
Operations Research

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# Letter to the Editor—Comments on a Paper by H. K. Weiss

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study phase but may have saved the over-all effort. Estimates of cost and time for implementation would have been more realistic and recommendations made on a sounder basis. One general conclusion, then, is that the operations researcher get closer to the operation that he is studying.

The second basic difficulty stemmed from the lack of training of the operating personnel who had to take over the procedures and make them work. In particular, one man was required to supervise and administer the system through which the implementation could take place. One way to provide the requisite training is to include this person on the team at the inception of the study. As a team member he will receive 'on-the-problem' training and be allowed to participate in the formulation and solution of the problem with which he will be associated. Thus, he will understand the underlying reasons for each component of the model as well as become familiar with its manipulation. Perhaps the inclusion of such a team member should be standard practice in every operations-research study!

Ackoff (p. 3) stated that "the domain of implementation has been almost untouched by scientific hands." I regard this as an admonition rather than a challenge.

## COMMENTS ON A PAPER BY H. K. WEISS

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(Received October 15, 1962)

IN H. K. WEISS' letter to the Editor,* Weiss pointed out that REAR ADMIRAL FISKE, USN, seemed to have foreshadowed by 11 years the developments that have come to be known as Lanchester's Square Law. Weiss stated that Fiske appears to have been the first to observe publicly that, when the damage each of two forces can inflict on the other is proportional to its own size, the change of size of each force was geometric rather than arithmetic.

It may be of interest to observe that the Fiske example quoted by Weiss is indeed a close analogue of the situation treated by Lanchester. The general equations characterizing Fiske's example may be written

$$M_0 = M, \qquad N_0 = N, \tag{1}$$

$$\begin{aligned} M_{i+1} &= M_i - aN_i, \\ N_{i+1} &= N_i - bM_i, \end{aligned} \tag{2}$$

where $M_0$, $N_0$ are the force sizes at the start of an engagement fought in distinct volleys, $M_i$, $N_i$ the numbers surviving the $i$th volley, $a$, ($b$) the number of troops on the side of $M$, ($N$), destroyed per man on the $N$, ($M$), side on each volley.

The closeness of this situation to the classic one treated by Lanchester may be

* "The Fiske Model of Warfare," *Opns. Res.* **10,** 569–571 (1962).

seen by rewriting equations (2) as

$$\begin{aligned} M(t+\Delta t) &= M(t) - N(t)\alpha\Delta t, \\ N(t+\Delta t) &= N(t) - M(t)\beta\Delta t, \end{aligned} \tag{3}$$

if the end of the $i$th volley is associated with the time, $t$, the $(i+1)$st volley with time $t+\Delta t$, $\alpha$, $(\beta)$ the rates with which each $N$, $(M)$ troop destroy $M$, $(N)$ troops so that $a=\alpha\Delta t$, $b=\beta\Delta t$

In the limit, of course, as $\Delta t$ approaches zero, the equations, (3), become the classical Lanchester differential equation

$$\begin{aligned} dM(t)/dt &= -\alpha N(t), \\ dN(t)/dt &= -\beta M(t) \end{aligned} \tag{4}$$

The Fiske equations (1) and (2) may easily be solved by rewriting the equations, (2), for $M_{i+2}$ and $N_{i+2}$, combining the equations for $M_{i+1}$, $M_{i+2}$ algebraically so as to eliminate $N_i$, $N_{i+1}$, obtaining

$$M_{i+2} = 2M_{i+1} - (1-ab)M_i \tag{5}$$

All solutions to (5) are determinable as linear combinations of powers of the roots of the associated equation

$$x^2 = 2x - (1-ab) \tag{5'}$$

These roots are

$$x = 1 + (ab)^{1/2} \qquad \text{or} \qquad x = 1 - (ab)^{1/2} \tag{6}$$

Hence the set of solutions is that linear combination satisfying the boundary conditions imposed by (1), so that if

$$M_i = u[1+(ab)^{1/2}]^i + v[1-(ab)^{1/2}]^i$$

it follows that we must have

$$\begin{aligned} u+v &= M, \\ (u+v)+(u-v)(ab)^{1/2} &= M - aN \end{aligned} \tag{7}$$

Solving for $u$, $v$, we obtain

$$\begin{aligned} u &= \tfrac{1}{2}\,[M - N(a/b)^{1/2}], \\ v &= \tfrac{1}{2}\,[M + N(a/b)^{1/2}] \end{aligned} \tag{8}$$

This yields immediately

$$M_i = \tfrac{1}{2}\,\{[M - N(a/b)^{1/2}][1+(ab)^{1/2}]^i + [M + N(a/b)^{1/2}][1-(ab)^{1/2}]^i\} \tag{9}$$

Similar results may be obtained for $N_i$

It is not surprising to observe that the solution to the Fiske equations illustrated above approaches the solution to the Lanchester equations as the time increment between successive volleys approaches zero For, if the same notation is used on (9) as was used in transforming equations (2) to equations (3), we see that, on passing to the limit as $\Delta t$ approaches zero, we obtain

$$M(t) = \lim_{\Delta t \to 0} \{\tfrac{1}{2}[M - N(\alpha/\beta)^{1/2}][1 + (\alpha\beta)^{1/2}\Delta t]^{t/\Delta t} + \tfrac{1}{2}[M + N(\alpha/\beta)^{1/2}][1 - (\alpha\beta)^{1/2}\Delta t]^{t/\Delta t}\}, \tag{10}$$

or

$$M(t) = \tfrac{1}{2}\{[M - N(\alpha/\beta)^{1/2}]e^{(\alpha\beta)^{1/2}t} + [M + N(\alpha/\beta)^{1/2}]e^{-(\alpha\beta)^{1/2}t}\},$$

which is the standard solution to (4)

The solution, (9), to the Fiske equations (1) and (2), or (5) with boundary conditions (1), may also be expanded by use of the binomial theorem into the computationally convenient forms

$$\begin{aligned} M_{2k} &= M\sum_{j=0}^{k} \frac{(2k)!(ab)^j}{(2j)!(2k-2j)!} - Na\sum_{j=0}^{k-1} \frac{(2k)!(ab)^j}{(2j+1)!(2k-2j-1)!}, \\ M_{2k+1} &= M\sum_{j=0}^{k} \frac{(2k+1)!(ab)^j}{(2j)!(2k+1-2j)!} - Na\sum_{j=0}^{k} \frac{(2k+1)!(ab)^j}{(2j+1)!(2k-2j)!} \end{aligned} \tag{9'}$$

In the above form, these expressions may easily be verified to produce the results yielded directly from the basic recursions, for example

$$\begin{aligned} M_0 &= M, \\ M_1 &= M - aN, \\ M_2 &= M[1 + ab] - 2aN, \\ M_3 &= M[1 + 3ab] - Na[3 + ab], \\ M_4 &= M[1 + 6ab + (ab)^2] - Na[4 + 4ab], \end{aligned}$$
etc

In either of the forms (9) or (9′), these expressions may easily be programmed for evaluation by a digital computer, and as such can be useful in calculations pertaining to situations covered by the Fiske (discrete volley) model

The Fiske model also has a 'square law' similar to the Lanchester Square Law From equation (9), and its counterpart solution for $N_i$, the reader can easily ascertain that the Fiske Square Law is

$$b[M^2(1 - ab)^i - M_i^2] = a[N^2(1 - ab)^i - N_i^2] \tag{11}$$

It may easily be determined that in the limit, the Fiske Square Law does converge toward the Lanchester Square Law as the time increment between successive volleys approaches zero Unfortunately, the presence of the factor, $(1 - ab)^i$, in equation (11) decreases its usefulness compared to the Lanchester Square Law, as the relation between $M_i$ and $N_i$ is, because of the presence of this factor, not independent of the number of volleys, $i$

Interestingly enough, however, the Fiske Square Law is still sufficiently powerful to permit a simple determination of which side will be victorious in an engagement fought until one side is completely destroyed One-volley battles are trivial and need not be considered In battles lasting more than one volley (i e , with both $M_i$, $N_i$ greater than zero), it may easily be seen that $ab<1$, whence the quantity, $1-(ab)>0$

The relation (11) together with (9) (and its counterpart for $N_i$) show that a graph of $M_i$ versus $N_i$ is a hyperbola-like figure (if $i$ is regarded as a continuous parameter)*

The values of $M_i$, $N_i$ for integral values of $i$ will be strung out in the positive quadrant of this figure, like beads on a necklace, during the real portion of the engagement Ascertaining whether the figure crosses the $M$ axis or the $N$ axis will determine whether the $M$ or $N$ side is victorious Thus if the $M$ side is victorious it follows that for some $f$ (not necessarily an integer) $N_f=0$, and $M_f$ is real (in fact positive since we are interested in the positive quadrant to begin with)

Then, from (11) it follows that

$$b[M^2(1-ab)^f-M_f^2]=a[N^2(1-ab)^f],$$

which yields

$$M_f^2=[bM^2-aN^2](1-ab)^f/b \tag{12}$$

Since $M_f$ is real, $M_f^2>0$ whence

| | | | |
|---|---|---|---|
| | $bM^2>aN^2$, | if the side $M$ is victorious, | |
| and similarily, | $aN^2>bM^2$, | if the side $N$ is victorious, | (13) |
| while, | $aN^2=bM^2$, | if the battle is a tie | |

Thus we see that the Fiske Square Law yields the same result as the Lanchester Square Law with respect to predicting the winner of the battle

"The victor will be the side with the greatest initial effectiveness, where initial effectiveness of a force is determined as the square of the initial number of troops on the force are multiplied by the rate at which one of its troops produces casualties on the other side"

It is clear that, in many ways, Fiske's model pursued to its logical conclusions, is closely akin to Lanchester's, and provides several similar results

* The diligent reader, interested in determining a nonparametric equation for this figure can do so One interesting form of such a relation follows

"The quantity

$$\log(b^{1/2}M_i+a^{1/2}N_i)\,\log[1+(ab)^{1/2}]-\log(b^{1/2}M_i-a^{1/2}N_i)\,\log[1-(ab)^{1/2}],$$

is invariant with $i$, hence equal throughout the battle to its value at the beginning when $i = 0$"