
# Operations Research



# Letter to the Editor—Kill Probability for Multiple Shots

Frank McNolty,

and $$\sigma^2(r_{[0.8n]}/A)=\sigma^2(r_{[0.8n]})/A^2.$$

The variance of our estimate is computed in Table I for values of $n=3$ to 20. These variances it should be noted are relatively close to those given by Moranda.

Since we have derived an estimate for $\sigma$ of the circular normal distribution, it can be now used to estimate the probable error as follows:

$$\hat{B}=(r_{[0.8n]}/A)\sqrt{2\ln(2)}.$$

## REFERENCES

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# KILL PROBABILITY FOR MULTIPLE SHOTS

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(Received August 1, 1966)

A point target is randomly located according to an offset circular normal distribution and remains in its unknown position throughout $N$ independent tosses of a lethal circle. The paper presents integral expressions for the probability of: (1) killing the target at least once in $N$ tosses of the lethal circle; (2) killing the target exactly $n$ times in $N$ tosses; (3) requiring less than or equal to $m$ shots to kill the target exactly once; (4) killing the target at least once in $N$ tosses when the bias (offset distance) is randomly distributed. The paper also presents formulas for the expected number of shots required to kill the target exactly once and the expected number of times the target is killed in $N$ tosses. A simple expression is also given for the single-shot kill probability for an offset ellipsoidal case when the lethal radius of the weapon is variable rather than fixed.

THE BRIEF discussion that follows presents integral expressions for killing a point target with a lethal circle when the point target remains in its unknown (fixed) position throughout $N$ tosses (shots) of the lethal circle.

The single point target $T(X_T, Y_T)$ is randomly distributed about the origin of the $X$, $Y$ plane in accordance with a circular normal density function with variance $\sigma_T^2$, while the center $(X_C, Y_C)$ of a lethal circle $C$ of radius $R$ is randomly distributed about an offset point $(a, b)$ in accordance with a circular normal distribution having variance $\sigma_C^2$.

When the lethal circle is tossed onto the $X$, $Y$ plane $N$ times, a common interpretation of the problem is to regard the point target as having been picked up and retossed prior to each toss of the circle. In the sequel it is assumed that the point target is thrown only once and remains in its unknown location throughout $N$ tosses of the circle.

## SINGLE-SHOT CASE

THE RANDOMLY located target $T(X_T, Y_T)$ is distributed in the $X$, $Y$ plane according to,

$$f_T(X, Y)\,dX\,dY = (1/2\pi\sigma_T^2)\exp[-(X^2+Y^2)/2\sigma_T^2]\,dX\,dY, \tag{1}$$

while the center of the lethal circle is distributed according to,

$$g_C(X, Y)\,dX\,dY = (1/2\pi\sigma_C^2)\exp[-(X-a)^2+(Y-b)^2]/2\sigma_C^2]\,dX\,dY. \tag{2}$$

The probability of killing the point target on a single toss of the circle is given by

$${}^1P_K = (1/\sigma^2)\exp(-r^2/2\sigma^2)\int_0^R \rho\exp(-\rho^2/2\sigma^2)I_0(\rho r/\sigma^2)\,d\rho, \tag{3}$$

where $\sigma^2=\sigma_C^2+\sigma_T^2$, $r^2=a^2+b^2$ and $I_0(X)$ is the modified Bessel function of the first kind of order zero. Expression (3), of course, is simply an integration of the non-central chi-square distribution with 2 degrees of freedom.

When the point target is picked up and retossed prior to each toss of the circle, one can write,

$${}^NE_K = \text{expected number of times the target is killed in } N \text{ shots} = N\cdot{}^1P_K.$$

## MULTIPLE-SHOTS CASE

SINCE THE point target remains in its unknown (fixed) location throughout $N$ tosses of the circle, one must consider all possible locations by integrating over the entire $X$, $Y$ plane. Thus, the probability ${}^NP_K$ of killing the target at least once in $N$ tosses becomes

$$\begin{aligned}{}^NP_K &= \frac{1}{2\pi\sigma_T^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left(-\frac{X^2+Y^2}{2\sigma_T^2}\right)\left\{1-\left[1-\frac{1}{\sigma_C^2}\exp\left(-\frac{(X-a)^2+(Y-b)^2}{2\sigma_C^2}\right)\right.\right.\\ &\qquad\left.\left.\cdot\int_0^R t\exp\left(-\frac{t^2}{2\sigma_C^2}\right)I_0\left(\frac{t\sqrt{(X-a)^2+(Y-b)^2}}{\sigma_C^2}\right)dt\right]^N\right\}dX\,dY\\ &= 1-\frac{1}{\sigma_T^2\sigma_C^{2N}}\exp\left(-\frac{r^2}{2\sigma_T^2}\right)\int_0^\infty s\exp\left[-\frac{s^2}{2}\left(\frac{N\sigma_T^2+\sigma_C^2}{\sigma_C^2\sigma_T^2}\right)\right]\\ &\qquad\cdot\left[\int_R^\infty t\exp\left(-\frac{t^2}{2\sigma_C^2}\right)I_0\left(\frac{ts}{\sigma_C^2}\right)dt\right]^N ds\end{aligned} \tag{4}$$

and one can show that expression (4) is not equal to $1-(1-{}^1P_K)^N$.

The probability ${}^N_nP_K$ of killing the target exactly $n$ times in $N$ trials is given by

$$\begin{aligned}{}^N_nP_K &= \frac{1}{\sigma_T^2\sigma_C^{2N}}C_n^N\exp\left(-\frac{r^2}{2\sigma_T^2}\right)\int_0^\infty s\exp\left[-\frac{s^2}{2}\left(\frac{\sigma_C^2+N\sigma_T^2}{\sigma_T^2\sigma_C^2}\right)\right]I_0\left(\frac{rs}{\sigma_T^2}\right)\\ &\qquad\cdot\left\{\left[\int_0^R t\exp\left(-\frac{t^2}{2\sigma_C^2}\right)I_0\left(\frac{ts}{\sigma_C^2}\right)dt\right]^n\cdot\left[\int_R^\infty t\exp\left(-\frac{t^2}{2\sigma_C^2}\right)I_0\left(\frac{ts}{\sigma_C^2}\right)dt\right]^{N-n}\right\}ds.\end{aligned} \tag{5}$$

It can be shown, in the multiple-shot case, that the expected number of times the target is killed in $N$ shots is given by $N\cdot{}^1P_K$.

One can also write an expression for the expected number $E_K(n)$ of tosses of the lethal circle required to kill the target exactly once,

$$E_K(n)=\frac{\sigma_C^2}{\sigma_T^2}\exp\left(-\frac{r^2}{2\sigma_T^2}\right)\int_0^\infty s\exp\left(-\frac{s^2}{2\sigma_T^2}+\frac{s^2}{2\sigma_C^2}\right)I_0\left(\frac{rs}{\sigma_T^2}\right)\cdot\left[\int_0^R t\exp\left(-\frac{t^2}{2\sigma_C^2}\right)I_0\left(\frac{ts}{\sigma_C^2}\right)dt\right]^{-1}ds, \tag{6}$$

where $E_K(n)\geqq 1$ for any value of $r$; $E_K(n)\to\infty$ as $r\to\infty$; $E_K(n)\to\infty$ as $R\to 0$ and $E_K(n)\to 1$ as $R\to\infty$.

The probability $P_K(\leqq m)$ that $\leqq m$ tosses will be required to kill the target exactly once is,

$$P_K(\leqq m)=\frac{1}{\sigma_T^2}\exp\left(-\frac{r^2}{2\sigma_T^2}\right)\sum_{\nu=0}^{m-1}\frac{1}{\sigma_C^{2(\nu+1)}}\int_0^\infty s\exp\left[-\frac{s^2}{2}\left(\frac{\sigma_C^2+(\nu+1)\sigma_T^2}{\sigma_T^2\sigma_C^2}\right)\right]\cdot I_0\left(\frac{rs}{\sigma_T^2}\right)\left\{\left[\int_R^\infty v\exp\left(-\frac{v^2}{2\sigma_C^2}\right)I_0\left(\frac{vs}{\sigma_C^2}\right)dv\right]^\nu\int_0^R t\exp\left(-\frac{t^2}{2\sigma_C^2}\right)I_0\left(\frac{ts}{\sigma_C^2}\right)dt\right\}ds. \tag{7}$$

## MULTIPLE SHOTS WITH A RANDOM BIAS

IF THE offset distance $r$ (also called the system bias) cannot be estimated with sufficient accuracy, one may assign a suitable distribution to it—for instance, the Rayleigh, Maxwell-Boltzmann, one-sided Gaussian, gamma, etc. The writer has found the following distribution useful in this regard

$$g(r)\,dr=[2/\Gamma(\lambda)](\lambda/\alpha)^\lambda r^{2\lambda-1}\exp(-\lambda/\alpha r^2)\,dr, \qquad (r\geqq 0) \tag{8}$$

where $$E(r)=\sqrt{\alpha/\lambda}\cdot[\Gamma(\lambda+\tfrac{1}{2})/\Gamma(\lambda)] \quad\text{and}\quad E(r^2)=\alpha.$$

(In most random-bias applications, $\lambda$ falls in the range $\frac{1}{2}\leqq\lambda\leqq 3$).

Expression (8) includes such distributions as the Rayleigh, Maxwell-Boltzmann, and one-sided Gaussian as special cases, and it does not appear to have been previously applied to the random bias problem. Introducing (8) into (4) yields, the probability of killing the point target at least once in $N$ trials when the bias is randomly distributed,

$${}^NP_K[g(r)]=1-\frac{1}{\sigma_T^2\sigma_C^{2N}\left(1+\frac{\alpha}{2\lambda\sigma_T^2}\right)^\lambda}\int_0^\infty s\exp\left[-\frac{s^2}{2}\left(\frac{N\sigma_T^2+\sigma_C^2}{\sigma_T^2\sigma_C^2}\right)\right]\cdot{}_1F_1\left[\lambda;1;\frac{\alpha s^2}{2\sigma_T^2(2\lambda\sigma_T^2+\alpha)}\right]\cdot\left[\int_R^\infty t\exp\left(-\frac{t^2}{2\sigma_C^2}\right)I_0\left(\frac{ts}{\sigma_C^2}\right)dt\right]^N ds, \tag{9}$$

where ${}_1F_1(a;b;X)$ is the confluent hypergeometric function.

When $N=1$, most of the preceding expressions reduce to simple forms. For instance, if in (9) one lets $\lambda=1$, $\alpha=2\sigma_r^2$ and $N=1$, ${}^NP_K[g(r)]$ reduces to

$${}^1P_K(\text{Rayleigh } r)=1-\exp[-R^2/2(\sigma_C^2+\sigma_T^2+\sigma_r^2)], \tag{10}$$

and similarly, for $r=0$ expression (4) yields the expected number of times the target is killed in $N$ shots,

$${}^NE_K=N\{1-\exp[-R^2/2(\sigma_C^2+\sigma_T^2)]\}. \tag{11}$$

When $N\neq 1$ and $r\neq 0$ a desk-calculator evaluation of expressions (4)–(9) can be effected only for fortuitous values of $R$, $\sigma_C^2$, and $\sigma_T^2$; otherwise the expressions are amenable only to digital treatment and are of interest mainly as they occur in calculations intermediate to other results.

## SINGLE-SHOT, OFFSET-ELLIPSOIDAL CASE

An interesting and useful single-shot (offset) situation occurs when the coordinates $X_T$, $Y_T$, $Z_T$ of a randomly located point target are distributed about the origin of the $X$, $Y$, $Z$ system in accordance with an ellipsoidal normal distribution $X_T$: $n(0, \sigma_{TX})$, $Y_T$: $n(0, \sigma_{TY})$, and $Z_T$: $n(0, \sigma_{TZ})$; while the coordinates of a lethal burst point are distributed about the offset point $a_1$, $a_2$, $a_3$ also according to an ellipsoidal normal distribution $X_B$: $n(a_1, \sigma_{BX})$, $Y_B$: $n(a_2, \sigma_{BY})$, and $Z_B$: $n(a_3, \sigma_{BZ})$.

We then consider the three independent, normally distributed variables $X_B-X_T=X$: $n(a_1, \sigma_1)$, $Y_B-Y_T=Y$: $n(a_2, \sigma_2)$, $Z_B-Z_T=Z$: $n(a_3, \sigma_3)$, where $\sigma_1^2=\sigma_{BX}^2+\sigma_{TX}^2$, $\sigma_2^2=\sigma_{BY}^2+\sigma_{TY}^2$ and $\sigma_3^2=\sigma_{BZ}^2+\sigma_{TZ}^2$. The probability density function, $h(s)\,ds$ of the random variable $s=X^2+Y^2+Z^2$ is then expressed in terms of its characteristic function,

$$h(s)\,ds=\frac{ds}{2\pi}\int_{-\infty}^{\infty}\frac{\exp\left(-its+\sum_{n=1}^{3}\mu_n\right)dt}{\prod_{n=1}^{3}\sqrt{1-2it\sigma_n^2}},\qquad \mu_n=\frac{ita_n^2}{1-2it\sigma_n^2},\tag{12}$$

and rather than employ a fixed radius lethal sphere (with center at $X_B$, $Y_B$, $Z_B$) we use a conditional probability $P_C(r)$ that a kill will occur given that the burst point is a specified distance $r$ from the target. Here

$$P_C(r)=\exp(-r^2/2\lambda^2),\qquad (0\leqq r<\infty)$$

and introducing $P_C(r)$ in expression (12) with a permissible change in order of integration, yields the probability of kill,

$$\begin{aligned}{}^1P_K&=\frac{1}{\pi}\int_{-\infty}^{\infty}\int_0^{\infty}\frac{r\exp\left[-\frac{r^2}{2\lambda^2}(1+2it\lambda^2)+\sum_{n=1}^{3}\mu_n\right]drdt}{\prod_{n=1}^{3}\sqrt{1-2it\sigma_n^2}}\\&=\frac{1}{2\pi i}\int_C\frac{\exp\left[\sum_{n=1}^{3}w_n\right]dw}{\left(w-\frac{i}{2\lambda^2}\right)\prod_{n=1}^{3}\sqrt{1-2iw\sigma_n^2}}=\frac{\lambda^3\exp\left[-\sum_{n=1}^{3}\frac{a_n^2}{2(\lambda^2+\sigma_n^2)}\right]}{\prod_{n=1}^{3}\sqrt{\lambda^2+\sigma_n^2}},\end{aligned}\tag{13}$$

where $w_n=iwa_n^2(1-2iw\sigma_n^2)$ and $C$ is a semicircular contour in the $w=t+iv$ plane.

In the excellent paper by W. Guenther and P. Terragno (see Editor's Note below) a result similar to expression (13) above is derived in a different manner for the ellipsoidal, nonoffset case.

## REMARKS

(1) The probability ${}^{N}P_K$ given by expression (4) is also equal to the expected coverage of a population of point objects when the population density is distributed in the same manner as the point target.

(2) For $N \geqq 2$, expression $(4) \leqq 1-(1-{}^{1}P_K)^N$ and for $N=1$ equality holds.

(3) The expected number of times in which the point target is killed in $N$ shots is the same for the two cases: (a) point target picked up after each toss of the lethal circle and (b) target remains in its fixed (unknown) position throughout the $N$ tosses.

(4) The expected number of tosses of the lethal circle required to kill the target exactly once is different for the two cases, i.e., expression $(6) \geqq (1/{}^{1}P_K)$ and, in some cases, (6) can be very much larger than $1/{}^{1}P_K$. To see this, set $\sigma_C^2=\sigma_T^2=1$, plug $R=0.1$ or 0.2 into (6) and compare with $1/{}^{1}P_K$.

## REFERENCES


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# EDITOR'S NOTE


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(Received October 15, 1966)

FROM time-to-time, Operations Research has published letters, such as the foregoing, on various aspects of the coverage problem. The literature on this problem is extensive and widely scattered. While there is no single source to which the reader can turn for a comprehensive classification of the many variations and special cases of the coverage problem, there are two less comprehensive sources that have come to the Editor's attention that should be of use in placing new developments in context and in locating techniques and mathematical tables for specific applications.

The first source is a paper by William C. Guenther and Paul J. Terragno titled "A Review of the Literature on a Class of Coverage Problems," which ap-