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Letter to the Editor—A Note on Cutting-Plane Methods Without Nested Constraint Sets

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A NOTE ON CUTTING-PLANE METHODS WITHOUT NESTED CONSTRAINT SETS

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This note shows that a framework I developed earlier can be used to give a simplified proof of conditions given by Eaves and Zangwill (which weaken the uniform concavity requirement on my earlier objective function) under which inactive constraints may be dropped after each subproblem in cutting-plane algorithms. Here the convergence rate I established previously as an extension of the results of Levitin and Polyak is improved and its application extended.

The PROBLEM considered is that of maximizing a real-valued continuous function f over a nonempty closed convex subset S of E^n. One is given a compact (this compactness assumption is relaxed in reference 4) convex set T containing S. The general algorithm to be considered proceeds by setting $T_0=E^n$, and, given T_k as the intersection of E^n and a finite set of closed half-spaces containing S, picking x_k to maximize f over $T_k\cap T$, stopping with x_k optimal if $x_k\epsilon S$, and otherwise letting S_k be the intersection of E^n and a subset of the half-spaces determining T_k such that x_k maximizes f over $S_k\cap T$, finding a certain closed half-space H_k containing S but not x_k, setting $T_{k+1}=S_k\cap H_k$, and continuing. If f is pseudo-concave on T, then S_k will satisfy these conditions if the constraints dropped are any of the constraints determining T_k that are inactive at x_k.

A real-valued function f is *uniformly concave*[3] on a convex set T if there exists a nondecreasing function $\delta(r)>0$ on $(0, \infty)$ such that

$$f[\tfrac{1}{2}(x+y)]\geqq \tfrac{1}{2}f(x)+\tfrac{1}{2}f(y)+\delta(|x-y|)$$

for all x, $y\epsilon T$. A uniformly concave function is *strongly concave*[3] if $\delta(r)=\gamma r^2$ for some $\gamma>0$.

I showed in reference 4 that if H_k is picked in a certain manner as below, and f is uniformly concave on T, then $\{x_k\}$ converges to the optimum. EAVES AND ZANGWILL[1] allowed the cuts to be certain closed convex sets (rather than just closed half-spaces) and only required (essentially) that f be quasi-concave on T and strictly quasi-concave on any convex subset of $T\sim S$ in proving the optimality of this procedure by using the notion of a separator. In Theorem 2, the result of Eaves and Zangwill (with the cuts here limited to certain closed half-spaces but with their weakened conditions on f) is established by a much simpler proof using the framework of reference 4. Theorem 3 generalizes and extends the convergence rate established in reference 4.

A mapping $[a(x), b(x)]$ from $T\sim S$ into E^{n+1} with $a(x)\epsilon E^n$ and $b(x)\epsilon E^1$ is a *limiting cutting-plane function* if $S\subseteq H(x)\equiv\{y:a(x)\cdot y\geqq b(x)\}$ for all $x\epsilon T\sim S$, $[a(x), b(x)]$ is bounded on $T\sim S$, and, for any $\{x_k:k=1, 2, \cdots\}\subseteq T\sim S$ with $\lim_{k\to\infty} x_k=\bar{x}\epsilon T\sim S$, the limit point $(\bar{a}, \bar{b})$ of any convergent subsequence of $\{[a(x_k), b(x_k)]\}$ satisfies $\bar{a}\cdot\bar{x}<\bar{b}$. A generalized version of this notion was introduced by ZANGWILL,[6] and examples are given in references 4 and 5.

The following was proved in reference 4.

THEOREM 1. *If $H(x)$ is determined by a limiting cutting-plane function, $H_k=H(x_k)$, and $\lim_{i\to\infty} x_{k_i}=\lim_{i\to\infty} x_{k_i+1}=\bar{x}$, then, $\bar{x}$ is optimal.*

THEOREM 2. *Suppose that $H(x)$ is determined by a limiting cutting-plane function, $H_k=H(x_k)$, and f is quasi-concave on T and strictly quasi-concave on any convex subset of $T\sim S$. Then the limit point of any convergent subsequence of $\{x_k\}$ is optimal.*

Proof. Let $\bar{x}$ be the limit point of any convergent subsequence of $\{x_k\}$. Since $\{x_k\}$ is bounded, there exists a subsequence $\{x_{k_i}\}$ such that $\lim_{i\to\infty} x_{k_i}=\bar{x}$ and $\lim_{i\to\infty} x_{k_i+1}=\bar{y}$. Now suppose $\bar{x}$ is not optimal. If $\bar{x}$ were feasible, it would be optimal,[4] so $\bar{x}\notin S$, and by Theorem 1, $\bar{x}\neq\bar{y}$. Since x_{k_i} maximizes f over the convex set $S_{k_i}\cap T$ and $x_{k_i+1}\epsilon T_{k_i+1}\cap T\subseteq S_{k_i}\cap T$ for all i, $f(x_{k_i})\geqq f(\alpha x_{k_i}+(1-\alpha)x_{k_i+1})$ for all i and all $\alpha\epsilon[0, 1]$. By continuity,

$$f(\bar{x})\geqq f[\alpha\bar{x}+(1-\alpha)\bar{y}] \qquad \text{for all} \quad \alpha\epsilon[0, 1]. \quad (1)$$

Since $f(x_k)$ is nonincreasing in k it is easily seen that $f(\bar{x})=f(\bar{y})$, so by quasi-concavity

$$f[\alpha\bar{x}+(1-\alpha)\bar{y}]\geqq\min\{f(\bar{x}), f(\bar{y})\}=f(\bar{x}) \qquad \text{for all} \quad \alpha\epsilon[0, 1]. \quad (2)$$

By (1) and (2),

$$f(\bar{x})=f[\alpha\bar{x}+(1-\alpha)\bar{y}] \qquad \text{for all} \quad \alpha\epsilon[0, 1]. \quad (3)$$

Since S is closed and $\bar{x}\notin S$, there exists $\gamma\epsilon[0, 1)$ such that $\alpha\bar{x}+(1-\alpha)\bar{y}\notin S$ for all $\alpha\epsilon[\gamma, 1]$. But by (3) and the strict quasi-concavity of f on the line segment joining $\gamma\bar{x}+(1-\gamma)\bar{y}$ and $\bar{x}$,

$$f[\alpha\bar{x}+(1-\alpha)\bar{y}]>\min\{f(\bar{x}), f[\gamma\bar{x}+(1-\gamma)\bar{y}]\}=f(\bar{x}) \qquad \text{for all} \quad \alpha\epsilon(\gamma, 1). \quad (4)$$

But (4) contradicts (3), so $\bar{x}$ must be optimal.

LEVITIN AND POLYAK[3] have established an arithmetic convergence rate for a cutting-plane algorithm that, when specialized to subsets of E^n, has $S_k = T_k$ (although their proof still holds if inactive constraints are dropped after each subproblem) and uses the cutting-plane method of Lemma 3 of reference 4 [i.e., $\alpha(x)=1$ always, below]. Here their algorithm and method of proof are generalized to show the same convergence rate for algorithms that allow inactive constraints to be dropped after each subproblem and for which the cutting plane at $x\epsilon T\sim S$ may be generated at some point $w(x)\epsilon\overline{T\sim S}$ other than x on the line segment joining x to a certain interior point of S (as in Lemma 4 of reference 4). The following result eliminates the restriction of the generalization given in reference 4 that $G[w(x)]\leqq \epsilon G(x)$ for some $\epsilon\epsilon(0, 1]$ and the dependence of the convergence rate on ϵ.

THEOREM 3. *Suppose that* $S=\{x:G(x)\geqq 0, x\epsilon T\}$, $G(x)$ *is concave and continuous on* T *and that there exists* $t\epsilon S$ *with* $G(t)>0$; *for* $x\epsilon T\sim S$ *define* $\lambda(x)=\sup\{\lambda:\lambda x+(1-\lambda)t\epsilon S\}$ *and set* $w(x)=\alpha(x)x+[1-\alpha(x)]t$ *for any* $\alpha(x)\epsilon[\lambda(x), 1]$. *Suppose also that there exists a function* $\mu(w)$ *from* $\overline{T\sim S}$ *into* E^n *with* $|\mu(w)|\leqq K$ *for all* $w\epsilon\overline{T\sim S}$ *and such that* $G(y)\leqq G(w)+\mu(w)\cdot(y-w)$ *for all* $w\epsilon\overline{T\sim S}$ *and* $y\epsilon S$, *and let* $H_k=\{x:0\leqq G[w(x_k)]+\mu[w(x_k)]\cdot[x-w(x_k)]\}$. *If* f *is strongly concave on* T *and differentiable on* S *and* $\bar{x}$ *is the unique maximum of* f *on* S, *then for* $k\geqq 1$,

$$f(x_k)-f(\bar{x})\leqq 1/a_1k \quad\text{and}\quad |x_k-\bar{x}|\leqq 1/a_2\sqrt{k},$$

where

$$a_1=2\gamma[G(t)/bdK]^2, \qquad a_2=2\gamma G(t)/bdK,$$

$d=\max\{|\nabla f(y)|:y\epsilon S\}$, $b=\max\{|y-t|:y\epsilon T\}$, *and* γ *is as in the definition of strong concavity.*

Proof. Let $\lambda_k=\lambda(x_k)$, $\alpha_k=\alpha(x_k)$, $w_k=w(x_k)$, and $\mu_k=\mu(w_k)$.

Clearly, $\lambda_kx_k+(1-\lambda_k)t=(\lambda_k/\alpha_k)w_k+(1-\lambda_k/\alpha_k)t$ and $G[\lambda_kx_k+(1-\lambda_k)t]=0$, so by concavity

$$0=G[(\lambda_k/\alpha_k)w_k+(1-\lambda_k/\alpha_k)t]\geqq(\lambda_k/\alpha_k)G(w_k)+(1-\lambda_k/\alpha_k)G(t),$$

and

$$(1-\alpha_k/\lambda_k)G(t)\geqq G(w_k). \tag{5}$$

Since $t\epsilon S$,

$$G(t)\leqq G(w_k)+\mu_k\cdot(t-w_k). \tag{6}$$

But it is easily seen that $x_k\notin H_k$, so $x_k\neq x_{k+1}\epsilon T_{k+1}\cap T=H_k\cap S_k\cap T$, and, by the strict concavity of f on T,

$$0=G(w_k)+\mu_k\cdot(x_{k+1}-w_k), \tag{7}$$

or,

$$0=G(w_k)+\alpha_k\mu_k\cdot(x_{k+1}-x_k)+(1-\alpha_k)\mu_k\cdot(x_{k+1}-t). \tag{8}$$

By (6) and (7),

$$-G(t)\geqq\mu_k\cdot(x_{k+1}-t), \tag{9}$$

so, by (8), (9), and (5),

$$\begin{aligned}0&\leqq G(w_k)+\alpha_k\mu_k\cdot(x_{k+1}-x_k)-(1-\alpha_k)G(t)\\&\leqq(1-\alpha_k/\lambda_k)G(t)+\alpha_kK|x_{k+1}-x_k|-(1-\alpha_k)G(t)\\&=-(\alpha_k/\lambda_k)(1-\lambda_k)G(t)+\alpha_kK|x_{k+1}-x_k|\\&\leqq-\alpha_k(1-\lambda_k)G(t)+\alpha_kK|x_{k+1}-x_k|,\end{aligned}$$

or

$$1-\lambda_k\leqq[K/G(t)]|x_{k+1}-x_k|. \tag{10}$$

By the concavity of f and the optimality of $\bar{x}$ on S,

$$\begin{aligned} f(x_k)-f(\bar{x}) &\leqq f(x_k)-f[\lambda_k x_k+(1-\lambda_k)t] \\ &\leqq (1-\lambda_k)(x_k-t)\cdot\nabla f[\lambda_k x_k+(1-\lambda_k)t] \\ &\leqq (1-\lambda_k)|x_k-t|\cdot|\nabla f[\lambda_k x_k+(1-\lambda_k)t]| \\ &\leqq (1-\lambda_k)bd. \end{aligned} \tag{11}$$

Combining (10) and (11), we obtain

$$f(x_k)-f(\bar{x})\leqq [bdK/G(t)]|x_{k+1}-x_k|. \tag{12}$$

Since x_k, $x_{k+1}\epsilon S_k\cap T$ and x_k maximizes the strongly concave function f over the convex set $S_k\cap T$, for some $\gamma>0$

$$f(x_k)>f[\tfrac{1}{2}\ (x_k+x_{k+1})]\geqq \tfrac{1}{2} f(x_k)+\tfrac{1}{2} f(x_{k+1})+\gamma|x_k-x_{k+1}|^2, \tag{13}$$

and from (13),

$$f(x_k)-f(x_{k+1})\geqq 2\gamma|x_k-x_{k+1}|^2. \tag{14}$$

Now let $D_k=f(x_k)-f(\bar{x})>0$. By (12) and (14),

$$D_k^2\leqq [bdK/G(t)]^2|x_{k+1}-x_k|^2\leqq [bdK/G(t)]^2(1/2\gamma)(D_k-D_{k+1}),$$

or

$$D_{k+1}\leqq D_k-a_1D_k^2=D_k(1-a_1D_k). \tag{15}$$

The arithmetic convergence rate for D_k then follows from (15) as in reference 2 by observing that

$$\begin{aligned} 1/D_{k+1} &\geqq (1/D_k)[1/(1-a_1D_k)]=(1/D_k)[\textstyle\sum_{i=0}^{i=\infty}\ (a_1D_k)^i] \\ &\geqq (1/D_k)(1+a_1D_k)=1/D_k+a_1, \end{aligned} \tag{16}$$

and, using induction on (16), we get $1/D_k\geqq 1/D_0+a_1k$, or

$$D_k\leqq 1/[1/D_0+a_1k]\leqq 1/a_1k. \tag{17}$$

As in (13) and (14), it follows that

$$D_k=f(x_k)-f(\bar{x})\geqq 2\gamma|x_k-\bar{x}|^2, \tag{18}$$

and, by (17) and (18),

$$|x_k-\bar{x}|\leqq 1/\sqrt{2\gamma a_1k}.$$

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SELECTING PRICE QUOTATIONS FOR AN INDUSTRIAL FIRM'S SALE OF INDIVIDUAL CONTRACT PROJECTS

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Firms in contract work have data available from their bidding experience that are useful in making bids on future work for marketing their plant capacity. This note describes a procedure, based upon probability principles, for estimating the demand curve for the firm's output from its record of successes and failures in bidding for orders.

TWO GENERAL rules for pricing industrial projects are widely recognized from printshops to shipyards: The seller quotes higher contract prices when his plant is operating near full capacity and lower prices when some plant capacity is idle; and he anticipates the probable reactions to his price quotations by remembering past successes or failures to get contracts.

Based upon probabilities of successful bids, what price should be quoted when a given per cent of plant capacity remains to be filled by orders? The relation between the proportions of successful bids at different price levels and the demand curve for the firm's output provides a key that has apparently been overlooked in Bayesian analysis of price bidding. Of course, price questions can be resolved better if the demand curve for a firm's production is known.[1]

The type of product now considered is one made to customer specifications and for which the price is negotiated, rather than a uniform one with the price the same to all buyers. To simplify, I shall assume that the firm sells a similar product or service from project to project, although details are tailored to customer requirements. Consumer surveys sold by market-research companies with which I am associated are in this category. For a numerical example to make arithmetic details explicit, I will draw upon this experience, although to my knowledge no firm is yet following the procedure that this note suggests.