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Pointwise Arbitrage Pricing Theory in Discrete Time

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Abstract. We develop a robust framework for pricing and hedging of derivative securities in discrete-time financial markets. We consider markets with both dynamically and statically traded assets and make minimal measurability assumptions. We obtain abstract (pointwise) fundamental theorem of asset pricing and pricing–hedging duality. Our results are general and, in particular, cover both the so-called model independent case as well as the classical probabilistic case of Dalang–Morton–Willinger. Our analysis is scenario-based: a model specification is equivalent to a choice of scenarios to be considered. The choice can vary between all scenarios and the set of scenarios charged by a given probability measure. In this way, our framework interpolates between a model with universally acceptable broad assumptions and a model based on a specific probabilistic view of future asset dynamics.

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Keywords: robust modelling approach • fundamental theorem of asset pricing • superhedging duality • semistatic optimization • pointwise stochastic analysis • arbitrage pricing theory • model ambiguity • Knightian uncertainty

1. Introduction

The *state preference model* or *asset pricing model* underpins most mathematical descriptions of financial markets. It postulates that the price of d financial assets is known at a certain initial time $t_0 = 0$ (today), and the price at future times $t > 0$ is unknown and is given by a certain random outcome. To formalize such a model, we only need to fix a quadruple $(X, \mathcal{F}, \mathbb{F}, S)$, where X is the set of scenarios, $\mathcal{F}$ a σ-algebra, and $\mathbb{F} := \{\mathcal{F}_t\}_{t\in I} \subseteq \mathcal{F}$ a filtration such that the d-dimensional process $S := (S_t)_{t\in I}$ is adapted. At this stage, no probability measure is required to specify the financial market model $(X, \mathcal{F}, \mathbb{F}, S)$.

One of the fundamental reasons for producing such models is to assign rational prices to contracts that are not liquid enough to have a market-determined price. Rationality here is understood via the economic principle of absence of arbitrage opportunities, stating that it should not be possible to trade in the market in a way to obtain a positive gain without taking any risk. Starting from this premise, the theory of pricing by no arbitrage has been successfully developed over the last 50 years. Its cornerstone result, known as the fundamental theorem of asset pricing (FTAP), establishes equivalence between absence of arbitrage and existence of risk-neutral pricing rules. The intuition for this equivalence can be accredited to de Finetti for his work on *coherence* and *previsions* (see de Finetti [19, 20]). The first systematic attempt to understand the absence of arbitrage opportunities in models of financial assets can be found in the works of Ross [44, 45] on capital pricing; see also Huberman [33]. The intuition underpinning the arbitrage theory for derivative pricing was developed by Samuelson [46], Black and Scholes [7], and Merton [38]. The rigorous theory was then formalized by Harrison and Kreps [26] and extended in Harrison and Pliska [27]; see also Kreps [36]. Their version of FTAP, in the case of a finite set of scenarios X, can be formulated as follows. Consider $X = \{\omega_1, \dots, \omega_n\}$, and let $s = (s^1, \dots, s^d)$ be the initial prices of d assets with random outcome $S(\omega) = (S^1(\omega), \ldots, S^d(\omega))$ for any $\omega \in X$. Then we have the following equivalence:

$$\begin{matrix} \nexists H \in \mathbb{R}^d \text{ such that } H \cdot s \le 0 \\ \text{and } H \cdot S(\omega) \ge 0 \text{ with} > \text{for some } \omega \in X \end{matrix} \iff \begin{matrix} \exists Q \in \mathcal{P} \text{ such that } Q(\omega_j) > 0 \text{ and} \\ E_Q[S^i] = s^i, \forall\, 1 \le j \le n, 1 \le i \le d, \end{matrix} \tag{1}$$

where $\mathcal{P}$ is the class of probability measures on X. In particular, no reference probability measure is needed, and impossible events are automatically excluded from the construction of the state space X. On the other hand, linear pricing rules consistent with the observed prices $s^1, \ldots, s^d$ and the no-arbitrage condition turn out to be (risk-neutral) probabilities with full support; that is, they assign positive measure to any state of the world. By introducing a *reference probability measure* P with full support and defining an arbitrage as a portfolio with $H \cdot s \le 0$, $P(H \cdot S(\omega) \ge 0) = 1$, and $P(H \cdot S(\omega) > 0) > 0$, the thesis in Equation (1) can be restated as

$$\text{There is No Arbitrage} \iff \exists Q \sim P \text{ such that } E_Q[S^i] = s^i \quad \forall i = 1, \ldots, d. \tag{2}$$

The identification suggested by Equation (2) allows nontrivial extensions of the FTAP to the case of a general space X with a fixed reference probability measure and was proven in the celebrated work Dalang et al. [16] by use of measurable selection arguments. It was then extended to continuous-time models by Delbaen and Schachermayer [21, 22].

The idea of introducing a reference probability measure to select scenarios proved very fruitful in the case of a general X and was instrumental for the rapid growth of the modern financial industry. It was pioneered by Samuelson [46] and Black and Scholes [7], who used it to formulate a continuous-time financial asset model with unique rational prices for all contingent claims. Such models with strong assumptions implying a unique derivative pricing rule are in stark contrast to a setting with little assumptions, for example, where the asset can follow any nonnegative continuous trajectory, which are consistent with many rational pricing rules. This dichotomy was described and studied in the seminal paper of Merton [38, p. 142], who referred to the latter as "assumptions sufficiently weak to gain universal support"[1] and pointed out that it typically generates outputs that are not specific enough to be of practical use. For that reason, it was the former approach with the reference probability measure interpreted as a probabilistic description of future asset dynamics that became the predominant paradigm in the field of quantitative finance. The original simple models were extended, driven by the need to capture additional features observed in the increasingly complex market reality, including, for example, local or stochastic volatility. Such extensions can be seen as enlarging the set of scenarios considered in the model and usually led to plurality of rational prices.

More recently and, in particular, in the wake of the financial crisis, the critique of using a single reference probability measure came from considerations of the so-called Knightian uncertainty, going back to Knight [35], and describing *the model risk* as contrasted with financial risks captured within a given model. The resulting stream of research aims at extending the probabilistic framework of Dalang et al. [16] to a framework that allows for a set of possible priors $\mathcal{R} \subseteq \mathcal{P}$. The class $\mathcal{R}$ represents a collection of plausible (probabilistic) models for the market. In continuous-time models, this led naturally to the theory of quasi-sure stochastic analysis as in Denis and Martini [23], Peng [40], and Soner et al. [47, 48] and many further contributions; see, for example, Dolinsky and Soner [25]. In discrete time, a general approach was developed by Bouchard and Nutz [8]. Under some technical conditions on the state space and the set $\mathcal{R}$, they provide a version of the FTAP as well as the superhedging duality. Their framework includes the two extreme cases: the classical case in which $\mathcal{R} = \{P\}$ is a singleton and, on the other extreme, the case of full ambiguity when $\mathcal{R}$ coincides with the whole set of probability measures and the description of the model becomes pathwise. Their setup has been used to study a series of related problems; see, for example, Bayraktar and Zhang [2] and Bayraktar and Zhou [3].

Describing models by specifying a family of probability measures $\mathcal{R}$ appears natural when starting from the dominant paradigm in which a reference measure P is fixed. However, it is not the only way and possibly not the simplest one to specify a model. Indeed, in this paper, we develop a different approach inspired by the original finite state-space model used in Harrison and Pliska [27] as well as the notion of *prediction set* in Mykland [39]; see also Hou and Obłój [32]. Our analysis is scenario based. More specifically, *agent's beliefs* or *a model* are equivalent to selecting a set of admissible scenarios that we denote by $\Omega \subseteq X$. The selection may be formulated, for example, in terms of behaviour of some market-observable quantities and may reflect both the information the agent has as well as the modelling assumptions the agent is prepared to make. Our approach clearly includes the "universally acceptable" case of considering all scenarios $\Omega = X$, but we also show that it subsumes the probabilistic framework of Dalang et al. [16]. Importantly, as we work under a minimal measurability requirement on Ω, our models offer a flexible way to interpolate between the two settings. The scenario-based specification of a model requires less sophistication than selection of a family of probability measures and appears particularly natural when considering (super)-hedging, which is a pathwise property.

Our first main result, Theorem 1, establishes a fundamental theorem of asset pricing for an arbitrary specification of a model Ω and gives equivalence between existence of a rational pricing rule (i.e., a calibrated martingale measure) and absence of suitably defined arbitrage opportunities. Interestingly the equivalence in Equation (1) does not simply extend to a general setting: specification of Ω that is inconsistent with any rational pricing rule does not imply existence of *one arbitrage strategy*. Ex post, this is intuitive: although all agents may agree that rational pricing is impossible, they may well disagree on why this is so. This discrepancy was first observed and illustrated with an example by Davis and Hobson [18]. The equivalence is only recovered under strong assumptions as shown by Riedel [41] in a topological one-period setup and by Acciaio et al. [1] in a general discrete time setup. A rigorous analysis of this phenomenon in the case $\Omega = X$ was subsequently given by Burzoni et al. [10], who also showed that several notions of arbitrage can be studied within the same framework. Here, we extend their result to an arbitrary $\Omega \subseteq X$ and to the setting with both dynamically and statically traded assets. We show that also in such cases agents' different views on arbitrage opportunities may be aggregated in a canonical way into a pointwise arbitrage strategy in an enlarged filtration. As special cases of our general FTAP, we recover results in Acciaio et al. [1] and Burzoni et al. [10] as well as the classical Dalang–Morton–Willinger theorem of Dalang et al. [16]. For the latter, we show that choosing a probability measure P on X is equivalent to fixing a suitable set of scenarios Ω^P, and our results then lead to probabilistic notions of arbitrage as well as the probabilistic version of the fundamental theorem of asset pricing.

Our second main result, Theorem 2, characterises the range of rational prices for a contingent claim. Our setting is comprehensive: we make no regularity assumptions on the model specification Ω, on the payoffs of traded assets both dynamic and static, or the derivative that we want to price. We establish a pricing–hedging duality result asserting that the infimum of prices of superhedging strategies is equal to the supremum of rational prices. As already observed in Burzoni et al. [11] and also in Beiglböck et al. [5] in the context of martingale optimal transport, it may be necessary to consider superhedging on a smaller set of scenarios than Ω to avoid a duality gap between rational prices and superhedging prices. In this paper, this feature is achieved through the set of *efficient* trajectories Ω^*_Φ that only depend on Ω and the market. The set Ω^*_Φ recollects all scenarios that are supported by some rational pricing rule. Its intrinsic and constructive characterisation is given in the FTAP, Theorem 1. Our duality generalizes the results of Burzoni et al. [11] to the setting of abstract model specification Ω as well as a generic finite set of statically traded assets. The flexibility of model choice is of particular importance as stressed previously. The universally acceptable setting $\Omega = X$ will typically produce a wide range of rational prices that may not be of practical relevance as already discussed by Merton [38]. However, as we shrink Ω from X to a set Ω^P, the range of rational prices shrinks accordingly, and in case Ω^P corresponds to a complete market model, the interval reduces to a single point. This may be seen as a quantification of the impact of modelling assumptions on rational prices and gives a powerful description of model risk.

We note that pricing–hedging duality results have a long history in the field of robust pricing and hedging. First contributions focused on obtaining explicit results, working in a setting with one dynamically traded risky asset and a strip of statically traded comaturing call options with all strikes. In his pioneering work, Hobson [29] devised a methodology based on Skorkohod embedding techniques and treated the case of lookback options. His approach was then used in a series of works focusing on different classes of exotic options; see Brown et al. [9], Cox and Obłój [13, 14], Cox and Wang [15], Henry-Labordère et al. [28], Hobson and Klimmek [30], and Hobson and Neuberger [31]. More recently, it has been recast as an optimal transportation problem along martingale dynamics and the focus shifted to establishing abstract pricing–hedging duality; see Beiglböck et al. [4], Davis et al. [17], Dolinsky and Soner [24], and Hou and Obłój [32].

The remainder of the paper is organised as follows. First, in Section 2, we present all the main results. We give the necessary definitions and, in Section 2.1, state our two main theorems: the fundamental theorem of asset pricing, Theorem 1, and the pricing–hedging duality, Theorem 2, which we also refer to as the superhedging duality. In Section 2.2, we generalize the results of Acciaio et al. [1] for a multidimensional noncanonical stock process. Here, suitable continuity assumptions and presence of a statically traded option ϕ_0 with a convex payoff with superlinear growth allow "lifting" superhedging from Ω^*_Φ to the whole Ω. Finally, in Section 2.3, we recover the classical probabilistic results of Dalang et al. [16]. The rest of the paper then discusses the methodology and the proofs. Section 3 is devoted to the construction of strategy and filtration that aggregate arbitrage opportunities seen by different agents. We first treat the case without statically traded options when the so-called arbitrage aggregator is obtained through a conditional backward induction. Then, when statically traded options are present, we devise a path-space partition scheme, which iteratively identifies the class of polar sets with respect to the calibrated martingale measure. Section 4 contains the proofs with some technical remarks relegated to the appendix.

2. Main Results

We work on a Polish space X and denote $\mathcal{B}_X$ as its Borel sigma-algebra and $\mathcal{P}$ the set of all probability measures on $(X, \mathcal{B}_X)$. If $\mathcal{G} \subseteq \mathcal{B}_X$ is a sigma algebra and $P \in \mathcal{P}$, we denote with $\mathcal{N}^P(\mathcal{G}) := \{N \subseteq A \in \mathcal{G} \mid P(A) = 0\}$ the class of P-null sets from $\mathcal{G}$. We denote with $\mathcal{F}^{\mathcal{A}}$ the sigma-algebra generated by the analytic sets of $(X, \mathcal{B}_X)$ and with $\mathcal{F}^{\mathrm{pr}}$ the sigma-algebra generated by the class Λ of projective sets of $(X, \mathcal{B}_X)$. The latter is required for some of our technical arguments, and we recall its properties in the appendix. In particular, under a suitable choice of set theoretical axioms, it is included in the universal completion of $\mathcal{B}_X$; see Remark A.2. As discussed in the introduction, we consider pointwise arguments and think of a model as a choice of universe of scenarios $\Omega \subseteq X$. Throughout, we assume that Ω is an analytic set.

Given a family of measures $\mathcal{R} \subseteq \mathcal{P}$, we say that a set is polar (with respect to $\mathcal{R}$) if it belongs to $\{N \subseteq A \in \mathcal{B}_X \mid Q(A) = 0\ \forall Q \in \mathcal{R}\}$, and a property is said to hold *quasi-surely* ($\mathcal{R}$-q.s.) if it holds outside a polar set. For those random variables g whose positive and negative part is not Q-integrable ($Q \in \mathcal{P}$), we adopt the convention $\infty - \infty = -\infty$ when we write $E_Q[g] = E_Q[g^+] - E_Q[g^-]$. Finally for any sigma-algebra $\mathcal{G}$, we denote by $\mathcal{L}(X, \mathcal{G}; \mathbb{R}^d)$ the space of $\mathcal{G}$-measurable d-dimensional random vectors. For a given set $A \subseteq X$ and $f, g \in \mathcal{L}(X, \mathcal{G}; \mathbb{R})$, we often refer to $f \leq g$ on A whenever $f(\omega) \leq g(\omega)$ for every $\omega \in A$ (similarly for $=$ and $<$).

We fix a time horizon $T \in \mathbb{N}$ and let $\mathbb{T} := \{0, 1, \ldots, T\}$. We assume the market includes both liquid assets, which can be traded dynamically through time, and less liquid assets, which are only available for trading at time $t = 0$. The prices of assets are represented by an $\mathbb{R}^d$-valued stochastic process $S = (S_t)_{t \in \mathbb{T}}$ on $(X, \mathcal{B}_X)$. In addition, we may also consider the presence of a vector of nontraded assets represented by an $\mathbb{R}^{\tilde{d}}$-valued stochastic process $Y = (Y_t)_{t \in \mathbb{T}}$ on $(X, \mathcal{B}_X)$ with Y_0 a constant, which may also be interpreted as market factors, or additional information available to the agent. The prices are given in units of some fixed numeraire asset S^0, which itself is, thus, normalized: $S_t^0 = 1$ for all $t \in \mathbb{T}$. In the presence of the additional factors Y, we let $\mathbb{F}^{S,Y} := (\mathcal{F}_t^{S,Y})_{t \in \mathbb{T}}$ be the natural filtration generated by S and Y (when $Y \equiv 0$, we have $\mathbb{F}^{S,0} = \mathbb{F}^S$, the natural filtration generated by S). For technical reasons, we also make use of the filtration $\mathbb{F}^{\mathrm{pr}} := (\mathcal{F}_t^{\mathrm{pr}})_{t \in \mathbb{T}}$, where $\mathcal{F}_t^{\mathrm{pr}}$ is the sigma-algebra generated by the projective sets of $(X, \mathcal{F}_t^{S,Y})$, namely $\mathcal{F}_t^{\mathrm{pr}} := \sigma((S_u, Y_u)^{-1}(L) \mid L \in \Lambda, u \leq t)$ (see the appendix for further details). Clearly, $\mathbb{F}^{S,Y} \subseteq \mathbb{F}^{\mathrm{pr}}$ and $\mathcal{F}_t^{\mathrm{pr}}$ is "nonanticipative" in the sense that the atoms of $\mathcal{F}_t^{\mathrm{pr}}$ and $\mathcal{F}_t^{S,Y}$ are the same. Finally, we let Φ denote the vector of payoffs of the statically traded assets. We consider the setting in which $\Phi = \{\phi_1, \ldots, \phi_k\}$ is finite and each $\phi \in \Phi$ is $\mathcal{F}^{\mathcal{A}}$-measurable. When there are no statically traded assets, we set $\Phi = 0$, which makes our notation consistent.

For any filtration $\mathbb{F}$, $\mathcal{H}(\mathbb{F})$ is the class of $\mathbb{F}$-predictable stochastic processes with values in $\mathbb{R}^d$, which represent admissible trading strategies. Gains from investing in S, adopting a strategy H, are given by $(H \circ S)_T := \sum_{t=1}^T \cdot \sum_{j=1}^d H_t^j (S_t^j - S_{t-1}^j) = \sum_{t=1}^T H_t \cdot \Delta S_t$. In contrast, ϕ_j can only be bought or sold at time $t = 0$ (without loss of generality with zero initial cost) and held until the maturity T so that trading strategies are given by $\alpha \in \mathbb{R}^k$ and generate payoff $\alpha\Phi := \sum_{j=1}^k \alpha_j \phi_j$ at time T. We let $\mathcal{A}_\Phi(\mathbb{F})$ denote the set of such $\mathbb{F}$-admissible trading strategies (α, H).

Given a filtration $\mathbb{F}$, universe of scenarios Ω, and set of statically traded assets Φ, we let

$$\mathcal{M}_{\Omega,\Phi}(\mathbb{F}) := \{Q \in \mathcal{P} \mid S \text{ is an } \mathbb{F}\text{-martingale under Q},\ \mathrm{Q}(\Omega) = 1 \text{ and } \mathrm{E_Q}[\phi] = 0\ \forall \phi \in \Phi\}.$$

The support of a probability measure Q is given by $\mathrm{supp(Q)} := \bigcap \{C \in \mathcal{B}_X \mid C \text{ closed},\ \mathrm{Q}(C) = 1\}$. We often consider measures with finite support and denote it with a superscript ^{f}; that is, for a given set $\mathcal{R}$ of probability measures, we put $\mathcal{R}^f := \{Q \in \mathcal{R} \mid \mathrm{supp(Q)} \text{ is finite}\}$. To wit, $\mathcal{M}_{\Omega,\Phi}^f(\mathbb{F})$ denotes finitely supported martingale measures on Ω that are calibrated to options in Φ. Define

$$\mathbb{F}^M := (\mathcal{F}_t^M)_{t \in \mathbb{T},} \quad \text{where } \mathcal{F}_t^M := \bigcap_{P \in \mathcal{M}_{\Omega,\Phi}(\mathbb{F}^{S,Y})} \mathcal{F}_t^{S,Y} \vee \mathcal{N}^P(\mathcal{F}_T^{S,Y}), \tag{3}$$

and we convene $\mathcal{F}_t^M$ is the power set of Ω whenever $\mathcal{M}_{\Omega,\Phi}(\mathbb{F}^{S,Y}) = \emptyset$.

Remark 1. In the results of this paper, we only consider filtrations $\mathbb{F}$ that satisfy $\mathbb{F}^{S,Y} \subseteq \mathbb{F} \subseteq \mathbb{F}^M$. All such filtrations generate the same set of martingale measures in the sense that any $Q \in \mathcal{M}_{\Omega,\Phi}(\mathbb{F})$ uniquely extends to a measure

$\hat{Q} \in \mathcal{M}_{\Omega,\Phi}(\mathbb{F}^M)$, and reciprocally, for any $\hat{Q} \in \mathcal{M}_{\Omega,\Phi}(\mathbb{F}^M)$, the restriction $\hat{Q}_{|\mathbb{F}}$ belongs to $\mathcal{M}_{\Omega,\Phi}(\mathbb{F})$. Accordingly, with a slight abuse of notation, we write $\mathcal{M}_{\Omega,\Phi}(\mathbb{F}) = \mathcal{M}_{\Omega,\Phi}(\mathbb{F}^M) = \mathcal{M}_{\Omega,\Phi}$.

In the subsequent analysis, the set of scenarios charged by martingale measures is crucial:

$$\Omega^*_\Phi := \{\omega \in \Omega \mid \exists Q \in \mathcal{M}^f_{\Omega,\Phi} \text{ such that } Q(\omega) > 0\} = \bigcup_{Q \in \mathcal{M}^f_{\Omega,\Phi}} \text{supp}(Q). \tag{4}$$

We have by definition that for every $Q \in \mathcal{M}^f_{\Omega,\Phi}$ its support satisfies $\text{supp}(Q) \subseteq \Omega^*_\Phi$. Notice that the key elements introduced so far, namely $\mathcal{M}_{\Omega,\Phi}$, $\mathcal{M}^f_{\Omega,\Phi}$, $\mathbb{F}^M$, and Ω^*_Φ, only depend on the four basic ingredients of the market: Ω, S, Y, and Φ. Finally, in all of these notations, we omit the subscript Ω when $\Omega = X$, and we omit the subscript Φ when $\Phi = 0$; for example, $\mathcal{M}^f$ denotes all finitely supported martingale measures on X.

2.1. Fundamental Theorem of Asset Pricing and Superhedging Duality

We now introduce different notions of arbitrage opportunities that play a key role in the statement of the pointwise fundamental theorem of asset pricing.

Definition 1. Fix a filtration $\mathbb{F}$, $\Omega \subseteq X$, and a set of statically traded options Φ.

- A *one-point arbitrage* (1p-arbitrage) is a strategy $(\alpha, H) \in \mathcal{A}_\Phi(\mathbb{F})$ such that $\alpha \cdot \Phi + (H \circ S)_T \geq 0$ on Ω with a strict inequality for some $\omega \in \Omega$.
- A *strong arbitrage* is a strategy $(\alpha, H) \in \mathcal{A}_\Phi(\mathbb{F})$ such that $\alpha \cdot \Phi + (H \circ S)_T > 0$ on Ω.
- A *uniformly strong arbitrage* is a strategy $(\alpha, H) \in \mathcal{A}_\Phi(\mathbb{F})$ such that $\alpha \cdot \Phi + (H \circ S)_T > \varepsilon$ on Ω for some $\varepsilon > 0$.

Clearly, these notions are relative to the inputs, and we often stress this and refer to an arbitrage *in* $\mathcal{A}_\Phi(\mathbb{F})$ and *on* Ω. We are now ready to state the pathwise version of the fundamental theorem of asset pricing. It generalizes theorem 1.3 in Burzoni et al. [10] in two directions: we include an analytic selection of scenarios Ω, and we include static trading in options as well as dynamic trading in S.

Theorem 1 (Pointwise FTAP on $\Omega \subseteq X$). *Fix Ω analytic and Φ a finite set of $\mathcal{F}^{\mathcal{A}}$-measurable, statically traded options. Then there exists a filtration $\widetilde{\mathbb{F}}$ that aggregates arbitrage views in that*

$$\textit{No Strong Arbitrage in } \mathcal{A}_\Phi(\widetilde{\mathbb{F}}) \textit{ on } \Omega \iff \mathcal{M}_{\Omega,\Phi}(\mathbb{F}^{S,Y}) \neq \emptyset \iff \Omega^*_\Phi \neq \emptyset$$

and $\mathbb{F}^{S,Y} \subseteq \widetilde{\mathbb{F}} \subseteq \mathbb{F}^M$. Further, Ω^*_Φ is analytic, and there exists a trading strategy $(\alpha^*, H^*) \in \mathcal{A}_\Phi(\widetilde{\mathbb{F}})$ that is an arbitrage aggregator in that $\alpha^* \cdot \Phi + (H^* \circ S)_T \geq 0$ on Ω and

$$\Omega^*_\Phi = \{\omega \in \Omega \mid \alpha^* \cdot \Phi(\omega) + (H^* \circ S)_T(\omega) = 0\}. \tag{5}$$

Moreover, one may take $\widetilde{\mathbb{F}}$ and (α^*, H^*) as constructed in Equations (21) and (20), respectively.

All the proofs of the results of Sections 2.1 and 2.2 are given in Section 4.

Remark 2. Several examples in which Theorem 1 may fail replacing $\widetilde{\mathbb{F}}$ with $\mathbb{F}^{S,Y}$ can be found in Burzoni et al. [10]. We stress that $\mathcal{M}_{\Omega,\Phi} = \emptyset$ does not imply existence of a strong arbitrage in $\mathcal{A}_\Phi(\mathbb{F}^{S,Y})$—this is true only under additional strong assumptions; see Theorem 3 and Example 1. In general, the former corresponds to a situation in which all agents agree that rational option pricing is not possible, but they may disagree on why this is so. Our result shows that any such arbitrage views can be aggregated into one strategy (α^*, H^*) in a filtration $\widetilde{\mathbb{F}}$ that does not perturb the calibrated martingale measures; see Remark 1. The proof of Theorem 1 relies on an explicit—up to a measurable selection—construction of the enlarged filtration $\widetilde{\mathbb{F}}$ and the arbitrage aggregator strategy (α^*, H^*). It also puts in evidence that these objects may require the use of projective sets, which is also explained in Remark 7.

We turn now to our second main result. For a given set of scenarios $A \subseteq X$, define the superhedging price on A:

$$\pi_{A,\Phi}(g) := \inf\{x \in \mathbb{R} \mid \exists(\alpha, H) \in \mathcal{A}_\Phi(\mathbb{F}^{\text{pr}}) \text{ such that } x + \alpha \cdot \Phi + (H \circ S)_T \geq g \text{ on } A\}. \tag{6}$$

Following the intuition in Burzoni et al. [11], we expect to obtain pricing–hedging duality only when considering superhedging on the set of scenarios visited by martingales; that is, we consider $\pi_{\Omega^*_\Phi,\Phi}(g)$. In Theorem 2, there is no need for the construction of a larger filtration $\tilde{\mathbb{F}}$ as explained. Indeed, in Theorem 1, such a filtration is used for aggregating arbitrage opportunities that, in particular, yield a positive gain on the set $(\Omega^*_\Phi)^C$. On the contrary, the aim of Theorem 2 is to show a pricing–hedging duality in which both the primal and dual elements depend only on the set of efficient paths Ω^*_Φ.

Theorem 2. *Fix Ω analytic and Φ a finite set of $\mathcal{F}^{\mathcal{A}}$-measurable, statically traded options. Then, for any $\mathcal{F}^{\mathcal{A}}$-measurable g,*

$$\pi_{\Omega^*_\Phi,\Phi}(g) = \sup_{Q\in\mathcal{M}^f_{\Omega,\Phi}} E_Q[g] = \sup_{Q\in\mathcal{M}_{\Omega,\Phi}} E_Q[g] \tag{7}$$

and, if finite, the left-hand side is attained by some strategy $(\alpha, H) \in \mathcal{A}_\Phi(\mathbb{F}^{\mathrm{pr}})$.

For the case with no options, it was claimed in Burzoni et al. [11] that the superhedging strategy is universally measurable. This is true under the set-theoretic axioms that guarantee that projective sets are universally measurable (see Remark A.2), but in general, Theorem 2 offers a correction and only asserts measurability with respect to $\mathbb{F}^{\mathrm{pr}}$. To prove our main results, we first deal with the case in which $\Phi = 0$ and then extend iterating on the number k of statically traded options. The proofs are intertwined, and we explain their logic at the beginning of Section 4. Further, for technical reasons, in Section 4 we need to show that some results stated here for Ω analytic, such as Proposition 1, also extend to $\Omega \in \Lambda$; see Remark 15.

The following proposition is important as it shows that there are no one-point arbitrages on Ω if and only if each $\omega \in \Omega$ is weighted by some martingale measure $Q \in \mathcal{M}^f_{\Omega,\Phi}$.

Proposition 1. *Fix Ω analytic. Then there are no one-point arbitrages on Ω with respect to $\mathbb{F}^{\mathrm{pr}}$ if and only if $\Omega = \Omega^*_\Phi$.*

Under a mild assumption, this situation has further equivalent characterisations:

Remark 3. Under the additional assumption that Φ is not perfectly replicable on Ω, the following are easily shown to be equivalent:

1. No one-point arbitrage on Ω with respect to $\mathbb{F}^{\mathrm{pr}}$.
2. For any $x \in \mathbb{R}^k$, when $\varepsilon_x > 0$ is small enough, $\mathcal{M}^f_{\Omega,\Phi+\varepsilon_x x} \neq \emptyset$.
3. When $\varepsilon > 0$ is small enough, for any $x \in \mathbb{R}^k$ such that $|x| < \varepsilon$, $\mathcal{M}^f_{\Omega,\Phi+x} \neq \emptyset$, where $\Phi + x = \{\phi_1 + x_1, \ldots, \phi_n + x_n\}$.

In particular, small uniform modifications of the statically traded options do not affect the existence of calibrated martingale measures.

Remark 4. In Bouchard and Nutz [8] the notion of no-arbitrage (NA($\mathcal{P}$)) depends on a reference class of probability measures $\mathcal{P}$. If we choose as $\mathcal{P}$ the set of all the probability measures on Ω, then NA($\mathcal{P}$) corresponds to the notion of no one-point arbitrage in this paper, and in this case, Proposition 1 is a reformulation of the first fundamental theorem showed therein. Notice that such theorem in Bouchard and Nutz [8] was proven under the assumption that Ω is equal to the T-fold product of a Polish space Ω_1 (where T is the time horizon). Proposition 1 extends this result to Ω equal to an analytic subset of a general Polish space.

Arbitrage de la Classe $\mathcal{S}$. In Burzoni et al. [10], a large variety of different notions of arbitrage were studied with respect to a given class of relevant measurable sets. To cover the present setting of statically traded options, we adapt the definition of arbitrage de la classe $\mathcal{S}$ in Burzoni et al. [10].

Definition 2. Let $\mathcal{S} \subseteq \mathcal{B}_X$ be a class of measurable subsets of Ω such that $\emptyset \notin \mathcal{S}$. Fix a filtration $\mathbb{F}$ and a set of statically traded options Φ. An arbitrage de la classe $\mathcal{S}$ on Ω is a strategy $(\alpha, H) \in \mathcal{A}_\Phi(\mathbb{F})$ such that $\alpha \cdot \Phi + (H \circ S)_T \geq 0$ on Ω and $\{\omega \in \Omega \mid \alpha \cdot \Phi + (H \circ S)_T > 0\}$ contains a set in $\mathcal{S}$.

Notice that (1) when $\mathcal{S} = \{\Omega\}$, then the arbitrage de la classe $\mathcal{S}$ coincides with the notion of strong arbitrage; (2) when the class $\mathcal{S}$ consists of all nonempty subsets of Ω, the arbitrage de la classe $\mathcal{S}$ coincides with the notion of $1p$-arbitrage.

We now apply our Theorem 1 to characterize no arbitrage de la classe $\mathcal{S}$ in terms of the structure of the set of martingale measures. In this way, we generalize Burzoni et al. [10] to the case of semistatic trading. Define $\mathcal{N}^M := \{A \subseteq \Omega \mid Q(A) = 0 \;\; \forall\, Q \in \mathcal{M}^f_{\Omega,\Phi}\}$.

Corollary 1 (FTAP for the Class $\mathcal{S}$). *Fix Ω analytic and Φ a finite set of $\mathcal{F}^{\mathcal{A}}$-measurable, statically traded options. Then, there exists a filtration $\widetilde{\mathbb{F}}$ such that*

$$\textit{No Arbitrage de la classe } \mathcal{S} \textit{ in } \mathcal{A}_\Phi(\widetilde{\mathbb{F}}) \textit{ on } \Omega \iff \mathcal{M}_{\Omega,\Phi} \neq \emptyset \textit{ and } \mathcal{N}^M \cap \mathcal{S} = \emptyset$$

and $\mathbb{F}^{S,Y} \subseteq \widetilde{\mathbb{F}} \subseteq \mathbb{F}^M$.

2.2. Pointwise FTAP for Arbitrary Many Options in the Spirit of Acciaio et al. [1]

In this section, we want to recover and extend the main results in Acciaio et al. [1]. A similar result can be also found in Cheridito et al. [12] under slightly different assumptions. We work in the same setup as previously described except that we can allow for a larger, possibly uncountable, set of statically traded options $\Phi = \{\phi_i : i \in I\}$. Trading strategies $(\alpha, H) \in \mathcal{A}_\Phi(\mathbb{F}^{\mathrm{pr}})$ correspond to dynamic trading in S using $H \in \mathcal{H}(\mathbb{F}^{\mathrm{pr}})$ combined with a static position in a finite number of options in Φ.

Assumption 1. *In this section, we assume that S takes values in $\mathbb{R}_+^{d\times(T+1)}$ and all the options $\phi \in \Phi$ are continuous derivatives on the underlying assets S; more precisely,*

$$\phi_i = g_i \circ S \quad \text{for some continuous } g_i : \mathbb{R}_+^{d\times(T+1)} \to \mathbb{R}, \quad \forall i \in I.$$

In addition, we assume $0 \in I$ and $\phi_0 = g_0(S_T)$ for a strictly convex superlinear function g_0 on $\mathbb{R}^d$, such that other options have a slower growth at infinity:

$$\lim_{|x|\to\infty} \frac{g_i(x)}{m(x)} = 0, \quad \forall\ i \in I/\{0\}, \quad \text{where} \quad m(x_0, \dots, x_T) := \sum_{t=0}^{T} g_0(x_t).$$

The option ϕ_0 can be only bought at time $t = 0$. Therefore, admissible trading strategies $\mathcal{A}_\Phi(\mathbb{F})$ consider only positive values for the static position in ϕ_0.

The presence of ϕ_0 has the effect of restricting nontrivial considerations to a compact set of values for S, and then the continuity of g_i allows aggregating different arbitrages without enlarging the filtration. This results in the following special case of the pathwise fundamental theorem of asset pricing. Denote by $\widetilde{\mathcal{M}}_{\Omega,\Phi} := \{Q \in \mathcal{M}_{\Omega,\Phi\setminus\{\phi_0\}} \mid E_Q[\phi_0] \le 0\}$.

Theorem 3. *Consider Ω analytic such that $\Omega = \Omega^*$, $\pi_{\Omega^*}(\phi_0) > 0$, and there exists $\omega^* \in \Omega$ such that $S_0(\omega^*) = S_1(\omega^*) = \dots = S_T(\omega^*)$. Under Assumption 1, the following are equivalent:*

1. *There is no uniformly strong arbitrage on Ω in $\mathcal{A}_\Phi(\mathbb{F}^{\mathrm{pr}})$.*
2. *There is no strong arbitrage on Ω in $\mathcal{A}_\Phi(\mathbb{F}^{\mathrm{pr}})$.*
3. *$\widetilde{\mathcal{M}}_{\Omega,\Phi} \neq \emptyset$.*

Moreover, when any of these holds, for any upper semicontinuous $g : \mathbb{R}_+^{d\times(T+1)} \to \mathbb{R}$ that satisfies

$$\lim_{|x|\to\infty} \frac{g^+(x)}{m(x)} = 0, \tag{8}$$

the following pricing–hedging duality holds:

$$\pi_{\Omega,\Phi}(g(S)) = \sup_{Q\in\widetilde{\mathcal{M}}_{\Omega,\Phi}} E_Q[g(S)]. \tag{9}$$

Remark 5. We show in Remark 11 that the pricing–hedging duality may fail in general when superreplicating on the whole set Ω as in Equation (9). This confirms the intuition that the existence of an option ϕ_0 that satisfies the hypothesis of Theorem 3 is crucial. However, as shown in Burzoni et al. [11, section 4], the presence of such ϕ_0 is not sufficient. In fact, the pricing–hedging duality (9) may fail if g is not upper semicontinuous.

2.3. Classical Model-Specific Setting and Its Selection of Scenarios

In this section, we are interested in the relation of our results with the classical Dalang, Morton, and Willinger approach from Dalang et al. [16]. For simplicity and ease of comparison, throughout this section we restrict to dynamic trading only: $\Phi = 0$ and $\mathcal{A}(\mathbb{F}^{\mathbb{P}}) = \mathcal{H}(\mathbb{F}^{\mathbb{P}})$. For any filtration $\mathbb{F}$, we let $\mathbb{F}^{\mathbb{P}}$ be the $\mathbb{P}$-completion of $\mathbb{F}$. Recall that a $(\mathbb{F},\mathbb{P})$-arbitrage is a strategy $H \in \mathcal{A}(\mathbb{F}^{\mathbb{P}})$ such that $(H \circ S)_T \ge 0$ $\mathbb{P}$-a.s. and $\mathbb{P}((H \circ S)_T > 0) > 0$, which is the classical notion of arbitrage.

Proposition 2. *Consider a probability measure $\mathbb{P} \in \mathcal{P}$ and let $\mathcal{M}^{\ll\mathbb{P}} := \{Q \in \mathcal{M} \mid Q \ll \mathbb{P}\}$. There exists a set of scenarios $\Omega^{\mathbb{P}} \in \mathcal{F}^{\mathcal{A}}$ and a filtration $\widetilde{\mathbb{F}}$ such that $\mathbb{F}^{S,Y} \subseteq \widetilde{\mathbb{F}} \subseteq \mathbb{F}^M$ and*

$$\text{No Strong Arbitrage in } \mathcal{A}(\widetilde{\mathbb{F}}) \text{ on } \Omega^{\mathbb{P}} \iff \mathcal{M}^{\ll\mathbb{P}} \neq \emptyset.$$

Further,

$$\text{No } (\mathbb{F}^{S,Y},\mathbb{P})\text{-arbitrage} \iff \mathbb{P}\big((\Omega^{\mathbb{P}})^*\big) = 1 \iff \mathcal{M}^{\sim\mathbb{P}} \neq \emptyset,$$

where $\mathcal{M}^{\sim\mathbb{P}} := \{Q \in \mathcal{M} \mid Q \sim \mathbb{P}\}$.

Proof. For $1 \le t \le T$, we denote χ_{t-1} the random set $\chi_{\mathcal{G}}$ from Equation (A.1) with $\xi = \Delta S_t$ and $\mathcal{G} = \mathcal{F}^{S,Y}_{t-1}$ (see Appendix A.1 for further details). Consider now the set

$$U := \bigcap_{t=1}^{T} \{\omega \in X \mid \Delta S_t(\omega) \in \chi_{t-1}(\omega)\}$$

and note that, by Lemma A.3 in the appendix, $U \in \mathcal{B}_X$ and $\mathbb{P}(U) = 1$. Consider now the set U^* defined as in Equation (4) (using U in the place of Ω and for $\Phi = 0$) and define

$$\Omega^{\mathbb{P}} := \begin{cases} U & \text{if } \mathbb{P}(U^*) > 0 \\ U \setminus U^* & \text{if } \mathbb{P}(U^*) = 0 \end{cases},$$

which satisfies $\Omega^{\mathbb{P}} \in \mathcal{F}^A$ and $\mathbb{P}(\Omega^{\mathbb{P}}) = 1$.

In the proofs of both sufficiency and necessity, the existence of the technical filtration is a consequence of Theorem 1. To prove sufficiency, let $Q \in \mathcal{M}^{\ll \mathbb{P}}$ and observe that, because $Q(\Omega^{\mathbb{P}}) = 1$, we have $\mathcal{M}_{\Omega^{\mathbb{P}}} \ne \emptyset$. Because necessarily $Q(U^*) > 0$, we have $\mathbb{P}(U^*) > 0$, and hence, $\Omega^{\mathbb{P}} = U \in \mathcal{B}_X$. From Theorem 1, we have no $(\Omega^{\mathbb{P}}, \tilde{\mathbb{F}})$ strong arbitrage.

To prove necessity, observe first that $(\Omega^{\mathbb{P}})^*$ is either equal to U^* if $\mathbb{P}(U^*) > 0$ or to the empty set otherwise. In the latter case, Theorem 1 with $\Omega = U$ would contradict no $(\Omega^{\mathbb{P}}, \tilde{\mathbb{F}})$ strong arbitrage. Thus, $(\Omega^{\mathbb{P}})^* \ne \emptyset$ and $\mathbb{P}((\Omega^{\mathbb{P}})^*) = \mathbb{P}(U^*) > 0$. Note now that, by considering $\tilde{\mathbb{P}}(\cdot) := \mathbb{P}(\cdot \mid (\Omega^{\mathbb{P}})^*)$, we have, by construction, $0 \in ri(\chi_{t-1})$ $\tilde{\mathbb{P}}-$*a.s.* for every $1 \le t \le T$, where $ri(\cdot)$ denotes the relative interior of a set. By Rokhlin [43], we conclude that $\tilde{\mathbb{P}}$ admits an equivalent martingale measure and, hence, the thesis. The last statement then also follows. □

3. Construction of the Arbitrage Aggregator and Its Filtration

3.1. The Case Without Statically Traded Options

The following lemma is an empowered version of lemma 4.4 in Burzoni et al. [10], which relies on measurable selection arguments instead of a pathwise explicit construction. In the following, we set $\Delta S_t = S_t - S_{t-1}$ and Σ^{ω}_{t-1} the level set of the trajectory ω up to time $t-1$ of both traded and nontraded assets; that is,

$$\Sigma^{\omega}_{t-1} = \{\tilde{\omega} \in X \mid S_{0:t-1}(\tilde{\omega}) = S_{0:t-1}(\omega) \text{ and } Y_{0:t-1}(\tilde{\omega}) = Y_{0:t-1}(\omega)\}, \tag{10}$$

where $S_{0:t-1} := (S_0, \dots, S_{t-1})$ and $Y_{0:t-1} := (Y_0, \dots, Y_{t-1})$. Moreover, by recalling that $\Lambda = \cup_{n \in \mathbb{N}} \Sigma^1_n$ (see the appendix), we define $\mathcal{F}^{\text{pr,n}}_t := \sigma((S_u, Y_u)^{-1}(L) \mid L \in \Sigma^1_n,\ u \le t)$.

Lemma 1. *Fix any $t \in \{1, \dots, T\}$ and $\Gamma \in \Lambda$. There exist $n \in \mathbb{N}$, an index $\beta \in \{0, \dots, d\}$, random vectors $H^1, \dots, H^{\beta} \in \mathcal{L}(X, \mathcal{F}^{\text{pr,n}}_{t-1}; \mathbb{R}^d)$, $\mathcal{F}^{\text{pr,n}}_t$-measurable sets $E^0, \dots, E^{\beta}$ such that the sets $B^i := E^i \cap \Gamma$, $i = 0, \dots, \beta$, form a partition of Γ satisfying:*

1. *If $\beta > 0$ and $i = 1, \dots, \beta$, then $B^i \ne \emptyset$; $H^i \cdot \Delta S_t(\omega) > 0$ for all $\omega \in B^i$ and $H^i \cdot \Delta S_t(\omega) \ge 0$ for all $\omega \in \cup_{j=i}^{\beta} B^j \cup B^0$.*
2. *$\forall H \in \mathcal{L}(X, \mathcal{F}^{\text{pr,n}}_{t-1}; \mathbb{R}^d)$ such that $H \cdot \Delta S_t \ge 0$ on B^0 we have $H \cdot \Delta S_t = 0$ on B^0.*

Remark 6. Clearly, if $\beta = 0$, then $B^0 = \Gamma$ (which includes the trivial case $\Gamma = \emptyset$). Notice also that for any $\Gamma \in \Lambda$ and $t = \{1, \dots, T\}$ we have that $H^i = H^{i,\Gamma}_t, B^i = B^{i,\Gamma}_t, \beta = \beta^{\Gamma}_t$ depend explicitly on t and Γ.

Remark 7. To appreciate why the use of projective sets is necessary, consider a market with $d \ge 2$ assets, $T \ge 2$ trading periods, and a Borel selection of paths $\Omega \in \mathcal{B}_X$. For a given $t > 0$ and a realized price S_t, an investor willing to exploit an arbitrage opportunity (if it exists) needs to analyze all the possible evolutions of the price S_{t+1} given the realized value S_t. Mathematically, this conditional set is the projection of ΔS_{t+1} at time t, and it is provided by Lemma 1. Such a projection does not preserve Borel measurability, and therefore, the arbitrage vector H^1 (if it exists) is an analytically measurable strategy. Now, if $B^1 = \{H^1 \Delta S_{t+1} > 0\} \ne \emptyset$, agents considering B^1 significant will call H^1 an arbitrage opportunity. On the other hand, even on the restricted set $A^1 := \Omega \setminus B^1 \in \mathcal{F}^{\mathcal{A}}$, there might be inefficiencies. Note that, in general, A^1 is neither Borel nor analytic but only in $\mathcal{F}^{\mathcal{A}}$. Its projection, obtained by a second application of Lemma 1, does not need to be analytic measurability, and the potential arbitrage strategy H^2 will be, in general, projective. Agents considering $B^2 = \{H^2 \Delta S_{t+1} > 0\} \ne \emptyset$ significant will call H_2 an arbitrage strategy. As shown by Lemma 1, the geometry of the finite dimensional space $\mathbb{R}^d$ imposes that the procedure terminates in a finite number of steps.

Proof of Lemma 1. Fix $t \in \{1, \dots, T\}$ and consider, for an arbitrary $\Gamma \in \Lambda$, the multifunction

$$\psi_{t,\Gamma} : \omega \in X \mapsto \left\{\Delta S_t(\tilde{\omega}) \mathbf{1}_{\Gamma}(\tilde{\omega}) \mid \tilde{\omega} \in \Sigma^{\omega}_{\mathbf{t-1}}\right\} \subseteq \mathbb{R}^d, \tag{11}$$

where Σ_{t-1}^{ω} is defined in Equation (10). By definition of Λ, there exists $\ell \in \mathbb{N}$ such that $\Gamma \in \Sigma_{\ell}^{1}$. We first show that $\psi_{t,\Gamma}$ is an $\mathcal{F}_{t-1}^{\mathrm{pr},\ell+1}$-measurable multifunction. Note that for any open set $O \subseteq \mathbb{R}^d$

$$\{\omega \in X \mid \psi_{t,\Gamma}(\omega) \cap O \neq \emptyset\} = (S_{0:t-1}, Y_{0:t-1})^{-1}((S_{0:t-1}, Y_{0:t-1})(B)),$$

where $B = (\Delta S_t \mathbf{1}_\Gamma)^{-1}(O)$. First, $\Delta S_t \mathbf{1}_\Gamma$ is an $\mathcal{F}^{\mathrm{pr},\ell}$-measurable random vector, then $B \in \mathcal{F}^{\mathrm{pr},\ell}$, the sigma-algebra generated by the ℓ-projective sets of X. Second, S_u, Y_u are Borel measurable functions for any $0 \le u \le t-1$ so that, from Lemma A.2, we have that $(S_{0:t-1}, Y_{0:t-1})(B)$ belongs to the sigma-algebra generated by the $(\ell+1)$-projective sets of $Mat((d+\tilde{d}) \times t; \mathbb{R})$ (the space of $(d+\tilde{d}) \times t$ matrices with real entries) endowed with its Borel sigma-algebra. Applying again Lemma A.2, we deduce that $(S_{0:t-1}, Y_{0:t-1})^{-1}((S_{0:t-1}, Y_{0:t-1})(B)) \in \mathcal{F}_{t-1}^{\mathrm{pr},\ell+1}$, and hence, the desired measurability for $\psi_{t,\Gamma}$.

Let $\mathbb{S}^d$ be the unit sphere in $\mathbb{R}^d$; by preservation of measurability (see Rockafellar and Wets [42], chapter 14-B), the following multifunction is a closed valued and $\mathcal{F}_{t-1}^{\mathrm{pr},\ell+1}$-measurable

$$\psi_{t,\Gamma}^{*}(\omega) := \{H \in \mathbb{S}^d \mid H \cdot y \ge 0 \quad \forall y \in \psi_{t,\Gamma}(\omega)\}.$$

It follows that it admits a Castaing representation (see theorem 14.5 in Rockafellar and Wets [42]); that is, there exists a countable collection of measurable functions $\{\xi_{t,\Gamma}^{n}\}_{n\in\mathbb{N}} \subseteq \mathcal{L}(X, \mathcal{F}_{t-1}^{\mathrm{pr},\ell+1}; \mathbb{R}^d)$ such that $\psi_{t,\Gamma}^{*}(\omega) = \overline{\{\xi_{t,\Gamma}^{n}(\omega) \mid n \in \mathbb{N}\}}$ for every ω such that $\psi_{t,\Gamma}^{*}(\omega) \neq \emptyset$ and $\xi_{t,\Gamma}^{n}(\omega) = 0$ for every ω such that $\psi_{t,\Gamma}^{*}(\omega) = \emptyset$. Recall that every $\xi_{t,\Gamma}^{n}$ is a measurable selector of $\psi_{t,\Gamma}^{*}$, and hence $\xi_{t,\Gamma}^{n} \cdot \Delta S_t \ge 0$ on Γ. Note moreover that

$$\forall\, \omega \in X, \bigcup_{\xi \in \psi_{t,\Gamma}^{*}(\omega)} \{y \in \mathbb{R}^d \mid \xi \cdot y > 0\} = \bigcup_{n\in\mathbb{N}} \{y \in \mathbb{R}^d \mid \xi_{t,\Gamma}^{n}(\omega) \cdot y > 0\}. \tag{12}$$

The inclusion $(\supseteq)$ is clear for the converse, note that if y satisfies $\xi_{t,\Gamma}^{n}(\omega) \cdot y \le 0$ for every $n \in \mathbb{N}$, then, by continuity, $\xi \cdot y \le 0$ for every $\xi \in \psi_{t,\Gamma}^{*}(\omega)$.

We now define the the conditional standard separator as

$$\xi_{t,\Gamma} := \sum_{n=1}^{\infty} \frac{1}{2^n} \xi_{t,\Gamma}^{n}, \tag{13}$$

which is $\mathcal{F}_{t-1}^{\mathrm{pr},\ell+1}$-measurable and, from Equation (12), satisfies the following maximality property: $\{\omega \in X \mid \xi(\omega) \cdot \Delta S_t(\omega) > 0\} \subseteq \{\omega \in X \mid \xi_{t,\Gamma}(\omega) \cdot \Delta S_t(\omega) > 0\}$ for any ξ measurable selector of $\psi_{t,\Gamma}^{*}$.

Step 0. We take $A^0 := \Gamma$ and consider the multifunction ψ_{t,A^0}^{*} and the conditional standard separator ξ_{t,A^0} in Equation (13). If $\psi_{t,A^0}^{*}(\omega)$ is a linear subspace of $\mathbb{R}^d$ (i.e., $H \in \psi_{t,A^0}^{*}(\omega)$ implies necessarily $-H \in \psi_{t,A^0}^{*}(\omega)$) for any $\omega \in A^0$, then set $\beta = 0$ and $A^0 = B^0$ (in this case, obviously $E^0 = X$).

Step 1. If there exists an $\omega \in A^0$ such that $\psi_{t,A^0}^{*}(\omega)$ is not a linear subspace of $\mathbb{R}^d$, then we set $H_1 = \xi_{t,A^0}$, $E^1 = \{\omega \in X \mid H_1 \Delta S_t > 0\}$, $B^1 = \{\omega \in A^0 \mid H_1 \Delta S_t > 0\} = E^1 \cap \Gamma$, and $A^1 = A^0 \setminus B^1 = \{\omega \in A^0 \mid H_1 \Delta S_t = 0\}$. If now $\psi_{t,A^1}^{*}(\omega)$ is a linear subspace of $\mathbb{R}^d$ for any $\omega \in A^1$, then we set $\beta = 1$ and $A^1 = B^0$. If this is not the case, we proceed iterating this scheme.

Step 2. Notice that for every $\omega \in A^1$ we have $\Delta S_t(\omega) \in R_1(\omega) := \{y \in \mathbb{R}^d \mid H_1(\omega) \cdot y = 0\}$, which can be embedded in a subspace of $\mathbb{R}^d$ whose dimension is $d-1$. We consider the case in which there exists one $\omega \in A^1$ such that $\psi_{t,A^1}^{*}(\omega)$ is not a linear subspace of $R_1(\omega)$: we set $H_2 = \xi_{t,A^1}$, $E^2 = \{\omega \in X \mid H_2 \Delta S_t > 0\}$, $B^2 = \{\omega \in A^0 \mid H_2 \Delta S_t > 0\} = E^2 \cap \Gamma$, and $A^2 = A^1 \setminus B^2 = \{\omega \in A^1 \mid H_2 \Delta S_t = 0\}$. If now $\psi_{t,A^2}^{*}(\omega)$ is a linear subspace of $R_1(\omega)$ for any $\omega \in A^2$, then we set $\beta = 2$ and $A^2 = B^0$. If this is not the case, we proceed iterating this scheme.

The scheme can be iterated and ends at most within d steps so that there exists $n \le \ell + 2d$ yielding the desired measurability. □

Define, for $\Omega \in \Lambda$,

$$\begin{aligned} \Omega_T &:= \Omega \\ \Omega_{t-1} &:= \Omega_t \setminus \bigcup_{i=1}^{\beta_t} B_t^i, \quad t \in \{1, \ldots, T\}, \end{aligned} \tag{14}$$

where $B_t^i := B_t^{i,\Gamma}$, $\beta_t := \beta_t^{\Gamma}$ are the sets and index constructed in Lemma 1 with $\Gamma = \Omega_t$ for $1 \le t \le T$. Note that we can iteratively apply Lemma 1 at time $t-1$ because $\Gamma = \Omega_t \in \Lambda$.

Corollary 2. *For any $t \in \{1,\dots,T\}$, Ω analytic, and $Q \in \mathcal{M}_\Omega$ we have, $\cup_{i=1}^{\beta_t} B_t^i$ is a subset of a Q-null set. In particular, $\cup_{i=1}^{\beta_t} B_t^i$ is an $\mathcal{M}_\Omega$ polar set.*

Proof. Let $\Gamma = \Omega$. First, observe that the map $\psi_{T,\Gamma}$ in Equation (11) is $\mathcal{F}_{T-1}^{\mathcal{A}}$-measurable. Indeed, the set $B = (\Delta S_t \mathbf{1}_\Gamma)^{-1}(O)$ is analytic because it is equal to $\Delta S_t^{-1}(O) \cap \Gamma$ if $0 \notin O$ or $\Delta S_t^{-1}(O) \cup \Gamma$ if $0 \in O$. The measurability of $\psi_{T,\Gamma}$ follows from Lemma A.1. As a consequence, H^1 and B^1 from Lemma 1 satisfy $H^1 \in \mathcal{L}(X, \mathcal{F}_{T-1}^{\mathcal{A}}; \mathbb{R}^d)$ and $B^1 = \{H^1 \cdot \Delta S_T > 0\} \in \mathcal{F}^{\mathcal{A}}$. Suppose $Q(B^1) > 0$. The strategy $H_u := H^1 \mathbf{1}_{T-1}(u)$ satisfies that

- H is $\mathbb{F}^Q$-predictable, where $\mathbb{F}^Q = \{\mathcal{F}_t^S \vee \mathcal{N}^Q(\mathcal{B}_X)\}_{t \in \{0,\dots,T\}}$;
- $(H \cdot S)_T \geq 0$ Q-a.s. and $(H \cdot S)_T > 0$ on B^1, which has positive probability.

Thus, H is an arbitrage in the classical probabilistic sense, which leads to a contradiction. Because B^1 is a Q-null set, there exists $\tilde{B}^1 \in \mathcal{B}_X$ such that $B^1 \subseteq \tilde{B}^1$ and $Q(\tilde{B}^1) = 0$. Consider now the Borel-measurable version of S_T given by $\tilde{S}_T = S_T \mathbf{1}_{X \setminus \tilde{B}^1} + S_{T-1} \mathbf{1}_{\tilde{B}^1}$. We iterate the previous procedure, replacing S with $\tilde{S}$ at each step up to time t. As in Lemma 1, the procedure ends in a finite number of steps yielding a collection $\{\tilde{B}_t^i\}_{i=1}^{\tilde{\beta}_t}$ such that $\cup_{i=1}^{\beta_t} B_t^i \subseteq \cup_{i=1}^{\tilde{\beta}_t} \tilde{B}_t^i$ with $Q(\cup_{i=1}^{\beta_t} \tilde{B}_t^i) = 0$. □

Corollary 3. *Let B_t^0 be the set provided by Lemma 1 for $\Gamma = \Omega_t$. For every $\omega \in B_t^0$, there exists $Q \in \mathcal{P}^f$ with $Q(\{\omega\}) > 0$ such that $\mathbb{E}_Q[S_t \mid \mathcal{F}_{t-1}^S](\omega) = S_{t-1}(\omega)$.*

Proof. Fix $\omega \in B_t^0$ and let Σ_{t-1}^ω be given as in Equation (10). We consider $D := \Delta S_t(\Sigma_{t-1}^\omega \cap B_t^0) \subseteq \mathbb{R}^d$ and $C := \{\lambda v \mid v \in conv(D),\ \lambda \in \mathbb{R}^+\}$, where $conv(D)$ denotes the convex hull of D. Denote by $ri(C)$ the relative interior of C. From Lemma 1, item 2, we have $H \cdot \Delta S_t(\widetilde{\omega}) \geq 0$ for all $\widetilde{\omega} \in \Sigma_{t-1}^\omega \cap B_t^0$ implies $H \cdot \Delta S_t(\widetilde{\omega}) = 0$ for all $\widetilde{\omega} \in \Sigma_{t-1}^\omega \cap B_t^0$, which is equivalent to $0 \in ri(C)$. From remark 4.8 in Burzoni et al. [10], we have that for every $x \in D$ there exists a finite collection $\{x_j\}_{j=1}^m \subseteq D$ and $\{\lambda_j\}_{j=1}^{m+1}$ with $0 < \lambda_j \leq 1$, $\sum_{j=1}^{m+1} \lambda_j = 1$, such that

$$0 = \sum_{j=1}^m \lambda_j x_j + \lambda_{m+1} x. \tag{15}$$

Choose now $x := \Delta S_t(\omega)$ and note that for every $j = 1, \dots m$ there exists $\omega_j \in \Sigma_{t-1}^\omega \cap B_t^0$ such that $\Delta S_t(\omega_j) = x_j$. Choose now $Q \in \mathcal{P}^f$ with conditional probability $Q(\cdot \mid \mathcal{F}_{t-1}^S)(\omega) := \sum_{j=1}^m \lambda_j \delta_{\omega_j} + \lambda_{m+1} \delta_\omega$, where $\delta_{\widetilde{\omega}}$ denotes the Dirac measure with mass point in $\widetilde{\omega}$. From Equation (15), we have the thesis. □

Lemma 2. *For $\Omega \in \Lambda$, the set Ω^* defined in Equation (4) with $\Phi = 0$ coincides with Ω_0 defined in Equation (14), and therefore, $\Omega^* \in \Lambda$. Moreover, if Ω is analytic, then Ω^* is analytic, and we have the following:*

$$\Omega^* \neq \emptyset \iff \mathcal{M}_\Omega \neq \emptyset \iff \mathcal{M}_\Omega^f \neq \emptyset.$$

Proof. The proof is analogous to that of proposition 4.18 in Burzoni et al. [10], but we give here a self-contained argument. Notice that $\Omega^* \subseteq \Omega_0$ follows from the definitions and Corollary 2. For the reverse inclusion, it suffices to show that for $\omega_* \in \Omega_0$ there exists a $Q \in \mathcal{M}_\Omega^f$ such that $Q(\{\omega_*\}) > 0$; that is, $\omega_* \in \Omega^*$. From Corollary 3, for any $1 \leq t \leq T$, there exists a finite number of elements of $\Sigma_{t-1}^\omega \cap B_t^0$ named $C_t(\omega) := \{\omega, \omega_1, \dots, \omega_m\}$, such that

$$S_{t-1}(\omega) = \lambda_t(\omega) S_t(\omega) + \sum_{j=1}^m \lambda_t(\omega_j) S_t(\omega_j), \tag{16}$$

where $\lambda_t(\omega) > 0$ and $\lambda_t(\omega) + \sum_{j=1}^m \lambda_t(\omega_j) = 1$.

Fix now $\omega_* \in \Omega_0$. We iteratively build a set Ω_f^T that is suitable for being the finite support of a discrete martingale measure (and contains ω_*).

Start with $\Omega_f^1 = C_1(\omega_*)$, which satisfies Equation (16) for $t = 1$. For any $t > 1$, given Ω_f^{t-1}, define $\Omega_f^t := \{C_t(\omega) \mid \omega \in \Omega_f^{t-1}\}$. Once Ω_f^T is settled, it is easy to construct a martingale measure via Equation (16):

$$Q(\{\omega\}) = \prod_{t=1}^T \lambda_t(\omega) \quad \forall \omega \in \Omega_f^T.$$

Because, by construction, $\lambda_t(\omega_*) > 0$ for any $1 \leq t \leq T$, we have $Q(\{\omega_*\}) > 0$ and $Q \in \mathcal{M}_\Omega^f$.

For the last assertion, suppose Ω is analytic. From remark 5.6 in Burzoni et al. [11], Ω^* is also analytic. In particular, if $\mathcal{M}_\Omega \neq \emptyset$ then, from Corollary 2, $Q(\Omega^*) = 1$ for any $Q \in \mathcal{M}_\Omega$. This implies $\Omega^* \neq \emptyset$. The converse implication is trivial. □

Lemma 3. *Suppose $\Phi = 0$. Then, no one-point arbitrage $\Leftrightarrow \Omega^* = \Omega$.*

Proof. We first show the $\Leftarrow$ implication. If $(H \circ S)_T \geq 0$ on Ω, then $(H \circ S)_T = 0$ Q-a.s. for every $Q \in \mathcal{M}_\Omega^f$. From the hypothesis, we have

$$\cup\{\mathrm{supp}(Q) \mid Q \in \mathcal{M}_\Omega^f\} = \Omega$$

from which the thesis follows. For the converse implication, let $1 \leq t \leq T$ and $\Gamma = \Omega_t$. Note that if β_t from Lemma 1 is strictly positive, then H^1 is a one-point arbitrage. We, thus, have $\beta_t = 0$ for any $1 \leq t \leq T$, and hence, $\Omega_0 = \Omega$. From Lemma 2, we have $\Omega^* = \Omega$. □

Definition 3. We call arbitrage aggregator the process

$$H_t^*(\omega) := \sum_{i=1}^{\beta_t} H_t^{i,\Omega_t}(\omega)\mathbf{1}_{B_t^{i,\Omega_t}}(\omega) \tag{17}$$

for $t \in \{1, \ldots, T\}$, where $H_t^{i,\Omega_t}, B_t^{i,\Omega_t}, \beta_t$ are provided by Lemma 1 with $\Gamma = \Omega_t$.

Remark 8. Observe that from Lemma 1, item 1, $(H^* \circ S)_T(\omega) \geq 0$ for all $\omega \in \Omega$, and from Lemma 2, $(H^* \circ S)_T(\omega) > 0$ for all $\omega \in \Omega \setminus \Omega^*$.

Remark 9. By construction, we have that H_t^* is $\mathcal{F}^{\mathrm{pr,n}}$-measurable for every $t \in \{1, \ldots, T\}$ for some $n \in \mathbb{N}$. Moreover, any B_t^{i,Ω_t} is the intersection of an $\mathcal{F}_t^{\mathrm{pr,n}}$-measurable set with Ω_t. As a consequence, we have that $(H_t^*)_{|\Omega_t} : \Omega_t \to \mathbb{R}^d$ is $(\mathcal{F}_t^{\mathrm{pr,n}})_{|\Omega_t}$-measurable.

Remark 10. In case there are no options to be statically traded, $\Phi = 0$, the enlarged filtration $\widetilde{\mathbb{F}}$ required in Theorem 1 is given by

$$\widetilde{\mathcal{F}}_t := \mathcal{F}_t^{S,Y} \vee \sigma(H_1^*, \ldots, H_{t+1}^*), t \in \{0, \ldots, T-1\}, \tag{18}$$

$$\widetilde{\mathcal{F}}_T := \mathcal{F}_T^{S,Y} \vee \sigma(H_1^*, \ldots, H_T^*), \tag{19}$$

so that the arbitrage aggregator from Equation (17) is predictable with respect to $\widetilde{\mathbb{F}} = \{\widetilde{\mathcal{F}}_t\}_{t \in \mathbb{T}}$.

3.2. The Case with a Finite Number of Statically Traded Options

Throughout this section, we consider the case of a finite set of options Φ. As in the previous section, we consider $\mathcal{F}_t^{\mathrm{pr,n}} := \sigma((S_u, Y_u)^{-1}(L) \mid L \in \Sigma_n^1, u \leq t)$, and $\mathbb{F}^{\mathrm{pr,n}} := (\mathcal{F}_t^{\mathrm{pr,n}})_{t \in \mathbb{T}}$.

Definition 4. A path-space partition scheme $\mathcal{R}(\alpha^\star, H^\star)$ of Ω is a collection of trading strategies $H^1, \ldots, H^\beta \in \mathcal{H}(\mathbb{F}^{\mathrm{pr,n}})$ for some $n \in \mathbb{N}$, $\alpha^1, \ldots, \alpha^\beta \in \mathbb{R}^k$, and arbitrage aggregators $\tilde{H}^0, \ldots, \tilde{H}^\beta$ for some $1 \leq \beta \leq k$, such that

1. α^i, $1 \leq i \leq \beta$, are linearly independent.
2. For any $i \leq \beta$,

$$(H^i \circ S)_T + \alpha^i \cdot \Phi \geq 0 \text{ on } A_{i-1}^*,$$

where $A_0 = \Omega$, $A_i := \{(H^i \circ S)_T + \alpha^i \cdot \Phi = 0\} \cap A_{i-1}^*$, and A_i^* is the set Ω^* in Equation (4) with $\Omega = A_i$ and $\Phi = 0$ for $1 \leq i \leq \beta$.

3. For any $i = 0, \ldots, \beta$, $\tilde{H}^i$ is the arbitrage aggregator as defined in Equation (17), substituting Ω with A_i.
4. If $\beta < k$, then either $A_\beta^* = \emptyset$, or for any $\alpha \in \mathbb{R}^k$ linearly independent from $\alpha^1, \ldots, \alpha^\beta$, there does not exist H such that

$$(H \circ S)_T + \alpha \cdot \Phi \geq 0 \text{ on } A_\beta^*.$$

We note that as defined in (2), each $A_i \in \Lambda$ so that $A_i^* \in \Lambda$ by Lemma 2. The purpose of a path-space partition scheme is to iteratively split the path-space Ω into subsets on which a strong arbitrage strategy can be identified. For the existence of a calibrated martingale measure, it will be crucial to see whether this procedure exhausts the path space or not. Note that on A_i we can perfectly replicate i linearly independent combinations of options $\alpha^j \cdot \Phi$, $1 \leq j \leq i$. In consequence, we make at most k such iterations, $\beta \leq k$, and if $\beta = k$, then all statically traded options are perfectly replicated on A_β^*, which reduces here to the setting without statically traded options.

Definition 5. A path-space partition scheme $\mathcal{R}(\alpha^\star, H^\star)$ is successful if $A_\beta^* \neq \emptyset$.

We illustrate now the construction of a successful path-space partition scheme.

Example 1. Let $X = \mathbb{R}^2$. Consider a financial market with one dynamically traded asset S and two options available for static trading $\Phi := (\phi_1, \phi_2)$. Let S be the canonical process; that is, $S_t(x) = x_t$ for $t = 1, 2$ with initial price $S_0 = 2$. Moreover, let $\phi_i := g_i(S_1, S_2) - c$ for $i = 1, 2$ with $c > 0$, $g_i := (x_2 - K_i)^+ \mathbf{1}_{[0,b]}(x_1) + c\mathbf{1}_{(b,\infty)}(x_1)$, and $K_1 > K_2$. Namely, ϕ_i is a knock-in call option on S with maturity $T = 2$, strike price K_i, knock-in value $b \geq 0$, and cost $c > 0$, but the cost is recovered if the option is not knocked in. Consider the path-space selection $\Omega = [0,4] \times [0,4]$.

Start with $A_0 := \Omega$ and suppose $0 < K_2 < b < 2$.

1. The process $(\tilde{H}_1^0, \tilde{H}_2^0) := (0, \mathbf{1}_{\{0\}}(S_1) - \mathbf{1}_{\{4\}}(S_1))$ is an arbitrage aggregator: when S_1 hits the values $\{0, 4\}$, the price process does not decrease or increase, respectively. It is easily seen that there are no more arbitrages on A_0 from dynamic trading only. Thus, $A_0^* = \{(0,4) \times [0,4]\} \cup \{\{0\} \times \{0\}\} \cup \{\{4\} \times \{4\}\}$.

2. Suppose now we have a semistatic strategy (H^1, α^1) such that

$$(H^1 \circ S)_T + \alpha^1 \cdot \Phi \geq 0 \text{ on } A_0^*,$$

where $\alpha^1 = (-\alpha, \alpha) \in \mathbb{R}^2$ for some $\alpha > 0$ (because $K_1 > K_2$ and ϕ_1, ϕ_2 have the same cost). Moreover, because $\Phi = 0$ if $S_1 \notin [0,b]$, we can choose $(H_1^1, H_2^1) = (0,0)$. A positive gain is obtained on $B_1 := [0,b] \times (K_2, 4]$ (see Figure 1(a)). Thus, $A_1 := A_0^* \setminus B_1$.

3. $(\tilde{H}_1^1, \tilde{H}_2^1) := (0, -\mathbf{1}_{[K_2,b]}(S_1))$ is an arbitrage aggregator on A_1: the price process does not increase, and $(\tilde{H}_1^1, \tilde{H}_2^1)$ yields a positive gain on $B_2 := \{K_2 \leq S_1 \leq b\} \setminus \{\{K_2\} \times \{K_2\}\}$ (see Figure 1(b)). Thus, $A_1^* = A_1 \setminus B_2$.

4. The null process H^2 and the vector $\alpha^2 := (-1, 0) \in \mathbb{R}^2$ satisfy

$$(H^2 \circ S)_T + \alpha^2 \cdot \Phi \geq 0 \text{ on } A_1^*.$$

A positive gain is obtained on $[0,b] \times [0,K_2]$. Thus, $A_2 := (b,4) \times [0,4] \cup \{\{4\} \times \{4\}\}$.

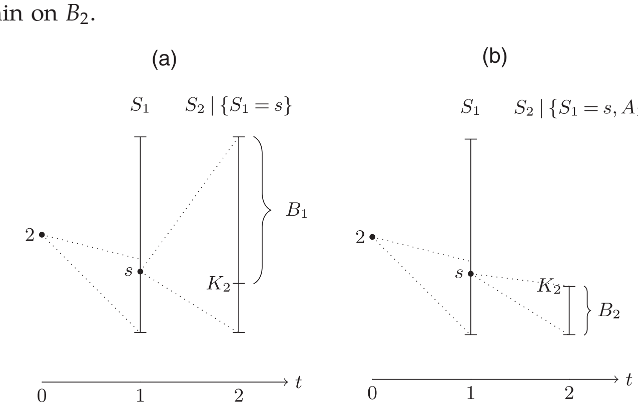
Obviously, there are no more semistatic 1p-arbitrage opportunities, and $A_2 = A_2^*$, $\beta = 2$. We set $\tilde{H}^2 \equiv 0$, and the partition scheme is successful with arbitrage aggregators $\tilde{H}^0, \tilde{H}^1, \tilde{H}^2$ and semistatic strategies (H^j, α^j) for $j = 1, 2$ as before.

Remark 11. The preceding example also shows that $\pi_{\Omega_\Phi^*,\Phi}(g) = \pi_{\Omega,\Phi}(g)$ is a rather exceptional case if we do not assume the existence of an option with dominating payoff as in Theorem 3. Consider indeed the market of Example 1 with $b := 4$, $0 < K_2 < 2$, which has the same features as the example on page 5 in Davis and Hobson [18]. Take $g \equiv 1$ and note that Ω is compact and S, Φ, and g are continuous functions on Ω. From this discussion, we see easily that $\Omega_\Phi^* = \emptyset$ and $\pi_{\Omega_\Phi^*,\Phi} = -\infty$. Nevertheless, by considering the path-space partition scheme, we see that although we can devise an arbitrage strategy on $B_1 = [0,4) \times (K_2, 4]$ its payoff is not bounded below by a positive constant, and in fact, we see that $\pi_{\Omega,\Phi}(g) = 1$.

Remark 12. Note that if a partition scheme is successful, then there are no one-point arbitrages on A_β^*. When $\beta < k$, this follows from *(iv)* in Definition 4. In the case $\beta = k$, suppose there is a one-point arbitrage $(\alpha, H) \in \mathcal{A}_\Phi(\mathbb{F}^{pr})$ so that, in particular, $(H \circ S)_T + \alpha \cdot \Phi \geq 0$ on A_β^*. Because the vectors α^i form a basis of $\mathbb{R}^k$, we get, for some $\lambda_i \in \mathbb{R}$,

$$(H \circ S)_T + \alpha \cdot \Phi = \sum_{i=1}^{k} \lambda_i \left[(H^i \circ S)_T + \alpha^i \cdot \Phi \right] + (\hat{H} \circ S)_T,$$

Figure 1. Some steps of the path-space partition scheme with $b = 1.5$, $K_2 = 1$. (a) The strategy (H_1, α^1), defined on A_0^*, has positive gain on B_1. As a consequence, the path space reduces to $A_1 = A_0^* \setminus B_1$. (b) For $S_1 \geq K_2$, the arbitrage aggregator $\tilde{H}_1$, defined on A_1, has positive gain on B_2.



where $\hat{H} := H - \sum_{i=1}^k \lambda_i H^i$. Because, by construction, $(H^i \circ S)_T + \alpha^i \cdot \Phi = 0$ on A^*_β for any $i = 1, \dots, \beta$, we obtain that $\hat{H}$ is a one-point arbitrage with $\Phi = 0$ on A^*_β. From Lemma 3, we have a contradiction.

Remark 13. As we see, Lemma 5 implies relative uniqueness of $\mathcal{R}(\alpha^\star, H^\star)$ in the sense that either every $\mathcal{R}(\alpha^\star, H^\star)$ is not successful or all $\mathcal{R}(\alpha^\star, H^\star)$ are successful, and then $A^*_\beta = \Omega^*_\Phi$.

Definition 6. Given a path-space partition scheme, we define the arbitrage aggregator as

$$(\alpha^\star, H^\star) = \left(\sum_{i=1}^\beta \alpha^i \mathbf{1}_{A^*_i}, \sum_{i=1}^\beta H^i \mathbf{1}_{A^*_i} + \sum_{i=1}^\beta \tilde{H}^i \mathbf{1}_{A_i \setminus A^*_i} \right), \tag{20}$$

with $(\alpha^\star, H^\star) = (0, \tilde{H}^0 \mathbf{1}_{\Omega\setminus\Omega^*})$, if $\beta = 0$.

To make this arbitrage aggregator predictable, we need to enlarge the filtration. We therefore introduce the arbitrage-aggregating filtration $\widetilde{\mathbb{F}}$ given by

$$\begin{aligned} \widetilde{\mathcal{F}}_t &= \mathbb{F}^{S,Y}_t \vee \{A_0, A^*_0, \dots, A_\beta, A^*_\beta\} \vee \sigma(\tilde{H}^0_1, \dots, \tilde{H}^\beta_1, \dots, \tilde{H}^0_{t+1}, \dots, \tilde{H}^\beta_{t+1}),\ t = 0, \dots, T-1, \\ \widetilde{\mathcal{F}}_T &= \mathbb{F}^{S,Y}_T \vee \{A_0, A^*_0, \dots, A_\beta, A^*_\beta\} \vee \sigma(\tilde{H}^0_1, \dots, \tilde{H}^\beta_1, \dots, \tilde{H}^0_T, \dots, \tilde{H}^\beta_T). \end{aligned} \tag{21}$$

It will follow, as a consequence of Corollary 2, that $\widetilde{\mathcal{F}}_t \subseteq \mathcal{F}^M_T$ for any $t = 0, \dots, T$, and in particular, as observed before, any $\mathbb{Q} \in \mathcal{M}_{\Omega,\Phi}(\mathbb{F}^S)$ extends uniquely to a measure in $\mathcal{M}_{\Omega,\Phi}(\widetilde{\mathbb{F}})$.

4. Proofs

We first describe the logical flow of our proofs, and we point out that we need to show some of the results for $\Omega \in \Lambda$ (and not only analytic). In particular, we show that $\Omega^*_\Phi \in \Lambda$ is involved. First, Theorem 1 and then Theorem 2 are established when $\Phi = 0$. Then we show Theorem 2 *under the further assumption* that $\Omega^*_{\Phi_i} \in \Lambda$ for all $1 \le i \le k$, where $\Phi_i = \{\phi_1, \dots, \phi_i\}$. Note that in the case with no statically traded options ($\Phi = 0$) and for $\Omega \in \Lambda$, the property $\Omega^*_\Phi = \Omega^* \in \Lambda$ follows from the construction and is shown in Lemma 2. This allows us to prove Proposition 1 for which we use Theorem 2 only when $\Omega^*_{\Phi_i} = \Omega$, which belongs to Λ by assumption. Proposition 1, in turn, allows us to establish Lemma 5, which implies that in all cases $\Omega^*_{\Phi_i} \in \Lambda$. This then completes the proofs of Theorems 1 and 2 in the general setting.

4.1. Proof of the FTAP and Pricing–Hedging Duality When No Options Are Statically Traded

Proof of Theorem 1 (When No Options Are Statically Traded). In this case, we consider $\Omega \in \Lambda$, the technical filtration as described in Remark 10, and the arbitrage aggregator H^* defined by Equation (17). We prove that

$$\exists \text{ Strong Arbitrage on } \Omega \text{ in } \mathcal{H}(\widetilde{\mathbb{F}}) \Leftrightarrow \mathcal{M}^f_\Omega = \emptyset.$$

Notice that if $H \in \mathcal{H}(\widetilde{\mathbb{F}})$ satisfies $(H \circ S)_T(\omega) > 0\ \forall \omega \in \Omega$, then if there exists $Q \in \mathcal{M}^f_\Omega$, we would get $0 < \mathbb{E}_Q[(H \circ S)_T] = 0$, which is a contradiction. For the opposite implication, let H^* be the arbitrage aggregator from Equation (17), and note that $(H^* \circ S)_T(\omega) \ge 0\ \forall \omega \in \Omega$ and $\{\omega \mid (H^* \circ S)_T(\omega) > 0\} = (\Omega^*)^c$. If $\mathcal{M}^f_\Omega = \emptyset$, then by Lemma 2, $(\Omega^*)^c = \Omega$, and H^* is therefore a strong arbitrage on Ω in $\mathcal{H}(\widetilde{\mathbb{F}})$. The last assertion, namely $\Omega^* = \{\omega \in \Omega \mid (H^* \circ S)_T(\omega) = 0\}$, follows straightforwardly from the definition of H^*. ◻

Proposition 3 (Superhedging on $\Omega \subseteq X$ Without Options). *Let $\Omega \in \Lambda$. We have that for any $g \in \mathcal{L}(X, \mathcal{F}^{\mathcal{A}}; \mathbb{R})$*

$$\pi_{\Omega^*}(g) = \sup_{Q \in \mathcal{M}^f_\Omega} E_Q[g], \tag{22}$$

with $\pi_{\Omega^*}(g) = \inf\{x \in \mathbb{R} \mid \exists H \in \mathcal{H}(\mathbb{F}^{\text{pr}})$ such that $x + (H \circ S)_T \ge g$ on $\Omega^*\}$. In particular, the left-hand side of Equation (22) is attained by some strategy $H \in \mathcal{H}(\mathbb{F}^{\text{pr}})$.

Proof. Note that by its definition in Equation (4), $\Omega^* = \emptyset$ if and only if $\mathcal{M}^f_\Omega = \emptyset$, and in this case, both sides in Equation (22) are equal to $-\infty$. We assume now that $\Omega^* \neq \emptyset$ and recall from Lemma 2 that we have $\Omega^* \in \Lambda$. By definition, there exists $n \in \mathbb{N}$ such that $\Omega^* \in \Sigma^1_n$. The second part of the statement follows with the same procedure proposed in the Burzoni et al. [11] proof of theorem 1.1. The reason can be easily understood recalling the following construction, which appears in step 1 of the proof. For any $\ell \in \mathbb{N}$, $D \in \mathcal{F}^{\mathrm{pr},\ell}$, $1 \le t \le T$, $G \in \mathcal{L}(X, \mathcal{F}^{\mathrm{pr},\ell})$, we define the multifunction

$$\psi_{t,G,D} : \omega \mapsto \{[\Delta S_t(\widetilde{\omega}); 1; G(\widetilde{\omega})]\mathbf{1}_D(\widetilde{\omega}) \mid \widetilde{\omega} \in \Sigma_{t-1}^{\omega}\} \subseteq \mathbb{R}^{d+2},$$

where $[\Delta S_t; 1; G]\mathbf{1}_D = [\Delta S_t^1 \mathbf{1}_D, \dots, \Delta S_t^d \mathbf{1}_D, \mathbf{1}_D, G\mathbf{1}_D]$ and Σ_t^{ω} is given as in Equation (10). We show that $\psi_{t,G,D}$ is an $\mathcal{F}_{t-1}^{\mathrm{pr},\ell+1}$-measurable multifunction. Let $O \subseteq \mathbb{R}^d \times \mathbb{R}^2$ be an open set and observe that

$$\{\omega \in X \mid \psi_{t,G,D}(\omega) \cap O \neq \emptyset\} = (S_{0:t-1}, Y_{0:t-1})^{-1}((S_{0:t-1}, Y_{0:t-1})(B)),$$

where $B = ([\Delta S_t; 1; G]\mathbf{1}_D)^{-1}(O)$. First, $[\Delta S_t, 1, G]\mathbf{1}_D$ is an $\mathcal{F}^{\mathrm{pr},\ell}$-measurable random vector; then $B \in \mathcal{F}^{\mathrm{pr},\ell}$, the sigma-algebra generated by the ℓ-projective sets of X. Second, (S_u, Y_u) is a Borel-measurable function for any $0 \le u \le t-1$ so that we have, as a consequence of Lemma A.2, that $(S_{0:t-1}, Y_{0:t-1})(B)$ belongs to the sigma-algebra generated by the $(\ell+1)$-projective sets in $Mat((d+\tilde{d}) \times t; \mathbb{R})$ (the space of $(d+\tilde{d}) \times t$ matrices with real entries) endowed with its Borel sigma-algebra. Applying again Lemma A.2, we deduce that $(S_{0:t-1}, Y_{0:t-1})^{-1}((S_{0:t-1}, Y_{0:t-1})(B)) \in \mathcal{F}_{t-1}^{\mathrm{pr},\ell+1}$ and, hence, the desired measurability for $\psi_{t,G,D}$.

The remaining steps 1–5 follow, replicating the argument in Burzoni et al. [11]. □

4.2. Proof of the FTAP and Pricing–Hedging Duality with Statically Traded Options

We first extend the results from Lemma 2 to the present case of nontrivial Φ.

Lemma 4. *Let Ω be analytic. For any $Q \in \mathcal{M}_{\Omega,\Phi}$, we have $Q(\Omega_\Phi^*) = 1$. In particular, $\mathcal{M}_{\Omega,\Phi} \neq \emptyset$ if and only if $\mathcal{M}_{\Omega,\Phi}^f \neq \emptyset$.*

Proof. Recall that Ω analytic implies that Ω_Φ^* is analytic from remark 5.6 in Burzoni et al. [11]. Let $\tilde{Q} \in \mathcal{M}_{\Omega,\Phi}$ and consider the extended market $(S, \tilde{S})$ with $\tilde{S}_t^j$ equal to a Borel-measurable version of $E_{\tilde{Q}}[\phi_j \mid \mathcal{F}_t^S]$ for any $j = 1, \dots, k$ and $t \in \mathbb{T}$ (see lemma 7.27 in Bertsekas and Shreve [6]). In particular, $\tilde{Q} \in \tilde{\mathcal{M}}_\Omega$, the set of martingale measures for $(S, \tilde{S})$, which are concentrated on Ω. Denote by $\tilde{\Omega}^*$ the set of scenarios charged by martingale measures for $(S, \tilde{S})$ as defined in Equation (4). From Corollary 2 and Lemma 2, we deduce that $\tilde{Q}(\tilde{\Omega}^*) = 1$. Because, obviously, $\tilde{\mathcal{M}}_\Omega^f \subseteq \mathcal{M}_{\Omega,\Phi}^f$, we also have $\tilde{\Omega}^* \subseteq \Omega_\Phi^*$. Because the former has full probability, the claim follows. □

As highlighted, we start with the proof of Theorem 2 under the further assumption that $\Omega_{\Phi_n}^* \in \Lambda$ for all $1 \le n \le k$, where $\Phi_n = \{\phi_1, \dots, \phi_n\}$. This assumption is then shown to hold at the end of this subsection.

Proof of Theorem 2 (Under the Assumption $\Omega_{\Phi_n}^* \in \Lambda$ for All $n \le k$). Similarly to the proof of Proposition 3, we note that the statement is clear when $\mathcal{M}_{\Omega,\Phi}^f = \emptyset$, so we may assume the contrary. For any $\mathcal{F}^{\mathcal{A}}$-measurable g, standard arguments imply

$$\sup_{Q \in \mathcal{M}_{\Omega,\Phi}^f} E_Q[g] \le \pi_{\Omega_\Phi^*,\Phi}(g),$$

so that it remains to show the converse inequality. We prove the statement by induction on the number of static options used for superhedging. For this, we consider the superhedging problem with additional options Φ_n on Ω_Φ^* and denote its superhedging cost by $\pi_{\Omega_\Phi^*,\Phi_n}(g)$, which is defined as in Equation (6) but with Φ_n replacing Φ.

Assume that $\Omega_{\Phi_n}^* \in \Lambda$ for all $n \le k$. The case $n = 0$ corresponds to the superhedging problem on Ω^* when only dynamic trading is possible. Because, by assumption $\Omega_\Phi^* \in \Lambda$, the pricing–hedging duality and the attainment of the infimum follow from Proposition 3. Now assume that for some $n < k$, for any $\mathcal{F}^{\mathcal{A}}$-measurable g, we have the following pricing–hedging duality:

$$\pi_{\Omega_\Phi^*,\Phi_n}(g) = \sup_{Q \in \mathcal{M}_{\Omega_\Phi^*,\Phi_n}^f} E_Q[g]. \tag{23}$$

We show that the same statement holds for $n+1$. Note that the attainment property is always satisfied. Indeed, using the notation of Bouchard and Nutz [8], we have $NA(\mathcal{M}_{\Omega_\Phi^*,\Phi_n}^f)$. As a consequence of theorem 2.3 in Bouchard and Nutz [8], which holds also in the setup of this paper, the infimum is attained whenever it is finite.

The proof proceeds in three steps.

Step 1. First observe that if ϕ_{n+1} is replicable on Ω_Φ^* by semistatic portfolios with the static hedging part restricted to Φ_n, that is, $x + h \cdot \Phi_n(\omega) + (H \circ S)_T(\omega) = \phi_{n+1}(\omega)$ for any $\omega \in \Omega_\Phi^*$, then necessarily $x = 0$ (otherwise, $\mathcal{M}_{\Omega,\Phi}^f = \emptyset$). Moreover, because any such portfolio has zero expectation under measures in $\mathcal{M}_{\Omega_\Phi^*,\Phi_n}^f$, we have that $E_Q[\phi_{n+1}] = 0 \;\; \forall Q \in \mathcal{M}_{\Omega_\Phi^*,\Phi_n}^f$. In particular, $\mathcal{M}_{\Omega_\Phi^*,\Phi_n}^f = \mathcal{M}_{\Omega_\Phi^*,\Phi_{n+1}}^f$, and Equation (23) holds for $n+1$.

Step 2. We now look at the more interesting case; that is, ϕ_{n+1} is not replicable. In this case, we show that

$$\sup_{Q\in\mathcal{M}^f_{\Omega^*_\Phi,\Phi_n}} E_Q[\phi_{n+1}]>0 \quad \text{and} \quad \inf_{Q\in\mathcal{M}^f_{\Omega^*_\Phi,\Phi_n}} E_Q[\phi_{n+1}]<0. \tag{24}$$

Inequalities $\geq$ and $\leq$ are obvious from the assumption $\mathcal{M}^f_{\Omega^*_\Phi,\Phi}\neq\emptyset$. From the inductive hypothesis, we only need to show that $\pi_{\Omega^*_\Phi,\Phi_n}(\phi_{n+1})$ is always strictly positive (the analogous argument applies to $\pi_{\Omega^*_\Phi,\Phi_n}(-\phi_{n+1})$). Suppose, by contradiction, that $\pi_{\Omega^*_\Phi,\Phi_n}(\phi_{n+1})=0$. Because the infimum is attained, there exists some $(\alpha,H)\in\mathbb{R}^n\times\mathcal{H}(\mathbb{F}^{\mathrm{pr}})$ such that

$$\alpha\cdot\Phi_n(\omega)+(H\circ S)_T(\omega)\geq\phi_{n+1}(\omega)\quad\forall\omega\in\Omega^*_\Phi.$$

Because ϕ_{n+1} is not replicable, this inequality is strict for some $\tilde\omega\in\Omega^*_\Phi$. Then, by taking expectation under $\tilde Q\in\mathcal{M}^f_{\Omega^*_\Phi,\Phi}$ such that $\tilde Q(\{\tilde\omega\})>0$, we obtain

$$0=E_{\tilde Q}[\alpha\cdot\Phi_n+(H\circ S)_T]>E_{\tilde Q}[\phi_{n+1}]=0, \tag{25}$$

which is clearly a contradiction.

Step 3. Given Equation (24), we now show that Equation (23) holds for $n+1$ also in the case that ϕ_{n+1} is not replicable. We first use a variational argument to deduce the following equalities:

$$\begin{aligned}\pi_{\Omega^*_\Phi,\Phi_{n+1}}(g) &= \inf_{l\in\mathbb{R}}\pi_{\Omega^*_\Phi,\Phi_n}(g-l\phi_{n+1})\\ &= \inf_{l\in\mathbb{R}}\sup_{Q\in\mathcal{M}^f_{\Omega^*_\Phi,\Phi_n}} E_Q[g-l\phi_{n+1}]\\ &= \inf_N\inf_{|l|\leq N}\sup_{Q\in\mathcal{M}^f_{\Omega^*_\Phi,\Phi_n}} E_Q[g-l\phi_{n+1}]\\ &= \inf_N\sup_{Q\in\mathcal{M}^f_{\Omega^*_\Phi,\Phi_n}}\inf_{|l|\leq N} E_Q[g-l\phi_{n+1}]\\ &= \inf_N\sup_{Q\in\mathcal{M}^f_{\Omega^*_\Phi,\Phi_n}} (E_Q[g]-N|E_Q[\phi_{n+1}]|).\end{aligned} \tag{26}$$

The first equality follows by definition, the second from the inductive hypothesis, the fourth is obtained with an application of min–max theorem (see corollary 2 in Terkelsen [49]), and the last one follows from an easy calculation.

We also observe that there exist $Q_{\sup}\in\mathcal{M}^f_{\Omega^*_\Phi,\Phi_n}$ and $Q_{\inf}\in\mathcal{M}^f_{\Omega^*_\Phi,\Phi_n}$ such that

$$E_{Q_{\sup}}[\phi_{n+1}]\geq\frac{1}{2}\left(\pi_{\Omega^*_\Phi,\Phi_n}(\phi_{n+1})\wedge 1\right)\quad\text{and}\quad E_{Q_{\inf}}[\phi_{n+1}]\leq-\frac{1}{2}\left(\pi_{\Omega^*_\Phi,\Phi_n}(-\phi_{n+1})\wedge 1\right).$$

From Equation (24) and the inductive hypothesis, $E_{Q_{\inf}}[\phi_{n+1}]<0<E_{Q_{\sup}}[\phi_{n+1}]$. We later use $Q_{\inf}$ and $Q_{\sup}$ for calibrating measures in $\mathcal{M}^f_{\Omega^*_\Phi,\Phi_n}$ to the additional option ϕ_{n+1}. Namely, for $Q\in\mathcal{M}^f_{\Omega^*_\Phi,\Phi_n}$, we might set $\tilde Q=Q_{\inf}$ if $E_Q[\phi_{n+1}]\geq 0$ and $Q_{\sup}$ otherwise to find $\lambda\in[0,1]$ such that

$$\hat Q=\lambda Q+(1-\lambda)\tilde Q\in\mathcal{M}^f_{\Omega^*_\Phi,\Phi_{n+1}}.$$

We can now distinguish two cases:

Case 1. Suppose first there exists a sequence $\{Q_m\}\subseteq\mathcal{M}^f_{\Omega^*_\Phi,\Phi_n}\setminus\mathcal{M}^f_{\Omega^*_\Phi,\Phi_{n+1}}$ such that

$$\lim_{m\to\infty}\frac{E_{Q_m}[g]}{|E_{Q_m}[\phi_{n+1}]|}=+\infty\quad\text{and}\quad\lim_{m\to\infty}E_{Q_m}[g]=+\infty. \tag{27}$$

Given $\{Q_m\}$ such that Equation (27) is satisfied, we can construct a sequence of calibrated measures $\{\hat Q_m\}\subseteq\mathcal{M}^f_{\Omega^*_\Phi,\Phi_{n+1}}$, as described, so that

$$E_{\hat Q_m}[\phi_{n+1}]=\lambda_m E_{Q_m}[\phi_{n+1}]+(1-\lambda_m)E_{\tilde Q_m}[\phi_{n+1}]=0,$$

for some $\{\lambda_m\} \subseteq [0,1]$. We stress that $\tilde{Q}_m$ can only be equal to $Q_{\inf}$ or $Q_{\sup}$, which do not depend on m. A simple calculation shows

$$\lambda_m = \frac{E_{\tilde{Q}_m}[\phi_{n+1}]}{E_{\tilde{Q}_m}[\phi_{n+1}] - E_{Q_m}[\phi_{n+1}]}.$$

From

$$E_{\hat{Q}_m}[g] = \lambda_m(E_{Q_m}[g] - E_{\tilde{Q}_m}[g]) + E_{\tilde{Q}_m}[g]$$

we have two cases: either $\lambda_m \to a > 0$ and from $E_{Q_m}[g] \to +\infty$ we deduce $E_{\hat{Q}_m}[g] \to +\infty$ or $\lambda_m \to 0$, which happens when $|E_{Q_m}[\phi_{n+1}]| \to \infty$. Nevertheless, in such a case, from Equation (27) we obtain again $E_{\hat{Q}_m}[g] \to +\infty$ as $m \to \infty$. Therefore, $\infty = \sup_{Q\in\mathcal{M}^f_{\Omega^*_\Phi,\Phi_{n+1}}} E_Q[g] \le \pi_{\Omega^*_\Phi,\Phi_{n+1}}(g)$ and, hence, the duality.

Case 2. We are only left with the case in which Equation (27) is not satisfied. For any $N \in \mathbb{N}$, we define the decreasing sequence $s_N := \sup_{Q\in\mathcal{M}^f_{\Omega^*_\Phi,\Phi_n}} (E_Q[g] - N|E_Q[\phi_{n+1}]|)$ and let $\{Q^m_N\}_{m\in\mathbb{N}}$, a sequence realizing the supremum. If there exists a subsequence s_{N_j} such that $|E_{Q^m_{N_j}}[\phi_{n+1}]| = 0$ for $m > \bar{m}(N_j)$, then the duality follows directly from Equation (26). Suppose this is not the case. We claim that we can find a sequence $\{Q_N\} \subseteq \mathcal{M}^f_{\Omega^*_\Phi,\Phi_n}$ such that

$$\lim_{N\to\infty}(E_{Q_N}[g] - N|E_{Q_N}[\phi_{n+1}]|) = \lim_{N\to\infty} s_N \qquad \text{and} \qquad \lim_{N\to\infty} |E_{Q_N}[\phi_{n+1}]| = 0. \tag{28}$$

Let indeed $c(N) := \limsup_{m\to\infty} E_{Q^m_N}[g]/|E_{Q^m_N}[\phi_{n+1}]|$. If $\sup_{N\in\mathbb{N}} c(N) = \infty$, because Equation (27) is not satisfied, there exists $m = m(N)$ such that $|E_{Q^{m(N)}_N}[\phi_{n+1}]|$ converges to zero as $N \to \infty$, from which the claim easily follows. Suppose now $\sup_{N\in\mathbb{N}} c(N) < \infty$. Then (by taking subsequences if needed),

$$s_N = \lim_{m\to\infty}(E_{Q^m_N}[g] - N|E_{Q^m_N}[\phi_{n+1}]|) \le (c(N) - N) \lim_{m\to\infty} |E_{Q^m_N}[\phi_{n+1}]|.$$

Note that $a_N := \lim_{m\to\infty} |E_{Q^m_N}[\phi_{n+1}]|$ satisfies $\lim_{N\to\infty} Na_N < \infty$; otherwise, from Equation (26), $\pi_{\Omega^*_\Phi,\Phi_{n+1}}(g) = -\infty$, which is not possible because, for any $Q \in \mathcal{M}^f_{\Omega^*_\Phi,\Phi}$, we have that $\pi_{\Omega^*_\Phi,\Phi_{n+1}}(g) \ge E_Q[g] > -\infty$. In particular, $\lim_{N\to\infty} a_N = 0$, and the claim easily follows.

Given a sequence as in Equation (28), we now conclude the proof. It follows from Equation (26) that

$$\begin{aligned}
\pi_{\Omega^*_\Phi,\Phi_{n+1}}(g) &= \inf_N \sup_{Q\in\mathcal{M}^f_{\Omega^*_\Phi,\Phi_n}} \inf_{|l|\le N} E_Q[g - l\phi_{n+1}] \\
&= \lim_{N\to\infty} E_{Q_N}[g] - N|E_{Q_N}[\phi_{n+1}]| \\
&\le \lim_{N\to\infty} E_{Q_N}[g].
\end{aligned}$$

The calibrating procedure described yields $\lambda_N \in [0,1]$ such that $\hat{Q}_N = \lambda_N Q_N + (1-\lambda_N)\tilde{Q}_N \in \mathcal{M}^f_{\Omega^*_\Phi,\Phi_{n+1}}$. Moreover, because $|E_{Q_N}[\phi_{n+1}]| \to 0$ and $\tilde{Q}_N$ can only be either $Q_{\inf}$ or $Q_{\sup}$, these λ_N satisfy $\lambda_N \to 1$. This implies $E_{\hat{Q}_N}[g] - E_{Q_N}[g] \to 0$ as $N \to \infty$, from which it follows

$$\pi_{\Omega^*_\Phi,\Phi_{n+1}}(g) \le \lim_{N\to\infty} E_{Q_N}[g] = \lim_{N\to\infty} E_{\hat{Q}_N}[g] \le \sup_{Q\in\mathcal{M}^f_{\Omega^*_\Phi,\Phi_{n+1}}} E_Q[g].$$

The converse inequality follows from standard arguments, and hence, we obtain $\pi_{\Omega^*_\Phi,\Phi_{n+1}}(g) = \sup_{Q\in\mathcal{M}^f_{\Omega^*_\Phi,\Phi_{n+1}}} E_Q[g]$ as required. □

We now prove Proposition 1 for the more general case of $\Omega \in \Lambda$. We use Theorem 2 only when $\Omega^*_\Phi = \Omega$, which belongs to Λ by assumption.

Proof of Proposition 1. The $\Leftarrow$ implication is clear because, if a strategy $(\alpha, H) \in \mathcal{A}_\Phi(\mathbb{F}^{\text{pr}})$ satisfies $\alpha\cdot\Phi + (H\circ S)_T \ge 0$ on Ω, then, by definition in Equation (4), for any $\omega \in \Omega^*_\Phi = \Omega$, we can take a calibrated martingale measure that assigns a positive probability to ω, which implies $\alpha\cdot\Phi + (H\circ S)_T = 0$ on ω. Because ω is arbitrary, we obtain the thesis. We prove $\Rightarrow$ by iteration on the number of options used for static trading. No one-point arbitrage using dynamic trading and Φ in particular means that there is no one-point arbitrage using only dynamic trading. From Lemma 3, we have $\Omega^* = \Omega$, and hence, for any $\omega \in \Omega$, there exists $Q \in \mathcal{M}^f_\Omega$ such that $Q(\{\omega\}) > 0$.

Note that if, for some $j \le k$, ϕ_j is replicable on Ω^* by dynamic trading in S, then there exist $n \in \mathbb{N}$ and $(x, H) \in \mathbb{R} \times \mathcal{H}(\mathbb{F}^{\mathrm{pr},n})$ such that $x + (H \circ S)_T = \phi_j$ on Ω^*. No one-point arbitrage implies $x = 0$, and hence, $E_Q[\phi_j] = 0$ for every $Q \in \mathcal{M}^f_\Omega$. With no loss of generality, we assume that $(\phi_1, \ldots, \phi_{k_1})$ is a vector of nonreplicable options on Ω^* with $k_1 \le k$. We now apply Theorem 2 in the case with $\Phi = 0$ to ϕ_1 and argue that

$$m_1 := \min\{\pi_{\Omega^*}(\phi_1), \pi_{\Omega^*}(-\phi_1)\} > 0.$$

Indeed, if $m_1 < 0$, then we would have a strong arbitrage, and if $m_1 = 0$, because the superhedging price is attained, there exists $H \in \mathcal{H}(\mathbb{F}^{\mathrm{pr}})$ such that, for example, $\phi_1 \le (H \circ S)_T$ on Ω. To avoid one-point arbitrage, we have to have $\phi_1 = (H \circ S)_T$ on Ω, which is a contradiction because ϕ_1 is not replicable. This shows that $m_1 > 0$, which, in turn, implies there exist $Q_1, Q_2 \in \mathcal{M}^f_\Omega$ such that $E_{Q_1}[\phi_1] > 0$ and $E_{Q_2}[\phi_1] < 0$. Then, for any $Q \in \mathcal{M}^f_\Omega$, there exist $\alpha, \beta, \gamma \in [0,1]$, $\alpha + \beta + \gamma = 1$, and $E_{\alpha Q_1 + \beta Q_2 + \gamma Q}[\phi_1] = 0$. Thus, for any $\omega \in \Omega^*$, there exists $Q \in \mathcal{M}^f_{\Omega,\phi_1}$ such that $Q(\{\omega\}) > 0$. In particular, $\Omega^*_{\phi_1} = \Omega$, and we may apply Theorem 2 with Ω and $\Phi = \{\phi_1\}$ (indeed $\Omega \in \Lambda$, and we can therefore apply the version of Theorem 2 proved in this section). Define now

$$m_{1,j} := \min\{\pi_{\Omega^*,\phi_1,}(\phi_j), \pi_{\Omega^*,\phi_1}(-\phi_j)\} \quad \forall j = 2, \ldots, k_1.$$

By absence of strong arbitrage, we necessarily have $m_{1,j} \ge 0$ for every $j = 2, \ldots, k_1$. Let $j \in I_2 = \{j = 2, \ldots, k_1 \mid m_{1,j} = 0\}$; by no one-point arbitrage, we have perfect replication of ϕ_j using semistatic strategies with ϕ_1 on Ω, and in consequence for any $Q \in \mathcal{M}^f_{\Omega,\phi_1}$, we have $E_Q[\phi_j] = 0$ for all $j \in I_2$. We may discard these options and, up to renumbering, assume that $(\phi_2, \ldots, \phi_{k_2})$ is a vector of the remaining options, nonreplicable on Ω with semistatic trading in ϕ_1 with $k_2 \le k_1$. If $k_2 \ge 2$, $m_{1,2} > 0$ by Theorem 2 and absence of one-point arbitrage using arguments as earlier. Hence, there exist $Q_1, Q_2 \in \mathcal{M}^f_{\Omega,\phi_1}$ such that $E_{Q_1}[\phi_2] > 0$ and $E_{Q_2}[\phi_2] < 0$. As before, this implies that $\Omega^*_{\{\phi_1,\phi_2\}} = \Omega^*_{\phi_1} = \Omega$. We can iterate the previous arguments, and the procedure ends after at most k steps showing $\Omega^*_\Phi = \Omega$ as required. □

The following lemma shows that the outcome of a successful partition scheme is the set Ω^*_Φ.

Lemma 5. *Recall the definition of Ω^*_Φ in Equation (4). For any $\mathcal{R}(\alpha^\star, H^\star)$, $A^*_i = \Omega^*_{\{\alpha^j \cdot \Phi : j \le i\}}$ for any $i \le \beta$. Moreover, if $\mathcal{R}(\alpha^\star, H^\star)$ is successful, then $A^*_\beta = \Omega^*_\Phi$.*

Proof. If $\Omega^* = \emptyset$, then the claim holds trivial. We now assume $\Omega^* \ne \emptyset$, fix a partition scheme $\mathcal{R}(\alpha^\star, H^\star)$, and prove the claim by induction on i. For simplicity of notation, let $\Omega^*_i := \Omega^*_{\{\alpha^j \cdot \Phi : j \le i\}}$ with $\Omega_0 = \Omega$. By definition of A_0, we have $A^*_0 = \Omega^* = \Omega^*_0$. Suppose now $A^*_{i-1} = \Omega^*_{i-1}$ for some $i \le \beta$. Then, by definition of Ω_i, we have $\Omega^*_i \subseteq \Omega^*_{i-1} = A^*_{i-1}$. Further, because $(H_i \circ S)_T + \alpha^i \cdot \Phi \ge 0$ on A^*_{i-1} with strict inequality on $A^*_{i-1} \setminus A_i$, it follows that $\Omega^*_i \subseteq A_i$. Finally, from $\mathcal{M}^f_{\Omega^*_i, \{\alpha^j \cdot \Phi : j \le i\}} \subseteq \mathcal{M}^f_{A_i, \{\alpha^j \cdot \Phi : j \le i\}} \subseteq \mathcal{M}^f_{A_i} = \mathcal{M}^f_{A^*_i}$, we also have $\Omega^*_i \subseteq A^*_i$. For the reverse inclusion, consider $\omega \in A^*_i$. By definition of A^*_i and Lemma 2, there exists $Q \in \mathcal{M}^f_{A^*_i}$ with $Q(\{\omega\}) > 0$. Because on A^*_i all options $\alpha^j \cdot \Phi$, $1 \le j \le i$, are perfectly replicated by the dynamic strategies $-H^j$, it follows that $Q \in \mathcal{M}^f_{A^*_i, \{\alpha^j \cdot \Phi : j \le i\}}$ so that $\omega \in \Omega^*_i$.

Suppose now $\mathcal{R}(\alpha^\star, H^\star)$ is successful. In the case $\beta = k$, because α^i forms a basis of $\mathbb{R}^k$, we have $\mathcal{M}^f_{A^*_\beta, \Phi} = \mathcal{M}^f_{A^*_\beta, \{\alpha^j \cdot \Phi : j \le \beta\}}$, and hence, $\Omega^*_\Phi = A^*_\beta$ from earlier. Suppose $\beta < k$ so that the earlier notation shows $\Omega^*_\Phi \subseteq \Omega^*_\beta = A^*_\beta$. Observe that because $A_\beta \in \Lambda$ from Lemma 2, we have $A^*_\beta \in \Lambda$; moreover, by Remark 12, there are no one-point arbitrages on A^*_β. Thus, each $\omega \in A^*_\beta$ is weighted by some $Q \in \mathcal{M}^f_{A^*_\beta, \Phi} \subseteq \mathcal{M}^f_{\Omega,\Phi}$ by Proposition 1 applied to A^*_β. Therefore, $A^*_\beta \subseteq \Omega^*_\Phi$, which concludes the proof. □

Remark 14. It follows from Lemma 2 that $A^*_i \equiv \Omega^*_{\{\alpha^j \cdot \Phi : j \le i\}}$, introduced in Lemma 5, belongs to Λ for any $i \le \beta$. In particular, Ω^*_Φ in Equation (4) is in Λ (see also the discussion after Definition 4).

Remark 15. Observe that in the proof of Lemma 5, we apply Proposition 1 with $\Omega = A^*_\beta$, which is only known to belong to Λ. For this reason, we need Proposition 1 to hold for a generic set in Λ (and not only analytic).

Proof of Theorem 1. We now prove the pointwise fundamental theorem of asset pricing when semistatic trading strategies in a finite number of options are allowed. Let Ω be analytic, $\mathcal{R}(\alpha^\star, H^\star)$ be a path-space partition scheme, and $\widetilde{\mathbb{F}}$ be given by Equation (21). We first show that the following are equivalent:

1. $\mathcal{R}(\alpha^\star, H^\star)$ is successful.
2. $\mathcal{M}^f_{\Omega,\Phi} \ne \emptyset$.
3. $\mathcal{M}_{\Omega,\Phi} \ne \emptyset$.
4. No strong arbitrage with respect to $\widetilde{\mathbb{F}}$.

(1) ⇒ (2) follows from Remark 13 and the definition of Ω^*_Φ in (4) because $A^*_\beta = \Omega^*_\Phi$. (2) ⇔ (3) follows from Lemma 4. To show (2) ⇒ (4), observe that under $Q \in \mathcal{M}^f_{\Omega,\Phi}$ the expectation of any admissible semistatic trading strategy is zero, which excludes the possibility of existence of a strong arbitrage. For the implication (4) ⇒ (1) note that, for $1 \le i \le \beta$, we have $(\tilde{H}^{i-1} \circ S)_T > 0$ on $A_{i-1} \setminus A^*_{i-1}$ from the properties of the arbitrage aggregator (see Remark 8) and $(H^i \circ S)_T + \alpha^i \cdot \Phi > 0$ on $A^*_{i-1} \setminus A_i$ by construction so that a positive gain is realized on $A_{i-1} \setminus A_i$. Finally, from $\Omega = A_0 = (\cup_{i=1}^{\beta} A_{i-1} \setminus A_i) \cup A_\beta$ and $(\tilde{H}^\beta \circ S)_T > 0$ on $A_\beta \setminus A^*_\beta$, we get

$$\sum_{i=1}^{\beta} (H^i \circ S)_T + \alpha^i \cdot \Phi + \sum_{i=0}^{\beta} (\tilde{H}^i \circ S)_T > 0 \quad \text{on } (A^*_\beta)^C , \tag{29}$$

and equal to zero otherwise. The hypothesis (4) implies, therefore, that A^*_β is nonempty, and hence, the path-space partition scheme is successful.

The existence of the technical filtration and arbitrage aggregator are provided explicitly by Equations (21) and (20). Moreover, from Lemma 5, $\Omega^*_\Phi = A^*_\beta$. Finally, Equation (5) follows from Equation (29). □

Proof of Corollary 1. Let $\widetilde{\mathbb{F}}$ be given by Equation (21). We prove that

$$\exists \text{ an Arbitrage de la classe } \mathcal{S} \text{ in } \mathcal{A}_\Phi(\widetilde{\mathbb{F}}) \iff \mathcal{M}^f_{\Omega,\Phi} = \emptyset \text{ or } \mathcal{N}^M \text{ contains sets of } \mathcal{S}.$$

(⇒): Let $(\alpha, H) \in \mathcal{A}_\Phi(\widetilde{\mathbb{F}})$ be an arbitrage de la classe $\mathcal{S}$. By definition, $\alpha \cdot \Phi + (H \circ S)_T \ge 0$ on Ω, and there exists $A \in \mathcal{S}$ such that $A \subseteq \{\omega \in \Omega \mid \alpha \cdot \Phi + (H \circ S)_T > 0\}$. Note now that for any $Q \in \mathcal{M}^f_{\Omega,\Phi}$ we have $E_Q[\alpha \cdot \Phi + (H \circ S)_T] = 0$, which implies $Q(\{\omega \in \Omega \mid \alpha \cdot \Phi + (H \circ S)_T > 0\}) = 0$. Thus, if $\{\omega \in \Omega \mid \alpha \cdot \Phi + (H \circ S)_T > 0\} = \Omega$, then $\mathcal{M}^f_{\Omega,\Phi} = \emptyset$; otherwise, $A \in \mathcal{N}^M \cap \mathcal{S}$.

(⇐): Consider the arbitrage aggregator (α^*, H^*) as constructed in Equation (21), which is predictable with respect to $\widetilde{\mathbb{F}}$ given by Equation (21). Let $A \in \mathcal{N}^M \cap \mathcal{S}$; then, from (5) in Theorem 1, $A \subseteq \{\omega \in \Omega \mid \alpha^* \cdot \Phi + (H^* \circ S)_T > 0\}$, which implies the thesis. □

Proof of Theorem 2 (Justification of the Assumption Previously Made). As a consequence of Lemma 5 (see also Remark 14) we obtain $\Omega^*_{\Phi_n} \in \Lambda$ for all $n \le k$. Therefore, the assumption made in the proof of Theorem 2 at the beginning of this subsection is always satisfied. Moreover, for Ω analytic, the equality between the suprema over $\mathcal{M}_{\Omega,\Phi}$ and over $\mathcal{M}^f_{\Omega,\Phi}$ may be deduced following the same arguments as in the proof of theorem 1.1, step 2, in Burzoni et al. [11]. The proof is complete.

4.3. Proof of Theorem 3

We recall that the option ϕ_0 can be only bought at time $t = 0$, and the notations are as follows: $\widetilde{\mathcal{A}}_{\phi_0}(\mathbb{F}^{\mathrm{pr}}) := \{(\alpha, H) \in \mathbb{R}_+ \times \mathcal{H}(\mathbb{F}^{\mathrm{pr}})\}$ and $\widetilde{\mathcal{M}}_{\Omega,\phi_0} := \{Q \in \mathcal{M}_\Omega \mid E_Q[\phi_0] \le 0\}$.

We first extend the results of Theorem 2 to the case in which only ϕ_0 is available for static trading.

Lemma 6. *Suppose $\widetilde{\mathcal{M}}^f_{\Omega,\phi_0} \neq \emptyset$, $\pi_{\Omega^*}(\phi_0) > 0$, and $\Omega^*_{\phi_0} \in \mathcal{F}^{\mathcal{A}}$. Then, for any $\mathcal{F}^{\mathcal{A}}$-measurable g,*

$$\pi_{\Omega^*_{\phi_0},\phi_0}(g) = \sup_{Q \in \widetilde{\mathcal{M}}^f_{\Omega,\phi_0}} E_Q[g] = \sup_{Q \in M^f_{\Omega,\phi_0}} E_Q[g].$$

Proof. The assumption $\pi_{\Omega^*}(\phi_0) > 0$ automatically implies $\sup_{Q \in \mathcal{M}^f_\Omega} E_Q[\phi_0] > 0$. Moreover, by assumption, $\widetilde{\mathcal{M}}^f_{\Omega,\phi_0} \neq \emptyset$, from which $\inf_{Q \in \mathcal{M}^f_\Omega} E_Q[\phi_0] \le 0$.

The idea of the proof is the same as that of Theorem 2. Suppose first that

$$\inf_{Q \in \mathcal{M}^f_\Omega} E_Q[\phi_0] < 0 . \tag{30}$$

Then it is easy to see that $\Omega^*_{\phi_0} = \Omega^*$. We use a variational argument to deduce the following equality:

$$\pi_{\Omega^*_{\phi_0},\phi_0}(g) = \pi_{\Omega^*,\phi_0}(g) = \inf_N \sup_{Q \in \mathcal{M}^f_\Omega} (E_Q[g] - N|E_Q[\phi_0]|)$$

obtained with an application of min–max theorem (see corollary 2 in Terkelsen [49]). The last step of the proof of Theorem 2 is only based on this variational equality and is analogous of Equation (30) joined with $\sup_{Q\in\mathcal{M}^f_\Omega} E_Q[\phi_0]>0$. By repeating the same argument, we obtain $\pi_{\Omega^*_{\phi_0},\phi_0}(g)=\sup_{Q\in\mathcal{M}^f_{\Omega,\phi_0}} E_Q[g]$. Because obviously

$$\pi_{\Omega^*_{\phi_0},\phi_0}(g)=\sup_{Q\in\mathcal{M}^f_{\Omega,\phi_0}} E_Q[g]\le\sup_{Q\in\widetilde{\mathcal{M}}^f_{\Omega,\phi_0}} E_Q[g]\le\pi_{\Omega^*_{\phi_0},\phi_0}(g), \tag{31}$$

we have the thesis.

Suppose now that $\inf_{Q\in\mathcal{M}^f_\Omega} E_Q[\phi_0]=0$. From Proposition 3, $\pi_{\Omega^*}(-\phi_0)=0$, and there exists a strategy $\bar H\in\mathcal{H}(\mathbb{F}^{\mathrm{pr}})$ such that $(\bar H\circ S)_T\ge-\phi_0$ on Ω^*. We claim that the inequality is actually an equality on $\Omega^*_{\phi_0}$ (which is nonempty by assumption). If indeed for some $\omega\in\Omega^*_{\phi_0}$ the inequality is strict, then any $Q\in\mathcal{M}^f_\Omega$ such that $Q(\{\omega\})>0$ satisfies $E_Q[\phi_0]>0$, which contradicts $\omega\in\Omega^*_{\phi_0}$. This implies that ϕ_0 is replicable on $\Omega^*_{\phi_0}$, and thus, $\widetilde{\mathcal{M}}^f_{\Omega,\phi_0}=\mathcal{M}^f_{\Omega,\phi_0}=\mathcal{M}^f_{\Omega^*_{\phi_0}}$. In such a case,

$$\pi_{\Omega^*_{\phi_0},\phi_0}(g)\le\pi_{\Omega^*_{\phi_0}}(g)=\sup_{Q\in\mathcal{M}^f_{\Omega^*_{\phi_0}}} E_Q[g]=\sup_{Q\in\mathcal{M}^f_{\Omega,\phi_0}} E_Q[g],$$

where the first equality follows from Proposition 3 because $\Omega^*_{\phi_0}\in\mathcal{F}^{\mathcal{A}}\subseteq\Lambda$ by assumption. The thesis now follows from standard arguments as before. □

Proposition 4. *Assume that Ω satisfies that there exists an ω^* such that $S_0(\omega^*)=S_1(\omega^*)=\ldots=S_T(\omega^*)$, $\Omega=\Omega^*$, and $\pi_{\Omega^*}(\phi_0)>0$. Then the following are equivalent:*

1. *There is no uniformly strong arbitrage on Ω in $\widetilde{\mathcal{A}}_{\phi_0}(\mathbb{F}^{\mathrm{pr}})$.*
2. *There is no strong arbitrage on Ω in $\widetilde{\mathcal{A}}_{\phi_0}(\mathbb{F}^{\mathrm{pr}})$.*
3. *$\widetilde{\mathcal{M}}_{\Omega,\phi_0}\ne\emptyset$.*
4. *$\widetilde{\mathcal{M}}^f_{\Omega,\phi_0}\ne\emptyset$.*

Moreover, when any of these holds, for any upper semicontinuous $g:\mathbb{R}_+^{d\times(T+1)}\to\mathbb{R}$ such that

$$\lim_{|x|\to\infty}\frac{g^+(x)}{m(x)}=0, \tag{32}$$

where $m(x_0,\ldots,x_T):=\sum_{t=0}^T g_0(x_t)$, we have the following pricing–hedging duality:

$$\pi_{\Omega^*,\phi_0}(g(S))=\sup_{Q\in\widetilde{\mathcal{M}}_{\Omega,\phi_0}} E_Q[g(S)]=\sup_{Q\in\mathcal{M}_{\Omega,\phi_0}} E_Q[g(S)]. \tag{33}$$

Remark 16. We observe that the assumption $\pi_{\Omega^*}(\phi_0)>0$ is not binding and can be removed. In fact, if $\pi_{\Omega^*}(\phi_0)\le 0$, (1) ⇒ (3) is obviously satisfied because $\widetilde{\mathcal{M}}_{\Omega,\phi_0}=\mathcal{M}_\Omega\ne\emptyset$. The difference is that the pricing–hedging duality (33) is (trivially) satisfied only in the first equation.

Proof of Proposition 4. (3) ⇒ (2) and (2) ⇒ (1) are obvious. (4) ⇔ (3) is an easy consequence of Theorem 1. To show (1) ⇒ (4), we suppose there is no uniformly strong arbitrage on Ω in $\widetilde{\mathcal{A}}_{\phi_0}(\mathbb{F}^{\mathrm{pr}})$.

We first show that the interesting case is $\pi_{\Omega^*}(\phi_0)>0$ and $\pi_{\Omega^*}(-\phi_0)=0$. The other cases follow trivially from Proposition 3 and Lemma 6:

- If $\pi_{\Omega^*}(-\phi_0)<0$, because the superhedging price is attained and $\Omega=\Omega^*$, there exist $H\in\mathbb{F}^{\mathrm{pr}}$ and $x<0$ such that

$$\phi_0(\omega)+(H\circ S)_T(\omega)\ge -x>0,\quad\forall\omega\in\Omega,$$

which is clearly a uniform strong arbitrage on Ω.

- If $\pi_{\Omega^*}(-\phi_0)>0$ and $\pi_{\Omega^*}(\phi_0)>0$, we have that zero is in the interior of the price interval formed by $\inf_{Q\in\mathcal{M}^f_\Omega} E_Q[\phi_0]$ and $\sup_{Q\in\mathcal{M}^f_\Omega} E_Q[\phi_0]$. Thus, $\widetilde{\mathcal{M}}^f_{\Omega,\phi_0}\supseteq\mathcal{M}^f_{\Omega,\phi_0}\ne\emptyset$, and it is straightforward to see that $\Omega^*_{\phi_0}=\Omega^*$.

Note that in all these cases $\Omega^*_{\phi_0}=\Omega^*=\Omega\in\mathcal{F}^{\mathcal{A}}$, and hence Equation (33) follows from Lemma 6.

The remaining case is $\pi_{\Omega^*}(\phi_0)>0$ and $\pi_{\Omega^*}(-\phi_0)=0$. In this case, by considering the ω^* such that $s_0=S_0(\omega^*)=S_1(\omega^*)=\ldots=S_T(\omega^*)$, we observe that the superreplication of $-\phi_0$ necessarily requires an initial capital of, at least, $-g_0(s_0)$. From $\pi_{\Omega^*}(-\phi_0)=0$, we can rule out the possibility that $g_0(s_0)<0$. Note now that by the convexity of g_0, for any $l \in 0, \ldots, T-1$, $g_0(S_T(\omega)) - \sum_{i=l+1}^{T} g_0'(S_{i-1}(\omega))\Big(S_i(\omega) - S_{i-1}(\omega)\Big) \geq g_0(S_l(\omega))$ for any $\omega \in \Omega$. In particular, when $l = 0$,

$$g_0(S_T(\omega)) - \sum_{i=1}^{T} g_0'(S_{i-1}(\omega))\Big(S_i(\omega) - S_{i-1}(\omega)\Big) \geq g_0(s_0) \quad \forall \omega \in \Omega. \tag{34}$$

Denote by $\bar{H}$ the dynamic strategy in Equation (34). If $g_0(s_0) > 0$, $(1, \bar{H})$ is a uniformly strong arbitrage on Ω and, hence, a contradiction to our assumption. Thus, $g_0(s_0) = 0$. In this case, it is obvious that the Dirac measure $\delta_{\omega^*} \in \mathcal{M}^f_{\Omega,\phi_0} \subseteq \widetilde{\mathcal{M}}^f_{\Omega,\phi_0}$, which is therefore nonempty.

Moreover, because $\delta_{\omega_*} \in \mathcal{M}^f_{\Omega,\phi_0}$,

$$\sup_{Q \in \widetilde{\mathcal{M}}^f_{\Omega,\phi_0}} E_Q[g(S)] \geq \sup_{Q \in \mathcal{M}^f_{\Omega,\phi_0}} E_Q[g(S)] \geq g(s_0, \ldots, s_0).$$

Case 1. Suppose g is bounded from above. We show that it is possible to superreplicate g with any initial capital larger than $g(s_0, \ldots, s_0)$. To see this, recall that, from the strict convexity of g_0, the inequality in Equation (34) is strict for any $s \in \mathbb{R}_+^{d \times (T+1)}$ such that s is not a constant path; that is, $s_i \neq s_0$ for some $i \in \{0, \ldots, T\}$. In fact, it is bounded away from zero outside any small ball of $(s_0, \ldots, s_0)$. Hence, because of the upper semicontinuity and boundedness of g, for any $\varepsilon > 0$, there exists a sufficiently large K such that

$$g(s_0, \ldots, s_0) + \varepsilon + K\Big\{g_0(S_T(\omega)) - \sum_{i=1}^{T} g_0'(S_{i-1}(\omega))(S_i(\omega) - S_{i-1}(\omega))\Big\} \geq g(S(\omega)) \quad \forall \omega \in \Omega^*.$$

Therefore, $\pi_{\Omega^*,\phi_0}(g) \leq g(s_0, \ldots, s_0) \leq \sup_{Q \in \mathcal{M}^f_{\Omega,\phi_0}} E_Q[g(S)] \leq \sup_{Q \in \widetilde{\mathcal{M}}^f_{\Omega,\phi_0}} E_Q[g(S)]$. The converse inequality is easy, and hence, we obtain $\pi_{\Omega^*,\phi_0}(g) = \sup_{Q \in \mathcal{M}^f_{\Omega,\phi_0}} E_Q[g(S)] = \sup_{Q \in \widetilde{\mathcal{M}}^f_{\Omega,\phi_0}} E_Q[g]$ as required.

Case 2. It remains to argue that the duality still holds true for any g that is upper semicontinuous and satisfies Equation (32). We first argue that any upper semicontinuous $g : \mathbb{R}_+^{d \times (T+1)} \to \mathbb{R}$, satisfying Equation (32), can be superreplicated on Ω^* by a strategy involving dynamic trading in S, static hedging in g_0, and cash. Define a synthetic option with payoff $\widetilde{m} : \mathbb{R}_+^{d \times (T+1)} \to \mathbb{R}$ by

$$\widetilde{m}(x_0, \ldots, x_T) = \sum_{l=0}^{T} \Big\{g_0(x_T) - \sum_{i=l+1}^{T} g_0'(x_{i-1})(x_i - x_{i-1})\Big\}. \tag{35}$$

By convexity of g_0, we know that

$$\widetilde{m}(x_0, \ldots, x_T) = \sum_{l=0}^{T} \Big\{g_0(x_T) - \sum_{i=l+1}^{T} g_0'(x_{i-1})(x_i - x_{i-1})\Big\} \geq \sum_{l=0}^{T} g_0(x_l) = m(x_0, \ldots, x_T).$$

Because we assume there is no uniform strong arbitrage, it is clear that $\pi_{\Omega^*,\phi_0}(\widetilde{m}(S)) = 0$.

From Equation (32) it follows that $g(S) - \widetilde{m}(S)$ is bounded from above. By sublinearity of $\pi_{\Omega^*,\phi_0}(\cdot)$, we have

$$\begin{aligned} \pi_{\Omega^*,\phi_0}(g) &\leq \pi_{\Omega^*,\phi_0}(g(S) - \widetilde{m}(S)) + \pi_{\Omega^*,\phi_0}(\widetilde{m}(S)) \\ &= \sup_{Q \in \mathcal{M}^f_{\Omega,\phi_0}} E_Q[g(S) - \widetilde{m}(S)] + 0 \\ &= \sup_{Q \in \mathcal{M}^f_{\Omega,\phi_0}} E_Q[g(S)] \leq \sup_{Q \in \widetilde{\mathcal{M}}^f_{\Omega,\phi_0}} E_Q[g(S)], \end{aligned}$$

where the first equality follows from the pricing–hedging duality for claims bounded from above, which we established in Case 1, and the fact that $\pi_{\Omega^*,\phi_0}(\widetilde{m}(S)) = 0$. Moreover, for $Q \in \mathcal{M}_{\Omega,\phi_0}$, $E_Q[\widetilde{m}(S)] = 0$, from which the second equality follows.

The converse inequality follows from standard arguments, and hence, we have obtained $\pi_{\Omega^*,\phi_0}(g) = \sup_{Q \in \mathcal{M}^f_{\Omega,\phi_0}} E_Q[g] = \sup_{Q \in \widetilde{\mathcal{M}}^f_{\Omega,\phi_0}} E_Q[g]$. The equality with the supremum over $\mathcal{M}_{\Omega,\phi_0}$ and $\widetilde{\mathcal{M}}_{\Omega,\phi_0}$ follows from the same argument for the proof of theorem 1.1, step 2, in Burzoni et al. [11]. □

Proof of Theorem 3. (3) ⇒ (2) and (2) ⇒ (1) are obvious.

Step 1. To show that (1) implies (3), suppose there is no uniformly strong arbitrage on Ω in $\mathcal{A}_\Phi(\widetilde{\mathbb{F}})$. We first know from Proposition 4 that $\mathcal{M}_{\Omega,\phi_0} \neq \emptyset$. We can use a variational argument to deduce the following equalities: fix an arbitrary $K > 0$ and let $\widetilde{m} : \mathbb{R}_+^{d\times(T+1)} \to \mathbb{R}$ defined as in Equation (35):

$$\begin{aligned}\pi_{\Omega^*,\Phi}(g(S)) &= \inf_{X\in\mathrm{Lin}(\Phi/\{\phi_0\})} \pi_{\Omega^*,\phi_0}(g(S)-X)\\ &= \inf_{X\in\mathrm{Lin}(\Phi/\{\phi_0\})} \sup_{Q\in\mathcal{M}_{\Omega,\phi_0}} E_Q[g(S)-X] \qquad (36)\\ &= \inf_{X\in\mathrm{Lin}(\Phi/\{\phi_0\})} \sup_{Q\in\mathcal{M}_{\Omega,\phi_0}} E_Q[g(S)-X-K\widetilde{m}(S)], \qquad (37)\end{aligned}$$

where the second equality follows from Proposition 4. Denote by $\mathfrak{Q}$ (respectively, $\widetilde{\mathfrak{Q}}$) the set of law of S under the measures $Q \in \mathcal{M}_{\Omega,\phi_0}$ (respectively, $Q \in \widetilde{\mathcal{M}}_{\Omega,\phi_0}$) and write $\mathrm{Lin}(\{\mathrm{g_i}\}_{\mathrm{i}\in\mathrm{I}/\{0\}})$ for the set of finite linear combinations of elements in $\{g_i\}_{i\in I/\{0\}}$. Observe that from step 1 of the proof of theorem 1.3 in Acciaio et al. [1] $\widetilde{\mathfrak{Q}}$ is weakly compact, and hence, the same is true for $\overline{\mathfrak{Q}}$ (the weak closure of $\mathfrak{Q}$).

By a change of variable, we have

$$\inf_{X\in\mathrm{Lin}(\Phi/\{\phi_0\})} \sup_{Q\in\mathcal{M}_{\Omega,\phi_0}} E_Q[g(S)-X-K\widetilde{m}(S)] = \inf_{G\in\mathrm{Lin}(\{\mathrm{g_i}\}_{\mathrm{i}\in\mathrm{I}/\{0\}})} \sup_{\mathbb{Q}\in\mathfrak{Q}} \mathbb{E}_{\mathbb{Q}}[\widetilde{g}(\mathbb{S})],$$

where $\mathbb{S} := (\mathbb{S}_t)_{t=0}^T$ is the canonical process on $\mathbb{R}_+^{d\times(T+1)}$ and $\widetilde{g} = g - G - K\widetilde{m}$. We aim at applying min–max theorem (see corollary 2 in Terkelsen [49]) to the compact convex set $\overline{\mathfrak{Q}}$, the convex set $\mathrm{Lin}(\{\mathrm{g_i}\}_{\mathrm{i}\in\mathrm{I}/\{0\}})$, and the function

$$f(\mathbb{Q},G) = \int_{\mathbb{R}_+^{d\times(T+1)}} \Big(g(s_0,\ldots,s_T) - G(s_0,\ldots,s_T) - K\widetilde{m}(s_0,\ldots,s_T)\Big)d\mathbb{Q}(s_0,\ldots,s_n).$$

Clearly, f is affine in each of the variables. Furthermore, we show that $f(\cdot,G)$ is upper semicontinuous on $\overline{\mathfrak{Q}}$. To see this, fix $G \in \mathrm{Lin}(\{\mathrm{g_i}\}_{\mathrm{i}\in\mathrm{I}/\{0\}})$. By definition of f, we have that

$$f(\mathbb{Q},G) = \mathbb{E}_{\mathbb{Q}}[\widetilde{g}(\mathbb{S})]. \tag{38}$$

It follows from Assumption 1 and Equation (8) that $\widetilde{g}$ is bounded from above. Hence, for every sequence of $\{\mathbb{Q}_n\}_n \in \overline{\mathfrak{Q}}$ with $\mathbb{Q}_n \to \mathbb{Q}$ as $n\to\infty$ for some $\mathbb{Q}$ weakly, we have

$$\lim_{n\to\infty} \mathbb{E}_{\mathbb{Q}_n}[\widetilde{g}^+(\mathbb{S})] \le \mathbb{E}_{\mathbb{Q}}[\widetilde{g}^+(\mathbb{S})]$$

by Portmanteau theorem and

$$\liminf_{n\to\infty} \mathbb{E}_{\mathbb{Q}_n}[\widetilde{g}^-(\mathbb{S})] \ge \mathbb{E}_{\mathbb{Q}}[\widetilde{g}^-(\mathbb{S})]$$

by Fatou's lemma, where $\widetilde{g} := \widetilde{g}^+ - \widetilde{g}^-$ with $\widetilde{g}^+ := \max\{\widetilde{g},0\}$, $\widetilde{g}^- := (-\widetilde{g})^+$. Then

$$\limsup_{n\to\infty} f(\mathbb{Q}_n,G) = \limsup_{n\to\infty} \mathbb{E}_{\mathbb{Q}_n}[\widetilde{g}(\mathbb{S})] \le \mathbb{E}_{\mathbb{Q}}[\widetilde{g}(\mathbb{S})] = f(\mathbb{Q},G).$$

Therefore, the assumptions of corollary 2 in Terkelsen [49] are satisfied, and we have, by recalling $\mathfrak{Q} \subseteq \overline{\mathfrak{Q}}$ and Equation (37),

$$\begin{aligned}\pi_{\Omega^*,\Phi}(g(S)) &= \inf_{X\in\mathrm{Lin}(\Phi\{\phi_0\})} \sup_{Q\in\mathcal{M}_{\Omega,\phi_0}} E_Q[g(S)-X-K\widetilde{m}(S)]\\ &= \inf_{G\in\mathrm{Lin}(\{g_i\}_{i\in I/\{0\}})} \sup_{\mathbb{Q}\in\mathfrak{Q}} \mathbb{E}_{\mathbb{Q}}[g(\mathbb{S})-G(\mathbb{S})-K\widetilde{m}(\mathbb{S})]\\ &\le \inf_{G\in\mathrm{Lin}(\{g_i\}_{i\in I/\{0\}})} \sup_{\mathbb{Q}\in\overline{\mathfrak{Q}}} \mathbb{E}_{\mathbb{Q}}[g(\mathbb{S})-G(\mathbb{S})-K\widetilde{m}(\mathbb{S})]\\ &= \sup_{\mathbb{Q}\in\overline{\mathfrak{Q}}} \inf_{G\in\mathrm{Lin}(\{g_i\}_{i\in I/\{0\}})} \mathbb{E}_{\mathbb{Q}}[g(\mathbb{S})-G(\mathbb{S})-K\widetilde{m}(\mathbb{S})]\\ &\le \sup_{\mathbb{Q}\in\widetilde{\mathfrak{Q}}} \inf_{G\in\mathrm{Lin}(\{g_i\}_{i\in I/\{0\}})} \mathbb{E}_{\mathbb{Q}}[g(\mathbb{S})-G(\mathbb{S})-K\widetilde{m}(\mathbb{S})]\\ &= \sup_{Q\in\widetilde{\mathcal{M}}_{\Omega,\phi_0}} \inf_{X\in\mathrm{Lin}(\Phi/\{\phi_0\})} E_Q[g(S)-X-K\widetilde{m}(S)]. \qquad (39)\end{aligned}$$

Take $g = 0$. If $\widetilde{\mathcal{M}}_{\Omega,\Phi} = \emptyset$, then

$$\sup_{Q\in\widetilde{\mathcal{M}}_{\Omega,\phi_0}} \inf_{X\in \mathrm{Lin}(\Phi/\{\phi_0\})} E_Q[-X - K\widetilde{m}(S)] = -\infty,$$

and hence, $\pi_{\Omega^*,\Phi}(0) = -\infty$, which contradicts the no-arbitrage assumption. Therefore, we have (1) implies (3).

Step 2. To show the pricing–hedging duality, suppose now $\widetilde{\mathcal{M}}_{\Omega,\Phi} \neq \emptyset$. If $Q \notin \widetilde{\mathcal{M}}_{\Omega,\Phi}$, then

$$\inf_{X\in \mathrm{Lin}(\Phi/\{\phi_0\})} E_Q[g(S) - X - K\widetilde{m}(S)] = -\infty.$$

Therefore, in Equation (39), it suffices to look at measures in $\widetilde{\mathcal{M}}_{\Omega,\Phi} \neq \emptyset$ only, and hence, we obtain

$$\begin{aligned}
\pi_{\Omega^*,\Phi}(g(S)) &\leq \sup_{Q\in\widetilde{\mathcal{M}}_{\Omega,\Phi}} E_Q[g(S) - K\widetilde{m}(S)]\\
&\leq \sup_{Q\in\widetilde{\mathcal{M}}_{\Omega,\Phi}} E_Q[g(S)] + K \sup_{Q\in\widetilde{\mathcal{M}}_{\Omega,\Phi}} E_Q[-\widetilde{m}(S)]\\
&= \sup_{Q\in\widetilde{\mathcal{M}}_{\Omega,\Phi}} E_Q[g(S)] + K(T+1) \sup_{Q\in\widetilde{\mathcal{M}}_{\Omega,\Phi}} E_Q[-g_0(S)].
\end{aligned}$$

Because $-g_0$ is bounded from above, the quantity $\sup_{Q\in\widetilde{\mathcal{M}}_{\Omega,\Phi}} E_Q[-g_0(S)]$ is finite, and by recalling that $K>0$ is arbitrary, we get the thesis for $K \downarrow 0$. $\square$

Appendix

Let X be a Polish space. The so-called projective hierarchy (see Kechris [34, chapter V]) is constructed as follows. The first level is composed of the analytic sets Σ_1^1 (projections of closed subsets of $X\times\mathbb{N}^{\mathbb{N}}$), the coanalytic sets Π_1^1 (complementary of analytic sets), and the Borel sets $\Delta_1^1 = \Sigma_1^1 \cap \Pi_1^1$. The subsequent levels are defined iteratively through the operations of projection and complementation, namely

$$\begin{aligned}
\Sigma_{n+1}^1 &= \text{projections of } \Pi_n^1 \text{ subsets of } X\times\mathbb{N}^{\mathbb{N}},\\
\Pi_{n+1}^1 &= \text{complementary of sets in } \Sigma_{n+1}^1,\\
\Delta_{n+1}^1 &= \Sigma_{n+1}^1 \cap \Pi_{n+1}^1.
\end{aligned}$$

From the definition it is clear that $\Sigma_n^1 \subseteq \Sigma_{n+1}^1$ for any $n\in\mathbb{N}$, and analogous inclusions hold for Π_n^1 and Δ_n^1. Sets in the union of the projective classes (also called *Lusin classes*) are called *projective sets*, which we denote by $\Lambda := \bigcup_{n=1}^\infty \Delta_n^1 = \bigcup_{n=1}^\infty \Sigma_n^1 = \bigcup_{n=1}^\infty \Pi_n^1$.

Remark A.1. We observe that $\Sigma_1^1 \cup \Pi_1^1$ is a sigma-algebra that actually coincides with $\mathcal{F}^{\mathcal{A}}$. Moreover, $\Sigma_1^1 \cup \Pi_1^1 = \mathcal{F}^{\mathcal{A}} \subseteq \Delta_2^1$.

We first recall the following result from Kechris [34] (see exercise 37.3).

Lemma A.1. *Let $f : X \mapsto \mathbb{R}^k$ be Borel measurable. For any $n\in\mathbb{N}$,*

1. $f^{-1}(\Sigma_n^1) \subseteq \Sigma_n^1$.
2. $f(\Sigma_n^1) \subseteq \Sigma_n^1$.

The following is a consequence of the previous lemma.

Lemma A.2. *Let $f : X \mapsto \mathbb{R}^k$ be Borel measurable. For any $n\in\mathbb{N}$,*

1. $f^{-1}(\sigma(\Sigma_n^1)) \subseteq \sigma(\Sigma_n^1)$.
2. $f(\sigma(\Sigma_n^1)) \subseteq \Sigma_{n+1}^1$.

Proof. From Lemma A.1, the first claim holds for Σ_n^1, which generates the sigma-algebra. In particular, Σ_n^1 is contained in

$$\{A \in \sigma(\Sigma_n^1) \,|\, f^{-1}(A) \in \sigma(\Sigma_n^1)\} \subseteq \sigma(\Sigma_n^1).$$

Because this set is a sigma-algebra, it also contains $\sigma(\Sigma_n^1)$, from which the claim follows. For the second assertion, we recall that Δ_n^1 is a sigma-algebra for any $n\in\mathbb{N}$ (see proposition 37.1 in Kechris [34]). In particular,

$$\sigma(\Sigma_n^1) \subseteq \sigma(\Sigma_n^1 \cup \Pi_n^1) \subseteq \Delta_{n+1}^1 \subseteq \Sigma_{n+1}^1.$$

Because, from Lemma A.1, $f(\Sigma_{n+1}^1) \subseteq \Sigma_{n+1}^1$, the thesis follows. $\square$

Remark A.2. We recall that under the axiom of projective determinacy the class Λ and, hence, also $\mathcal{F}^{\text{pr}}$ is included in the universal completion of $\mathcal{B}_X$ (see theorem 38.17 in Kechris [34]). This axiom has been thoroughly studied in set theory, and it is implied, for example, by the existence of infinitely many Woodin cardinals (see, e.g., Martin and Steel [37]).

A.1. Remark on Conditional Supports

Let $\mathcal{G} \subseteq \mathcal{B}_X$ be a countably generated sub σ-algebra of $\mathcal{B}_X$. Then there exists a proper regular conditional probability, that is, a function $\mathbb{P}_{\mathcal{G}}(\cdot,\cdot) : (X, \mathcal{B}_X) \mapsto [0,1]$ such that

1. For all $\omega \in \Omega$, $\mathbb{P}_{\mathcal{G}}(\omega, \cdot)$ is a probability measure on $\mathcal{B}_X$.
2. For each (fixed) $B \in \mathcal{B}_X$, the function $\mathbb{P}_{\mathcal{G}}(\cdot, B)$ is $\mathcal{G}$-measurable and a version of $E_{\mathbb{P}}[\mathbf{1}_B \mid \mathcal{G}](\cdot)$ (here the null set on which they differ depends on B).
3. There exists $N \in \mathcal{G}$ with $\mathbb{P}(N) = 0$ such that $\mathbb{P}_{\mathcal{G}}(\omega, B) = \mathbf{1}_B(\omega)$ for $\omega \in X \backslash N$ and $B \in \mathcal{G}$ (here the null set on which they differ does not depend on B); moreover, for all $\omega \in X \setminus N$, we have $\mathbb{P}_{\mathcal{G}}(\omega, A_\omega) = 1$, where $A_\omega = \bigcap\{A : \omega \in A,\ A \in \mathcal{G}\} \in \mathcal{G}$.
4. For every $A \in \mathcal{G}$ and $B \in \mathcal{B}_X$, we have $\mathbb{P}(A \cap B) = \int_A \mathbb{P}_{\mathcal{G}}(\omega, B)\mathbb{P}(d\omega)$.

We now consider a measurable $\xi : X \to \mathbb{R}^d$ and $P_\xi : X \times \mathcal{B}_{\mathbb{R}^d} \to [0,1]$ defined by

$$P_\xi(\omega, B) := \mathbb{P}_{\mathcal{G}}(\omega, \{\tilde{\omega} \in A_\omega \mid \xi(\tilde{\omega}) \in B\}),$$

and observe that from (1) and with N as in (3), for any $\omega \in X \setminus N$, $P_\xi(\omega, \cdot)$ is a probability measure on $(\mathbb{R}^d, \mathcal{B}_{\mathbb{R}^d})$. Finally, we let $B_\varepsilon(x)$ denote the ball of radius ε with center in x, and we introduce the closed valued random set

$$\omega \longrightarrow \chi_{\mathcal{G}}(\omega) := \{x \in \mathbb{R}^d \mid P_\xi(\omega, B_\varepsilon(x)) > 0\ \forall\, \varepsilon > 0\}, \tag{A.1}$$

for $\omega \in X \setminus N$ and $\mathbb{R}^d$ otherwise. Then $\chi_{\mathcal{G}}$ is $\mathcal{G}$-measurable because, for any open set $O \subseteq \mathbb{R}^d$, we have

$$\{\omega \in X \mid \chi_{\mathcal{G}}(\omega) \cap O \neq \emptyset\} = N \cup \{\omega \in X \setminus N \mid P_\xi(\omega, O) > 0\} = N \cup \{\omega \in X \setminus N \mid \mathbb{P}_{\mathcal{G}}(\omega, \xi^{-1}(O) \cap A_\omega) > 0\},$$

with the latter belonging to $\mathcal{G}$ from (2) and (3). By definition, $\chi_{\mathcal{G}}(\omega)$ is the support of $P_\xi(\omega, \cdot)$, and therefore, for every $\omega \in X$, $P_\xi(\omega, \chi_{\mathcal{G}}(\omega)) = 1$. Notice that because the map $\chi_{\mathcal{G}}$ is $\mathcal{G}$-measurable, then for $\omega \in X$ we have $\chi_{\mathcal{G}}(\omega) = \chi_{\mathcal{G}}(\tilde{\omega})$ for all $\tilde{\omega} \in A_\omega$.

Lemma A.3. *Under the previous assumption, we have* $\{\omega \in X \mid \xi(\omega) \in \chi_{\mathcal{G}}(\omega)\} \in \mathcal{B}_X$ *and* $\mathbb{P}(\{\omega \in X \mid \xi(\omega) \in \chi_{\mathcal{G}}(\omega)\}) = 1$.

Proof. Set $B := \{\omega \in X \mid \xi(\omega) \in \chi_{\mathcal{G}}(\omega)\}$. $B \in \mathcal{B}_X$ follows from the measurability of ξ and $\chi_{\mathcal{G}}$. From the properties of regular conditional probability, we have

$$\mathbb{P}(B) = \mathbb{P}(B \cap X) = \int_X \mathbb{P}_{\mathcal{G}}(\omega, B)\mathbb{P}(d\omega).$$

Consider the atom $A_\omega = \cap\{A : \omega \in A,\ A \in \mathcal{G}\}$. From property (3), we have $\mathbb{P}_{\mathcal{G}}(\omega, A_\omega) = 1$ for any $\omega \in X \setminus N$. Therefore, for every $\omega \in X \setminus N$, we deduce

$$\begin{aligned}\mathbb{P}_{\mathcal{G}}(\omega, B) &= \mathbb{P}_{\mathcal{G}}(\omega, B \cap A_\omega) = \mathbb{P}_{\mathcal{G}}(\omega, \{\tilde{\omega} \in A_\omega \mid \xi(\tilde{\omega}) \in \chi_{\mathcal{G}}(\tilde{\omega})\}) \\ &= \mathbb{P}_{\mathcal{G}}(\omega, \{\tilde{\omega} \in A_\omega \mid \xi(\tilde{\omega}) \in \chi_{\mathcal{G}}(\omega)\}) = P_\xi(\omega, \chi_{\mathcal{G}}(\omega)) = 1.\end{aligned}$$

Therefore,

$$\mathbb{P}(B) = \int_X \mathbb{P}_{\mathcal{G}}(\omega, B)\mathbb{P}(d\omega) = \int_X \mathbf{1}_{X\setminus N}(\omega)\mathbb{P}(d\omega) = 1. \quad \square$$

Endnote

[1] This setting has been often described as "model independent," but we see it as a modelling choice with very weak assumptions.

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