of Corollary 3.1 are easily verified:

- (i) The set $S_0 = \text{Fix } P_0 = \{(-1/3, 0)\}$.
- (ii) There is a unique fixed point $\bar{x} = (\bar{x}_1, \bar{x}_2, \bar{x}_3) = ((-1/3, 0), (-1/3, 2/\sqrt{3}), (2/3, 1/\sqrt{3}))$.
- (iii) The set of difference vectors is a singleton:

$$Z = \{\bar{\zeta}\} = \{(\bar{\zeta}_1, \bar{\zeta}_2, \bar{\zeta}_3)\} = \{((0, -2/\sqrt{3}), (-1, 1/\sqrt{3}), (1, 1/\sqrt{3}))\}.$$

- (iv) The sets S_1, S_2 , and S_3 are given by

$$S_1 = S_0 - \bar{\zeta}_1 = \{(-1/3, 2/\sqrt{3})\}, \quad S_2 = S_0 - \bar{\zeta}_1 - \bar{\zeta}_2 = \{(2/3, 1/\sqrt{3})\}, \quad S_3 = S_0 = \{(-1/3, 0)\}.$$

- (v) Condition (81a) is satisfied and condition (81b) is satisfied with $U_j = \mathbb{R}^2$ ($j = 1, 2, 3$).
- (vi) For $j \in \{1, 2, 3\}$, Ω_j is convex and hence elementally regular at $\bar{x}_j$ with constant $\varepsilon_j = 0$ Kruger et al. [44, proposition 4].
- (vii) The mapping Ψ is metrically subregular for $\bar{\zeta}$ on $(\mathbb{R}^2)^3$ with constant $\kappa = \sqrt{2}$ relative to $W(\bar{\zeta})$:

$$\text{dist}(x, \Psi^{-1}(\bar{\zeta}) \cap W(\bar{\zeta})) \leq \sqrt{2} \text{dist}(\bar{\zeta}, \Psi(x)), \quad \forall x \in (\mathbb{R}^2)^3.$$

- (viii) For all $x \in W(\bar{\zeta})$, the inequality $\text{dist}(\bar{\zeta}, \Psi(x)) \leq \sigma \text{dist}(0, \Phi_{\bar{\zeta}}(x))$ holds with $\sigma = 4\sqrt{2}/9$.

The assumptions of Corollary 3.1 are satisfied. Furthermore, Proposition 3.4 shows that the mapping $\Phi_{\bar{\zeta}}$ is metrically subregular for 0 on $(\mathbb{R}^2)^3$ relative to $W(\bar{\zeta})$ with constant $\bar{\kappa} = \kappa\sigma = \sqrt{2} \times 4\sqrt{2}/9 = 8/9$. Altogether, Corollary 3.1 yields that, from any starting point, the cyclic projections method converges linearly to $\bar{u}$ with rate at most $c = \sqrt{37}/8$.

The next example is new and rather unexpected.

Example 3.5 (Two Nonintersecting Circles). Fix $r > 0$ and consider the problem specified by the following two sets in $\mathbb{R}^2$

$$\Omega_1 = \{x \in \mathbb{R}^2 \mid \|x\| = 1\}, \quad \Omega_2 = \{x \in \mathbb{R}^2 \mid \|x + (0, 1/2 + r)\| = 2 + r\}. \quad (87)$$

In this example, we focus on (local) behavior around the point $\bar{u} = (0, 1)$. For U_1 , a sufficiently small neighborhood of $\bar{u}$, the following statements regarding the assumptions of Theorem 3.2 can be verified:

- (i) $S_0 = \text{Fix } P_0 \cap U_1 = \{\bar{u}\} = \{(0, 1)\}$;
- (ii) $\bar{x} = (\bar{x}_1, \bar{x}_2) = (\bar{u}, (0, 3/2)) = ((0, 1), (0, 3/2))$;
- (iii) $\mathcal{Z} = \{\bar{\zeta}\} = \{(\bar{\zeta}_1, \bar{\zeta}_2)\} = \{((0, -1/2), (0, 1/2))\}$;
- (iv) the sets S_1 and S_2 are given by

$$S_1 = S_0 - \bar{\zeta}_1 = \{(0, 1/2)\}, \quad S_2 = S_0 - \bar{\zeta}_1 - \bar{\zeta}_2 = \{(0, 1)\};$$

(v) (81a) is satisfied, and (81b) holds with U_1 already given and U_2 equal to a Scaled translate of U_1 —more precisely, U_1 and U_2 are related by

$$U_2 = \frac{2+r}{\text{dist}(\bar{u}, (0, -1/2-r))} U_1 + (0, 1/2);$$

(vi) $L = \mathbb{R}^2 \times \mathbb{R}^2$;

(vii) for $j \in \{1, 2\}$, Ω_j is uniformly elementally regular at $\bar{x}_j$ for any $\varepsilon_j \in (0, 1)$ Kruger et al. [44, example 2(b)].

To verify the remaining conditions of Theorem 3.2, we use the following parametrization: any double $x = (x_1, x_2) \in W(\bar{\zeta})$ with $x_1 \in \Omega_1$ may be expressed in the form $x_1 = (b, \sqrt{1-b^2}) \in \Omega_1$, where $b \in \mathbb{R}$ is a parameter.

(viii) $\{\Omega_1, \Omega_2\}$ is subtransversal at $\bar{x}$ relative to $W(\bar{\zeta})$, i.e., Ψ is metrically subregular at $\bar{x}$ for $\bar{\zeta}$ on U (metrically regular at $(\bar{x}, \bar{\zeta})$ on $U \times \{\bar{\zeta}\}$ relative to $W(\bar{\zeta})$ with constant

$$\kappa > \lim_{b \rightarrow 0} \frac{\text{dist}(x, \Psi^{-1}(\bar{\zeta}) \cap W(\bar{\zeta}))}{\text{dist}(\bar{\zeta}, \Psi(x))} = \frac{3(2r+3)}{\sqrt{2r^2+6r+9}}.$$

(ix) For any $\rho > 0$ such that

$$\rho = \frac{\sqrt{2r^2+6r+9}}{2r+1} > \lim_{b \rightarrow 0} \frac{\text{dist}(\bar{\zeta}, \Psi(x))}{\text{dist}(0, \Phi_{\bar{\zeta}}(x))} = \frac{\sqrt{2}\sqrt{2r^2+6r+9}(2r+3)}{2\sqrt{4r^2+12r+13}(r+2)},$$

the following inequality holds:

$$\text{dist}(\bar{\zeta}, \Psi(x)) \leq \rho \text{dist}(0, \Phi_{\bar{\zeta}}(x))$$

for all $x \in W(\bar{\zeta})$ sufficiently close to $\bar{x}$.

The assumptions of Theorem 3.2 are satisfied. Furthermore, the proof of Proposition 3.4 shows that the mapping $\Phi_{\bar{\zeta}}$ is metrically subregular at $\bar{x}$ for 0 relative to $W(\bar{\zeta})$ on U with the constant $\bar{\kappa}$ equal to the product of constant of subtransversality κ in (viii) and ρ . That is,

$$\bar{\kappa} > \frac{3(3r+3)}{2r+1} = \frac{3\sqrt{2}(2r+3)^2}{2\sqrt{4r^2+12r+13}(r+2)}.$$

Altogether, Theorem 3.2 yields that, for any c with

$$1 > c > \sqrt{1 - \frac{(2r+1)^2}{18(2r+3)^2}},$$

there exists a neighborhood of $\bar{u}$ such that the cyclic projections method converges linearly to $\bar{u}$ with rate c .

Remark 3.2 (Non-Intersecting Circle and Line). A similar analysis to Example 3.5 can be performed for the case in which the second circle Ω_2 is replaced with the line $(0, 3/2) + \mathbb{R}(1, 0)$. Formally, this corresponds to setting the parameter $r = +\infty$ in Example 3.5. Although there are some technicalities involved to make such an argument fully rigorous, a separate computation has verified the constants obtained in this way agree with those obtained from a direct computation. When the circle and line are tangent, then Example 2.4 shows how sublinear convergence of alternating projections can be quantified.

Example 3.6 (Phase Retrieval). In the discrete version of the phase retrieval problem (Luke et al. [52], Bauschke et al. [13], Burke and Luke [23], Luke [47, 49], Hesse et al. [35], Luke [50]), the constraint sets are of the form

$$\Omega_j = \{x \in \mathbb{C}^n \mid |(A_j x)_k|^2 = b_{jk}, k = 1, 2, \dots, n\}, \quad (88)$$

where $A_j: \mathbb{C}^n \rightarrow \mathbb{C}^n$ is a unitary linear operator (a Fresnel or Fourier transform depending on whether the data is in the near field or far field, respectively) for $j = 1, 2, \dots, m$, possibly together with an additional support/support-nonnegativity constraint, Ω_0 . It is elementary to show that the sets Ω_j are elementally regular (indeed, they are semi-algebraic (Hesse et al. [35, proposition 3.5]) and prox-regular (Kruger et al. [44, proposition 4]), and Ω_0 is convex) therefore condition (a) of Theorem 3.2 is satisfied for each Ω_j with some violation ε_j on local neighborhoods. Subtransversality of the collection of sets at a fixed point $\bar{x}$ of P_0 can only be violated when the sets are locally parallel at $\bar{x}$ for the corresponding difference vector. It is beyond the focus of this paper to show that this cannot happen in almost all practical instances, establishing that condition (b) of Theorem 3.2 holds. The

remaining conditions (c)–(d) are technical and Example 3.5—which essentially captures the geometry of the sets in the phase retrieval problem—shows that these assumptions are satisfied. Theorem 3.2 then shows that near *stable* fixed points (defined as those that correspond to local best approximation points Luke [48, definition 3.3]) the method of alternating projections *must* converge linearly. In particular, the cyclic projections algorithm can be expected to converge linearly on neighborhoods of stable fixed points *regardless of whether or not the phase sets intersect*. This improves, in several ways, the local linear convergence result obtained in Luke [49, theorem 5.1], which established local linear convergence of *approximate* alternating projections to local solutions with more general gauges for the case of two sets: first, the present theory handles more than two sets, which is relevant for wave front sensing (Luke [52], Hesse et al. [35]); second, it does not require that the intersection of the constraint sets (which are expressed in terms of noisy, incomplete measurement data) be nonempty. This is in contrast to recent studies of the phase retrieval problem (of which there are too many to cite here), which require the assumption of feasibility, despite evidence, numerical and experimental, to the contrary. Indeed, according to elementary noncrystallographic diffraction theory, since the experimental measurements—the constants b_{jk} in the sets Ω_j defined in (88)—are finite samples of the continuous Fourier/Fresnel transform, there can be no overlap between the set of points satisfying the measurements and the set of compactly supported objects specified by the constraint Ω_0 . Adding another layer to this fundamental inconsistency is the fact that the measurements are noisy and inexact. The presumption that these sets have nonempty intersection is neither reasonable nor necessary. Regarding approximate/inexact evaluation of the projectors studied in Luke [49], we see no obvious impediment to such an extension and this would indeed be a valuable endeavor, again, beyond the scope of this work. Toward global convergence results, Theorem 3.2 indicates that the focus of any such effort should be on determining when the set of *difference vectors* is unique rather than focusing on uniqueness of the intersection as proposed in Candès et al. [24], Hesse et al. [34].

3.2. Structured (Nonconvex) Optimization

We consider next the problem

$$\underset{x \in \mathbb{E}}{\text{minimize}} \quad f(x) + g(x) \quad (\mathcal{P})$$

under different assumptions on the functions f and g . At the very least, we will assume that these functions are proper, lower semicontinuous (l.s.c.) functions.

3.2.1. Forward-Backward. We begin with the ubiquitous *forward-backward* algorithm: given $x^0 \in \mathbb{E}$, generate the sequence $(x^k)_{k \in \mathbb{N}}$ via

$$x^{k+1} \in T_{\text{FB}}(x^k) := \text{prox}_{1,g}(x^k - t\nabla f(x^k)). \quad (89)$$

We keep the step-length t fixed for simplicity. This is a reasonable strategy, obviously, when f is continuously differentiable with Lipschitz continuous gradient and when g is convex (not necessarily smooth), which we will assume throughout this subsection. For the case that g is the indicator function of a set C ; that is, $g = \iota_C$, then (89) is just the projected gradient algorithm for constrained optimization with a smooth objective. For simplicity, we will take the proximal parameter $\lambda = 1$ and use the notation prox_g instead of $\text{prox}_{1,g}$. The following discussion uses the property of hypomonotonicity (Definition 2.3(b)).

Proposition 3.6 (Almost Averaged: Steepest Descent). *Let U be a nonempty open subset of $\mathbb{R}^n$. Let $f: \mathbb{R}^n \rightarrow \mathbb{R}$ be a continuously differentiable function with calm gradient at $\bar{x}$ and calmness modulus L on the neighborhood U of $\bar{x}$. In addition, let ∇f be pointwise hypomonotone at $\bar{x}$ with violation constant τ on U . Choose $\beta > 0$ and let $t \in (0, \beta)$. Then, the mapping $T_{t,f} := \text{Id} - t\nabla f$ is pointwise almost averaged at $\bar{x}$ with averaging constant $\alpha = t/\beta \in (0, 1)$ and violation constant $\varepsilon = \alpha(2\beta\tau + \beta^2L^2)$ on U . If ∇f is pointwise strongly monotone at $\bar{x}$ with modulus $|\tau| > 0$ (that is, pointwise hypomonotone with constant $\tau < 0$) and calm with modulus L on U and $t < 2|\tau|/L^2$, then $T_{t,f}$ is pointwise averaged at $\bar{x}$ with averaging constant $\alpha = tL^2/(2|\tau|) \in (0, 1)$ on U .*

Proof. Noting that

$$\text{Id} - (\alpha\beta)\nabla f = (1 - \alpha)\text{Id} + \alpha(\text{Id} - \beta\nabla f), \quad (90)$$

by definition, $T_{t,f} = \text{Id} - (\alpha\beta)\nabla f$ is pointwise almost averaged at $\bar{x}$ with violation $\varepsilon = \alpha(2\beta\tau + \beta^2L^2)$ and averaging constant $\alpha \in (0, 1)$ on U if and only if $\text{Id} - \beta\nabla f$ is pointwise almost nonexpansive at $\bar{x}$ with violation constant $\varepsilon/\alpha = 2\beta\tau + \beta^2L^2$ on U .

Define $T_{\beta,f} := \text{Id} - \beta\nabla f$. Then, since f is continuously differentiable with calm gradient at $\bar{x}$ and calmness modulus L on U , and the gradient ∇f is pointwise hypomonotone at $\bar{x}$ with violation τ on U ,

$$\begin{aligned} \|T_{\beta,f}(x) - T_{\beta,f}(\bar{x})\|^2 &= \|x - \bar{x}\|^2 - 2\beta\langle x - \bar{x}, \nabla f(x) - \nabla f(\bar{x}) \rangle + \beta^2\|\nabla f(x) - \nabla f(\bar{x})\|^2 \\ &\leq (1 + 2\beta\tau + \beta^2L^2)\|x - \bar{x}\|^2, \quad \forall x \in U. \end{aligned} \quad (91)$$

This proves the first statement.

In addition, if ∇f is pointwise strongly monotone (pointwise hypomonotone with $\tau < 0$) at $\bar{x}$, then from (91), $2\beta\tau + \beta^2L^2 \leq 0$ whenever $\beta \leq 2|\tau|/L^2$ —that is, $T_{\beta,f}$ is *nonexpansive*—on U where equality holds when $\beta = 2|\tau|/L^2$. Choose $\beta = 2|\tau|/L^2$ and set $\alpha = t/\beta = tL^2/(2|\tau|) \in (0, 1)$ since $t < 2|\tau|/L^2$. The first statement then yields the result for this case and completes the proof. $\square$

Note the trade-off between the step-length and the averaging property: the smaller the step, the smaller the averaging constant. In the case that ∇f is not monotone, the violation constant of nonexpansivity can also be chosen to be arbitrarily small by choosing β arbitrarily small, regardless of the size of the hypomonotonicity constant τ or the Lipschitz constant L . This will be exploited in Theorem 3.3. If ∇f is strongly monotone, the theorem establishes an upper limit on the stepsize for which nonexpansivity holds, but this does not rule out the possibility that, even for nonexpansive mappings, it might be more efficient to take a larger step that technically renders the mapping only *almost nonexpansive*. As we have seen in Theorem 2.2, if the fixed point set is *attractive* enough, then linear convergence of the iteration can still be guaranteed, even with this larger stepsize. This yields a local justification of *extrapolation* or excessively large stepsizes.

Proposition 3.7 (Almost Averaged: Nonconvex Forward-Backward). *Let $g: \mathbb{R}^n \rightarrow (-\infty, +\infty]$ be proper and l.s.c. with nonempty, pointwise Type-I nonmonotone subdifferential at all points on $S'_g \subset U'_g$ with violation τ_g on U'_g ; that is, at each $w \in \partial g(v)$ and $v \in S'_g$, the inequality*

$$-\tau_g \|(u+z) - (v+w)\|^2 \leq \langle z-w, u-v \rangle \quad (92)$$

holds whenever $z \in \partial g(u)$ for $u \in U'_g$. Let $f: \mathbb{R}^n \rightarrow \mathbb{R}$ be a continuously differentiable function with calm gradient (modulus L), which is also pointwise hypomonotone at all $\bar{x} \in S_f \subset U_f$ with violation constant τ_f on U_f . For $T_{t,f} := \text{Id} - t\nabla f$, suppose that $T_{t,f}U_f \subset U_g$, where $U_g := \{u+z \mid u \in U'_g, z \in \partial g(u)\}$ and that $T_{t,f}S_f \subset S_g$, where $S_g := \{v+w \mid v \in S'_g, w \in \partial g(v)\}$. Choose $\beta > 0$ and $t \in (0, \beta)$. Then, the forward-backward mapping $T_{\text{FB}} := \text{prox}_g(\text{Id} - t\nabla f)$ is pointwise almost averaged at all $\bar{x} \in S_f$ with violation constant $\varepsilon = (1 + 2\tau_g)(1 + t(2\tau_f + \beta L^2)) - 1$ and averaging constant α on U_f , where

$$\alpha = \begin{cases} \frac{2}{3} & \text{for all } \alpha_0 \leq \frac{1}{2}, \\ \frac{2\alpha_0}{\alpha_0 + 1} & \text{for all } \alpha_0 > \frac{1}{2}, \end{cases} \quad \text{and} \quad \alpha_0 = \frac{t}{\beta}. \quad (93)$$

Proof. The proof follows from Propositions 2.4 and 3.6. Indeed, by Proposition 3.6, the mapping $T_{t,f}$ is pointwise almost averaged at $\bar{x}$ with the violation constant $\varepsilon_f = \alpha_0(2\beta\tau_f + \beta^2L^2)$ and the averaging constant $\alpha_0 = t/\beta \in (0, 1)$ on U_f for $t < \beta$. It is more convenient to write the violation in terms of t as $\varepsilon_f = t(2\tau_f + \beta L^2)$. By Proposition 2.3 and Definition 2.3(a), prox_g is pointwise almost firmly nonexpansive at points $\bar{y} \in S_g$ with violation $\varepsilon_g = 2\tau_g$ on U_g , since prox_g is the resolvent of ∂g , which by assumption satisfies (92) at points in S'_g with constant τ_g on U'_g . Also, by assumption, $T_{t,f}U_f \subset U_g$ and $T_{t,f}S_f \subset S_g$, therefore we can apply Proposition 2.4(iii) to conclude that T_{FB} is pointwise averaged at $\bar{x} \in S_f$ with the violation constant $(1 + 2\tau_g)(1 + t(2\tau_f + \beta L^2)) - 1$ and the averaging constant α , which is given by (93) on U_f whenever $t < \beta$, as claimed. $\square$

Corollary 3.2 (Almost Averaged: Semi-Convex Forward-Backward). *Let $g: \mathbb{R}^n \rightarrow (-\infty, +\infty]$ be proper, l.s.c. and convex. Let $f: \mathbb{R}^n \rightarrow \mathbb{R}$ be a continuously differentiable function with calm gradient (calmness modulus L), which is also pointwise hypomonotone at all $\bar{x} \in S_f \subset U_f$ with violation constant τ_f on U_f . Choose $\beta > 0$ and $t \in (0, \beta)$. Then, the forward-backward mapping $T_{\text{FB}} := \text{prox}_g(\text{Id} - t\nabla f)$ is pointwise almost averaged at all $\bar{x} \in S_f$ with violation constant $\varepsilon = t(2\tau_f + \beta L^2)$ and averaging constant α given by (93) on U_f .*

Proof. This is a specialization of Proposition 3.7 to the case where g is convex. In this setting, ∂g is a maximally monotone mapping (Minty [55], Moreau [58]), and hence is Type-I nonmonotone on $\mathbb{R}^n$ with no violation (i.e., $\tau_g = 0$). The assumptions $T_{t,f}U_f \subset U_g$, where $U_g := \mathbb{R}^n$, and $T_{t,f}S_f \subset S_g$, where $S_g := \mathbb{R}^n$ of Proposition 3.7 are obviously automatically satisfied. $\square$

As the above proposition shows, the almost averaging property comes relatively naturally. A little more challenging is to show that Assumption (b) of Theorem 2.2 holds for a given application. The next theorem is formulated in terms of metric subregularity, but for the forward-backward iteration, the graphical derivative characterization given in Proposition 2.5 can allow for a direct verification of the regularity assumptions.

Theorem 3.3 (Local Linear Convergence: Forward-Backward). *Let $f: \mathbb{R}^n \rightarrow \mathbb{R}$ be a continuously differentiable function with calm gradient (modulus L), which is also pointwise hypomonotone at all $\bar{x} \in \text{Fix } T_{\text{FB}} \subset U_f$ with violation constant τ_f on U_f . Let $g: \mathbb{R}^n \rightarrow (-\infty, +\infty]$ be proper and l.s.c. with nonempty, pointwise Type-I nonmonotone subdifferential at all $v \in S'_g \subset U'_g$, with violation τ_g on U'_g whenever $z \in \partial g(u)$ for $u \in U'_g$. For $T_{t,f} := \text{Id} - t\nabla f$, let $T_{t,f}U_f \subset U_g$, where $U_g := \{u + z \mid u \in U'_g, z \in \partial g(u)\}$ and let $T_{t,f}\text{Fix } T_{\text{FB}} \subset S_g$, where $S_g := \{v + w \mid v \in S'_g, w \in \partial g(v)\}$. If for all $t \geq 0$ small enough, $\Phi_{\text{FB}} := T_{\text{FB}} - \text{Id}$ is metrically subregular for 0 on U_f with modulus $\kappa \leq \bar{\kappa} < 1/(2\sqrt{\tau_g})$, then for all t small enough, the forward-backward iteration $x^{k+1} \in T_{\text{FB}}x^k$ satisfies $\text{dist}(x^k, \text{Fix } T_{\text{FB}}) \rightarrow 0$ at least linearly for all x^0 close enough to $\text{Fix } T_{\text{FB}}$. In particular, if g is convex, and $\bar{\kappa}$ is finite, then the distance of the iterates to $\text{Fix } T_{\text{FB}}$ converges linearly to zero from any initial point x^0 close enough to $\text{Fix } T_{\text{FB}}$ provided that the stepsize t is sufficiently small.*

Proof. Denote the averaging constant of the inner forward mapping $T_{t,f}$ by α_0 . Since, by Proposition 3.6, the stepsize t , α_0 , and β are all relative, for convenience, we fix $\alpha_0 = 1/2$ so that $t = \beta/2$. From Proposition 3.7, it then holds that the forward-backward mapping T_{FB} is pointwise almost averaged at all $\bar{x} \in \text{Fix } T_{\text{FB}}$ with the violation constant $\varepsilon = (1 + 2\tau_g)(1 + \beta/2(2\tau_f + \beta L^2)) - 1$ and the averaging constant $\alpha = 2/3$ (given by (93)) on U_f . Hence Assumption (a) of Theorem 2.2 is satisfied with $S = \text{Fix } T_{\text{FB}}$. By assumption, for all t (hence β) small enough, Φ_{FB} is metrically subregular for 0 on U_f with modulus at most $\bar{\kappa}$, therefore by Corollary 2.3, for all x close enough to $\text{Fix } T_{\text{FB}}$

$$\text{dist}(x^+, \text{Fix } T_{\text{FB}}) \leq c \text{dist}(x, \text{Fix } T_{\text{FB}}), \quad (94)$$

where $x^+ \in T_{\text{FB}}x$ and $c := \sqrt{1 + \varepsilon - 1/(2\bar{\kappa}^2)}$. By assumption, the constant $\bar{\kappa}$ is suitable for all t small enough, but the violation $\varepsilon = 2\tau_g + o(t)$ can be made arbitrarily close to $2\tau_g$ simply by taking the stepsize $t = \beta/2$ small enough. Hence $c < 1$ for all $\beta > 0$ with $2\tau_g + \beta/2(2\tau_f + \beta L^2) + o(\beta^2) < 1/\bar{\kappa}^2$. In other words, for all x^0 close enough to $\text{Fix } T_{\text{FB}}$, and all t (or β) small enough, convergence of the forward-backward iteration is at least linear with rate at most $c := \sqrt{1 + \varepsilon - (1 - \alpha)/(\kappa^2\alpha)} < 1$.

If g is convex, then as in Corollary 3.2, $\tau_g = 0$, therefore it suffices simply to have $\bar{\kappa}$ bounded. $\square$

Corollary 3.3 (Global Linear Convergence: Convex Forward-Backward). *Let $f: \mathbb{R}^n \rightarrow \mathbb{R}$ be a continuously differentiable function with calm gradient (modulus L), which is also pointwise strongly monotone at all $\bar{x} \in \text{Fix } T_{\text{FB}}$ on $\mathbb{R}^n$. Let $g: \mathbb{R}^n \rightarrow (-\infty, +\infty]$ be proper, convex, and l.s.c. Let $T_{t,f}\text{Fix } T_{\text{FB}} \subset S_g$, where $S_g := \{v + w \mid v \in S'_g, w \in \partial g(v)\}$. If for all $t \geq 0$ small enough, $\Phi_{\text{FB}} := T_{\text{FB}} - \text{Id}$ is metrically subregular for 0 on $\mathbb{R}^n$ with modulus $\kappa \leq \bar{\kappa} < +\infty$, then for all fixed step-length t small enough, the forward-backward iteration $x^{k+1} = T_{\text{FB}}x^k$ satisfies $\text{dist}(x^k, \text{Fix } T_{\text{FB}}) \rightarrow 0$ at least linearly for all $x^0 \in \mathbb{R}^n$.*

Proof. Note that ∇f being pointwise strongly monotone is equivalent to ∇f being pointwise hypomonotone with violation $\tau_f < 0$. Proposition 3.7 then establishes that the forward-backward mapping T_{FB} is pointwise almost averaged at all $\bar{x} \in \text{Fix } T_{\text{FB}}$ with the violation constant $\varepsilon = \beta/2(2\tau_f + \beta L^2)$ and the averaging constant $\alpha = 2/3$ (given by (93)) on $\mathbb{R}^n$. For all stepsizes small enough, or equivalently for all β small enough, it holds that $2\tau_f + \beta L^2 < 0$ and T_{FB} is, in fact, pointwise averaged. Additionally, for all t (hence β) small enough, Φ_{FB} is metrically subregular for 0 on $\mathbb{R}^n$ with modulus at most $\bar{\kappa} < \infty$, therefore by Corollary 2.3, for all x

$$\text{dist}(T_{\text{FB}}x, \text{Fix } T_{\text{FB}}) \leq c \text{dist}(x, \text{Fix } T_{\text{FB}}), \quad (95)$$

where $c := \sqrt{1 - 1/2\bar{\kappa}^2} < 1$. This completes the proof. $\square$

Remark 3.3 (Extrapolation). In the proof of Corollary 3.3, it is not necessary to choose the stepsize small enough that T_{FB} is pointwise averaged. It suffices to choose the stepsize t small enough that $c := \sqrt{1 + \varepsilon - 1/2\bar{\kappa}^2} < 1$, where $\varepsilon = \beta/2(2\tau_f + \beta L^2)$. In this case, T_{FB} is only almost pointwise averaged with violation ε on $\mathbb{R}^n$.

Remark 3.4. Optimization problems involving the sum of a smooth function and a nonsmooth function are commonly found in applications and accelerations to forward-backward algorithms have been a subject of intense study (Beck and Teboulle [19], Attouch and Peypouquet [4], Nesterov [59], Chambolle and Dossal [25]). To this point, the theory on quantitative convergence of the iterates is limited to the convex setting under the additional assumption of strong convexity/strong monotonicity. Theorem 3.3 shows that locally, convexity of the smooth function plays no role in the convergence of the iterates or the order of convergence, and strong convexity, much less convexity, of the function g is also not crucial—it is primarily the regularity of the fixed points that matters locally. This agrees nicely with recent global linear convergence results of a primal-dual method for saddle point problems that use *pointwise quadratic supportability* in place of the much stronger strong convexity assumption (Luke and Shefi [51]). Moreover, local linear convergence is guaranteed by metric subregularity on an appropriate set without any fine tuning of the only algorithm parameter t , other than assuring that this parameter is small enough. When the nonsmooth term is the indicator function of some constraint set, then the regularity assumption can be replaced by the characterization in terms of the graphical derivative (54) to yield a familiar constraint qualification at fixed points.

If the functions in $(\mathcal{P})$ are piecewise linear quadratic, then the forward-backward mapping has polyhedral structure (Proposition 3.8), which, following Proposition 2.7, allows for easy verification of the conditions for linear convergence (Proposition 3.9).

Definition 3.3 (Piecewise Linear Quadratic Functions). A function $f: \mathbb{R}^n \rightarrow [-\infty, +\infty]$ is called *piecewise linear quadratic* if $\text{dom } f$ can be represented as the union of finitely many polyhedral sets, relative to each of which $f(x)$ is given by an expression of the form $\frac{1}{2}\langle x, Ax \rangle + \langle a, x \rangle + \alpha$ for some scalar $\alpha \in \mathbb{R}$ vector $a \in \mathbb{R}^n$, and symmetric matrix $A \in \mathbb{R}^{n \times n}$. If f can be represented by a single linear quadratic equation on $\mathbb{R}^n$, then f is said to be linear quadratic.

For instance, if f is piecewise linear quadratic, then the subdifferential of f and its proximal mapping prox_f are polyhedral (Rockafellar and Wets [72, proposition 12.30]).

Proposition 3.8 (Polyhedral Forward-Backward). Let $f: \mathbb{E} \rightarrow \mathbb{R}$ be quadratic and let $g: \mathbb{E} \rightarrow (-\infty, +\infty]$ be proper, l.s.c. and piecewise linear quadratic convex. The mapping T_{FB} defined by (89) is single-valued and polyhedral.

Proof. Since the functions f and g are piecewise linear quadratic, the mappings $\text{Id} - \nabla f$ and ∂g are polyhedral. Moreover, since g is convex, the mapping prox_g (that is, the resolvent of ∂g) is single-valued and polyhedral (Rockafellar and Wets [72, proposition 12.30]). The mapping $\text{Id} - \nabla f$ is clearly single-valued, therefore $T_{\text{FB}} = \text{prox}_g(\text{Id} - \nabla f)$ is also single-valued and polyhedral as the composition of single-valued polyhedral maps. $\square$

Proposition 3.9 (Linear Convergence of Polyhedral Forward-Backward). Let $f: \mathbb{E} \rightarrow \mathbb{R}$ be quadratic and let $g: \mathbb{E} \rightarrow (-\infty, +\infty]$ be proper, l.s.c. and piecewise linear quadratic convex. Suppose $\text{Fix } T_{\text{FB}}$ is an isolated point $\{\bar{x}\}$, where $T_{\text{FB}} := \text{prox}_g(\text{Id} - t\nabla f)$. Suppose also that the modulus of metric subregularity κ of $\Phi := T_{\text{FB}} - \text{Id}$ at $\bar{x}$ for 0 is bounded above by some constant $\bar{\kappa}$ for all $t > 0$ small enough. Then, for all t small enough, the forward-backward iteration $x^{k+1} = T_{\text{FB}}(x^k)$ converges at least linearly to $\bar{x}$ whenever x^0 is close enough to $\bar{x}$.

Proof. By Corollary 3.2, the mapping T_{FB} is pointwise almost averaged with violation ε proportional to the stepsize t . By Proposition 3.8 T_{FB} is polyhedral and by Proposition 2.6 metrically subregular at $\bar{x}$ for 0 with constant κ on some neighborhood U of $\bar{x}$. Since the violation ε can be made arbitrarily small by taking t arbitrarily small, and since the modulus of metric subregularity $\kappa \leq \bar{\kappa} < \infty$ for all t small enough, the result follows by Proposition 2.7. $\square$

Example 3.7 (Iterative Soft-Thresholding). Let $f(x) = x^T A x + x^T b$, and $g(x) = \alpha \|Bx\|_1$ for $A \in \mathbb{R}^{n \times n}$ symmetric and $B \in \mathbb{R}^{m \times n}$ full rank. The forward-backward algorithm applied to the problem *minimize* $f(x) + g(x)$ is the iterative soft-thresholding algorithm (Daubechies et al. [28]) with fixed step-length t in the forward step $x - t\nabla f(x) = x - t(2Ax + b)$. The function g is piecewise linear, therefore prox_g is polyhedral hence the forward-backward fixed point mapping T_{FB} is single-valued and polyhedral. As long as $\text{Fix } T_{\text{FB}}$ is an isolated point relative to the affine hull of the iterates $x^{k+1} = T_{\text{FB}}x^k$, and the modulus of metric subregularity is independent of the stepsize t for all t small enough, then by Proposition 3.9 for small enough stepsize t , the iterates x^k converge linearly to $\text{Fix } T_{\text{FB}}$ for all starting points close enough to $\text{Fix } T_{\text{FB}}$. If A is positive definite (i.e., f is convex), then the set of fixed points is a singleton and convergence is linear from any starting point x^0 .

3.2.2. Douglas-Rachford and Relaxations. The Douglas-Rachford algorithm is commonly encountered in one form or another for solving feasibility problems and structured optimization. In the context of problem $(\mathcal{P})$, the iteration takes the form

$$x^{k+1} \in T_{\text{DR}}(x^k) := \frac{1}{2}(R_f R_g + \text{Id})(x^k), \quad (96)$$

where $R_f := 2\text{prox}_f - \text{Id}$ (i.e., the proximal reflector) and R_g is similarly given.

Revisiting the setting of Luke [48], we use the tools developed in the present paper to show when one can expect local linear convergence of the Douglas-Rachford iteration. For simplicity, as in Luke [48], we will assume that f is convex to arrive at a clean final statement, though convexity is not needed for local linear convergence.

Proposition 3.10. Let $g = \iota_\Omega$ for $\Omega \subset \mathbb{R}^n$ a manifold, and let $f: \mathbb{R}^n \rightarrow \mathbb{R}$ be convex and linear quadratic. Fix $\bar{x} \in \text{Fix } T_{\text{DR}}$. Then, for any $\varepsilon > 0$ small enough, there exists $\delta > 0$ such that T_{DR} is single-valued and almost firmly nonexpansive with violation $\varepsilon_g = 4\varepsilon + 4\varepsilon^2$ on $\mathbb{B}_\delta(\bar{x})$.

Proof. Suppose that $g = \iota_\Omega$ for $\Omega \subset \mathbb{R}^n$ a manifold. In the language of Definition 3.1(iii), at each point $\bar{x} \in \Omega$, for any $\varepsilon > 0$, there is a δ such that Ω is elementally regular at $\bar{x}$ for all $(a, v) \in \text{gph } N_\Omega$, where $a \in \mathbb{B}_\delta(\bar{x})$ with constant ε and neighborhood $\mathbb{B}_\delta(\bar{x})$. In other words, Ω is *prox-regular* (Kruger et al. [44, proposition 4(vi)]). By Theorem 3.1(v), the reflector R_g is then almost firmly nonexpansive with violation $\varepsilon_g := 4\varepsilon + 4\varepsilon^2$ on $\mathbb{B}_\delta(\bar{x})$.

Another characterization of prox-regular sets is that the projector P_Ω is locally single-valued (Poliquin et al. [69]). We can furthermore conclude that R_g is single-valued on $\mathbb{B}_\delta(\bar{x})$. Next, the function $f: \mathbb{R}^n \rightarrow \mathbb{R}$ is quadratic convex, therefore R_f is firmly nonexpansive and single-valued as the reflected resolvent of the (maximal monotone) subdifferential of f . By Proposition 2.4(ii), the composition of reflectors $R_f R_g$ is therefore almost nonexpansive with violation ε_g on $\mathbb{B}_\delta(\bar{x})$. Then, by the definition of averaged mappings, the Douglas-Rachford mapping T_{DR} is almost firmly nonexpansive with violation ε_g on $\mathbb{B}_\delta(\bar{x})$. $\square$

Theorem 3.4. Let $g = \iota_\Omega$ for $\Omega \subset \mathbb{R}^n$ a manifold and let $f: \mathbb{R}^n \rightarrow \mathbb{R}$ be linear quadratic convex. Let $(x^k)_{k \in \mathbb{N}}$ be iterates of the Douglas-Rachford (96) algorithm and let $\Lambda = \text{aff}(x^k)$. If $T_{DR} - \text{Id}$ is metrically subregular at all points $\bar{x} \in \text{Fix } T_{DR} \cap \Lambda \neq \emptyset$ relative to Λ , then for all x^0 close enough to $\text{Fix } T_{DR} \cap \Lambda$, the sequence x^k converges linearly to a point in $\text{Fix } T \cap \Lambda$ with constant at most $c = \sqrt{1 + \varepsilon - 1/\kappa^2} < 1$, where κ is the constant of metric subregularity for $\Phi := T_{DR} - \text{Id}$ on some neighborhood U containing the sequence and ε is the violation of almost firm nonexpansiveness on the neighborhood U .

Proof. $T_{DR} - \text{Id}$ is metrically subregular at all points in $\text{Fix } T_{DR} \cap \Lambda$ with constant κ on some neighborhood U' . By Proposition 3.10, there exists a neighborhood $U \subset U'$ on which T_{DR} is single-valued and almost firmly nonexpansive with violation ε satisfying $\varepsilon < 1/\kappa^2$. By Corollary 2.3 the sequence $x^{k+1} = T_{DR} x^k$ then converges linearly to a point in $\text{Fix } T_{DR} \cap \Lambda$ with rate at most $c = \sqrt{1 + \varepsilon - 1/\kappa^2} < 1$. $\square$

Remark 3.5. Assuming that the fixed points, restricted to the affine hull of the iterates, are isolated points, polyhedrality was used in Aspelmeier et al. [3] to verify that the Douglas-Rachford mapping is indeed metrically subregular at the fixed points. While in principle the graphical derivative formulas (see Proposition 2.5) could be used for more general situations, it is not easy to compute the graphical derivative of the Douglas-Rachford operator, even in the simple setting above. This is a theoretical bottleneck for the practical applicability of metric subregularity for more general algorithms.

Example 3.8 (Relaxed Alternating Averaged Reflections for Phase Retrieval). Applied to feasibility problems, the Douglas-Rachford algorithm is also described as averaged alternating reflections (Bauschke et al. [15]). Here, $f = \iota_A$ and $g = \iota_B$ are the indicator functions of individual constraint sets. When the sets A and B are sufficiently regular, as they certainly are in the phase retrieval problem, and intersect transversally, local linear convergence of the Douglas-Rachford algorithm in this instance can be deduced from Phan [68]. As discussed in Example 2.4, however, for any phase retrieval problem arising from a physical noncrystallographic diffraction experiment, the constraint sets cannot intersect when finite support is required of the reconstructed object. This fact, seldom acknowledged in the phase retrieval literature, is borne out in the observed instability of the Douglas-Rachford algorithm applied to phase retrieval (Luke [47]): it cannot converge when the constraint sets do not intersect (Bauschke et al. [15, theorem 3.13]).

To address this issue, a relaxation for nonconvex feasibility was studied in Luke [47, 48] that amounts to (96) where f is the Moreau envelope of a nonsmooth function and g is the indicator function of a sufficiently regular set. Optimization problems with this structure are guaranteed to have solutions. In particular, when f is the Moreau envelope to ι_A with parameter λ , the corresponding iteration given by (96) can be expressed as a convex combination of the underlying basic Douglas-Rachford operator and the projector of the constraint set encoded by g (Luke [48, proposition 2.5]):

$$x^{k+1} \in T_{DR\lambda} x^k := \frac{1}{2(\lambda+1)}(R_A R_B + \text{Id})(x^k) + \frac{\lambda}{\lambda+1} P_B x^k \quad (97)$$

where $R_A = 2P_A - \text{Id}$ and $R_B = 2P_B - \text{Id}$. In Luke [47] and the physics literature, this is known as relaxed alternating averaged reflections (RAAR). As noted in Example 3.6, the phase retrieval problem in its many different manifestations in photonic imaging has exactly the structure of the functions in Theorem 3.4. If, in addition, the fixed point operator $T_{DR\lambda}$ is metrically subregular at its fixed points relative to the affine hull of the iterates, then according to Theorem 3.4, for λ large enough and for all starting points close enough to the set of fixed points, the algorithm (97) applied to the phase retrieval problem converges locally linearly to a fixed point. In contrast to the usual Douglas-Rachford algorithm and its variants (Bauschke et al. [14]), the RAAR method does not require that the constraint sets intersect. Still, it is an open problem to determine whether $T_{DR\lambda}$ is usually (in some appropriate sense) metrically subregular for phase retrieval.

Endnote

¹We learned from Alexander Kruger that Gurin's name was misprinted as Gubin in the translation of his work into English. Thanks to Anna Martins for pointing out an error in Example 3.5.

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