



Mathematics of Operations Research

Publication details, including instructions for authors and subscription information:
<http://pubsonline.informs.org>

Optimal Budget Aggregation with Star-Shaped Preference Domains

Felix Brandt, Matthias Greger, Erel Segal-Halevi, Warut Suksompong

To cite this article:

Felix Brandt, Matthias Greger, Erel Segal-Halevi, Warut Suksompong (2025) Optimal Budget Aggregation with Star-Shaped Preference Domains. *Mathematics of Operations Research*

Published online in Articles in Advance 17 Dec 2025

<https://doi.org/10.1287/moor.2024.0723>

This work is licensed under a Creative Commons Attribution 4.0 International License. You are free to copy, distribute, transmit and adapt this work, but you must attribute this work as “*Mathematics of Operations Research*.” Copyright © 2025 The Author(s). <https://doi.org/10.1287/moor.2024.0723>, used under a Creative Commons Attribution License: <https://creativecommons.org/licenses/by/4.0/>.”

Copyright © 2025 The Author(s)

Please scroll down for article—it is on subsequent pages



With 12,500 members from nearly 90 countries, INFORMS is the largest international association of operations research (O.R.) and analytics professionals and students. INFORMS provides unique networking and learning opportunities for individual professionals, and organizations of all types and sizes, to better understand and use O.R. and analytics tools and methods to transform strategic visions and achieve better outcomes. For more information on INFORMS, its publications, membership, or meetings visit <http://www.informs.org>

Optimal Budget Aggregation with Star-Shaped Preference Domains

 Felix Brandt,^{a,*} Matthias Greger,^b Erel Segal-Halevi,^c Warut Suksompong^d

^aSchool of Computation, Information and Technology, Technical University of Munich, 85748 Munich, Germany; ^bLAMSADE, Université Paris Dauphine-PSL, 75016 Paris, France; ^cDepartment of Computer Science, Ariel University, Ariel 40700, Israel; ^dSchool of Computing, National University of Singapore, Singapore 117418, Singapore

*Corresponding author

Contact: brandt@tum.de,  <https://orcid.org/0000-0002-4179-9897> (FB); matthias.greger0815@gmail.com,

 <https://orcid.org/0000-0002-3584-8656> (MG); erelsg1@gmail.com,  <https://orcid.org/0000-0002-7497-5834> (ES-H);

warut@comp.nus.edu.sg,  <https://orcid.org/0000-0001-8973-2539> (WS)

Received: September 30, 2024

Revised: June 25, 2025

Accepted: October 5, 2025

Published Online in *Articles in Advance*:
December 17, 2025

MSC2020 Subject Classifications:

Primary: 91B03, 91B14;


Secondary: 91B08, 91B18, 91B32

<https://doi.org/10.1287/moor.2024.0723>

Copyright: © 2025 The Author(s)

Abstract. We study the problem of aggregating distributions, such as budget proposals, into a collective distribution. An ideal aggregation mechanism would be Pareto efficient, strategyproof, and fair. Most previous work assumes that agents evaluate budgets according to the ℓ_1 distance to their ideal budget. We investigate and compare different models from the larger class of *star-shaped utility functions*—a multidimensional generalization of single-peaked preferences. For the case of two alternatives, we extend existing results by proving that under very general assumptions, the *uniform phantom mechanism* is the only strategyproof mechanism that satisfies proportionality—a minimal notion of fairness introduced in prior work. Moving to the case of more than two alternatives, we establish sweeping impossibilities for ℓ_1 and ℓ_∞ disutilities: no mechanism satisfies efficiency, strategyproofness, and proportionality. We then propose a new kind of star-shaped utility based on evaluating budgets by the *ratios* of shares between a given budget and an ideal budget. For these utilities, efficiency, strategyproofness, and fairness become compatible. In particular, we prove that the mechanism that maximizes the Nash product of individual utilities is characterized by group-strategyproofness and a core-based fairness condition.

History: This manuscript was accepted for the Special Issue on Mathematics of Market Design.

 **Open Access Statement:** This work is licensed under a Creative Commons Attribution 4.0 International License. You are free to copy, distribute, transmit and adapt this work, but you must attribute this work as “*Mathematics of Operations Research*. Copyright © 2025 The Author(s). <https://doi.org/10.1287/moor.2024.0723>, used under a Creative Commons Attribution License: <https://creativecommons.org/licenses/by/4.0/>.”

Funding: This work was supported by the Israel Science Foundation [Grants 712/20 and 1092/24], Singapore Ministry of Education [Grant MOE-T2EP20221-0001], Deutsche Forschungsgemeinschaft [Grants BR 2312/11-2 and BR 2312/12-1], and an NUS start-up grant.

Keywords: mechanism design • collective decision making • portioning • star-shaped preferences

1. Introduction

Social choice theory is concerned with the aggregation of individual preferences into a collective outcome (e.g., Arrow et al. [1, 2]). An important special case arises when the potential collective outcomes are *distributions* over a fixed set of alternatives. These distributions may represent how a budget should be divided among public projects in a city or among departments in an organization. Alternatively, they may reflect how time or space ought to be allotted between different types of activities at a social event. This scenario is sometimes referred to as *budget aggregation* or *portioning* and falls under the framework of *participatory budgeting*, which has received increasing interest in recent years (Aziz and Shah [4], De Vries et al. [18]).

In order to reason about the agents’ satisfaction with the collective outcome, one needs to make some assumptions about their preferences. Importantly, in our setting, the realized outcome is a distribution. Therefore, restricting attention to rankings over alternatives is insufficient, as an agent’s most-preferred outcome is typically a nondegenerate distribution over the alternatives. This is particularly evident in participatory budgeting problems, where even if an agent has a favorite project, she normally also likes other projects and does not want them to be left completely unfunded. This is in contrast to probabilistic social choice (see, e.g., Brandt [14]), where the final outcome is a single alternative picked at random from the distribution, so typically, the agents’ most-preferred distributions are degenerate.

In this paper, consistent with previous research in this domain, we consider utility models where agents' preferences are completely determined by their favorite distribution: their "peak." This keeps the amount of required information from each agent at a manageable level. We assume that each agent's utility decreases as the actual distribution moves away from her peak (see the formal definition in Section 3). Such utility functions are called *star-shaped* (Border and Jordan [12]).

When there are only two alternatives, the set of outcomes can be represented by the unit interval $[0, 1]$. Then, star-shaped utilities are equivalent to *single-peaked preferences*. According to a famous characterization by Moulin [33], there is a rather restrictive class of mechanisms that are strategyproof for all single-peaked preferences, the so-called *generalized median* rules. Moulin's characterization leaves open the possibility that for restricted subdomains of single-peaked preferences, other mechanisms than generalized median rules are strategyproof. In Section 7, we obtain characterizations of continuous mechanisms that hold not only for single-peaked but also for *any* subdomain of single-peaked preferences. Our characterizations refine results by Aziz et al. [6] and Freeman et al. [22], which are valid only for specific subdomains.

When there are $m \geq 3$ alternatives, the set of outcomes is a multidimensional simplex, and the space of possible utility functions is much richer. For example, for every $p \geq 1$, the ℓ_p metric induces a utility function $u_i(\mathbf{q}) = -(\sum_{j=1}^m |p_{i,j} - q_j|^p)^{1/p}$, where $\mathbf{q} = (q_1, \dots, q_m)$ is any distribution over m alternatives, and $\mathbf{p}_i = (p_{i,1}, \dots, p_{i,m})$ is agent i 's peak. Note that all these functions are equivalent when $m = 2$ but not when $m \geq 3$. Still, most of the previous works on budget aggregation assume ℓ_1 utilities (e.g., Caragiannis et al. [16], Freeman and Schmidt-Kraepelin [21], Freeman et al. [22], Goel et al. [25], Lindner et al. [30]). Under this utility model, Goel et al. [25] and Lindner et al. [30] showed that a mechanism that maximizes utilitarian welfare (i.e., minimizes the sum of agents' disutilities) is both strategyproof and efficient; however, this mechanism has a tendency to overweight majority preferences. Freeman et al. [22] proposed a mechanism, called the *independent markets mechanism*, which satisfies strategyproofness along with a weak fairness notion dubbed *proportionality*. Proportionality requires that the collective distribution is given by the uniform distribution over the agents' peaks whenever all peaks are degenerate. The independent markets mechanism violates efficiency, and Freeman et al. [22] raised the question of whether there are mechanisms that satisfy all three properties simultaneously. In Section 5, we settle this question by proving that no such mechanism exists under ℓ_1 as well as under ℓ_∞ preferences.

Using ℓ_1 distances to define preferences over distributions has some shortcomings when aiming for a suitable representation of alternatives in the collective distribution. For instance, if the agents are deciding the amount of time that should be allotted to three countries at an international conference, an agent with an ideal distribution of (10%, 40%, 50%) would find the outcome (0%, 45%, 55%) to be quite desirable according to the ℓ_1 distance, despite the fact that this outcome leaves the first country completely unrepresented. Moreover, any distance-based metric that aggregates coordinate-wise differences, such as ℓ_p for any $p \geq 1$, has similar shortcomings. For example, if a citizen believes that the two larger districts deserve 40% of the city budget each and the two smaller districts deserve 10% each, then ℓ_p preferences state that she is indifferent between the distributions (50%, 30%, 10%, 10%) and (40%, 40%, 20%, 0%) because for both distributions, the multiset of coordinate-wise differences is $\{0\%, 0\%, 10\%, 10\%\}$. But intuitively, the latter distribution is worse, as it leaves the last district without any funds. As a consequence, a different type of utility function is necessary to capture the representation of alternatives with respect to the ideal distribution.

We introduce a new class of star-shaped utility functions to the budget aggregation setting, where each agent's utility for a distribution equals the *smallest quotient*, over all alternatives, that the distribution preserves in comparison with the ideal distribution. Formally, agent i 's utility for a distribution \mathbf{q} is given by $\min_j q_j/p_{i,j}$, where the minimum is taken over all alternatives j for which $p_{i,j} > 0$. These utility functions are a special case of Leontief utility functions as commonly studied in economics, especially in consumer theory, with goods corresponding to alternatives (see, e.g., Goel et al. [24], Li and Xue [28], Nicoló [34]). In our setting, the total amount of goods is fixed, which allows us to interpret the relative distribution of Leontief weights as an ideal distribution. Agents then want all alternatives to receive as large a fraction as possible of their ideal amounts. We will refer to these utility functions as Leontief utility functions in the following. Leontief utilities are arguably more suitable than ℓ_1 preferences in applications where the representation of the alternatives is crucial—indeed, for both examples in the previous paragraph, distributions that allocate none of the budget to some alternative are least preferred among all distributions according to Leontief utilities. Not surprisingly, mechanisms that provide desirable properties such as strategyproofness with respect to ℓ_1 preferences may fail to do so with Leontief utilities.¹ Therefore, one needs to find different mechanisms when dealing with Leontief utilities. In Section 6, we show that maximizing Nash welfare, that is, the product of agents' utilities, results in a mechanism with several desirable properties. In fact, the impossibility for ℓ_1 preferences established in Section 5 can be turned into a complete characterization

for Leontief utilities: only the Nash product rule satisfies group-strategyproofness and a natural core-based fairness notion, which strengthens both efficiency and proportionality. Thus, in contrast to ℓ_1 preferences, Leontief utilities allow for the efficient, strategyproof, and fair aggregation of budgets via a unique attractive mechanism.

2. Related Work

Various streams of research have investigated the strategyproof aggregation of preferences when the space of alternatives is a subset of multidimensional Euclidean space.

2.1. Public Goods Provision

One setting that calls for such an aggregation occurs when the society has to decide on the amounts of m public goods. In this context, Barberà and Peleg [8] proved that any mechanism that is strategyproof for all continuous utilities, and whose range has more than two alternatives, is a dictatorship. Their result strongly hinges on the richness of the domain of preferences.

Zhou [37] proved an analogous theorem for the much more restricted class of *quadratic preferences*, a generalization of ℓ_2 -based preferences that is a special case of star-shaped preferences. Specifically, he proved that when the space of alternatives is an arbitrary convex subset of some finite-dimensional Euclidean space and the admissible domain contains all quadratic preferences, every strategyproof mechanism whose image has dimension at least two is dictatorial. Barberà and Jackson [7] complemented Zhou's result by characterizing the strategyproof mechanisms whose image has dimension one.

2.2. Probabilistic Social Choice

Any model in which preferences over the set of all lotteries Δ^m are aggregated to a collective lottery, including literature on probabilistic social choice (e.g., Gibbard [23]) and fair mixing (e.g., Bogomolnaia et al. [11]), can be interpreted in the context of budget aggregation. However, the underlying assumptions of linear preferences (vNM utilities) are hardly applicable to budget aggregation. This is because when utility functions are linear, the maximum utility must be attained in a corner of the simplex (at least when there is a unique maximum), which corresponds to a degenerate ideal distribution.

2.3. Peak-Based Preferences

To the best of our knowledge, Intriligator [26] was the first to consider a model in which each agent has a unique ideal distribution, and the considered mechanisms only need to aggregate individual distributions into a collective distribution. He proposed three simple axioms for this setting, which characterize the rule that returns the average (i.e., the arithmetic mean) of all individual distributions. Intriligator was not concerned with strategyproofness and therefore did not consider agents' preferences over distributions, but it is obvious that the average rule is manipulable for almost any reasonable definition of preferences.

Border and Jordan [12] studied separable star-shaped preferences when the space of alternatives is the m -dimensional Euclidean space \mathbb{R}^m . They focused on quadratic preferences and showed that strategyproof mechanisms can be decomposed into one-dimensional mechanisms. However, their results do not carry over to budget aggregation because their space of alternatives is the full Euclidean space and does not have a "budget constraint."

2.4. ℓ_1 -Based Preferences

Lindner et al. [30] initiated the study of strategyproof mechanisms for the space of all distributions Δ^m when preferences are based on the ℓ_1 -norm (see also Goel et al. [25], Lindner [29]). They showed that the utilitarian rule (i.e., minimizing the sum of ℓ_1 distances) satisfies strategyproofness and efficiency when breaking ties appropriately.

Freeman et al. [22] expanded the idea of Moulin's generalized median rules for two alternatives to strategyproof *moving phantom mechanisms* for larger numbers of alternatives m . Intuitively, the $n + 1$ phantoms are not "fixed" like in Moulin's characterization but increase continuously from zero to one over time. For any point in time and any alternative j , the mechanism computes the median of $p_{1,j}, \dots, p_{n,j}$ and the phantom voters. Freeman et al. [22] then showed that there exists a well-defined point in time where the m medians sum up to one and thus form a valid distribution. Furthermore, they proved that within this class, maximizing utilitarian welfare is the unique efficient mechanism. A different mechanism in this class, the *independent markets* mechanism, is inefficient but satisfies a fairness notion they called *proportionality*: when all voters have degenerate peaks, the collective distribution is the arithmetic mean of these peaks. Freeman et al. observed an "inherent tradeoff between

Pareto optimality and proportionality” for strategyproof mechanisms. We prove this trade-off formally in Theorem 1, which shows that all three properties are incompatible. Proportionality was generalized by Caragiannis et al. [16], who measured the “disproportionality” of a mechanism as the worst-case ℓ_1 distance between the mechanism outcome and the mean. Similarly, Freeman and Schmidt-Kraepelin [21] measured disproportionality using the ℓ_∞ distance. Both papers present variants of moving phantom mechanisms that guarantee low disproportionality. Elkind et al. [19] defined various axioms for budget aggregation with ℓ_1 disutilities, analyzed the implications between axioms, and determined which axioms are satisfied by common aggregation rules.

2.5. Belief Aggregation

Belief aggregation is a setting in which several experts have different beliefs, expressed as probability density functions over a set of potential outcomes. The goal is to construct a single aggregated distribution. Technically, the problem is similar to budget aggregation; however, the utility functions are often different. Varloot and Laraki [36] assumed that the outcomes are linearly ordered (for example, outcome j is an earthquake of magnitude j). Then, an expert whose belief is 3 with probability one would prefer the outcome 6 with probability one to the outcome 9 with probability one, even though the ℓ_1 distance is two in both cases. Varloot and Laraki suggested preferences based on distance between the *cumulative* distribution functions and characterized aggregation rules satisfying appropriate strategyproofness and proportionality axioms.

2.6. Donor Coordination

Brandt et al. [15] studied *donor coordination*, where individual monetary contributions by agents are distributed on projects based on the agents’ preferences. Assuming Leontief preferences, they proposed the *equilibrium distribution rule (EDR)*, which maximizes Nash welfare and distributes the contributions of the donors in such a way that no subset of donors has an incentive to redistribute their contributions. EDR can be interpreted as a budget aggregation mechanism by setting the contribution of each agent to $1/n$ (where n is the number of agents, so the total contribution is one) and setting the ideal distribution of an agent to the distribution given by the relative proportions of the Leontief weights. This allows the transfer of positive results concerning efficiency and strategyproofness of EDR from donor coordination to budget aggregation (see Section 6 and Appendix E). However, other properties considered by Brandt et al., such as contribution monotonicity and being in equilibrium, are irrelevant in budget aggregation, whereas properties such as core fair share and proportionality have not been considered in donor coordination.

3. Preliminaries

Let $N = [n]$ be a set of agents and $M = [m]$ be a set of alternatives, where $[k] := \{1, \dots, k\}$ for each positive integer k . We denote by Δ^m the standard simplex with m vertices, that is, the set of vectors $\mathbf{q} = (q_1, \dots, q_m)$ with nonnegative entries q_j such that $\sum_{j \in M} q_j = 1$. Every element $\mathbf{q} \in \Delta^m$ is called a *distribution*; here, q_j denotes the fraction of a public resource (e.g., money or time) allocated to alternative j . The support of distribution \mathbf{q} is given by $\text{supp}(\mathbf{q}) := \{j \in M : q_j > 0\}$. For a set of alternatives $T \subseteq M$, we denote $\mathbf{q}(T) := \sum_{j \in T} q_j$.

Each agent i has a utility function u_i over distributions; we denote by \mathcal{U} the set of all possible utility functions and leave this set unspecified for now.

The *ideal distribution* or *peak* of agent i is denoted by $\mathbf{p}_i := \arg \max_{\mathbf{q} \in \Delta^m} u_i(\mathbf{q})$, which is assumed to be unique. We further assume that “walking” toward an agent’s peak strictly increases her utility. Formally, if agent i has peak \mathbf{p}_i and utility function u_i , then, for any distribution $\mathbf{q} \neq \mathbf{p}_i$, the agent prefers to \mathbf{q} any distribution on the line between \mathbf{q} and \mathbf{p}_i . That is, for all $\lambda \in (0, 1)$,

$$u_i(\mathbf{p}_i) > u_i(\lambda \mathbf{p}_i + (1 - \lambda) \mathbf{q}) > u_i(\mathbf{q}). \tag{1}$$

This constitutes a generalization of single-peakedness (Black [10]) and is referred to as *star-shaped* or *star-convex* preferences (e.g., Border and Jordan [12]).²

The utility functions considered in this paper belong to a subclass of star-shaped preferences that we refer to as *peak-linear*. A utility function is peak-linear if, for any distribution \mathbf{q} and $\lambda \in [0, 1]$,

$$u_i(\lambda \mathbf{p}_i + (1 - \lambda) \mathbf{q}) = \lambda u_i(\mathbf{p}_i) + (1 - \lambda) u_i(\mathbf{q}). \tag{2}$$

Every distribution $\mathbf{q} \in \Delta^m$ lies on a line between \mathbf{p}_i and some point \mathbf{q}^B on the boundary of Δ^m ; that is, there is a unique λ such that $\mathbf{q} = \lambda \mathbf{p}_i + (1 - \lambda) \mathbf{q}^B$. Therefore, peak-linear preferences are completely characterized by an agent’s peak \mathbf{p}_i and by how much utility she assigns to these boundary distributions \mathbf{q}^B (in fact, it is sufficient to know how much utility she assigns to boundary distributions with $\text{supp}(\mathbf{p}_i) \not\subseteq \text{supp}(\mathbf{q}^B)$).

Another important property of preferences is *convexity*. An agent has convex preferences if, for every three outcomes \mathbf{q} , \mathbf{q}' , and \mathbf{q}'' where \mathbf{q}' and \mathbf{q}'' are both weakly preferred to \mathbf{q} , any mixture $\lambda\mathbf{q}' + (1 - \lambda)\mathbf{q}''$ with $\lambda \in [0, 1]$ is also weakly preferred to \mathbf{q} . For *strict convexity*, any mixture of \mathbf{q}' and \mathbf{q}'' with $\lambda \in (0, 1)$ has to be strictly preferred to \mathbf{q} unless $\mathbf{q}' = \mathbf{q}''$ (see, e.g., Mas-Colell et al. [31]). All strictly convex preferences are star-shaped, but this does not hold for all convex preferences: assuming only convexity is neither sufficient to guarantee a unique peak, nor does it imply that an agent's utility is *strictly* increasing when moving toward her peak. The preferences we consider here (specifically, ℓ_1 , ℓ_∞ , and Leontief) are star-shaped and convex, but not strictly convex. For example, Leontief preferences indicate indifference between all distributions \mathbf{q} with $\text{supp}(\mathbf{p}_i) \not\subseteq \text{supp}(\mathbf{q})$. Therefore, we use star-shaped/peak-linear preferences as a collective term for all investigated utility models.

Unless explicitly stated otherwise, we further assume that \mathbf{p}_i completely determines u_i , so we can identify a profile $P \in \mathcal{P} := (\Delta^m)^n$ with the matrix $(p_{i,j})_{i \in N, j \in M}$ containing the peaks $\mathbf{p}_1, \dots, \mathbf{p}_n$ as rows.

A *mechanism* $f : \mathcal{P} \rightarrow \Delta^m$ aggregates individual peaks into a collective distribution. In the following, we define desirable properties of aggregated distributions and mechanisms.

3.1. Properties of Distributions

Two important properties of distributions are efficiency and fairness. We consider two efficiency properties.

Definition 1. A distribution $\mathbf{q} \in \Delta^m$ satisfies (*Pareto*) *efficiency* if there does not exist a distribution $\mathbf{q}' \in \Delta^m$ such that $u_i(\mathbf{q}') \geq u_i(\mathbf{q})$ for all $i \in N$ and $u_i(\mathbf{q}') > u_i(\mathbf{q})$ for at least one $i \in N$.

Definition 2. A distribution $\mathbf{q} \in \Delta^m$ is *range-respecting* for profile P if $\min_{i \in N} p_{i,j} \leq q_j \leq \max_{i \in N} p_{i,j}$ for all $j \in M$.

With ℓ_1 utilities, range-respect is weaker than Pareto efficiency (see, e.g., Elkind et al. [19], Freeman et al. [21]).

Our first fairness axiom is inspired by the core in cooperative game theory and was transferred to participatory budgeting by Fain et al. [20] and to fair mixing by Aziz et al. [5] under the name of core fair share. We adapt the definition to account for the fact that in the end, we still need to choose a probability distribution \mathbf{p} (and not just a partial distribution $(|N'|/n)\mathbf{p}$) and use the weak core rather than the strong one. In fact, this leads us back to the original definition of the α -core due to Aumann [3] and Scarf [35].³

Definition 3. A group of agents $N' \subseteq N$ is called a *blocking coalition* for a distribution \mathbf{q} if there exists a distribution $\mathbf{q}' \in \Delta^m$ such that for every $\mathbf{q}'' \in \Delta^m$,

$$u_i((|N'|/n)\mathbf{q}' + (1 - |N'|/n)\mathbf{q}'') > u_i(\mathbf{q}) \quad \text{for all } i \in N'.$$

A distribution \mathbf{q} satisfies *core fair share* if it has no blocking coalition.

Intuitively, a blocking coalition N' can take their share of the decision power $(|N'|/n)$ and redistribute it via \mathbf{q}' so that all members of N' gain utility compared with \mathbf{q} , even if the remainder is distributed in the worst possible way \mathbf{q}'' (e.g., on an alternative that no agent from N' values).

We consider another, weaker fairness axiom that is only informative on specific profiles in the next subsection.

3.2. Properties of Aggregation Mechanisms

Definition 4. A mechanism f satisfies *efficiency* (respectively, *core fair share*) if, for every profile $P \in \mathcal{P}$, $f(P)$ satisfies efficiency (respectively, *core fair share*).

The next axioms ensure that agents and alternatives are treated independently of their identities.

Definition 5. A mechanism f satisfies *anonymity* if, for every profile $P \in \mathcal{P}$ and permutation π of the agents in P , it holds that $f(P) = f(\pi \circ P)$.

Definition 6. A mechanism f satisfies *neutrality* if, for every profile $P \in \mathcal{P}$ and permutation π of the alternatives resulting in profile P' , it holds that $f(P') = \pi \circ f(P)$.

As agents report a peak in Δ^m , we do not want small perturbations of the peaks arising from uncertainties of the agents about their exact peak or inaccuracies during the aggregation process to have a large influence on the outcome.

Definition 7. A mechanism f satisfies *continuity* if

$$\forall P \in \mathcal{P} : \forall \varepsilon > 0 : \exists \delta > 0 : \forall P' \in \mathcal{P} : \|P - P'\|_1 < \delta \implies \|f(P) - f(P')\|_1 < \varepsilon.$$

For simplicity, we define continuity using the ℓ_1 distance, but note that because of the norm equivalence on finite-dimensional vector spaces, our results are the same for every other norm-based distance.

As $\mathcal{P} = (\Delta^m)^n$ is compact with respect to the ℓ_1 distance (or other equivalent norms), the Heine–Cantor theorem implies that a continuous mechanism f is also *uniformly continuous*; that is,

$$\forall \varepsilon > 0 : \exists \delta > 0 : \forall P, P' \in \mathcal{P} : \|P - P'\|_1 < \delta \implies \|f(P) - f(P')\|_1 < \varepsilon.$$

This insight will play an important role in the Proof of Theorem 3.

Another common goal is to prevent agents from misreporting their peaks on purpose.

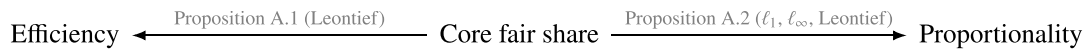
Definition 8. A mechanism f satisfies *group-strategyproofness* if, for all $N' \subseteq N$ and all $P, P' \in \mathcal{P}$ with $\mathbf{p}_j = \mathbf{p}'_j$ for all $j \notin N'$, either $u_i(f(P)) > u_i(f(P'))$ for at least one $i \in N'$ or $u_i(f(P)) = u_i(f(P'))$ for all $i \in N'$, where u_i refers to the utility function of agent i with peak at \mathbf{p}_i . The mechanism f satisfies *strategyproofness* if the above statement holds for $|N'| = 1$.

Finally, we consider another fairness property called *proportionality* by Freeman et al. [22]. It restricts the set of outcomes only on profiles where all agents are “single-minded,” thus representing a rather weak form of “traditional” proportionality considered, for example, in fair division.

Definition 9. A profile $P \in \mathcal{P}$ is called *single-minded* if $p_{i,j} \in \{0, 1\}$ for all $i \in N$ and $j \in M$.

Definition 10. A mechanism f satisfies *proportionality* if, for all single-minded profiles $P \in \mathcal{P}$, it holds that $f(P)_j = \sum_{i \in N} p_{i,j} / n$.

The following diagram shows logical relationships between efficiency and the fairness notions we consider. The proofs of Propositions A.1 and A.2 can be found in Appendix A.



4. Specific Star-Shaped Preference Domains

We initially discuss some typical star-shaped utility functions, for which we will present results in the next sections.

4.1. ℓ_p Preferences

A natural approach for specifying a utility function based on a single peak is to measure the distance to the peak using some metric $d : \Delta^m \times \Delta^m \rightarrow \mathbb{R}_{\geq 0}$. Given an agent with peak \mathbf{p}_i , her utility for a distribution \mathbf{q} is then defined as $u_i(\mathbf{q}) = -d(\mathbf{p}_i, \mathbf{q})$.

Among such models, ℓ_p norms, given by $\|\mathbf{q}\|_p := (\sum_{j \in M} |q_j|^p)^{1/p}$ for $p \geq 1$, are the most studied utility functions. In particular, the special case of $p = 1$ has received considerable attention.

Definition 11. An agent i with peak \mathbf{p}_i has ℓ_p preferences if $u_i(\mathbf{q}) = -\|\mathbf{p}_i - \mathbf{q}\|_p$.

We will also sometimes refer to these preferences as ℓ_p disutilities. In this paper, we focus on ℓ_1 preferences ($u_i(\mathbf{q}) = -\sum_{j \in M} |p_{i,j} - q_j|$) and ℓ_∞ preferences ($u_i(\mathbf{q}) = -\max_{j \in M} |p_{i,j} - q_j|$). It can be easily shown that ℓ_p preferences are star-shaped for $p \geq 1$. This does not hold for arbitrary metrics; for example, consider the *trivial metric*, $d(x, y) = 0$ if $x = y$, and $d(x, y) = 1$ otherwise. Moreover, ℓ_p preferences are peak-linear. The utility of every point in Δ^m can be computed using (2) with $u_i(\mathbf{p}_i) = 0$ and $u_i(\mathbf{q}) = -\|\mathbf{p}_i - \mathbf{q}\|_p$ for all distributions \mathbf{q} on the boundary of Δ^m .

4.2. Leontief Utilities

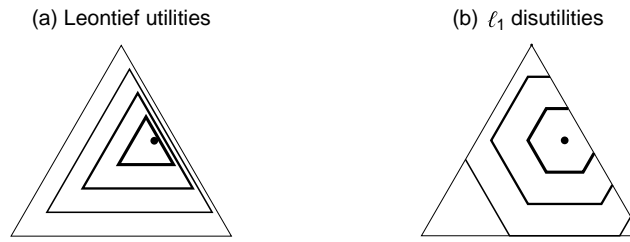
In contrast to ℓ_p preferences, Leontief utilities are not based on a metric. In particular, they are not symmetric and also not based on disutilities. As discussed in the introduction, metric-based preferences fail to capture important aspects of certain practical situations, notably the need to guarantee that all alternatives are adequately represented. This requirement is captured by Leontief utilities.

Let $M_i := \{j \in M : p_{i,j} > 0\}$ be the set of alternatives to which i wants to allocate a positive amount; note that $M_i \neq \emptyset$. The *Leontief utility* that agent i derives from a distribution \mathbf{q} is given by

$$u_i(\mathbf{q}) = \min_{j \in M_i} \frac{q_j}{p_{i,j}}.$$

Observe that $0 \leq u_i(\mathbf{q}) \leq 1$ for all distributions \mathbf{q} . Moreover, $u_i(\mathbf{q}) = 1$ if and only if $\mathbf{q} = \mathbf{p}_i$, and $u_i(\mathbf{q}) = 0$ if and only if $q_j = 0$ for some $j \in M_i$. As discussed in Section 1, Leontief utilities are based on the assumption that agents want all alternatives to receive as large a fraction of their ideal amounts as possible. The indifference curves of ℓ_1 preferences and Leontief utilities are illustrated in Figure 1.

Figure 1. Illustration of indifference classes for Leontief utilities and ℓ_1 disutilities for three alternatives.



Notes. The ideal distribution $(0.1, 0.4, 0.5)$ is represented by the black point. The main triangle is the simplex of distributions among three alternatives. Its vertices represent the degenerate distributions $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$. For each type of utility, the peak forms an indifference class by itself, and three other indifference classes are displayed with different line widths.

Leontief utility functions are peak-linear with $u_i(\mathbf{p}_i) = 1$ and $u_i(\mathbf{q}) = 0$ for all boundary distributions \mathbf{q} . In fact, as explained in the preliminaries, Leontief utilities are characterized by these properties: they are the only peak-linear utilities that assign utility one to the peak and utility zero to all boundary distributions. Some useful concepts and insights on Leontief utilities are collected in Appendix A.

It is possible to refine Leontief utilities further by considering the *leximin* over the quotients—that is, breaking ties in the smallest quotient using the second-smallest quotient, and so on. This refinement is discussed in Appendix G.

5. Impossibilities for ℓ_1 and ℓ_∞ Preferences

In this section, we show that efficiency, strategyproofness, and the rather weak fairness condition of proportionality (cf. Proposition A.2) are incompatible when agents have ℓ_1 or ℓ_∞ preferences.

5.1. ℓ_1 Preferences

Under ℓ_1 preferences, Freeman et al. [22] observed that the utilitarian welfare maximizing mechanism is the only efficient mechanism in their class of moving phantom mechanisms. However, maximizing utilitarian welfare violates weak fairness axioms such as proportionality. We prove that this trade-off between efficiency and fairness is inevitable in the presence of strategyproofness.

Theorem 1. *With ℓ_1 preferences, no mechanism satisfies efficiency, strategyproofness, and proportionality when $m \geq 3$ and $n \geq 3$.*

For the proof of this theorem, we consider disutilities d_i (the ℓ_1 distance to an agent’s peak p_i) instead of utilities; that is, $d_i(\mathbf{q}) = \|\mathbf{p}_i - \mathbf{q}\|_1$. It is also important to keep the following simple observations in mind.

Observation 1. With ℓ_1 preferences, if some agent i has $p_{i,j} = 1$, then for any distribution \mathbf{q} , $d_i(\mathbf{q}) = 2 - 2q_j$, regardless of the distribution on alternatives other than j . Therefore, agent i is indifferent if some amount is moved between alternatives other than j .

Observation 2. With ℓ_1 preferences, $d_i(\mathbf{q}) \geq 2 \cdot |p_{i,j} - q_j|$ for all $j \in M$ and $i \in N$.

Proof of Theorem 1. We start with the case $m = 3$ and $n = 3$. For $m = 3$, we set $M = \{a, b, c\}$ and write $\mathbf{q} = (q_a, q_b, q_c)$. For simplicity, we number profiles by a superscript (k) . We denote the disutility function of agent i in profile k by $d_i^{(k)}$ and the returned distribution in profile k by $\mathbf{q}^{(k)}$. Sometimes, we cannot determine $\mathbf{q}^{(k)}$ completely. In these cases, we give lower or upper bounds on the entries of $\mathbf{q}^{(k)}$.

Consider first the following two profiles. The outcome in profile 2 must be $(1/3, 1/3, 1/3)$ by proportionality. For profile 1, the last row of the table gives lower bounds on $\mathbf{q}_a^{(1)}, \mathbf{q}_b^{(1)}$ and an upper bound on $\mathbf{q}_c^{(1)}$, which we justify below.

Profile 1			Profile 2		
a	b	c	a	b	c
1/2	1/2	0	1	0	0
0	1	0	0	1	0
0	0	1	0	0	1
$\mathbf{q}^{(1)} \geq 1/6$	$\geq 1/2$	$\leq 1/3$	$\mathbf{q}^{(2)} 1/3$	$1/3$	$1/3$

As agent 1 can manipulate between profile 1 and profile 2, strategyproofness requires that agent 1 does not gain from either manipulation. This implies

$$d_1^{(1)}(\mathbf{q}^{(1)}) \leq d_1^{(1)}(\mathbf{q}^{(2)}) = 2/3, \text{ and} \quad (3)$$

$$d_1^{(2)}(\mathbf{q}^{(1)}) \geq d_1^{(2)}(\mathbf{q}^{(2)}) = 4/3. \quad (4)$$

By (3) and Observation 2, $q_a^{(1)} \geq 1/6$ (implying $q_b^{(1)} \leq 5/6$), $q_b^{(1)} \geq 1/6$, and $q_c^{(1)} \leq 1/3$. By (4) and Observation 1, $q_a^{(1)} \leq 1/3$, implying $q_b^{(1)} + q_c^{(1)} \geq 2/3$, and thus, $q_b^{(1)} \geq 1/3$.

Figure 2(a) illustrates both inequalities. The light-gray area corresponds to the set of distributions $\mathbf{q}^{(1)}$ with $d_1^{(1)}(\mathbf{q}^{(1)}) \leq 2/3$, whereas the medium-gray area consists of all distributions $\mathbf{q}^{(1)}$ satisfying (4). Strategyproofness requires $\mathbf{q}^{(1)}$ to be inside the intersection of the two areas, that is, the dark-gray region. Hence,

$$1/6 \leq q_a^{(1)} \leq 1/3, \quad 1/3 \leq q_b^{(1)} \leq 5/6, \quad \text{and} \quad 0 \leq q_c^{(1)} \leq 1/3.$$

By efficiency, we can even show that $q_b^{(1)} \geq 1/2$. Otherwise, as $q_a^{(1)} > 0$, some small amount could be moved from a to b . Agent 3 is indifferent because of Observation 1, and agent 2 strictly gains. Furthermore, this does not change agent 1's disutility as $q_b^{(1)} < 1/2$.

Next, we consider the following two profiles.

The outcome in profile 4 follows from proportionality. We now prove that the outcome in profile 3 must be identical. As agent 1 can manipulate between profile 3 and profile 4, strategyproofness requires that agent 1 does

Profile 3				Profile 4			
	a	b	c		a	b	c
	1/4	3/4	0		0	1	0
	0	1	0		0	1	0
	0	0	1		0	0	1
$\mathbf{q}^{(3)}$	0	2/3	1/3	$\mathbf{q}^{(4)}$	0	2/3	1/3

not gain from either manipulation. This implies that

$$d_1^{(3)}(\mathbf{q}^{(3)}) \leq d_1^{(3)}(\mathbf{q}^{(4)}) = 2/3, \text{ and} \quad (5)$$

$$d_1^{(4)}(\mathbf{q}^{(3)}) \geq d_1^{(4)}(\mathbf{q}^{(4)}) = 2/3. \quad (6)$$

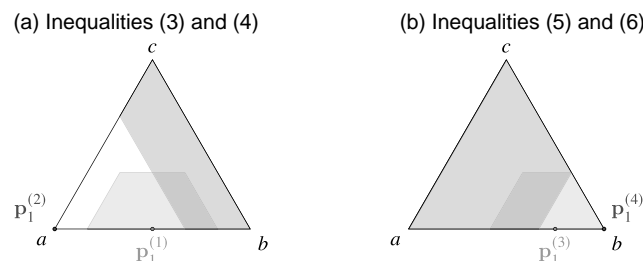
By (5), $q_c^{(3)} \leq 1/3$, implying $q_a^{(3)} + q_b^{(3)} \geq 2/3$. By (6), $q_b^{(3)} \leq 2/3$. Graphically, strategyproofness for agent 1 implies that $\mathbf{q}^{(3)}$ must be in the dark-gray region in Figure 2(b).

However, by efficiency, if $q_a^{(3)} > 0$, then $q_b^{(3)} \geq 3/4$. Otherwise, some small amount can be moved from a to b . Agent 3 is indifferent because of Observation 1, and agent 2 strictly gains. Furthermore, this does not change agent 1's disutility as $q_b^{(3)} < 3/4$. Therefore, $q_a^{(3)} = 0$ must hold, and the only outcome compatible with strategyproofness is $\mathbf{q}^{(3)} = (0, 2/3, 1/3)$.

Now that we know $\mathbf{q}^{(3)}$, we consider a manipulation of agent 1 from profile 3 to profile 1. Strategyproofness implies

$$d_1^{(3)}(\mathbf{q}^{(1)}) \geq d_1^{(3)}(\mathbf{q}^{(3)}) = 2/3.$$

Figure 2. Illustration for the Proof of Theorem 1.



But the bounds we already have for $\mathbf{q}^{(1)}$ imply that $d_1^{(3)}(\mathbf{q}^{(1)}) \leq 2/3$ as $q_a^{(1)} \geq 1/6$ and $q_b^{(1)} \geq 1/2$. Therefore, $d_1^{(3)}(\mathbf{q}^{(1)}) = 2/3$ together with $q_a^{(1)} = 1/6$ and $q_b^{(1)} = 1/2$. Hence, $\mathbf{q}^{(1)} = (1/6, 1/2, 1/3)$.

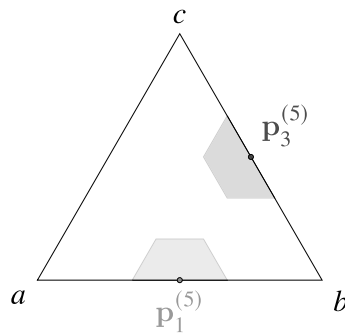
Finally, we consider profiles 5 and 6.

Profile 5			Profile 6		
a	b	c	a	b	c
1/2	1/2	0	1	0	0
0	1	0	0	1	0
0	1/2	1/2	0	1/2	1/2
$\mathbf{q}^{(5)}$			$\mathbf{q}^{(6)}$		
			1/3	1/2	1/6

The distribution $\mathbf{q}^{(6)}$ is determined by arguments analogous to those for $\mathbf{q}^{(1)}$, reasoning about agent 3 instead of agent 1.

We now consider a manipulation of agent 1 from profile 5 to profile 6. It follows from strategyproofness that $d_1^{(5)}(\mathbf{q}^{(5)}) \leq d_1^{(5)}(\mathbf{q}^{(6)}) = 1/3$, which implies that $q_c^{(5)} \leq 1/6$. Similarly, we consider a manipulation of agent 3 from profile 5 to profile 1. It follows from strategyproofness that $d_3^{(5)}(\mathbf{q}^{(5)}) \leq d_3^{(5)}(\mathbf{q}^{(1)}) = 1/3$, which implies that $q_c^{(5)} \geq 1/2 - 1/6 = 1/3$, a contradiction.

Graphically, both inequalities are shown in the figure below. The light-gray area at the bottom contains the points satisfying the first inequality, and the medium-gray area on the right contains the points satisfying the second inequality. It is evident that the two inequalities cannot be satisfied simultaneously.



This example can be extended to arbitrary numbers of alternatives and agents in the following way.

To increase the number of alternatives, simply add alternatives j^+ with $p_{i,j^+} = 0$ for all agents i . These new alternatives do not affect the argument, as efficiency implies range-respect, which ensures that none of them ever receives a positive amount.

Adding agents is more involved, as our proof relies on explicit distributions induced by proportionality and thus depends on the number of agents. However, we note that throughout the proof, agent 2 always has the same peak, which puts all mass on alternative b . Therefore, when adding agents i^+ with $p_{i^+,b} = 1$, we can run through the same proof but with adapted distributions. The generalized proof can be found in Appendix B. \square

Remark 1. The bounds $m \geq 3$ and $n \geq 3$ in Theorem 1 are tight. Indeed, there exists a moving phantom mechanism that satisfies strategyproofness, proportionality, and range-respect (Freeman et al. [22, p. 22]), and it is known that range-respect and efficiency coincide when $m = 2$ or $n = 2$ (Elkind et al. [19, section 8]).

The three axioms required for the impossibility are independent. Indeed, efficiency and strategyproofness (without proportionality) are satisfied by the mechanism that maximizes utilitarian welfare (Lindner et al. [30]), strategyproofness and proportionality (without efficiency) are satisfied by the independent markets mechanism (Freeman et al. [22]), and proportionality and efficiency (without strategyproofness) are satisfied by a natural generalization of the maximum Nash welfare mechanism studied by Aziz et al. [6].

Freeman et al. [22, p. 30] posed the question of whether every anonymous, neutral, continuous, and strategy-proof mechanism can be represented as a moving phantom mechanism. Whereas such a characterization might have the potential to simplify the previous proof, it does not hold in general; see Appendix C.

5.2. ℓ_∞ Preferences

ℓ_1 preferences take a special role among ℓ_p disutilities in terms of efficiency: indifference curves partially move along distributions with a constant sum on “approved” ($p_{i,j} > 0$) alternatives. As an example, consider $M = \{a, b, c, d\}$ and an agent i with peak $\mathbf{p}_i = (1/2, 1/2, 0, 0)$. With ℓ_1 preferences, she is indifferent between all distributions \mathbf{q} with $q_a + q_b = 1/2$. This implies that if we have two agents and the second agent i' has $\mathbf{p}_{i'} = (0, 0, 0, 1)$, every efficient distribution \mathbf{q} with $q_a + q_b = 1/2$ (equivalently, $q_c + q_d = 1/2$) must put zero on alternative c and $1/2$ on alternative d . By contrast, when considering, for example, ℓ_2 preferences, it also matters for agent i how $1/2$ is distributed on c and d . As a result, more distributions become efficient, which weakens the role of efficiency for a potential impossibility when $p > 1$.

We proceed by proving an impossibility for the preference model at the other end of the spectrum: ℓ_∞ preferences. These preferences behave similarly to ℓ_1 disutilities (Observation 1), which is helpful when arguing about efficiency.

Observation 3. With ℓ_∞ preferences, if some agent i has $p_{i,j} = 1$, then, for any distribution \mathbf{q} , $d_i(\mathbf{q}) = 1 - q_j$, regardless of the distribution on alternatives other than j . Therefore, agent i is indifferent if some amount is moved between alternatives other than j .

Theorem 2. With ℓ_∞ preferences, no mechanism satisfies efficiency, strategyproofness, and proportionality when $m \geq 3$ and $n \geq 3$.

The Proof of Theorem 2 can be found in Appendix D. It uses the same profiles as the one for Theorem 1 but needs more involved arguments when reasoning about efficiency and extending the argument to $m > 3$ alternatives. The reason is that with ℓ_∞ disutilities (in contrast to ℓ_1), efficiency does not imply range-respect, and an efficient distribution might allocate a positive amount to an alternative to which all agents allocate zero. For example, let $m = 4$ and $n = 2$ with peaks $(1/2, 1/4, 1/4, 0)$ and $(1/4, 1/2, 1/4, 0)$. Then, $(3/8, 3/8, 1/8, 1/8)$ is efficient, as the maximal distance is $1/8$ for both agents, and if agent 1 is better off in distribution \mathbf{q} , this means $q_1 > 3/8$, which decreases agent 2’s utility.

Remark 2. Theorem 2 requires $m \geq 3$ because for $m = 2$, ℓ_1 and ℓ_∞ preferences are single-peaked, and there are mechanisms that satisfy all requirements (see Remark 1). Moreover, $n \geq 3$ is required because for $m = 3$, the ℓ_∞ and ℓ_1 metrics are equivalent—the ℓ_1 distance is always twice the ℓ_∞ distance. Hence, the lower bound on n has to be the same as in Theorem 1.

Similar to the impossibility for ℓ_1 preferences, we expect all axioms to be independent. However, to the best of our knowledge, this does not follow from existing results, as ℓ_∞ preferences have been studied significantly less than ℓ_1 .

We conjecture that the incompatibility of efficiency, strategyproofness, and weak fairness conditions holds for ℓ_p disutilities for any $1 \leq p \leq \infty$ when $m \geq 3$ and $n \geq 3$.

In the next section, we demonstrate that the impossibility does not generalize to arbitrary peak-linear utility functions.

6. The Nash Product Rule for Leontief Preferences

In this section, we assume that preferences are representable by Leontief utilities. Inspired by the positive results obtained by maximizing the product of utilities in similar contexts (in particular, Brandt et al. [15]), we define the *Nash product rule* for budget aggregation as follows. For any $P \in \mathcal{P}$,

$$\text{NASH}[P] = \arg \max_{\mathbf{q} \in \Delta^m} \prod_{i \in N} u_i(\mathbf{q}).$$

NASH is well defined, as it always returns exactly one distribution (Brandt et al. [15]).

6.1. Properties

In this section, we investigate properties of *NASH* for budget aggregation.

Anonymity follows immediately from the fact that multiplication is commutative. Neutrality is also straightforward, as *NASH* does not take into account the identities of the alternatives. Moreover, *NASH* satisfies continuity, which will be important for the axiomatic characterization we give in the next section.

Proposition 1. With Leontief utilities, *NASH* is continuous.

In addition, *NASH* is efficient and group-strategyproof (Brandt et al. [15]).⁴ Furthermore, *NASH* satisfies core fair share (and thus, also proportionality), showing that the impossibilities for ℓ_1 and ℓ_∞ preferences observed in Section 5 do not apply for Leontief preferences.

Proposition 2. *With Leontief utilities, NASH satisfies core fair share.*

The proofs of Propositions 1 and 2 can be found in Appendix E.1.

6.2. Characterization

In fact, *NASH* even admits an appealing characterization via strategyproofness and fairness.

Theorem 3. *With Leontief utilities, NASH is the only continuous mechanism that satisfies group-strategyproofness and core fair share.*

The Proof of Theorem 3 can be found in Appendix E.2. In light of the theorem, *NASH* appears to be the ideal mechanism for aggregating Leontief preferences.

As to the independence of the axioms, it is easy to see that core fair share is required for Theorem 3 because any constant mechanism satisfies continuity and group-strategyproofness. The necessity of (group-)strategyproofness can be shown by slightly perturbing the outcome of *NASH*. For example, consider $n = m = 2$ and the two peaks at $1/4$ and $3/4$, respectively. *NASH* returns $\mathbf{q} = (1/2, 1/2)$ with $u_i(\mathbf{q}) = 2/3$. However, core fair share only guarantees utility $1/2$ for agent i (Lemma A.2). So any distribution that puts at least $3/8$ on both alternatives satisfies core fair share. This “gap” can be used to slightly change the outcome without violating core fair share. Defining such changes in a continuous way (note that if both agents have the same peak, that change is zero) results in a different continuous mechanism satisfying core fair share.

We conjecture that continuity is required for the characterization as well.

The condition of group-strategyproofness is used only at one point of the proof (in Lemma E.4). If we assume that agents have *Leximin–Leontief preferences* (that is, subject to maximizing the smallest ratio, they maximize the second-smallest ratio, etc.), then this condition can be weakened to ordinary strategyproofness; the proof is given in Appendix G.

Theorem 4. *With Leximin–Leontief preferences, NASH is the only continuous mechanism that satisfies strategyproofness and core fair share.*

7. The Case of Two Alternatives

Notably, the impossibility results for ℓ_1 and ℓ_∞ preferences require $m > 2$. For $m = 2$, the set of possible outcomes becomes one-dimensional, and the class of all star-shaped utilities coincides with the well-studied class of *single-peaked utilities*.

For two alternatives $M = \{a, b\}$, the set of outcomes can be identified with the unit interval $[0, 1]$, where the endpoints 0 and 1 correspond to allocating the entire budget to alternatives a and b , respectively.

Denote by \mathcal{U}^{SP} the set of all single-peaked utility functions. We denote agent i 's peak p_i as a scalar in $[0, 1]$ representing her favorite distribution $[1 - p_i : a, p_i : b]$.

In this section, we relax our assumption that \mathcal{U} has to contain *exactly* one utility function per peak in $[0, 1]$; we only demand that \mathcal{U} contains *at least* one utility function per peak and implicitly assume that property from now on for all $\mathcal{U} \subseteq \mathcal{U}^{SP}$. The “at least one” requirement is needed to allow agents to misreport their peak to any other peak in $[0, 1]$. Note that we still require our mechanisms to be *tops-only*, that is, depend only on the agents' peaks. This generalization is possible because the mechanisms we characterize satisfy strategyproofness and further desirable properties without relying on any knowledge of the agents' utility functions except their peaks.

Under the assumption that $\mathcal{U} = \mathcal{U}^{SP}$, Moulin [33] characterized the set of all strategyproof mechanisms as generalized median rules. This characterization assumes that the rules have to handle *all* profiles in \mathcal{U}^{SP} . As a consequence, it no longer holds when restricting \mathcal{U} to a strict subset of \mathcal{U}^{SP} , which often occurs in practical applications, for example, when preferences are symmetric around the peak. In principle, allowing rules to handle only a subset of \mathcal{U}^{SP} may enable a greater selection of strategyproof rules. This possibility has been studied in some later works (Aziz et al. [6], Berga and Serizawa [9], Border and Jordan [12], Freeman et al. [22], Massó and de Barreda [32]), which extended Moulin's result for some alternative axioms and to some specific subdomains of \mathcal{U}^{SP} .

We substantially generalize these results by proving that Moulin's characterization holds for *any* subdomain $\mathcal{U} \subseteq \mathcal{U}^{SP}$ when assuming continuity of the mechanism in addition. The proofs are deferred to Appendix F.

Theorem 5. For $m = 2$ and arbitrary domain $\mathcal{U} \subseteq \mathcal{U}^{SP}$, a continuous mechanism f satisfies anonymity and strategyproofness if and only if there exist $\alpha_0 \leq \alpha_1 \leq \dots \leq \alpha_n$ in $[0, 1]$ such that

$$f(P) = \text{med}(p_1, \dots, p_n, \alpha_0, \dots, \alpha_n).$$

The *uniform phantom rule* is a special case of the generalized median rule in which the peaks are distributed uniformly in $[0, 1]$; that is, $\alpha_k = k/n$ for $k \in \{0, \dots, n\}$. It is range-respecting, as at least $n + 1$ arguments to the median (namely, $\alpha_0, p_1, \dots, p_n$) are at most $\max_i p_i$, and at least $n + 1$ arguments to the median (namely, $\alpha_n, p_1, \dots, p_n$) are at least $\min_i p_i$.

Freeman et al. [22] showed that the uniform phantom rule is the only mechanism that ensures proportionality in addition to all axioms from Theorem 5. Aziz et al. [6] strengthened this result by pointing out that continuity, strategyproofness, and proportionality suffice for characterizing the uniform phantom mechanism for symmetric single-peaked preferences. Note that proportionality already contains some form of anonymity: when all agents have peaks at $[1:a]$ or $[1:b]$, proportionality requires picking a specific distribution that is independent of the agents' identities.

Recently, Jennings et al. [27] showed that the uniform phantom rule is the unique mechanism that satisfies strategyproofness and proportionality among all rules defined on \mathcal{U}^{SP} . Again, we present a characterization that holds for every subset of \mathcal{U}^{SP} , whether symmetric or not.

Theorem 6. For $m = 2$ and arbitrary domain $\mathcal{U} \subseteq \mathcal{U}^{SP}$, the only continuous mechanism that satisfies strategyproofness and proportionality is the uniform phantom rule.

Theorem 6 entails an interesting connection of NASH to the independent markets mechanism for ℓ_1 preferences, which follows immediately from the mechanisms' axiomatic properties.

Corollary 1. With two alternatives, NASH for Leontief utilities is equivalent to the uniform phantom mechanism for ℓ_1 preferences.

Proof. For $m = 2$, Leontief utilities as well as ℓ_1 preferences are subsets of \mathcal{U}^{SP} , each of which contains one utility function per peak. Both mechanisms are continuous, strategyproof, and proportional. Therefore, they need to be equivalent on their respective utility models by Theorem 6. \square

8. Conclusion

Aggregating individual distributions into a collective distribution constitutes an important problem in social choice theory. Our work shows that understanding how agents form their preferences has crucial implications on the possibility of optimal aggregation mechanisms.

When agents' utilities are based on metrics such as ℓ_1 and ℓ_∞ , no rule simultaneously guarantees strategyproofness, efficiency, and proportionality. However, when agents' utilities are nonmetric and based on quotients (Leontief utilities), the Nash product rule guarantees group-strategyproofness and core fair share, which implies both efficiency and proportionality. Moreover, this rule is characterized by group-strategyproofness, core fair share, and continuity.

The Nash product rule satisfies further desirable properties such as *reinforcement* and *participation*. The former states that when aggregating distributions for two disjoint electorates results in the same distribution, the mechanism should return the same distribution for the union of both electorates. The latter requires that agents are never better off by not participating in the aggregation mechanism. Both statements follow trivially from the definition of the Nash product rule and hold for arbitrary utility models with nonnegative utilities.

It would be interesting to identify other sensible utility models for which the Nash product rule is a most attractive aggregation mechanism and to pinpoint domain conditions that cause impossibilities similar to Theorems 1 and 2. Some concrete open questions are:

- Does Theorem 3 also hold when weakening group-strategyproofness to strategyproofness? Is continuity required for the characterization of the Nash product rule?
- Are there classes of peak-linear or star-shaped utility functions, other than Leontief, for which mechanisms satisfying core fair share and strategyproofness exist? In particular, for ℓ_p preferences with $1 \leq p \leq \infty$ such as ℓ_2 , are there strategyproof mechanisms that satisfy core fair share?

Acknowledgments

The authors thank anonymous referees as well as the participants of the 25th ACM Conference on Economics and Computation (July 2024); the Workshop on Voting, Matching, and Preference Aggregation at the National University of

Singapore (December 2024); and the COMSOC Video Seminar (February 2025) for their insightful comments, stimulating discussions, and encouraging feedback.

Appendix A. Properties of Leontief Utilities

A useful concept when dealing with Leontief utilities is that of *critical alternatives*.

Definition A.1. Given a distribution \mathbf{q} , we define the set of agent i 's *critical alternatives*

$$T_{\mathbf{q},i} := \arg \min_{j \in M_i} \frac{q_j}{p_{i,j}}.$$

Critical alternatives allow for a characterization of efficient distributions.

Lemma A.1 (Brandt et al. [15, Lemma 2]). *With Leontief utilities, a distribution \mathbf{q} is efficient if and only if every alternative j with $q_j > 0$ is critical for some agent.*

It is easy to see that core fair share implies a weak version of efficiency, which merely rules out the existence of distributions that make *every* agent better off. For Leontief preferences, core fair share also implies the stronger version of efficiency used in this paper.

Proposition A.1. *For Leontief utilities, core fair share implies efficiency.*

Proof. Assume that \mathbf{q} violates efficiency. So there is a distribution \mathbf{q}' for which

$$\begin{aligned} u_i(\mathbf{q}') &\geq u_i(\mathbf{q}) && \text{for all } i \in N, \text{ and} \\ u_{i'}(\mathbf{q}') &> u_{i'}(\mathbf{q}) && \text{for at least one } i' \in N. \end{aligned} \tag{A.1}$$

The latter condition implies that every alternative $j \in T_{\mathbf{q},i'}$ is allocated strictly more in \mathbf{q}' than in \mathbf{q} .

We now construct a new distribution \mathbf{q}^* from \mathbf{q}' by taking a small amount $m\varepsilon$ from some $j_0 \in T_{\mathbf{q},i'}$ such that $q_{j_0}^* > q_{j_0}$ still holds and then adding ε to every $j \in M$. Now, we have

$$\begin{aligned} q_{j_0}^* &> q_{j_0}, \\ q_j^* &> q_j' && \text{for all } j \neq j_0. \end{aligned} \tag{A.2}$$

We claim that $u_i(\mathbf{q}^*) > u_i(\mathbf{q})$ for all $i \in N$. Indeed, for all $i \in N$, if $j_0 \in M_i$:

$$\begin{aligned} u_i(\mathbf{q}^*) &= \min_{j \in M_i} \frac{q_j^*}{p_{i,j}} && \text{(by definition of Leontief utilities)} \\ &= \min \left(\frac{q_{j_0}^*}{p_{i,j_0}}, \min_{j \in M_i, j \neq j_0} \frac{q_j^*}{p_{i,j}} \right) && \text{(by min properties)} \\ &> \min \left(\frac{q_{j_0}}{p_{i,j_0}}, \min_{j \in M_i, j \neq j_0} \frac{q_j'}{p_{i,j}} \right) && \text{(by (A.2))} \\ &\geq \min \left(\frac{q_{j_0}}{p_{i,j_0}}, \min_{j \in M_i} \frac{q_j'}{p_{i,j}} \right) && \text{(by min properties)} \\ &= \min \left(\frac{q_{j_0}}{p_{i,j_0}}, u_i(\mathbf{q}') \right) && \text{(by definition of Leontief utilities)} \\ &\geq \min \left(\frac{q_{j_0}}{p_{i,j_0}}, u_i(\mathbf{q}) \right) && \text{(by (A.1))} \\ &= u_i(\mathbf{q}) && \text{(by definition of Leontief utilities).} \end{aligned}$$

If $j_0 \notin M_i$, we can repeat a similar argument to arrive again at $u_i(\mathbf{q}^*) > u_i(\mathbf{q})$. Hence, the distribution \mathbf{q}^* shows that \mathbf{q} violates core fair share. \square

Remark A.1. A similar proof (where the set of all agents N in (A.1) is replaced by a subset $N' \subseteq N$ of a blocking coalition) shows that for Leontief utilities, core fair share is equivalent to the strong core fair share condition used by Aziz et al. [5], where only one member of a blocking coalition has to become strictly better.

Another useful feature of Leontief utilities is that core fair share provides a uniform lower bound on all agents' utilities.

Lemma A.2. *With Leontief utilities, if \mathbf{q} satisfies core fair share, then $u_i(\mathbf{q}) \geq 1/n$ for all $i \in [n]$.*

Proof. If $u_i(\mathbf{q}) < 1/n$ for some $i \in [n]$, then $\{i\}$ is a blocking coalition by taking $\mathbf{q}' = \mathbf{p}_i$, as any distribution that gives at least $p_{i,j}/n$ for all $j \in [m]$ gives i a utility of at least $1/n$. \square

We now give a sufficient condition and another necessary condition for core fair share with Leontief utilities. For any subset of agents $G \subseteq N$, let $T_{\mathbf{q},G} := \bigcup_{i \in G} T_{\mathbf{q},i}$ denote the set of alternatives critical to at least one agent from G .

Lemma A.3. *With Leontief utilities, if $\mathbf{q}(T_{\mathbf{q},G}) \geq |G|/n$ for all subsets $G \subseteq N$, then \mathbf{q} satisfies core fair share.*

Proof. Assume for contradiction that \mathbf{q} violates core fair share for some $G \subseteq N$. Then, there exists $\mathbf{q}' \in \Delta^m$ such that for every $\mathbf{q}'' \in \Delta^m$,

$$u_i((|G|/n)\mathbf{q}' + (1 - |G|/n)\mathbf{q}'') > u_i(\mathbf{q}) \quad \text{for all } i \in G.$$

Note that $T_{\mathbf{q},G} = M$ cannot hold; otherwise, by Lemma A.1, \mathbf{q} would be efficient not only for N but already for G , contradicting that \mathbf{q} does not satisfy core fair share for G . Therefore, there exists a distribution \mathbf{q}'' with $q_j'' = 0$ for every $j \in T_{\mathbf{q},G}$. Choosing such a distribution \mathbf{q}'' in the above inequality shows that $(|G|/n)q_j' > q_j$ for all $j \in T_{\mathbf{q},G}$. Thus, $\mathbf{q}(T_{\mathbf{q},G}) := \sum_{j \in T_{\mathbf{q},G}} q_j < (|G|/n) \cdot \sum_{j \in T_{\mathbf{q},G}} q_j' \leq |G|/n$. \square

The opposite direction of Lemma A.3 does not hold even for $n = 2$ and $m = 2$. For example, suppose $\mathbf{p}_1 = (1/2, 1/2)$, $\mathbf{p}_2 = (1/3, 2/3)$, and $\mathbf{q} = (1/3, 2/3)$. Then, \mathbf{q} satisfies core fair share, as the utility of each agent is at least $1/2$, but $\mathbf{q}(T_{\mathbf{q},1}) = 1/3 < 1/2$.

Lemma A.4. *With Leontief utilities, if \mathbf{q} satisfies core fair share, then $q_j = 0$ if and only if $p_{i,j} = 0$ for all $i \in N$.*

Proof. If $q_j = 0$ for some $j \in M$, then $u_i(\mathbf{q}) = 0$ for all agents with $p_{i,j} > 0$, meaning that core fair share is violated for each of these agents by Lemma A.2.

Conversely, $p_{i,j} = 0$ for all $i \in N$ implies that $q_j = 0$ for every efficient mechanism, where efficiency follows from Proposition A.1. \square

We conclude this section by showing that core fair share is a stronger fairness axiom than proportionality for Leontief utilities, as well as for ℓ_1 and ℓ_∞ preferences.

Proposition A.2. *With ℓ_1 preferences, ℓ_∞ preferences, or Leontief utilities, core fair share implies proportionality.*

Proof. Assume that a mechanism f is not proportional for some single-minded profile $P \in \mathcal{P}$. Denote $\mathbf{q} := f(P)$, and let $N' \subseteq N$ be a maximal subset of agents where all agents in N' allocate 1 to the same alternative j^* and proportionality is violated; that is, $q_{j^*} < r$ for $r := |N'|/n$.

Let \mathbf{q}' be the peak of all agents $i \in N'$; that is, $q_{j^*}' = 1$ and $q_j' = 0$ for all $j \neq j^*$. We claim that $u_i(r\mathbf{q}' + (1 - r)\mathbf{q}'') > u_i(\mathbf{q})$ for all distributions $\mathbf{q}'' \in \Delta^m$.

With ℓ_1 , ℓ_∞ , and Leontief preferences, for all $i \in N'$, $u_i(\mathbf{q})$ depends only on q_{j^*} , and it is an increasing function of q_{j^*} . Specifically, with ℓ_1 preferences $u_i(\mathbf{q}) = -2(1 - q_{j^*})$, with ℓ_∞ preferences $u_i(\mathbf{q}) = -(1 - q_{j^*})$, and with Leontief preferences $u_i(\mathbf{q}) = q_{j^*}$.

Because $(r\mathbf{q}' + (1 - r)\mathbf{q}'')_{j^*} \geq r > q_{j^*}$ for all $\mathbf{q}'' \in \Delta^m$, we have $u_i(r\mathbf{q}' + (1 - r)\mathbf{q}'') > u_i(\mathbf{q})$, so f violates core fair share. \square

As the proof shows, core fair share implies a property even stronger than proportionality: we do not need that $p_{i,j} \in \{0, 1\}$ for all $i \in N$ but the guarantee is for every single-minded agent group, independently of the other agents' preferences.

Remark A.2. The proof of Proposition A.2 does not work for ℓ_2 preferences. For example, suppose $n = m = 3$ and some two agents have their peak at $(1, 0, 0)$. A rule that returns $\mathbf{q} = f(P) = (0.64, 0.18, 0.18)$ violates proportionality but does not violate core fair share, as for both $i \in N'$, $u_i(\mathbf{q}) > -\sqrt{0.2}$, but for $\mathbf{q}'' = (0, 1, 0)$, we have $r\mathbf{q}' + (1 - r)\mathbf{q}'' = (2/3, 1/3, 0)$, which leads to $u_i = -\sqrt{2/9} < -\sqrt{0.2}$. In fact, proportionality does not seem to be a very natural notion for such preferences, as the proportionality guarantee for single-minded agents, who put one on alternative j , concerns only the distribution on alternative j , whereas ℓ_2 agents care also about the distribution on alternatives other than j .

Appendix B. Impossibility of Efficiency, Strategyproofness, and Proportionality for ℓ_1 Preferences and Arbitrary $n \geq 3$

In this section, we present the Proof of Theorem 1 for arbitrary $n \geq 3$ but still fixed $m = 3$. Again, we set $M = \{a, b, c\}$ and write $\mathbf{q} = (q_a, q_b, q_c)$. Note that in contrast to the case $n = 3$, we now also need to denote the number of agents with certain peaks in a profile.

Consider first the following two profiles.

The outcome in profile 2 must be $(1/n, (n - 2)/n, 1/n)$ by proportionality. We now justify the bounds on the outcome in profile 1. As agent 1 can manipulate between profile 1 and profile 2, strategyproofness requires that agent 1 does not gain

Profile 1				Profile 2			
Number of agents	a	b	c	Number of agents	a	b	c
1	$3/2n$	$(2n-3)/2n$	0	1	1	0	0
$n-2$	0	1	0	$n-2$	0	1	0
1	0	0	1	1	0	0	1
$\mathbf{q}^{(1)}$	$\geq 1/2n$	$\geq (2n-3)/2n$	$\leq 1/n$	$\mathbf{q}^{(2)}$	$1/n$	$(n-2)/n$	$1/n$

from either manipulation. This implies that

$$d_1^{(1)}(\mathbf{q}^{(1)}) \leq d_1^{(1)}(\mathbf{q}^{(2)}) = 2/n, \tag{B.1}$$

$$d_1^{(2)}(\mathbf{q}^{(1)}) \geq d_1^{(2)}(\mathbf{q}^{(2)}) = (2n-2)/n. \tag{B.2}$$

By (B.1), $q_a^{(1)} \geq 1/2n$ (implying $q_b^{(1)} \leq (2n-1)/2n$), $q_b^{(1)} \geq (2n-5)/2n$, and $q_c^{(1)} \leq 1/n$. By (B.2), $q_a^{(1)} \leq 1/n$, implying $q_b^{(1)} + q_c^{(1)} \geq (n-1)/n$, and thus, $q_b^{(1)} \geq (n-2)/n$.

By efficiency, we can even show that $q_b^{(1)} \geq (2n-3)/2n$. Otherwise, as $q_a^{(1)} > 0$, some small amount could be moved from a to b . Agent n is indifferent because of Observation 1, and agents $2, \dots, n-1$ strictly gain. Furthermore, this does not change agent 1's disutility as $q_b^{(1)} < (2n-3)/2n$.

Profile 3				Profile 4			
Number of agents	a	b	c	Number of agents	a	b	c
1	$1/(n+1)$	$n/(n+1)$	0	1	0	1	0
$n-2$	0	1	0	$n-2$	0	1	0
1	0	0	1	1	0	0	1
$\mathbf{q}^{(3)}$	0	$(n-1)/n$	$1/n$	$\mathbf{q}^{(4)}$	0	$(n-1)/n$	$1/n$

Next, we consider profiles 3 and 4. The outcome in profile 4 follows from proportionality. We now prove that the outcome in profile 3 must be the same. As agent 1 can manipulate between profile 3 and profile 4, strategyproofness requires that agent 1 does not gain from either manipulation. This implies that

$$d_1^{(3)}(\mathbf{q}^{(3)}) \leq d_1^{(3)}(\mathbf{q}^{(4)}) = 2/n, \tag{B.3}$$

$$d_1^{(4)}(\mathbf{q}^{(3)}) \geq d_1^{(4)}(\mathbf{q}^{(4)}) = 2/n. \tag{B.4}$$

By (B.3), $q_c^{(3)} \leq 1/n$, implying $q_a^{(3)} + q_b^{(3)} \geq (n-1)/n$. By (B.4), $q_b^{(3)} \leq (n-1)/n$.

However, by efficiency, if $q_a^{(3)} > 0$, then $q_b^{(3)} \geq n/(n+1)$. Otherwise, some small amount can be moved from a to b . Agent n is indifferent because of Observation 1, and agents $2, \dots, n-1$ strictly gain. Furthermore, this does not change agent 1's disutility as $q_b^{(3)} < n/(n+1)$. Therefore, $q_a^{(3)} = 0$ must hold, and the only outcome compatible with strategyproofness is $\mathbf{q}^{(3)} = (0, (n-1)/n, 1/n)$.

Now that we know $\mathbf{q}^{(3)}$, we consider a manipulation of agent 1 from profile 3 to profile 1. Strategyproofness implies

$$d_1^{(3)}(\mathbf{q}^{(1)}) \geq d_1^{(3)}(\mathbf{q}^{(3)}) = 2/n.$$

But the bounds we already have for $\mathbf{q}^{(1)}$ imply that $d_1^{(3)}(\mathbf{q}^{(1)}) \leq 2/n$ as $q_a^{(1)} \geq 1/2n$ and $q_b^{(1)} \geq (2n-3)/2n$. Therefore, $d_1^{(3)}(\mathbf{q}^{(1)}) = 2/n$ together with $q_a^{(1)} = 1/2n$ and $q_b^{(1)} = (2n-3)/2n$. Hence, $\mathbf{q}^{(1)} = (1/2n, (2n-3)/2n, 1/n)$.

Finally, we consider the following two profiles.

Profile 5				Profile 6			
Number of agents	a	b	c	Number of agents	a	b	c
1	$3/2n$	$(2n-3)/2n$	0	1	1	0	0
$n-2$	0	1	0	$n-2$	0	1	0
1	0	$(2n-3)/2n$	$3/2n$	1	0	$(2n-3)/2n$	$3/2n$
$\mathbf{q}^{(5)}$	$\geq 1/2n$	$\geq (2n-3)/2n$	$\leq 1/n$	$\mathbf{q}^{(6)}$	$1/n$	$(2n-3)/2n$	$1/2n$

The distribution $\mathbf{q}^{(6)}$ is determined by arguments analogous to those for $\mathbf{q}^{(1)}$, reasoning about agent n instead of agent 1.

We now consider a manipulation of agent 1 from profile 5 to profile 6. It follows from strategyproofness that

$$d_1^{(5)}(\mathbf{q}^{(5)}) \leq d_1^{(5)}(\mathbf{q}^{(6)}) = 1/n,$$

which implies that $q_c^{(5)} \leq 1/2n$. Similarly, we consider a manipulation of agent n from profile 5 to profile 1. It follows from strategyproofness that

$$d_n^{(5)}(\mathbf{q}^{(5)}) \leq d_n^{(5)}(\mathbf{q}^{(1)}) = 1/n,$$

which implies that $q_c^{(5)} \geq 3/2n - 1/2n = 1/n$, a contradiction. \square

Appendix C. Not Every Anonymous, Neutral, Continuous, and Strategyproof Rule Is a Moving Phantom Mechanism

For a formal definition of moving phantom mechanisms, see Freeman et al. [22, definition 6].

Proposition C.1. *With ℓ_1 preferences, not every anonymous, neutral, continuous, and strategyproof mechanism can be represented as a moving phantom mechanism for any $n \geq 1$ and $m \geq 3$ and for any number of phantom functions.*

Proof. We first prove the claim for $n = 1$ and $m = 3$. Consider the mechanism that, in general, returns the agent’s peak but cannot put more than 0.9 on an alternative. If (without loss of generality) $p_{1,1} > 0.9$, the mechanism returns $q_1 = 0.9$, $q_2 = p_{1,2} + (p_{1,1} - 0.9)/2$, and $q_3 = p_{1,3} + (p_{1,1} - 0.9)/2$. Because this outcome minimizes the ℓ_1 distance of agent 1 among all “legal” distributions, the mechanism is strategyproof.

Anonymity is trivially satisfied, as there is only one agent. For neutrality, if $p_{1,j} = p_{1,k}$, then both alternatives receive the same probability share. In particular, if $p_{1,l} > 0.9$ for the third alternative l , then j and k both receive $p_{1,j} + (p_{1,l} - 0.9)/2$, and the distribution of the surplus does not depend on the identity of the alternatives. For continuity, the only “critical” points are those where $p_{1,j} > 0.9$ approaches 0.9 from above. For such peaks, $q_j = 0.9$ is constant, and the “surplus” $p_{1,j} - 0.9$ is distributed on the other two alternatives in a continuous manner. Thus, the mechanism also satisfies continuity.

Suppose by contradiction that the above mechanism can be represented as a moving phantom mechanism with phantom functions \mathbf{h} ; let k be the number of phantoms. Let $\mathbf{p}_1 = (0.91, 0.08, 0.01)$. Given this profile, the mechanism returns $\mathbf{q} = (0.9, 0.085, 0.015)$. This implies that for some $t \in [0, 1]$, 0.085 is the median of 0.08 and $h_1(t), \dots, h_k(t)$, so the number of phantoms larger than or equal to 0.085 should be at least $k/2 + 1$ (for even k) or $(k + 3)/2$ (for odd k).⁵ By similar considerations, because 0.015 is the median of 0.01 and $h_1(t), \dots, h_k(t)$, the number of phantoms smaller than or equal to 0.015 should be at least $k/2$ (for even k) or $(k - 1)/2$ (for odd k). These two observations are contradictory as there are only k phantoms in total.

A similar construction also works for larger m . Moreover, because the considered properties do not relate instances with different n , such a construction can be extended to a rule for arbitrary n by using this mechanism when $n = 1$ and a moving phantom mechanism when $n \geq 2$. \square

Note that the proof of Proposition C.1 does not assume continuity or any other property of the phantom functions.

After constructing this counterexample, we learned that de Berg et al. [17] independently came up with a similar construction with a more natural extension to larger n that does not coincide with a moving phantom mechanism for $n > 1$.

Appendix D. Impossibility of Efficiency, Strategyproofness, and Proportionality for ℓ_∞ Preferences and $n, m \geq 3$

In this section, we present the Proof of Theorem 2 for arbitrary $n \geq 3$ and $m \geq 3$. We start by fixing $m = 3$ and considering $n \geq 3$.

Lemma D.1. *With ℓ_∞ preferences, no mechanism satisfies efficiency, strategyproofness, and proportionality when $m = 3$ and $n \geq 3$.*

Proof. We use the same notation as in the Proof of Theorem 1.

	Profile 1				Profile 2		
Number of agents	a	b	c	Number of agents	a	b	c
1	$3/2n$	$(2n - 3)/2n$	0	1	1	0	0
$n - 2$	0	1	0	$n - 2$	0	1	0
1	0	0	1	1	0	0	1
$\mathbf{q}^{(1)}$	$\geq 1/2n$	$\geq (2n - 3)/2n$	$\leq 1/n$	$\mathbf{q}^{(2)}$	$1/n$	$(n - 2)/n$	$1/n$

Consider first profiles 1 and 2. The outcome in profile 2 must be $(1/n, (n - 2)/n, 1/n)$ by proportionality. We now justify the bounds on the outcome in profile 1. As agent 1 can manipulate between profile 1 and profile 2, strategyproofness requires

that agent 1 does not gain from either manipulation. This implies that

$$d_1^{(1)}(\mathbf{q}^{(1)}) \leq d_1^{(1)}(\mathbf{q}^{(2)}) = 1/n, \tag{D.1}$$

$$d_1^{(2)}(\mathbf{q}^{(1)}) \geq d_1^{(2)}(\mathbf{q}^{(2)}) = (n-1)/n. \tag{D.2}$$

By (D.1), $q_a^{(1)} \geq 1/2n$ (implying $q_b^{(1)} \leq (2n-1)/2n$), $q_b^{(1)} \geq (2n-5)/2n$, and $q_c^{(1)} \leq 1/n$. By (D.2), $q_a^{(1)} \leq 1/n$, implying $q_b^{(1)} + q_c^{(1)} \geq (n-1)/n$, and thus, $q_b^{(1)} \geq (n-2)/n$.

By efficiency, we can even show that $q_b^{(1)} \geq (2n-3)/2n$. Otherwise, $q_b^{(1)} < (2n-3)/2n$ and $q_c^{(1)} + q_a^{(1)} > 3/2n$, and some small amount can be moved from a to b . Agent n is indifferent because of Observation 3, and agents $2, \dots, n-1$ strictly gain. Furthermore, this does not increase agent 1's disutility as $d_1^{(1)}(\mathbf{q}^{(1)}) \geq q_c^{(1)} > 3/2n - q_a^{(1)}$ and $q_b^{(1)} < (2n-3)/2n$. Hence, $q_b^{(1)} \geq (2n-3)/2n$.

Profile 3				Profile 4			
Number of agents	a	b	c	Number of agents	a	b	c
1	$1/(n+1)$	$n/(n+1)$	0	1	0	1	0
$n-2$	0	1	0	$n-2$	0	1	0
1	0	0	1	1	0	0	1
$\mathbf{q}^{(3)}$	$\leq 1/n$	$\geq (n-1)/n$	$\geq 1/n$	$\mathbf{q}^{(4)}$	0	$(n-1)/n$	$1/n$

Assume for contradiction that $q_c^{(1)} \leq 3/4n$.

Consider a manipulation of agent 1 from profile 3 to profile 1. Note that $d_1^{(3)}(\mathbf{q}^{(1)}) \leq 3/4n$ with the bounds established for $\mathbf{q}^{(1)}$. By strategyproofness for agent 1, $q_c^{(3)} \leq 3/4n$.

By efficiency, $q_b^{(3)} \geq n/(n+1)$. Otherwise, $q_b^{(3)} < n/(n+1)$ and $q_a^{(3)} > 1/(n+1) - q_c^{(3)}$, and some small amount can be moved from a to b . Agent n is indifferent because of Observation 3, and agents $2, \dots, n-1$ strictly gain. Furthermore, this does not increase agent 1's disutility as $d_1^{(3)}(\mathbf{q}^{(3)}) \geq q_c^{(3)} > 1/(n+1) - q_a^{(3)}$ and $q_b^{(3)} < n/(n+1)$. Hence, $q_b^{(3)} \geq n/(n+1)$. However, as $n/(n+1) > (n-1)/n$, this contradicts strategyproofness for agent 1 manipulating from profile 4 to profile 3, where $\mathbf{q}^{(4)} = (0, (n-1)/n, 1/n)$ follows from proportionality.

Profile 5				Profile 6			
Number of agents	a	b	c	Number of agents	a	b	c
1	1	0	0	1	$3/2n$	$(2n-3)/2n$	0
$n-2$	0	1	0	$n-2$	0	1	0
1	0	$(2n-3)/2n$	$3/2n$	1	0	$(2n-3)/2n$	$3/2n$
$\mathbf{q}^{(5)}$	$\leq 1/n$	$\geq (2n-3)/2n$	$\geq 1/2n$	$\mathbf{q}^{(6)}$	0	$(2n-3)/2n$	$3/2n$

Therefore, $q_c^{(1)} > 3/4n$ has to hold. By analogous arguments with reversed roles of agents 1 and n , the same bounds for $\mathbf{q}^{(5)}$ as well as $q_a^{(5)} > 3/4n$ hold.

Because $q_b^{(1)} \geq (2n-3)/2n$, we must have $q_a^{(1)} < 3/4n$. Consider a manipulation of agent n from profile 6 to profile 1. Note that $d_n^{(6)}(\mathbf{q}^{(1)}) < 3/4n$, as $q_b^{(1)} \geq (2n-3)/2n$ and $q_c^{(1)} > 3/4n$. By strategyproofness for agent n , we have $d_n^{(6)}(\mathbf{q}^{(6)}) \leq d_n^{(6)}(\mathbf{q}^{(1)}) < 3/4n$, and thus, $q_a^{(6)} < 3/4n$. Finally, consider a manipulation of agent 1 from profile 6 to profile 5. Analogously, $d_1^{(6)}(\mathbf{q}^{(6)}) \leq d_1^{(6)}(\mathbf{q}^{(5)}) < 3/4n$, which implies $q_a^{(6)} > 3/2n - 3/4n = 3/4n$, a contradiction. \square

We finish the Proof of Theorem 2 by showing that this argument remains valid under the addition of alternatives j^+ with $p_{i,j^+} = 0$ for all agents $i \in N$.

Proof of Theorem 2. We prove that no efficient mechanism puts positive probability on new alternatives j^+ with $p_{i,j^+} = 0$ for all agents $i \in N$ in any of the six profiles used for the proof of Lemma D.1. Together with Lemma D.1, this completes the Proof of Theorem 2.

Proportionality directly implies that adding such an alternative j^+ to profiles 2 and 4 does not change the distribution.

Next, consider profile 1. If $|(2n-3)/2n - q_b^{(1)}| < d_1^{(1)}(\mathbf{q}^{(1)})$, or $|(2n-3)/2n - q_b^{(1)}| = d_1^{(1)}(\mathbf{q}^{(1)})$ and $q_b^{(1)} < (2n-3)/2n$, moving some amount of probability from j^+ to b cannot increase agent 1's disutility, does not change agent n 's disutility by Observation 3, and decreases the disutilities of all other agents. Therefore, such a redistribution would correspond to a Pareto improvement. Note that $|(2n-3)/2n - q_b^{(1)}| = d_1^{(1)}(\mathbf{q}^{(1)})$ and $q_b^{(1)} = (2n-3)/2n$ cannot hold simultaneously, as $q_{j^+}^{(1)} > 0$. Hence,

the only remaining case we need to consider is $|(2n-3)/2n - q_b^{(1)}| = d_1^{(1)}(\mathbf{q}^{(1)})$ and $q_b^{(1)} > (2n-3)/2n$. This implies $3/2n - q_a^{(1)} \leq d_1^{(1)}(\mathbf{q}^{(1)})$ and thus, $q_a^{(1)} + q_b^{(1)} \geq 3/2n - d_1^{(1)}(\mathbf{q}^{(1)}) + (2n-3)/2n + d_1^{(1)}(\mathbf{q}^{(1)}) = 1$, contradicting $q_{j^+}^{(1)} > 0$. The argument for profile 5 works analogously.

For profile 3, moving some amount of probability from j^+ to b can only potentially increase the disutility of one agent, namely, agent 1, if $q_b^{(3)} \geq n/(n+1)$ and $d_1^{(3)}(\mathbf{q}^{(3)}) = q_b^{(3)} - n/(n+1)$. But then, $1/(n+1) - q_a^{(3)} \leq d_1^{(3)}(\mathbf{q}^{(3)})$, and again, $q_a^{(3)} + q_b^{(3)} = 1$, contradicting $q_{j^+}^{(3)} > 0$.

Finally, for profile 6, moving some amount of probability from j^+ to b can only potentially increase the disutilities of two agents, namely, agents 1 and n , if $q_b^{(6)} \geq (2n-3)/2n$ and $d_k^{(6)}(\mathbf{q}^{(6)}) = q_b^{(6)} - (2n-3)/2n$ holds for at least one $k \in \{1, n\}$, without loss of generality for $k = 1$. But then, $3/2n - q_a^{(6)} \leq d_1^{(6)}(\mathbf{q}^{(6)})$ and again, $q_a^{(6)} + q_b^{(6)} = 1$, contradicting $q_{j^+}^{(6)} > 0$. \square

Appendix E. Proofs for Section 6

Analogously to its application in donor coordination (Brandt et al. [15]), we can also interpret the outcome of *NASH* as a Nash equilibrium where each agent i reports a vector $\mathbf{s}_i \in \Delta^m$ and the outcome is determined by adding up the score $\sum_{i \in N} s_{i,j}$ of each alternative j . Here, the strategy set is the set of preferences. This interpretation will be useful for proving certain properties of *NASH*.

Definition E.1 (Decomposition). A decomposition of a distribution \mathbf{q} is a vector of nonnegative score vectors $(\mathbf{s}_i)_{i \in N}$ with

$$\begin{aligned} \sum_{i \in N} s_{i,j} &= q_j && \text{for all } j \in M; \\ \sum_{j \in M} s_{i,j} &= \frac{1}{n} && \text{for all } i \in N. \end{aligned}$$

An alternative characterization of the *NASH* outcome uses the notion of critical alternatives. Recall the definition of this notion from Appendix A.

Lemma E.1 (Brandt et al. [15, Section 4.3]). A distribution \mathbf{q} maximizes the Nash product if and only if it has a decomposition $(\mathbf{s}_i)_{i \in N}$ such that $s_{i,j} = 0$ for every alternative $j \notin T_{\mathbf{q},i}$.

Another fact to keep in mind is that utilities and efficient outcomes admit a one-to-one correspondence.

Lemma E.2 (Brandt et al. [15, Lemma 3]). Let \mathbf{q} and \mathbf{q}' be efficient distributions inducing the same utility vector; that is, $u_i(\mathbf{q}) = u_i(\mathbf{q}')$ for all $i \in N$. Then, $\mathbf{q} = \mathbf{q}'$.

With ℓ_1 preferences, every efficient distribution \mathbf{q} must be range-respecting (that is, q_j must be between $\min_i p_{i,j}$ and $\max_i p_{i,j}$ for all $j \in M$). However, with Leontief utilities, this is not the case. In fact, *NASH* is efficient but not range-respecting. Intuitively, *NASH* may prefer to decrease the distribution for one alternative below the minimum peak in order to increase the distribution for other alternatives whose ratio is smaller. Nevertheless, an efficient distribution always satisfies one direction of range-respect.

Lemma E.3. With Leontief utilities, if \mathbf{q} is an efficient distribution, then $q_j \leq \max_i p_{i,j}$ for all $j \in M$. In particular, this holds for any distribution returned by *NASH*.

Proof. Suppose by contradiction that for some $j \in M$, it holds that $q_j > p_{i,j}$ for all $i \in N$. Construct a new distribution \mathbf{q}' from \mathbf{q} by removing some amount from j and allocating it equally among the other alternatives such that $q_j > p_{i,j}$ for all $i \in N$ still holds.

Note that for any agent i with Leontief utilities, $u_i(\mathbf{q}) \leq 1$ for all \mathbf{q} . Therefore, the decrease in q_j does not decrease the Leontief utility of any agent, as $q_j/p_{i,j} > 1$. But the increase in allocation to other alternatives must increase the utilities of all agents. Therefore, \mathbf{q}' is a Pareto improvement of \mathbf{q} , contradicting efficiency. \square

E.1. Properties of *NASH*

Proposition 1. With Leontief utilities, *NASH* is continuous.

Proof. Suppose we are given a sequence of profiles P^1, P^2, \dots converging to P^* ; that is, $\lim_{k \rightarrow \infty} p_{i,j}^k = p_{i,j}^*$ for every agent $i \in N$ and alternative $j \in M$. Denote by u_i^k the utility of agent i in profile P^k , and u_i^* the utility in profile P^* . Denote $\mathbf{q}^k = \text{NASH}[P^k]$ for every $k \in \mathbb{N}$ and $\mathbf{q}^* = \text{NASH}[P^*]$. By boundedness, it suffices to show that every convergent subsequence of $\mathbf{q}^1, \mathbf{q}^2, \dots$ converges to \mathbf{q}^* . Take such a subsequence, which must exist by the Bolzano–Weierstrass theorem, and denote its limit by \mathbf{q}^L . With abuse of notation, we now refer to this subsequence as $\mathbf{q}^1, \mathbf{q}^2, \dots$. Our goal is to show that $\mathbf{q}^L = \mathbf{q}^*$.

Case 1: $q_j^* > 0$ for all alternatives $j \in M$. Denote by $\text{NASH}[\mathbf{q}, P]$ the Nash welfare of the outcome \mathbf{q} when the profile is P . By definition of *NASH* on P^k , we have

$$\text{NASH}[\mathbf{q}^*, P^k] \leq \text{NASH}[\mathbf{q}^k, P^k]$$

for every k . We take the limit of both sides as $k \rightarrow \infty$.

- The left-hand side is a product of utilities $u_i^k(\mathbf{q}^*)$, where the distribution is fixed and only the utility functions change. Each utility is a minimum of ratios $q_j^*/p_{i,j}^k$, where all numerators are at least ε , for some $\varepsilon > 0$. Because the minimum is always at most one, elements with $p_{i,j}^k < \varepsilon$ do not affect the minimum and can be ignored. Therefore, the minimum is determined only by ratios with $p_{i,j}^k \geq \varepsilon$. In this domain, the ratios are continuous functions of $p_{i,j}^k$, and their minimum is continuous too. Therefore, $\lim_{k \rightarrow \infty} u_i^k(\mathbf{q}^*) = u_i^*(\mathbf{q}^*)$, so the limit of the product at the left-hand side equals $NASH[\mathbf{q}^*, P^*]$.

- The right-hand side is a product of utilities $u_i^k(\mathbf{q}^k)$, where both the distribution and the utility functions change. There may be agents i and alternatives j for which both q_j^k and $p_{i,j}^k$ approach zero as $k \rightarrow \infty$, so the limit of $u_i^k(\mathbf{q}^k)$ may differ from $u_i^*(\mathbf{q}^L)$. For example, if $\mathbf{p}_i^k = (2/k, 1 - 2/k)$ and $\mathbf{q}^k = (1/k, 1 - 1/k)$, then $u_i^k(\mathbf{q}^k) = 1/2$ for all k , but $\mathbf{q}^L = (0, 1)$, so $u_i^*(\mathbf{q}^L) = 1$. However, any alternative j for which $p_{i,j}^* = \lim_{k \rightarrow \infty} p_{i,j}^k = 0$ is removed from the minimum, so the minimum at the limit profile can only be larger than the limit of minima. Therefore, $\lim_{k \rightarrow \infty} u_i^k(\mathbf{q}^k) \leq u_i^*(\mathbf{q}^L)$, and the limit of the product at the right-hand side is at most $NASH[\mathbf{q}^L, P^*]$.

Therefore, we have $NASH[\mathbf{q}^*, P^*] \leq NASH[\mathbf{q}^L, P^*]$. By definition and uniqueness of $NASH$ on P^* , we get $\mathbf{q}^L = \mathbf{q}^*$.

Case 2: $q_j^* = 0$ for some alternatives $j \in M$. Define $Z = \{j \in M : q_j^* = 0\}$. As \mathbf{q}^* maximizes Nash welfare for P^* , we have $p_{i,j}^* = 0$ for every $i \in N$ and $j \in Z$; otherwise, an agent i with $p_{i,j}^* > 0$ would receive zero utility, causing the whole product to become zero. Consequently, $\lim_{k \rightarrow \infty} p_{i,j}^k = 0$ for every $i \in N, j \in Z$. By Lemma E.3, the amount allocated to alternatives in Z by \mathbf{q}^k also tends to zero for $k \rightarrow \infty$. Hence, $q_j^* = 0$ implies $q_j^L = 0$.

Now, we will measure utilities and Nash welfare only with respect to the alternatives outside Z . Let w^k be the total amount allocated to alternatives outside Z in \mathbf{q}^k , so we know that w^k converges to one for $k \rightarrow \infty$. By definition of $NASH$ on P^k , we have

$$NASH[w^k \cdot \mathbf{q}^*, P^k] \leq NASH[\mathbf{q}^k, P^k].$$

As in Case 1, we take the limit of both sides: the left-hand side equals $NASH[\mathbf{q}^*, P^*]$, and the right-hand side is at most $NASH[\mathbf{q}^L, P^*]$. Because in P^* all agents assign zero to alternatives in Z , these two quantities remain the same even if we take the alternatives in Z back into account for the Nash welfare. Hence, as in Case 1, we get $NASH[\mathbf{q}^*, P^*] \leq NASH[\mathbf{q}^L, P^*]$. By definition and uniqueness of $NASH$ on P^* , we conclude that $\mathbf{q}^L = \mathbf{q}^*$. \square

Proposition 2. *With Leontief utilities, NASH satisfies core fair share.*

Proof. Let G be any subset of agents. By Lemma E.1, the $NASH$ distribution can be decomposed in such a way that every agent from G only contributes her share of $1/n$ to alternatives in $T_{\mathbf{q},G}$. Thus, $\mathbf{q}(T_{\mathbf{q},G}) \geq |G|/n$. By Lemma A.3, $NASH$ satisfies core fair share. \square

E.2. Characterization of NASH

Theorem 3. *With Leontief utilities, NASH is the only continuous mechanism that satisfies group-strategyproofness and core fair share.*

Let f be a mechanism satisfying the properties in the theorem statement. The proof is divided into three lemmas and has the following structure. Starting at an arbitrary profile P , we first show in Lemma E.4 that moving to a “key” profile P^* cannot change the outcome: $f(P^*) = f(P)$. Then, Lemma E.5 states that $f(P^*) = NASH[P^*]$. Finally, Lemma E.6 proves $NASH[P^*] = NASH[P]$, which completes the proof, as we then have $f(P) = NASH[P]$.

Let $\mathbf{q} := f(P)$. By core fair share and Lemma A.4, $q_j = 0$ if and only if $p_{i,j} = 0$ for all $i \in N$.

Denote by P^* the profile with peaks

$$p_{i,j}^* = \begin{cases} p_{i,j}/\mathbf{p}_i(T_{\mathbf{q},i}) & \text{for } j \in T_{\mathbf{q},i}; \\ 0 & \text{for } j \notin T_{\mathbf{q},i}; \end{cases}$$

where $\mathbf{p}_i(T_{\mathbf{q},i}) := \sum_{j \in T_{\mathbf{q},i}} p_{i,j}$. That is, in P^* , each agent moves her peak so that it is nonzero only on alternatives critical for her under \mathbf{q} . For example, suppose $\mathbf{p}_i = (0.1, 0.2, 0.3, 0.4)$ and $\mathbf{q} = (0.1, 0.1, 0.6, 0.2)$. Then, $T_{\mathbf{q},i} = \{2, 4\}$, $\mathbf{p}_i(T_{\mathbf{q},i}) = 0.6$, and $\mathbf{p}_i^* = (0, 1/3, 0, 2/3)$. Note that for an agent i with $T_{\mathbf{q},i} = M$, it holds that $\mathbf{p}_i = \mathbf{p}_i^*$.

Lemma E.4. *With Leontief utilities, if f is a continuous mechanism satisfying group-strategyproofness and efficiency, then*

- the outcome does not change; that is, $f(P^*) = f(P) = \mathbf{q}$;
- the sets of critical alternatives do not change; that is, $T_{\mathbf{q},i} = T_{\mathbf{q},i}^*$ for every $i \in N$.

Proof. (a) We move the peak of each agent in turn. For each agent i , we change \mathbf{p}_i toward \mathbf{p}_i^* gradually to some $\widehat{\mathbf{p}}_i := \lambda \mathbf{p}_i^* + (1 - \lambda) \mathbf{p}_i$ for some $\lambda \in [0, 1]$ to be computed later. Then, we proceed along this line until we reach \mathbf{p}_i^* . In the above example, $\lambda = 0.3$ gives $\widehat{\mathbf{p}}_i = (0.07, 0.24, 0.21, 0.48)$. If $\mathbf{p}_i = \mathbf{p}_i^*$, it is clear that the outcome does not change, so assume that $\mathbf{p}_i \neq \mathbf{p}_i^*$. The change from \mathbf{p}_i to $\widehat{\mathbf{p}}_i$ has a simple structure:

- $\widehat{p}_{i,j} > p_{i,j}$ for all $j \in T_{\mathbf{q},i}$, and the ratio $\widehat{p}_{i,j}/p_{i,j} = \lambda/\mathbf{p}_i(T_{\mathbf{q},i}) + (1 - \lambda) =: \lambda^+$, a constant independent of j (in the example, $\lambda^+ = 1.2$);
- $\widehat{p}_{i,j} < p_{i,j}$ for all $j \notin T_{\mathbf{q},i}$, and the ratio $\widehat{p}_{i,j}/p_{i,j} = (1 - \lambda) =: \lambda^-$, again independent of j (in the example, $\lambda^- = 0.7$).

Now, consider the ratios $q_j/p_{i,j}$ versus the ratios $q_j/\widehat{p}_{i,j}$. For each $j \in T_{q,i}$, we have $q_j/p_{i,j} > q_j/\widehat{p}_{i,j}$ because $\widehat{p}_{i,j} > p_{i,j}$, whereas for each $j \notin T_{q,i}$, we have $q_j/p_{i,j} < q_j/\widehat{p}_{i,j}$ because $\widehat{p}_{i,j} < p_{i,j}$. Furthermore, for all $j \in T_{q,i}$, the ratios $q_j/\widehat{p}_{i,j}$ remain equal (as $\widehat{p}_{i,j}/p_{i,j}$ is constant) when moving from \mathbf{p}_i to $\widehat{\mathbf{p}}_i$, and they remain the smallest ratios for agent i . This implies that $\widehat{T}_{q,i} = T_{q,i}$.

Moreover, the entire ordering of alternatives by the ratio $q_j/p_{i,j}$ is identical to the ordering of alternatives by the ratio $q_j/\widehat{p}_{i,j}$, as the smallest ratio is divided by $\lambda^+ > 1$, and the other ratios are divided by $\lambda^- < 1$. In other words, suppose we partition the alternatives into subsets according to the ratio $q_j/p_{i,j}$, and denote the subset with the smallest ratio by $T_{q,i,1} \equiv T_{q,i}$, the subset with the second-smallest ratio by $T_{q,i,2}$, etc., then $T_{q,i,r} = \widehat{T}_{q,i,r}$ for all $r \geq 1$.

Computing λ . We pick λ sufficiently small such that no new alternative becomes critical for i in the new distribution yielded by f . Specifically, set

$$\varepsilon := \min_{j \in T_{q,i}, j' \notin T_{q,i}} (q_{j'} p_{i,j} - q_j p_{i,j'}) \leq \min_{j \in T_{q,i}, j' \notin T_{q,i}} \frac{q_{j'} p_{i,j} - q_j p_{i,j'}}{p_{i,j} + p_{i,j'}}.$$

Note that $\varepsilon > 0$, as $q_{j'}/p_{i,j'} > q_j/p_{i,j}$ by definition of critical alternatives.

By uniform continuity of f , there exists $\delta > 0$ such that $\|f(P) - f(P')\|_1 < 2\varepsilon$ for all P' with $\|P - P'\|_1 \leq \delta$. Set

$$\lambda := \min\left(1, \frac{\delta}{\|\mathbf{p}_i - \mathbf{p}_i^*\|_1}\right),$$

and define \widehat{P} as a profile identical to P except that i changes her peak from \mathbf{p}_i to $\widehat{\mathbf{p}}_i := \lambda \mathbf{p}_i^* + (1 - \lambda) \mathbf{p}_i$. Note that $\|P - \widehat{P}\|_1 = \lambda \|\mathbf{p}_i - \mathbf{p}_i^*\|_1 \leq \delta$, so $\|\mathbf{q} - \widehat{\mathbf{q}}\|_1 < 2\varepsilon$, where $\mathbf{q} = f(P)$ and $\widehat{\mathbf{q}} = f(\widehat{P})$.

The choice of ε ensures that $T_{\widehat{q},i} \subseteq T_{q,i}$, as for arbitrary $j \in T_{q,i}$ and $j' \notin T_{q,i}$, it holds that $\widehat{q}_j < q_j + \varepsilon$ and $\widehat{q}_{j'} > q_{j'} - \varepsilon$, so

$$\begin{aligned} \frac{\widehat{q}_{j'}}{p_{i,j'}} &> \frac{q_{j'} - \varepsilon}{p_{i,j'}} \geq \frac{q_{j'}}{p_{i,j'}} - \frac{q_{j'} p_{i,j} - q_j p_{i,j'}}{p_{i,j'}(p_{i,j} + p_{i,j'})} = \frac{q_{j'} p_{i,j'} + q_j p_{i,j'}}{p_{i,j'}(p_{i,j} + p_{i,j'})} = \frac{q_{j'} + q_j}{p_{i,j} + p_{i,j'}} \\ &= \frac{q_j p_{i,j} + q_{j'} p_{i,j}}{p_{i,j}(p_{i,j} + p_{i,j'})} = \frac{q_j}{p_{i,j}} + \frac{q_{j'} p_{i,j} - q_j p_{i,j'}}{p_{i,j}(p_{i,j} + p_{i,j'})} \geq \frac{q_j + \varepsilon}{p_{i,j}} > \frac{\widehat{q}_j}{p_{i,j}}. \end{aligned}$$

So every j' that is not critical for i under \mathbf{q} cannot be critical for i under $\widehat{\mathbf{q}}$. Therefore,

$$T_{\widehat{q},i} \subseteq T_{q,i} = \widehat{T}_{q,i}.$$

Proving that the outcome does not change. Consider a manipulation of agent i who manipulates between reporting \mathbf{p}_i and $\widehat{\mathbf{p}}_i$. Strategyproofness for i implies both $u_i(\mathbf{q}) \geq u_i(\widehat{\mathbf{q}})$ and $\widehat{u}_i(\widehat{\mathbf{q}}) \geq \widehat{u}_i(\mathbf{q})$.

The latter condition implies that for every alternative $j \in T_{q,i}$,

$$\begin{aligned} \frac{q_j}{\widehat{p}_{i,j}} &= \widehat{u}_i(\mathbf{q}) && \text{because } j \in T_{q,i} = \widehat{T}_{q,i}, \\ &\leq \widehat{u}_i(\widehat{\mathbf{q}}) && \text{by strategyproofness,} \\ &\leq \frac{\widehat{q}_j}{\widehat{p}_{i,j}} && \text{by the definition of Leontief utilities.} \end{aligned}$$

So $q_j \leq \widehat{q}_j$ for each alternative $j \in T_{q,i}$. Together with $T_{\widehat{q},i} \subseteq T_{q,i}$, this implies $u_i(\mathbf{q}) \leq u_i(\widehat{\mathbf{q}})$. Therefore, $u_i(\mathbf{q}) = u_i(\widehat{\mathbf{q}})$. Furthermore, if $\widehat{u}_i(\widehat{\mathbf{q}}) > \widehat{u}_i(\mathbf{q})$, then $\widehat{q}_j > q_j$ for all $j \in \widehat{T}_{q,i} \supseteq T_{\widehat{q},i}$, which means that $u_i(\mathbf{q}) < u_i(\widehat{\mathbf{q}})$, contradicting $u_i(\mathbf{q}) = u_i(\widehat{\mathbf{q}})$. Thus, $\widehat{u}_i(\widehat{\mathbf{q}}) = \widehat{u}_i(\mathbf{q})$.

Moreover, if the utility of some other agent i' increases, group-strategyproofness is violated for the pair $\{i, i'\}$, as this pair could profitably manipulate from \mathbf{q} to $\widehat{\mathbf{q}}$. Similarly, if the utility of some other agent i' decreases, group-strategyproofness is again violated for the pair $\{i, i'\}$, as this pair could profitably manipulate from $\widehat{\mathbf{q}}$ to \mathbf{q} . Thus, $u_r(\mathbf{q}) = u_r(\widehat{\mathbf{q}})$ for all $r \in N$. Because \mathbf{q} is efficient with respect to P , so is $\widehat{\mathbf{q}}$. By Lemma E.2, $\mathbf{q} = \widehat{\mathbf{q}}$.

Applying this argument repeatedly, we get a sequence of profiles (P^k) with $P^0 = P$, where \mathbf{p}_i^k lies on the line $\lambda \mathbf{p}_i^* + (1 - \lambda) \mathbf{p}_i$ for every k . It remains to show that (\mathbf{p}^k) reaches \mathbf{p}_i^* after a finite number of steps. For that, consider the expression in the definition of ε :

$$\min_{j \in T_{q,i}, j' \notin T_{q,i}} (q_{j'} p_{i,j} - q_j p_{i,j'}).$$

As \mathbf{p}_i comes closer to \mathbf{p}_i^* , $p_{i,j}$ increases and $p_{i,j'}$ decreases while \mathbf{q} and $T_{q,i}$ stay the same, so overall, the expression increases. Thus, we can take the ε (and the corresponding δ) from the first step for every step. Furthermore, $\|P^k - P^{k+1}\|_1 = \delta$ (unless $\lambda = 1$, but then we have reached \mathbf{p}_i^*), implying that we reach \mathbf{p}_i^* after at most $\lceil \|\mathbf{p}_i - \mathbf{p}_i^*\|_1 / \delta \rceil$ steps as we move on a line of length $\|P^k - P^{k'}\|_1 = \sum_{\ell=k}^{k'-1} \|P^\ell - P^{\ell+1}\|_1$ for $k' \geq k$.

After the first agent has reached her desired peak \mathbf{p}_i^* , we turn to the next agent and repeat the procedure. In that way, we eventually arrive at P^* .

(b) To see that $T_{q,i}^* = T_{q,i}$ for all $i \in N$, note that for every noncritical alternative $j \notin T_{q,i}$, we have $p_{i,j}^* = 0$, so $j \notin T_{q,i}^*$. Furthermore, for any critical alternative $j \in T_{q,i}$ and any other $j' \in T_{q,i}$,

$$\frac{q_j}{p_{i,j}^*} = \frac{q_j \cdot \mathbf{P}_i(T_{q,i})}{p_{i,j}} = \frac{q_{j'} \cdot \mathbf{P}_i(T_{q,i})}{p_{i,j'}} = \frac{q_{j'}}{p_{i,j'}^*},$$

so $j \in T_{q,i}^*$. Therefore, $T_{q,i} = T_{q,i}^*$. \square

Lemma E.5. Let P^* be a profile and \mathbf{q} be a distribution in which every agent values every noncritical alternative at zero ($j \notin T_{q,i}^*$ implies $p_{i,j}^* = 0$ for any agent i). If \mathbf{q} satisfies core fair share, then $\mathbf{q} = \text{NASH}[P^*]$.

Proof. Let P^* be an arbitrary profile, and let $\mathbf{q} \neq \text{NASH}[P^*]$ be a distribution such that $j \notin T_{q,i}^*$ implies $p_{i,j}^* = 0$. In particular, \mathbf{q} does not maximize Nash welfare. By lemma 11 of Brandt et al. [15], there exists a group N^- of agents such that the total amount given to alternatives critical for some agent from N^- is less than $|N^-|/n$. That is,

$$\mathbf{q}(T_{q,N^-}^*) < \frac{|N^-|}{n}, \tag{E.1}$$

where $T_{q,N^-}^* := \cup_{i \in N^-} T_{q,i}^*$.

We will now show that core fair share is violated for this N^- . This is clear if $\mathbf{q}(T_{q,N^-}^*) = 0$, so assume that $\mathbf{q}(T_{q,N^-}^*) > 0$.

Define a new distribution in which only alternatives in T_{q,N^-}^* are funded:

$$\mathbf{q}' := \begin{cases} q_j / \mathbf{q}(T_{q,N^-}^*) & \text{for } j \in T_{q,N^-}^*; \\ 0 & \text{for } j \notin T_{q,N^-}^*. \end{cases}$$

For every $i \in N^-$, as $p_{i,j}^* = 0$ for $j \notin T_{q,N^-}^* \supseteq T_{q,i}^*$, the utility $u_i^*(\mathbf{q}')$ equals $u_i^*(\mathbf{q}) / \mathbf{q}(T_{q,N^-}^*)$, which is larger than $u_i^*(\mathbf{q}) / (|N^-|/n)$ by (E.1). Therefore, the utility $u_i^*((|N^-|/n)\mathbf{q}' + (1 - |N^-|/n)\mathbf{q}'')$ is at least $(|N^-|/n)u_i^*(\mathbf{q}') > u_i^*(\mathbf{q})$ for every $\mathbf{q}'' \in \Delta^m$, contradicting core fair share for N^- . \square

Lemma E.6. Let P^* and P be profiles where $T_{q,i}^* = T_{q,i}$ for $\mathbf{q} = \text{NASH}[P^*]$ and all $i \in N$. Then, $\text{NASH}[P] = \mathbf{q}$.

Proof. As \mathbf{q} maximizes Nash welfare in P^* , by Lemma E.1, there exists a decomposition $(s_i)_{i \in N}$ such that $s_{i,j} = 0$ for every $i \in N$ and $j \notin T_{q,i}^*$. Because of $T_{q,i}^* = T_{q,i}$, the same decomposition proves that \mathbf{q} also maximizes Nash welfare in P by Lemma E.1; thus, $\text{NASH}[P] = \mathbf{q}$. \square

Proof of Theorem 3. Let P be an arbitrary profile and P^* a modified profile defined as in Lemma E.4. Then,

$$f(P) \stackrel{\text{Lemma E.4}}{=} f(P^*) \stackrel{\text{Lemma E.5}}{=} \text{NASH}[P^*] \stackrel{\text{Lemma E.6}}{=} \text{NASH}[P],$$

where Lemma E.6 uses the fact that the sets of critical alternatives under \mathbf{q} did not change when moving from P to P^* . \square

Appendix F. Proofs for Section 7

We start with a characterization for a single agent; this will be used in further characterizations.

Lemma F.1. For $m = 2$ and $n = 1$, a mechanism f on a domain $\mathcal{U} \subseteq U^{SP}$ is continuous and strategyproof if and only if there exist $\alpha_0 \leq \alpha_1$ in $[0, 1]$ such that

$$f(p) = \text{med}(p, \alpha_0, \alpha_1).$$

Proof. The “if” direction is obvious; we focus on the “only if.”

Let f be a continuous strategyproof mechanism for $n = 1$ and any m . Let $S := f(\Delta^m)$ be the image of f . By continuity, S is a closed set. By strategyproofness, for all $\mathbf{p} \in \Delta^m$, we have $f(\mathbf{p}) \in \arg \max_{\mathbf{q} \in S} u(\mathbf{q})$. Moreover, S must be convex because if a segment with endpoints in S is not contained in S , then some internal point of this segment would be a discontinuity point of $\arg \max$.

For $n = 1$ and $m = 2$, this boils down to S being a closed interval, $S = [\alpha_0, \alpha_1]$ for some $\alpha_0 \leq \alpha_1$ in $[0, 1]$, and f being a function that maps each p to the point nearest to p in $[\alpha_0, \alpha_1]$. This is equivalent to $f(p) = \text{med}(p, \alpha_0, \alpha_1)$. \square

Note that continuity is essential for the above characterization. For example, the following discontinuous mechanism is strategyproof:

$$f(p) := \begin{cases} 0 & p < 0.5; \\ 1 & p \geq 0.5. \end{cases}$$

We now move on to mechanisms for any number of agents. The following lemma shows that for $m = 2$, any continuous strategyproof mechanism is completely determined by its outcomes on single-minded profiles.

Lemma F.2. For $m = 2$ and arbitrary domain $\mathcal{U} \subseteq \mathcal{U}^{SP}$, if two continuous and strategyproof mechanisms yield the same distribution for all single-minded profiles, then they yield the same distribution for all profiles.

Proof. Let f and g be two continuous strategyproof mechanisms. Let P be a profile for which $f(P) \neq g(P)$. We prove that there is a single-minded profile P' for which $f(P') \neq g(P')$.

At a high level, the proof works as follows. Step by step, each agent with peak on the left side of $f(P)$ moves her peak closer and closer to zero, and each agent with peak on the right side moves to one. Continuity and strategyproofness imply that $f(P)$ cannot change in the process. Finally, for all agents with peaks at $f(P)$, move their peaks to the alternative that is not “separated” from the peak by $g(P)$. In the process, $f(P)$ can only move further away from $g(P)$.

In detail, assume that $f(P) < g(P)$; the case $f(P) > g(P)$ can be handled analogously. Denote $q := f(P)$.

Partition the set of agents into four groups: $N = N^{01} \cup N^- \cup N^= \cup N^+$, where $N^{01} = \{i \in N : p_i \in \{0, 1\}\}$, $N^- = \{i \in N \setminus N^{01} : p_i < q\}$, $N^= = \{i \in N \setminus N^{01} : p_i = q\}$, and $N^+ = \{i \in N \setminus N^{01} : p_i > q\}$. Our overall goal is to “move” all agents to N^{01} while keeping the chosen distribution different from $g(P)$.

Take any agent $i \in N^-$, and consider the function $F : [0, 1] \rightarrow [0, 1]$ defined by $F(p) := f(p, P_{-i})$. Because f is continuous and strategyproof, so is F as a mechanism for a single agent. Hence, by Lemma F.1, $F(p) = \text{med}(p, \alpha_0, \alpha_1)$ for some constants $\alpha_0 \leq \alpha_1$. Note that $F(p_i) = f(P) > p_i$ as $i \in N^-$, so $p_i < \text{med}(p_i, \alpha_0, \alpha_1)$. The median properties imply that $F(p) = F(p_i)$ also for all $p < p_i$. In particular, $F(0) = F(p_i) = f(P)$.

Denote the profile where agent i changed her peak to zero by $P^{(i)}$; then, $f(P^{(i)}) = F(0) = f(P)$. The same argument applies to all other agents from N^- , so $f(P^{N^-}) = f(P)$, where P^{N^-} denotes the profile resulting from P after all agents in N^- moved their peak to zero. Also, $g(P^{N^-}) = g(P)$, as all agents from N^- were also on the left side of $g(P)$ because of $f(P) < g(P)$, so moving them farther left does not change the distribution returned by g .

For an agent $i \in N^+$, define $F(p) := f(p, P_{-i}^{N^+})$. One can show analogously that $F(p) = F(p_i) = f(P^{N^+}) = f(P)$ for all $p \geq p_i$, so the outcome remains $f(P)$ when i moves her peak to one. Therefore, $f(P^{N^- \cup N^+}) = f(P)$, where $P^{N^- \cup N^+}$ denotes the profile resulting from P after all agents in N^- moved their peak to zero and all agents in N^+ moved their peak to one. Also, $g(P^{N^- \cup N^+}) \geq g(P^{N^-})$, as moving peaks to the right can only increase the median returned by g . Therefore, $f(P^{N^- \cup N^+}) < g(P^{N^- \cup N^+})$ still holds.

We now consider an agent $i \in N^=$ for whom $p_i = f(P) < g(P) \leq g(P^{N^- \cup N^+})$. Define $F'(p) := f(p, P_{-i}^{N^- \cup N^+})$. As it is continuous and strategyproof, Lemma F.1 implies that $F'(p) = \text{med}(p, \alpha_0, \alpha_1)$ for some constants $\alpha_0 \leq \alpha_1$. As the median is a weakly monotone function of its arguments, $F'(p) \leq F'(p_i)$ for $p \leq p_i$. Thus, with $P^{N^- \cup N^+ \cup \{i\}}$ denoting the profile where agent i moved her peak to zero, $f(P^{N^- \cup N^+ \cup \{i\}}) \leq f(P^{N^- \cup N^+}) < g(P^{N^- \cup N^+}) = g(P^{N^- \cup N^+ \cup \{i\}})$. If $f(P^{N^- \cup N^+ \cup \{i\}}) = f(P^{N^- \cup N^+})$, repeat the procedure with the next agent from $N^=$. If $f(P^{N^- \cup N^+ \cup \{i\}}) < f(P^{N^- \cup N^+})$, all remaining agents from $N^=$ now have their peak on the right side of $f(P^{N^- \cup N^+ \cup \{i\}})$ and can move their peak to one without changing the chosen distribution, as in the case $i \in N^+$. Again, the outcome from g can only move to the right or stay fixed.

Let $P^{N^- \cup N^+ \cup N^=}$ denote the profile after all agents in $N^- \cup N^+ \cup N^=$ have moved their peaks. This profile is single-minded, as all agents have their peaks at zero or one, and $f(P^{N^- \cup N^+ \cup N^=}) < g(P^{N^- \cup N^+ \cup N^=})$, as required. \square

Remark F.1. Border and Jordan [12] considered a property called *uncompromisingness*, which states that the outcome cannot change when agents from N^- and N^+ move their peaks to zero and one, respectively (i.e., there is no compromise with agents who express extreme preferences). They showed that uncompromisingness implies continuity. By contrast, we assume continuity and obtain uncompromisingness. As a result, all our characterizations hold if we replace continuity with uncompromisingness.

Using Lemma F.2, we can now prove several characterizations.

F.1. Characterizing Generalized Median Rules

Theorem 5. For $m = 2$ and arbitrary domain $\mathcal{U} \subseteq \mathcal{U}^{SP}$, a continuous mechanism f satisfies anonymity and strategyproofness if and only if there exist $\alpha_0 \leq \alpha_1 \leq \dots \leq \alpha_n$ in $[0, 1]$ such that

$$f(P) = \text{med}(p_1, \dots, p_n, \alpha_0, \dots, \alpha_n).$$

Proof. The “if” direction is obvious; we focus on the “only if.”

For any $k \in \{0, \dots, n\}$, let P_k be a single-minded profile in which some k agents have their peak at one and the other $n - k$ have their peak at zero. Let $\alpha_k := f(P_k)$; because of the anonymity of f , it holds that α_k does not depend on the selection of P_k .

Because f is strategyproof, $\alpha_k \leq \alpha_{k+1}$ for all $k \in \{0, \dots, n - 1\}$; otherwise, in profile P_k , some agent with peak at zero could gain from reporting a peak at one.

Let $g(P) := \text{med}(p_1, \dots, p_n, \alpha_0, \dots, \alpha_n)$. Then, for any $k \in \{0, \dots, n\}$, $g(P_k) = \alpha_k$, as n arguments of the median are at most α_k , and n arguments are at least α_k . This means that f and g agree on all single-minded profiles. By Lemma F.2, $f \equiv g$. \square

Two additional characterizations can be obtained in a similar way (in the first, we add efficiency, whereas in the second, we remove anonymity):

1. A continuous mechanism f satisfies anonymity, strategyproofness, and efficiency if and only if there exist $\alpha_1 \leq \dots \leq \alpha_{n-1}$ in $[0, 1]$ such that

$$f(P) = \text{med}(p_1, \dots, p_n, \alpha_1, \dots, \alpha_{n-1}).$$

2. A continuous mechanism f satisfies strategyproofness if and only if there exist 2^n constants, $\alpha_G \in [0, 1]$ for all $G \subseteq N$ such that

$$f(P) = \max_{G \subseteq N} \min(\alpha_G, \min_{i \in G} p_i).$$

F.2. Characterizing the Uniform Phantom Rule

Theorem 6. For $m = 2$ and arbitrary domain $U \subseteq U^{SP}$, the only continuous mechanism that satisfies strategyproofness and proportionality is the uniform phantom rule.

Proof. Let g be the uniform phantom mechanism. Then, g is proportional. Because proportionality completely specifies the outcomes for all single-minded profiles, any proportional mechanism agrees with g on all single-minded profiles. By Lemma F.2, any such mechanism must equal g . \square

For $m = 2$, an outcome is efficient for a profile P if and only if it is range-respecting, which means that the uniform phantom rule is efficient. Thus, for only two alternatives, there is a unique way to aggregate utilities in an efficient, strategyproof, and fair manner, even without knowledge of the specific underlying utility model.

Appendix G. Leximin–Leontief Preferences

Leontief utilities, as defined in Section 4.2, assume that agents rank distributions only by the smallest ratio, $\min_{j \in M_i} q_j / p_{i,j}$. In this section, we assume that agents rank distributions with the same smallest ratio by the second-smallest ratio, and distributions with the same smallest and second-smallest ratio by the third-smallest ratio, and so on. We call these preferences *Leximin–Leontief preferences*. We denote the strict Leximin–Leontief preferences of each agent i by \succ_i^{lex} , and the weak relation by \succeq_i^{lex} . When we want to emphasize that the leximin relation uses a specific peak \mathbf{p}_i , we write $\succ_{\mathbf{p}_i}^{lex}$ and $\succeq_{\mathbf{p}_i}^{lex}$.

We still define the *NASH* rule based on the minimum ratio only, which we continue to denote by $u_i(\mathbf{q})$. Therefore, the *NASH* distribution remains a continuous function of the peaks (even though the Leximin–Leontief preferences are not continuous). However, the change of preferences may potentially affect some properties of the rule. In particular, $\mathbf{q} \succeq_i^{lex} \mathbf{q}'$ implies $u_i(\mathbf{q}) \geq u_i(\mathbf{q}')$, but for the strict relation, the opposite direction is true: $u_i(\mathbf{q}) > u_i(\mathbf{q}')$ implies $\mathbf{q} \succ_i^{lex} \mathbf{q}'$. Therefore, properties defined by the weak relation only, such as strategyproofness, are stronger with Leximin–Leontief preferences than with Leontief utilities. However, properties defined by both the weak and the strict relations, such as group-strategyproofness, core fair share, and efficiency, are not a priori stronger or weaker with Leximin–Leontief preferences than with Leontief utilities.

First, we claim that Lemma A.1 still holds, where the critical alternatives are defined as in Definition A.1 (based on the minimum ratio only).

Lemma G.1. With Leximin–Leontief preferences, an outcome \mathbf{q} is efficient if and only if every alternative j with $q_j > 0$ is critical for some agent.

Proof Sketch of Lemma G.1. \Rightarrow : Suppose that some alternative j with $q_j > 0$ is not critical for any agent. We can construct a new outcome \mathbf{q}' by removing a sufficiently small amount from j and distributing it equally among all other alternatives. This increases $\min_j q_j / p_{i,j}$ for all agents and thus makes the new distribution strictly leximin-better for all agents. Hence, \mathbf{q} is not efficient.

\Leftarrow : Suppose that every alternative j with $q_j > 0$ is critical for some agent. Let \mathbf{q}' be any outcome different than \mathbf{q} , and let y be an alternative with $q'_y < q_y$. As $q_y > 0$, the assumption implies that y is critical to some agent; denote one such agent by i_y . Then,

$$\begin{aligned} \min_j \frac{q'_j}{p_{i_y,j}} &\leq \frac{q'_y}{p_{i_y,y}} < \frac{q_y}{p_{i_y,y}} \\ &= \min_j \frac{q_j}{p_{i_y,j}}, \end{aligned} \quad (\text{as } y \text{ is critical for } i_y \text{ under } \mathbf{q})$$

so \mathbf{q}' is leximin-worse for i than \mathbf{q} . Hence, \mathbf{q}' does not Pareto dominate \mathbf{q} . This holds for all \mathbf{q}' , which implies that \mathbf{q} is efficient. \square

Lemma G.1 implies that \mathbf{q} is efficient for Leximin–Leontief preferences if and only if it is efficient for the corresponding Leontief utilities. In particular, *NASH* remains efficient. Moreover, *NASH* is still neutral, and Lemma E.3 (efficiency implies one-sided range-respect) remains valid as well.

Next, we show that *NASH* remains group-strategyproof too. We need a lemma.

Lemma G.2. Let \mathbf{q}' and \mathbf{q}'' be two distributions and $i \in N$ an agent. If $\mathbf{q}'' \succeq_i^{lex} \mathbf{q}'$, then every alternative in $T_{\mathbf{q}',i}$ receives at least as much in \mathbf{q}'' as in \mathbf{q}' ; that is, $q''_y \geq q'_y$ for all $y \in T_{\mathbf{q}',i}$.

Proof. For every alternative $y \in T_{\mathbf{q}',i}$:

$$\begin{aligned} q'_y &= p_{i,y} \cdot u_i(\mathbf{q}') && \text{(as } y \text{ is critical for } i \text{ under } \mathbf{q}') \\ &\leq p_{i,y} \cdot u_i(\mathbf{q}'') && \text{(because } \mathbf{q}'' \succeq_i^{\text{lex}} \mathbf{q}' \text{ implies } u_i(\mathbf{q}'') \geq u_i(\mathbf{q}')) \\ &= p_{i,y} \cdot \min_{j \in M_i} \frac{q''_j}{p_{i,j}} && \text{(by definition of } u_i) \\ &\leq p_{i,y} \cdot \frac{q''_y}{p_{i,y}} && \text{(because } y \in T_{\mathbf{q}',i} \subseteq M_i) \\ &= q''_y, \end{aligned}$$

completing the proof. \square

Theorem G.1. *With Leximin–Leontief preferences, NASH is group-strategyproof.*

Proof. Assume for contradiction that there exist profiles P and P' with NASH distributions $\mathbf{q} \neq \mathbf{q}'$ respectively, and an inclusion-maximal group of agents $G \subseteq N$ that do not lose from the manipulation from P to P' . Let $T_{\mathbf{q},G} := \bigcup_{i \in G} T_{\mathbf{q},i}$ be the set of alternatives critical to at least one agent from G under \mathbf{q} . As no agent from G loses from the manipulation, Lemma G.2 implies that $q'_x \geq q_x$ for all $x \in T_{\mathbf{q},G}$.

As $\mathbf{q}' \neq \mathbf{q}$, there is an alternative $y \in M$ for which $q'_y > q_y$. Denote $B := T_{\mathbf{q},G} \cup \{y\}$ (it is possible that $y \in T_{\mathbf{q},G}$). Then, $\mathbf{q}'(B) > \mathbf{q}(B)$.

We now consider the decompositions of \mathbf{q} and \mathbf{q}' guaranteed to exist by Lemma E.1. Because $\mathbf{q}'(B) > \mathbf{q}(B)$, there exists an agent $j \in N$ who contributes more to B in the decomposition of \mathbf{q}' than in the decomposition of \mathbf{q} . This implies that in the decomposition of \mathbf{q} , agent j contributes some of her share of $1/n$ to alternatives not in B . It follows that $T_{\mathbf{q},j} \not\subseteq B$, so $j \notin G$, and thus, $u_j = u'_j$ (as j is not a part of the manipulating group).

In the decomposition of \mathbf{q}' , agent j must contribute a positive amount to some alternative $x \in B$, which means that x is critical for j under \mathbf{q}' . Because $u_j = u'_j$, we have $u_j(\mathbf{q}') = u'_j(\mathbf{q}') = q'_x/p'_{j,x} \geq q_x/p'_{j,x} \geq q_x/p_{j,x} \geq u_j(\mathbf{q})$. Therefore, all agents in $G \cup \{j\}$ do not lose from the manipulation, which contradicts the maximality of G . \square

We now extend Proposition 2 to Leximin–Leontief preferences.

Proposition G.1. *With Leximin–Leontief preferences, NASH satisfies core fair share.*

Proof. Assume for contradiction that there exists $P \in \mathcal{P}$ such that $\mathbf{q} := \text{NASH}[P]$ does not satisfy core fair share for some $G \subseteq N$. Then, there exists $\mathbf{q}' \in \Delta^m$ such that for every $\mathbf{q}'' \in \Delta^m$,

$$(|G|/n)\mathbf{q}' + (1 - |G|/n)\mathbf{q}'' \succeq_i^{\text{lex}} \mathbf{q} \quad \text{for all } i \in G.$$

Let $T_{\mathbf{q},G} := \bigcup_{i \in G} T_{\mathbf{q},i}$ be the set of alternatives critical to at least one agent from G .

Note that $T_{\mathbf{q},G} = M$ cannot hold. Otherwise, by Lemma G.1, \mathbf{q} would be efficient not only for N but already for G , contradicting that \mathbf{q} does not satisfy core fair share for G . Therefore, there exists a distribution \mathbf{q}'' with $q''_j = 0$ for every $j \in T_{\mathbf{q},G}$. Choosing such a distribution \mathbf{q}'' shows that $(|G|/n)q'_j \geq q_j$; otherwise, some agent from G for whom j is critical would have a smaller utility. Thus, $\mathbf{q}(T_{\mathbf{q},G}) := \sum_{j \in T_{\mathbf{q},G}} q_j \leq (|G|/n) \cdot \sum_{j \in T_{\mathbf{q},G}} q'_j \leq |G|/n$.

By Lemma E.1, the NASH distribution can be decomposed in such a way that every agent from G only contributes her share of $1/n$ to alternatives in $T_{\mathbf{q},G}$. Thus, $\mathbf{q}(T_{\mathbf{q},G}) \geq |G|/n$. All in all, $\mathbf{q}(T_{\mathbf{q},G}) = |G|/n$, and $(|G|/n)q'_j = q_j$ for every $j \in T_{\mathbf{q},G}$. But this also implies that $(|G|/n) \cdot \mathbf{q}'(T_{\mathbf{q},G}) = |G|/n$, so $\mathbf{q}'(T_{\mathbf{q},G}) = 1$. This means that \mathbf{q}' only allocates to alternatives in $T_{\mathbf{q},G}$. As the allocation to alternatives in $T_{\mathbf{q},G}$ is the same in \mathbf{q} and $(|G|/n)\mathbf{q}'$, no agent in G can have a better leximin vector in $(|G|/n)\mathbf{q}'$ than in \mathbf{q} . \square

We now consider the characterization (Theorem 3) for Leximin–Leontief preferences. It turns out that group-strategyproofness can be weakened to strategyproofness.

As in the Proof of Theorem 3, to show the statement, we would like to change P gradually to P^* , where each agent's peak puts zero on noncritical alternatives. However, in order to exploit the fact that Leximin–Leontief preferences constitute a refinement of Leontief utilities, which allows us to weaken group-strategyproofness to strategyproofness, we need to adapt the proof. We first show that f coincides with NASH on all *strictly-positive profiles*, that is, all profiles $P \in \mathcal{P}^+$, where $\mathcal{P}^+ := \{P \in \mathcal{P} : p_{i,j} > 0 \text{ for all } i \in N, j \in M\}$.

Lemma G.3. *If f is a continuous mechanism satisfying strategyproofness, then when moving from P to P^* ,*

- the outcome does not change; that is, $f(P^*) = f(P) = \mathbf{q}$ for $P \in \mathcal{P}^+$;*
- the sets of critical alternatives do not change; that is, $T_{\mathbf{q},i} = T_{\mathbf{q},i}^*$ for every $i \in N$.*

Proof. We first show the statement for a slightly perturbed \tilde{P}_ε^* with $\tilde{\mathbf{p}}_i^* = (1 - \tilde{\varepsilon})\mathbf{p}_i^* + \tilde{\varepsilon}\mathbf{p}_i$ and arbitrary small but fixed $\tilde{\varepsilon} > 0$. Note that P^* may not be in \mathcal{P}^+ , but \tilde{P}_ε^* is always in \mathcal{P}^+ . Note also that for an agent i with $T_{\mathbf{q},i} = M$, it holds that $\mathbf{p}_i = \tilde{\mathbf{p}}_i^* = \mathbf{p}_i^*$.

Again, we move the peak of each agent in turn. For each agent i , we change \mathbf{p}_i toward $\tilde{\mathbf{p}}_i^*$ gradually, to some $\hat{\mathbf{p}}_i := \lambda\tilde{\mathbf{p}}_i^* + (1 - \lambda)\mathbf{p}_i$ for some $\lambda \in [0, 1]$ to be computed later. Then, we proceed along this line until we reach $\lambda = 1$ and $\tilde{\mathbf{p}}_i^*$.

Given the outcome $\mathbf{q} = f(P)$, we partition the alternatives of each agent i into *critical classes*, that is, subsets with the same ratio $q_j/p_{i,j}$. Here, we use the fact that $P \in \mathcal{P}^+$, so $p_{i,j} > 0$ for all i, j . Denote the subset with the smallest ratio by $T_{\mathbf{q},i,1} \equiv T_{\mathbf{q},i}$, the subset with the second-smallest ratio by $T_{\mathbf{q},i,2}$, and so on, up to $T_{\mathbf{q},i,w}$, where w is the number of different ratios. Also, for $r \in [w]$, denote $T_{\mathbf{q},i,\leq r} := T_{\mathbf{q},i,1} \cup \dots \cup T_{\mathbf{q},i,r}$, and define $T_{\mathbf{q},i,>r}$ analogously.

As $\hat{\mathbf{p}}_i$ lies along the line between \mathbf{p}_i and $\tilde{\mathbf{p}}_i^*$, the change from \mathbf{p}_i to $\hat{\mathbf{p}}_i$ has a simple structure:

- $\hat{p}_{i,j} > p_{i,j}$ for all $j \in T_{\mathbf{q},i,1}$, and the ratio $\hat{p}_{i,j}/p_{i,j} = \lambda(1 - \tilde{\varepsilon})/(\mathbf{p}_i(T_{\mathbf{q},i,1}) + \lambda\tilde{\varepsilon} + (1 - \lambda)) =: \lambda^+$, a constant independent of j ;
- $\hat{p}_{i,j} < p_{i,j}$ for all $j \in T_{\mathbf{q},i,>1}$, and the ratio $\hat{p}_{i,j}/p_{i,j} = \lambda\tilde{\varepsilon} + (1 - \lambda) =: \lambda^-$, again independent of j .

Computing λ . We pick λ sufficiently small such that no new alternative becomes critical for i , and moreover, critical classes do not mix; that is, $q'_j/p_{i,j'} > q_j/p_{i,j}$ implies $\hat{q}'_j/p_{i,j'} > \hat{q}_j/p_{i,j}$ for all $j, j' \in M$. Specifically, set

$$\varepsilon := \min_{j \in T_{\mathbf{q},i,r}, j' \in T_{\mathbf{q},i,s}} \frac{q_{j'}p_{i,j} - q_jp_{i,j'}}{p_{i,j} + p_{i,j'}}$$

where the minimum is taken over all $r, s \in [w]$ and $s > r$. Note that $\varepsilon > 0$, as $q_{j'}/p_{i,j'} > q_j/p_{i,j}$ by definition of critical classes.

By uniform continuity of f , there exists $\delta > 0$ such that $\|f(P) - f(P')\|_1 < 2\varepsilon$ for all P' with $\|P - P'\|_1 \leq \delta$. Set

$$\lambda := \min\left(1, \frac{\delta}{\|\mathbf{p}_i - \tilde{\mathbf{p}}_i^*\|_1}\right),$$

and define \hat{P} as a profile identical to P except that i changes her peak from \mathbf{p}_i to $\hat{\mathbf{p}}_i := \lambda\tilde{\mathbf{p}}_i^* + (1 - \lambda)\mathbf{p}_i$. Note that $\|P - \hat{P}\|_1 = \lambda\|\mathbf{p}_i - \tilde{\mathbf{p}}_i^*\|_1 \leq \delta$, so $\|\mathbf{q} - \hat{\mathbf{q}}\|_1 < 2\varepsilon$, where $\mathbf{q} = f(P)$ and $\hat{\mathbf{q}} = f(\hat{P})$.

The choice of ε ensures that for arbitrary $r, s \in [w]$ with $s > r$, $j \in T_{\mathbf{q},i,r}$, and $j' \in T_{\mathbf{q},i,s}$,

$$\begin{aligned} \frac{\hat{q}_{j'}}{p_{i,j'}} &> \frac{q_{j'} - \varepsilon}{p_{i,j'}} \geq \frac{q_{j'}}{p_{i,j'}} - \frac{q_{j'}p_{i,j} - q_jp_{i,j'}}{p_{i,j'}(p_{i,j} + p_{i,j'})} = \frac{q_{j'}p_{i,j'} + q_jp_{i,j'}}{p_{i,j'}(p_{i,j} + p_{i,j'})} = \frac{q_{j'} + q_j}{p_{i,j} + p_{i,j'}} \\ &= \frac{q_jp_{i,j} + q_{j'}p_{i,j}}{p_{i,j}(p_{i,j} + p_{i,j'})} = \frac{q_j}{p_{i,j}} + \frac{q_{j'}p_{i,j} - q_jp_{i,j'}}{p_{i,j}(p_{i,j} + p_{i,j'})} \geq \frac{q_j + \varepsilon}{p_{i,j}} > \frac{\hat{q}_j}{p_{i,j}}. \end{aligned}$$

Therefore, we have $T_{\mathbf{q},i,r} = \hat{T}_{\mathbf{q},i,r}$ for all $r \in [w]$.

Proving that the outcome does not change. Consider a manipulation of agent i who manipulates between reporting \mathbf{p}_i and $\hat{\mathbf{p}}_i$. Strategyproofness for agent i implies both $\mathbf{q} \succeq_{\mathbf{p}_i}^{\text{lex}} \hat{\mathbf{q}}$ and $\hat{\mathbf{q}} \succeq_{\hat{\mathbf{p}}_i}^{\text{lex}} \mathbf{q}$.

We now prove, by induction on r , that $q_j = \hat{q}_j$ for all $j \in T_{\mathbf{q},i,r}$. For the base case $r = 1$, consider the alternatives in $T_{\mathbf{q},i,1}$.

• As all alternatives in $T_{\mathbf{q},i,1}$ are at the bottom of the ordering by $q_j/p_{i,j}$ (by definition) as well as by $\hat{q}_j/p_{i,j}$ (by the choice of ε), the relation $\mathbf{q} \succeq_{\mathbf{p}_i}^{\text{lex}} \hat{\mathbf{q}}$ implies the same relation among the subvectors corresponding to the alternatives in $T_{\mathbf{q},i,1}$; that is,

$$[q_j | j \in T_{\mathbf{q},i,1}] \succeq_{\mathbf{p}_i}^{\text{lex}} [\hat{q}_j | j \in T_{\mathbf{q},i,1}]. \tag{G.1}$$

• Similarly, all alternatives in $\hat{T}_{\mathbf{q},i,1} = T_{\mathbf{q},i,1}$ are at the bottom of the ordering by $q_j/\hat{p}_{i,j}$ by construction. Therefore, the relation $\hat{\mathbf{q}} \succeq_{\hat{\mathbf{p}}_i}^{\text{lex}} \mathbf{q}$ implies

$$[\hat{q}_j | j \in T_{\mathbf{q},i,1}] \succeq_{\hat{\mathbf{p}}_i}^{\text{lex}} [q_j | j \in T_{\mathbf{q},i,1}]. \tag{G.2}$$

• But because $\hat{p}_{i,j}$ differs from $p_{i,j}$ by a constant factor λ^+ for all $j \in T_{\mathbf{q},i,1}$, (G.2) implies the same inequality with $\succeq_{\hat{\mathbf{p}}_i}^{\text{lex}}$ instead of $\succeq_{\mathbf{p}_i}^{\text{lex}}$. Combining this with (G.1), we get

$$[\hat{q}_j | j \in T_{\mathbf{q},i,1}] \simeq_{\mathbf{p}_i}^{\text{lex}} [q_j | j \in T_{\mathbf{q},i,1}].$$

As $q_j/p_{i,j}$ is constant for $j \in T_{\mathbf{q},i,1}$, lexicographic equivalence with respect to \mathbf{p}_i implies $q_j/p_{i,j} = \hat{q}_j/p_{i,j}$ for all $j \in T_{\mathbf{q},i,1}$. Thus, $q_j = \hat{q}_j$ must hold for all $j \in T_{\mathbf{q},i,1}$.

For the induction step, assume that $q_j = \hat{q}_j$ holds for all $j \in T_{\mathbf{q},i,\leq r}$, for some $r \in [w - 1]$. Next, consider the alternatives in $T_{\mathbf{q},i,r+1} = \hat{T}_{\mathbf{q},i,r+1}$.

• As $q_j = \hat{q}_j$ holds for all other alternatives with smaller ratios, the relation $\mathbf{q} \succeq_{\mathbf{p}_i}^{\text{lex}} \hat{\mathbf{q}}$ implies the same relation for the subset $T_{\mathbf{q},i,r+1}$; that is, $[q_j | j \in T_{\mathbf{q},i,r+1}] \succeq_{\mathbf{p}_i}^{\text{lex}} [\hat{q}_j | j \in T_{\mathbf{q},i,r+1}]$.

• Similarly, the relation $\hat{\mathbf{q}} \succeq_{\hat{\mathbf{p}}_i}^{\text{lex}} \mathbf{q}$ implies $[\hat{q}_j | j \in T_{\mathbf{q},i,r+1}] \succeq_{\hat{\mathbf{p}}_i}^{\text{lex}} [q_j | j \in T_{\mathbf{q},i,r+1}]$.

• But because $\widehat{p}_{i,j}$ differs from $p_{i,j}$ by a constant factor λ^- for all $j \in T_{q,i,r+1}$, $\widehat{q} \succeq_{\widehat{p}_i}^{lex} q$ implies $\widehat{q} \succeq_{p_i}^{lex} q$, so all in all, $[\widehat{q}_j | j \in T_{q,i,r+1}] \succeq_{p_i}^{lex} [q_j | j \in T_{q,i,r+1}]$ must hold. This means that $q_j = \widehat{q}_j$ must hold for all $j \in T_{q,i,r+1}$.

Therefore, $\widehat{q} = q$.

Applying this argument repeatedly, we get a sequence of profiles (P^k) with $P^0 = P$, where P^k lies on the line $\lambda \widehat{p}_i^* + (1 - \lambda)P$ for every k . It remains to show that (P^k) reaches \widehat{p}_i^* after a finite number of steps. For that, consider the expression in the definition of ε :

$$\min_{j \in T_{q,i,r}, j' \in T_{q,i,s}} \frac{q_{j'} p_{i,j} - q_j p_{i,j'}}{p_{i,j} + p_{i,j'}}$$

For $r = 1$, as P comes closer to \widehat{p}_i^* , $p_{i,j}$ increases and $p_{i,j'}$ decreases, while q and the critical classes stay the same, so overall, the expression increases. For $s > r > 1$, note that $p_{i,j}$ and $p_{i,j'}$ both decrease by the same factor λ^- , while q and the critical classes stay the same. Thus, we can take the ε (and the corresponding δ) from the first step for every step. Furthermore, $\|P^k - P^{k+1}\|_1 = \delta$ (unless $\lambda = 1$, but then we have reached \widehat{p}_i^*) implying that we reach \widehat{p}_i^* after at most $\lceil \|P - \widehat{p}_i^*\|_1 / \delta \rceil$ steps, because we move on, a line of length $\|P^k - P^{k+1}\|_1 = \sum_{\ell=k}^{k-1} \|P^\ell - P^{\ell+1}\|_1$ for $k \geq k$.

After the first agent has reached her desired peak \widehat{p}_i^* , we turn to the next agent and repeat the procedure. In that way, we eventually arrive at $\widehat{P}_\varepsilon^*$.

As ε was chosen arbitrarily and we have $\lim_{\varepsilon \rightarrow 0} \widehat{P}_\varepsilon^* = P^*$ for arbitrary $P \in \mathcal{P}^+$, continuity of f implies $f(P) = f(P^*)$.

Statement (b) now follows analogously to the one in Lemma E.4. \square

Proof of Theorem 4. As Lemmas E.5 and E.6 still hold for Leximin–Leontief preferences, $f(P) = \text{NASH}[P]$ for all $P \in \mathcal{P}^+$. Noting that \mathcal{P}^+ is dense in \mathcal{P} (and f and NASH are continuous), f has to coincide with NASH on all profiles in \mathcal{P} . \square

Endnotes

¹ Concretely, suppose that there are $n = 2$ agents and $m = 3$ alternatives. The agents' ideal distributions are $p_1 = (0.8, 0.2, 0)$ and $p_2 = (0.8, 0, 0.2)$, respectively. The independent markets mechanism of Freeman et al. [22] returns the distribution $q = (0.6, 0.2, 0.2)$. However, if the first agent reports $p_1' = (0.82, 0.18, 0)$ instead, the mechanism returns $q' = (0.62, 0.18, 0.2)$, which the agent prefers to q under Leontief utilities.

² Braga de Freitas et al. [13] use a weaker definition to star-shaped preferences: for every distribution q , the set of distributions that are weakly preferred to q is a star domain. This definition is implied by the definition we use, as for any distribution q , if q' is preferred to q , then by transitivity, all distributions on the line from p_i to q' are also preferred to q , so the set of distributions weakly preferred to q is a star domain with respect to p_i . Their definition is weaker as it allows the domain to be a star with respect to another point.

³ For Leontief preferences, the weak and the strong version of the core are equivalent (see Remark A.1). For dichotomous utilities, our definition is weaker than that of Aziz et al. [5], and thus, their positive result about the Nash product rule carries over. Scarf [35] has shown the nonemptiness of the weak α -core under very general conditions. By contrast, the strong α -core is already empty for dichotomous utilities.

⁴ Our notion of group-strategyproofness is slightly different, as agents do not have variable contributions. Specifically, we only need to consider manipulations of the individual peaks. However, this is a subset of all manipulations considered by Brandt et al. [15], so NASH also satisfies our notion of group-strategyproofness.

⁵ We assume that when k is odd, the median of $k + 1$ elements is the $((k + 1)/2)$ -th element.

References

- [1] Arrow KJ, Sen A, Suzumura K, eds. (2002) *Handbook of Social Choice and Welfare*, vol. 1 (North Holland, Amsterdam).
- [2] Arrow KJ, Sen A, Suzumura K, eds. (2011) *Handbook of Social Choice and Welfare*, vol. 2 (North Holland, Amsterdam).
- [3] Aumann RJ (1961) The core of a cooperative game without side payments. *Trans. Amer. Math. Soc.* 98(3):539–552.
- [4] Aziz H, Shah N (2021) Participatory budgeting: Models and approaches. Rudas T, Péli G, eds. *Pathways Between Social Science and Computational Social Science: Theories, Methods, and Interpretations*, Computational Social Sciences (Springer, Cham, Switzerland), 215–236.
- [5] Aziz H, Bogomolnaia A, Moulin H (2020) Fair mixing: The case of dichotomous preferences. *ACM Trans. Econom. Comput.* 8(4):18:1–18:27.
- [6] Aziz H, Lam A, Lee BE, Walsh T (2025) Proportionality-based fairness and strategyproofness in the facility location problem. *J. Math. Econom.* 119:103129.
- [7] Barberà S, Jackson MO (1994) A characterization of strategy-proof social choice functions for economies with pure public goods. *Soc. Choice Welfare* 11(3):241–252.
- [8] Barberà S, Peleg B (1990) Strategy-proof voting schemes with continuous preferences. *Soc. Choice Welfare* 7(1):31–38.
- [9] Berga D, Serizawa S (2000) Maximal domain for strategy-proof rules with one public good. *J. Econom. Theory* 90(1):39–61.
- [10] Black D (1948) On the rationale of group decision-making. *J. Political Econom.* 56(1):23–34.
- [11] Bogomolnaia A, Moulin H, Stong R (2005) Collective choice under dichotomous preferences. *J. Econom. Theory* 122(2):165–184.
- [12] Border KC, Jordan JS (1983) Straightforward elections, unanimity and phantom voters. *Rev. Econom. Stud.* 50(1):153–170.
- [13] Braga de Freitas S, Orrillo J, Sosa W (2020) From Arrow–Debreu condition to star shape preferences. *Optimization* 69(11):2405–2419.
- [14] Brandt F (2017) Rolling the dice: Recent results in probabilistic social choice. Endriss U, ed. *Trends in Computational Social Choice* (AI Access), 3–26.
- [15] Brandt F, Greger M, Segal-Halevi E, Suksompong W (2025) Coordinating charitable donations with Leontief preferences. *J. Econom. Theory* 230:106096.
- [16] Caragiannis I, Christodoulou G, Protopapas N (2024) Truthful aggregation of budget proposals with proportionality guarantees. *Artificial Intelligence* 335:104178.

- [17] de Berg M, Freeman R, Schmidt-Kraepelin U, Utke M (2024) Truthful budget aggregation: Beyond moving-phantom mechanisms. Preprint, submitted July 25, <https://arxiv.org/abs/2405.20303>.
- [18] De Vries MS, Nemeč J, Špaček D (2022) *International Trends in Participatory Budgeting: Between Trivial Pursuits and Best Practices*, Governance and Public Management (Palgrave Macmillan, Cham, Switzerland).
- [19] Elkind E, Suksompong W, Teh N (2023) Settling the score: Portioning with cardinal preferences. Gal K, Nowé A, Nalepa GJ, Fairstein R, Rădulescu R, eds. *Proc. 26th Eur. Conf. Artificial Intelligence (ECAI)* (IOS Press, Amsterdam), 621–628.
- [20] Fain B, Goel A, Munagala K (2016) The core of the participatory budgeting problem. Cai Y, Vetta A, eds. *Proc. 12th Internat. Conf. Web Internet Econom.* (WINE 2016), vol. 10123 (Springer-Verlag, Berlin, Heidelberg), 384–399.
- [21] Freeman R, Schmidt-Kraepelin U (2024) Project-fair and truthful mechanisms for budget aggregation. Wooldridge M, Dy J, Natarajan S, eds. *Proc. 38th AAAI Conf. Artificial Intelligence* (AAAI Press, Washington, DC), 9704–9712.
- [22] Freeman R, Pennock DM, Peters D, Wortman Vaughan J (2021) Truthful aggregation of budget proposals. *J. Econom. Theory* 193:105234.
- [23] Gibbard A (1977) Manipulation of schemes that mix voting with chance. *Econometrica* 45(3):665–681.
- [24] Goel A, Hulett R, Plaut B (2019) Markets beyond Nash welfare for Leontief utilities. *Proc. 15th Internat. Conf. Web Internet Econom.* (New York), 340.
- [25] Goel A, Krishnaswamy AK, Sakshuwong S, Aitamurto T (2019) Knapsack voting for participatory budgeting. *ACM Trans. Econom. Comput.* 7(2):8:1–8:27.
- [26] Intriligator MD (1973) A probabilistic model of social choice. *Rev. Econom. Stud.* 40(4):553–560.
- [27] Jennings AB, Laraki R, Puppe C, Varloot EM (2024) New characterizations of strategy-proofness under single-peakedness. *Math. Programming* 203(1–2):207–238.
- [28] Li J, Xue J (2013) Egalitarian division under Leontief preferences. *Econom. Theory* 54(3):597–622.
- [29] Lindner T (2011) Zur Manipulierbarkeit der Allokation öffentlicher Güter: Theoretische Analyse und Simulationsergebnisse. PhD thesis, Karlsruhe Institute of Technology, Karlsruhe, Germany.
- [30] Lindner T, Nehring K, Puppe C (2008) Allocating public goods via the midpoint rule. *Proc. 9th Internat. Meeting Soc. Soc. Choice Welfare* (Society of Social Choice and Welfare, Caen, France).
- [31] Mas-Colell A, Whinston MD, Green JR (1995) *Microeconomic Theory* (Oxford University Press, Oxford, UK).
- [32] Massó J, de Barreda IM (2011) On strategy-proofness and symmetric single-peakedness. *Games Econom. Behav.* 72(2):467–484.
- [33] Moulin H (1980) On strategy-proofness and single peakedness. *Public Choice* 35(4):437–455.
- [34] Nicoló A (2004) Efficiency and truthfulness with Leontief preferences. A note on two-agent, two-good economies. *Rev. Econom. Design* 8(4):373–382.
- [35] Scarf HE (1971) On the existence of a cooperative solution for a general class of n -person games. *J. Econom. Theory* 3(2):169–181.
- [36] Varloot EM, Laraki R (2022) Level-strategyproof belief aggregation mechanisms. *EC'22: Proc. 23rd ACM Conf. Econom. Comput.* (Association for Computing Machinery, New York), 335–369.
- [37] Zhou L (1991) Impossibility of strategy-proof mechanisms in economies with pure public goods. *Rev. Econom. Stud.* 58(1):107–119.