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Methods

Regular Variable Returns to Scale Production Frontier and Efficiency Measurement

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Abstract. The most frequently used empirical production frontier in data envelopment analysis, the variable returns to scale frontier, has a convex technology set and displays a special structure in economics, called the regular variable returns to scale in this paper; the production technology exhibits increasing returns to scale at the beginning of the production process followed by constant returns to scale and decreasing returns to scale. When the assumption of convexity is relaxed, modeling regular variable returns to scale becomes difficult, and currently, no satisfactory solution is available in multioutput production. Overcoming these difficulties, this paper adopts a suggestion in literature to incorporate regular variable returns to scale into the free disposal hull frontier under multiple outputs. We establish a framework for analyzing regular variable returns to scale and recommend an empirical production frontier for measuring technical efficiency with such pattern and multiple outputs. In the presence of regular variable returns to scale without convexity, the value of the technical efficiency measure computed from this new frontier is closer to the "true" value than that from the free disposal hull frontier, and the conventional variable returns to scale frontier may cause misleading implications.

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Keywords: regular variable returns to scale • strong disposability • nonconvex production technology

1. Introduction

In production theory, when only one input is variable, it is believed that the marginal product of this input rises first, levels off, and finally falls (see Ozawa 1971 and microeconomics textbooks for discussions of these three stages of production). This short-run pattern is implied by the law of variable proportion. The curve drawn is usually S shaped. When all inputs are variable and there are multiple outputs, the S-shaped pattern is reflected by the curve on the production frontier for proportional changes of inputs and outputs. An S-shaped curve is also called a *sigmoid* function or *convex-concave* function. It is initially ray convex and finally, ray concave. The S-shaped curve is a special case of a more general production structure: exhibiting increasing returns to scale (IRS) first, then constant returns to scale (CRS), and finally decreasing returns to scale (DRS). This structure is called *regular variable returns to scale* (RVRS) in this paper.

The importance of the S-shaped pattern can be seen in the following four aspects. (i) It is believed to be one possibility of production technology. This pattern is implicit in many discussions of production as seen in microeconomic textbooks and the references in this paper. Examples of previous attempts to incorporate it into production are rare. Some examples are Banker and Maindiratta (1986), Olesen and Petersen (2013), and An et al. (2023). (ii) This curve has been applied to many fields. It has been applied to engineering and mathematics; see Muu and Oettli (1991) and Kucharavy and De Guio (2011). In theoretical and empirical economic studies, S-shaped production function (Kumasaka 2011, Olesen et al. 2016) and S-shaped utility function (Fishburn and Kochenberger 1979, Tversky and Kahneman 1992) are frequently assumed. Notably, the S-shaped curve has significant relevance in economic growth theories (Skiba 1978) and ecological studies (Lu et al. 2016, Bunn et al. 2023) because of its ability to model complex patterns of development and resource utilization. (iii) There are reasons that lead to the existence of RVRS. Lipsey (2018) provided a detailed discussion on the sources of positive and negative "scale effects." (iv) The S-shaped pattern has been observed in real-world data. Destefanis and Storti (2002) and Li et al. (2020) found large differences between values of technical efficiency with and without convexity when RVRS is assumed. Olesen and Ruggiero (2014, p. 802) claimed that the technical efficiency computed from the conventional variable returns to scale (VRS) frontier "works well" for an S-shaped production function in the decreasing returns to scale region. Li et al. (2024, proposition 10) proved that claim. Thus, the large differences must appear in the increasing returns to scale region, and they provided some support for the existence of the S-shaped pattern. In the commercial sector, researchers found an S-curve pattern of performance when some companies implemented "production systems." This pattern is related to the low-hanging fruits argument, the theory of the performance frontier, and the theory of the learning curve (Netland and Ferdows 2014, 2016).

Although some researchers want to incorporate the S-shaped pattern into their studies, it is not an easy task in production analysis. It is even more difficult for multioutput production. As explained below, the methods used in the references in the previous paragraph are not complete. There are two approaches to deal with modeling S-shaped patterns. The first is to find a nonparametric production frontier that is able to capture the important properties of RVRS with an S-shaped pattern. The second is to find a parametric frontier that fits the data most. This paper adopts the first approach. The findings fill a gap in the literature of production. Furthermore, these results are also useful to the practitioners in the other fields mentioned previously.

The technology set with RVRS can be convex or not convex. Surprisingly, there are relatively few studies on the theoretical and empirical modeling of RVRS. One exception is in the literature on data envelopment analysis (DEA), in which RVRS is incorporated in the empirical production technology commonly known as the "variable returns to scale frontier." This technology was introduced to the literature in one-output production by Afriat (1972) and by Banker et al. (1984) in multioutput production. One characteristic of the VRS technology is that it must be convex. Some researchers, however, called for relaxing the convexity assumption (see Cherchye et al. 2000 and Briec et al. 2004 for examples). We agree with them that convexity may not be necessary in empirical studies. This paper seeks to relax the assumption of convexity on the VRS technology while the feature of RVRS is maintained.

A few directions of modeling nonconvexity while maintaining strong disposability of inputs and outputs draw our attention: (i) abandoning convexity and returns to scale completely (Afriat 1972, Deprins et al. 1984); (ii) modeling partial convexity without imposing any type of returns of scale (Diewert and Parkan 1983, Dekker and Post 2001); (iii) modeling partial convexity jointly with some types of returns to scale (Petersen 1990, Post 2001a, p. 140); and (iv) incorporating some types of returns to scale without convexity (Kerstens and Vanden Eeckaut 1999). The first direction introduces the free disposal hull (FDH) production frontier, which avoids convexity successfully. Direction (ii) retains some convexity. However, the models in (i) and (ii) do not follow any specific pattern of returns to scale. Directions (iii) and (iv) do consider some types of returns to scale. None of the studies in these works have been proven to model RVRS. Indeed, the only successful model of production technology with RVRS in DEA is the VRS frontier, in which RVRS and convexity jointly appear in the production technology. In other words, modeling RVRS is not separated from convexity in the current literature on DEA. If a researcher opts to impose RVRS but abandons the assumption of convexity, no satisfactory tools are available in the current literature.

In general, once the assumption of convexity is relaxed, modeling RVRS becomes extremely difficult when there are multiple outputs and multiple inputs. The first challenging issue of modeling RVRS is to define this concept in general multioutput production technology. The concepts of nondecreasing returns to scale (NDRS), constant returns to scale, and nonincreasing returns to scale (NIRS) have been defined vigorously in the literature (see Färe 1988, appendix A for an example). However, a vigorous definition of RVRS is not available for multioutput production. Briec et al. (2004, p. 163) defined "variable returns to scale" as not exhibiting any one of constant returns to scale, nondecreasing returns to scale, and nonincreasing returns to scale. Yet, this includes patterns that are not RVRS. Li et al. (2024) provided a vigorous definition of RVRS in one-output production without a multioutput version.

Another challenging issue of modeling RVRS is the identification of the nonconvex subset of the technology set. One major tool in DEA is linear programming. In linear programming problems, the feasible set must be convex. In efficiency analysis, the technology set is the feasible set for measuring technical efficiency. When we allow nonconvexity in the technology set, the feasible set may not be convex. In general, linear programming techniques per se are not sufficient to measure productive efficiency under the nonconvex technology set. What are the additional steps required to model RVRS in DEA? To solve this problem, one approach is to use iterations. Maintaining some convexity, Petersen (1990) suggested some formulae for nondecreasing returns to scale and nonincreasing returns to scale. Bogetoft (1996), Bogetoft et al. (2000), and Agrell et al. (2005) attempted to tackle the issue of nonconvexity by iterative computation, but a complete solution has not appeared yet. A better solution was suggested by Olesen and Ruggiero (2014). The limitations of their method are the assumption of homotheticity and one-output production.

Alternatively, the intersection of technology sets under different types of returns to scale can be used to construct the RVRS technology. Kerstens and Vanden Eeckaut (1999) incorporated nondecreasing and nonincreasing returns to scale into the FDH frontier to form two empirical technology sets. Destefanis (2002) and Briec et al. (2004) suggested using their intersection to form a technology set with "variable returns to scale." Applications of this technology can be found in Destefanis and Storti (2002), Destefanis and Sena (2005), and Coppola and Destefanis (2015). However, these studies have not shown that such an intersection exhibits RVRS. Adopting this intersection approach, Li (2019) constructed an empirical technology using the intersection of the formulae of Petersen (1990) of nondecreasing and nonincreasing returns to scale. He showed that such technology exhibits RVRS with convex input sets. Employing this empirical frontier, Li et al. (2020) estimated the technical efficiency of travel agents and hotels, and they found that the conventional VRS frontier in DEA overestimated the technical inefficiency of small firms. As shown by Li et al. (2024, proposition 4), the empirical frontier introduced by Li (2019) is the only formula that can satisfactorily model RVRS with strong disposability and partial convexity in the sense that it is the minimal extrapolation under the relevant assumptions. His formula's weakness is one-output production.

It is important to model the nonconvexity region with RVRS in production analysis. Destefanis and Storti (2002) have shown that the assumption of convexity has limited empirical support when applied to crosscountry technology sets. When there are many small firms, they are likely producing in the increasing returns to scale region if RVRS is true. If firms are technically inefficient, then strengthening internal management and the production process is sufficient to improve productive performance. If firms are technically efficient under increasing returns to scale, then enlarging the production scale is the only way to enhance productive performance. If the nonconvexity of the increasing returns to scale is not modeled appropriately, this paper shows that the conventional VRS frontier in DEA is prone to overestimating the technical inefficiency of small firms. This can potentially result in misleading policy implications.

This paper improves upon the current literature on RVRS by developing a framework for analyzing RVRS without convexity. We point out that an appropriate model of RVRS is crucial for measuring technical efficiency. The paper is organized as follows. Section 2 provides a vigorous definition of RVRS under multiple outputs and other relevant concepts. Section 3 explores the relations between production technology and its various extensions. Section 4 shows that RVRS production technology can be constructed by the intersection of two different technologies. Section 5 argues that the suggested technology set of Briec et al. (2004) is a suitable nonconvex empirical production technology for efficiency measurement. Section 6 illustrates the nonconvex RVRS frontier by measuring the technical efficiency of firms in two Chinese industries. Section 7 concludes this paper.

2. Returns to Scale Under Multiple Outputs

A special pattern of production is that the production process initially shows increasing returns to scale and finally shows decreasing returns to scale. This is called "Regular Ultra-Passum Law" by Frisch (1965, definition 8.1, p. 120). For a vigorous definition and further discussions of the Regular Ultra-Passum Law, see Olesen and Petersen (2013), Olesen and Ruggiero (2014), and Li (2019). The Regular Ultra-Passum Law was defined in terms of elasticity of scale in one-output production only. Thus, differentiability is assumed, and this law under multiple outputs is not available. Further, Frisch (1965) considered general input changes, which need not be proportional. When inputs increase in different proportions, the response of the output level involves two components: the effect of a larger scale and the effect of a different input mix. This section modified the Regular Ultra-Passum Law in three ways. (i) There are multiple outputs in the production process. (ii) The focus is on proportional changes of inputs and outputs only. (iii) Differentiability is not assumed. Furthermore, some properties of this production structure are explored.

Let $\boldsymbol{x} \in \mathbb{R}_+^N$ and $\boldsymbol{q} \in \mathbb{R}_+^M$ be input and output vectors, respectively. The production technology can be modeled by the technology set $\Im := \{(\boldsymbol{x}, \boldsymbol{q}) : \boldsymbol{x}$ can produce $\boldsymbol{q}\}$, the input set for output vector $\boldsymbol{q}$: $L(\boldsymbol{q}) := \{\boldsymbol{x} : \boldsymbol{x}$ can produce $\boldsymbol{q}\}$, or the output set for input vector $\boldsymbol{x}$: $P(\boldsymbol{x}) := \{\boldsymbol{q} : \boldsymbol{x}$ can produce $\boldsymbol{q}\}$. The maintained assumptions of this paper are as follows. (1) Inputs and outputs are strongly disposable. If $(\boldsymbol{x}, \boldsymbol{q}) \in \Im$ and $(-\boldsymbol{x}, \boldsymbol{q}) \leqq (-\boldsymbol{x}', \boldsymbol{q}')$, then $(\boldsymbol{x}', \boldsymbol{q}') \in \Im$. (2) The technology set is closed. (3) $(\mathbf{0}_N, \mathbf{0}_M) \in \Im$ and $(\mathbf{0}_N, \boldsymbol{q}) \in \Im \Longrightarrow \boldsymbol{q} = \mathbf{0}_M$. (4) $P(\boldsymbol{x})$ is bounded. These conditions are assumed throughout the whole paper on the theoretical production technology.

This paper characterizes returns to scale with unchanged input and output mixes. When inputs change proportionally, all inputs increase or decrease with the same input mix. This is called a *change in the pure input scale of production*. When outputs change proportionally, this is called a *change in the pure output scale (or returns) of production*. Pure scale changes are consistent with most discussions of returns to scale in the literature. Focusing on the effect of pure scale change without the influences of different input and output mixes is sufficient to study returns to scale.

To deal with changes of *pure* input scale and *pure* output scale, we note that any input vector and its proportional changes can be seen as points on the same ray from the origin in the input space. Similarly, any output vector and its proportional changes can be seen as points on the same ray from the origin in the output space. To study the returns to scale of production, we can focus on rays in the input and output spaces. Thus, for any $(\boldsymbol{x}^0, \boldsymbol{q}^0) \in \mathbb{R}_+^N \times \mathbb{R}_+^M$, where $\boldsymbol{x}^0 \neq \mathbf{0}_N$ and $\boldsymbol{q}^0 \neq \mathbf{0}_M$, any feasible activity $(\boldsymbol{x}, \boldsymbol{q})$ that is formed by proportional changes of $\boldsymbol{x}^0$ and $\boldsymbol{q}^0$ and is in the technology set can be expressed as $(\boldsymbol{x}, \boldsymbol{q}) = (s\boldsymbol{x}^0, t\boldsymbol{q}^0) \in \Im$ for some $s > 0$ and some $t > 0$. The change of t with respect to a change of s reflects the returns to scale from $(\boldsymbol{x}, \boldsymbol{q})$. Consider the following function with respect to the production activity $(\boldsymbol{x}^0, \boldsymbol{q}^0)$ for $\boldsymbol{x}^0 \geq \mathbf{0}_N$ and $\boldsymbol{q}^0 \geq \mathbf{0}_M$ (for $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}_+^N$, $\boldsymbol{a} \geq \boldsymbol{b}$ means $a_n \geqq b_n$ for all n but $\boldsymbol{a} \neq \boldsymbol{b}$):

$$g(s; \boldsymbol{x}^0, \boldsymbol{q}^0) := \max_{t>0} \{t : (s\boldsymbol{x}^0, t\boldsymbol{q}^0) \in \Im\} \text{ for } s > 0. \tag{1}$$

When outputs expand proportionally to the frontier of the technology for the given input vector $\boldsymbol{x} = s\boldsymbol{x}^0$, the maximum expanded vector is $g(s; \boldsymbol{x}^0, \boldsymbol{q}^0)\boldsymbol{q}^0$. Thus, this function describes production activities along the frontier of $\Im$. Banker (1984) first used this function to characterize local returns to scale, and it is a useful tool to discuss returns to scale (see, for example, Balk 2001, Banker et al. 2004, Førsund et al. 2007).

Instead of representing mixes using arbitrary input and output vectors as in (1), we limit our attention to the vectors with a magnitude of one. To proceed, define two sets that represent all possible input mixes and output mixes: $\boldsymbol{I}^x := \{v \in \mathbb{R}_+^N : \|v\| = 1\}$ and $\boldsymbol{I}^y := \{\boldsymbol{u} \in \mathbb{R}_+^M : \|\boldsymbol{u}\| = 1\}$, respectively. For $\boldsymbol{v} \in \boldsymbol{I}^x$ and $\boldsymbol{u} \in \boldsymbol{I}^y$, we call $\boldsymbol{v}$ an *input unit mix* and $\boldsymbol{u}$ an *output unit mix*. Thus, for any nonzero input vector $\boldsymbol{x}^0 \geq \mathbf{0}_N$ and nonzero output vector $\boldsymbol{q}^0 \geq \mathbf{0}_M$, it is clear that there must exist $\boldsymbol{v}^0 \in \boldsymbol{I}^x$ and $\boldsymbol{u}^0 \in \boldsymbol{I}^y$ such that $(\boldsymbol{x}^0, \boldsymbol{q}^0) = (s\boldsymbol{v}^0, t\boldsymbol{u}^0)$ for some $s > 0$ and some $t > 0$. Hereafter, we describe the behavior of production activities along the ray passing through input vector v^0 and the ray passing through output vector $\boldsymbol{u}^0$. We rewrite the function in (1) as follows. For $\boldsymbol{v}^0 \in \boldsymbol{I}^x$, $\boldsymbol{u}^0 \in \boldsymbol{I}^y$, and $s > 0$,

$$g(s; \boldsymbol{v}^0, \boldsymbol{u}^0) := \max_{t>0} \{t : (s\boldsymbol{v}^0, t\boldsymbol{u}^0) \in \Im\}. \tag{1$'$}$$

Each element of $\boldsymbol{I}^x$ and $\boldsymbol{I}^y$ can be regarded as a unit vector in the input and output spaces, respectively. Then, the values of s and t can be seen as the "quantities" of inputs and outputs, although these quantities cannot be compared between different mixes. Note that when there is only one input and there is only one output such that $v^0 = 1$ and $u^0 = 1$, we have $x = sv^0 = s$ and $q = tu^0 = t$. Let $f(x)$ be a production function in usual sense. The function becomes $f(x) = \max_{q>0} \{q : (x, q) \in \Im\} = \max_{t>0} \{t : (sv^0, tu^0) \in \Im\} = g(s; v^0, u^0) = g(x; 1, 1)$, which is the same as the conventional production function. Thus, one can regard this function as the counterpart of production function when there are multiple outputs. Hence, we call the function in (1′) the *ray production function*.

Consider the output measure of technical efficiency: $te_o(\boldsymbol{x}^0, \boldsymbol{q}^0) = \max_{\theta > 0} \{\theta : (\boldsymbol{x}^0, \theta\boldsymbol{q}^0) \in \Im\}$. Let $\boldsymbol{v}^0 = \boldsymbol{x}^0 / \|\boldsymbol{x}^0\|$ and $\boldsymbol{u}^0 = \boldsymbol{q}^0 / \|\boldsymbol{q}^0\|$. Then, $(v^0, \boldsymbol{u}^0) \in \boldsymbol{I}^x \times \boldsymbol{I}^y$ and $(\boldsymbol{x}^0, \boldsymbol{q}^0) = (s^0\boldsymbol{v}^0, t^0\boldsymbol{u}^0)$, where $s^0 = \|\boldsymbol{x}^0\|$ and $t^0 = \|\boldsymbol{u}^0\|$. Using (1′), one can find that $te_o(\boldsymbol{x}^0, \boldsymbol{q}^0) = g(s^0; \boldsymbol{v}^0, \boldsymbol{u}^0)/t^0$. However, $te_o(\boldsymbol{x}^0, \boldsymbol{q}^0)$ is an existing measure of technical efficiency. Its attention is $(\boldsymbol{x}^0, \boldsymbol{q}^0)$ relative to the production frontier. Although one can replace $g(s^0; \boldsymbol{v}^0, \boldsymbol{u}^0)$ by $te_o(\boldsymbol{x}^0, \boldsymbol{q}^0)$, we think that the ray production function in the current form can draw our attention to the proportional movements of outputs along the frontier in response to a proportional change of inputs. Defining the ray production function can facilitate discussions throughout the paper. Because the reference vectors $\boldsymbol{v}$ and $\boldsymbol{u}$ are arbitrary and fixed during the discussion, the function is essentially a function of one variable. To simplify expressions, one may ignore $\boldsymbol{v}$ and $\boldsymbol{u}$ by writing $g(s; .)$ or simply $g(s)$. Under the maintained assumptions, $g(s; \boldsymbol{v}, \boldsymbol{u})$ is well defined if $(s\boldsymbol{v}, t\boldsymbol{u}) \in \Im$ for some $t > 0$.

The ray production function $g(s; \boldsymbol{v}, \boldsymbol{u})$ inherits properties from $\Im$. The following lemma states results, like the properties of production function.

Lemma 1. *Let $(v^0, \boldsymbol{u}^0) \in \boldsymbol{I}^x \times \boldsymbol{I}^y$ and s^0 and t^0 be any two positive numbers such that $(s^0\boldsymbol{v}^0, t^0\boldsymbol{u}^0) \in \Im$. The following holds for the ray production function.*

a. $g(s^0; \boldsymbol{v}^0, \boldsymbol{u}^0) \geqq t^0$.

b. $g(s;\boldsymbol{v}^0,\boldsymbol{u}^0)$ *is weakly increasing and quasiconcave in* s.
c. $g(s;\boldsymbol{v}^0,\boldsymbol{u}^0)$ *is continuous from above in* s.
d. *If* $\Im$ *is a convex set, then* $g(s;\boldsymbol{v}^0,\boldsymbol{u}^0)$ *is a concave function in* s.

Proof. See Appendix A. □

In this paper, we call $(\boldsymbol{v},\boldsymbol{u}) \in \boldsymbol{I}^x \times \boldsymbol{I}^y$ a feasible (productive) inout unit mix of $\Im$ if there exist $s>0$ and $t \geqq (>) 0$ such that $(s\boldsymbol{v}, t\boldsymbol{u}) \in \Im$. The numbers s and t are called the feasible input factor and the feasible output factor of $(\boldsymbol{v},\boldsymbol{u})$ in $\Im$, respectively. By definition, $(s\boldsymbol{v}, g(s;\boldsymbol{v},\boldsymbol{u})\boldsymbol{u}) \in \Im$. Furthermore, if $g(s;\boldsymbol{v},\boldsymbol{u})>0$, then s is called a productive input factor of $(\boldsymbol{v},\boldsymbol{u})$ in $\Im$. Thus, if $(\boldsymbol{v},\boldsymbol{u})$ is a productive inout unit mix, then there exists a productive input factor s such that $g(s;\boldsymbol{v},\boldsymbol{u})>0$. To avoid trivial cases of zero outputs, the attention of this paper is on productive inout unit mixes only.

Different types of global returns to scale are defined in terms of $g(s;\boldsymbol{v},\boldsymbol{u})$ as follows.

Definition 1. Let $g(s;\boldsymbol{v},\boldsymbol{u})$ be a well-defined ray production function. Let $(\boldsymbol{v}^0,\boldsymbol{u}^0)$ be any productive inout unit mix and $s>0$ be any productive input factor of $(\boldsymbol{v}^0,\boldsymbol{u}^0)$.

a. The production technology exhibits nondecreasing returns to scale with respect to $(\boldsymbol{v}^0,\boldsymbol{u}^0)$ if $g(\theta s;\boldsymbol{v}^0,\boldsymbol{u}^0) \geqq \theta g(s;\boldsymbol{v}^0,\boldsymbol{u}^0)$ for $\theta>1$. It exhibits strictly increasing returns to scale (SIRS) with respect to $(\boldsymbol{v}^0,\boldsymbol{u}^0)$ if the inequality is always strict.

b. The production technology exhibits nonincreasing returns to scale with respect to $(\boldsymbol{v}^0,\boldsymbol{u}^0)$ if $g(\theta s;\boldsymbol{v}^0,\boldsymbol{u}^0) \leqq \theta g(s;\boldsymbol{v}^0,\boldsymbol{u}^0)$ for $\theta>1$. It exhibits strictly decreasing returns to scale (SDRS) with respect to $(\boldsymbol{v}^0,\boldsymbol{u}^0)$ if the inequality is always strict.

c. The production technology exhibits constant returns to scale with respect to $(\boldsymbol{v}^0,\boldsymbol{u}^0)$ if $g(\theta s;\boldsymbol{v}^0,\boldsymbol{u}^0) = \theta g(s;\boldsymbol{v}^0,\boldsymbol{u}^0)$ for $\theta>0$.

For $rts = NDRS, SIRS, NIRS, SDRS, CRS$, if a production technology exhibits returns to scale rts with respect to all productive inout unit mixes, then we just say that it exhibits rts.

Explicit expressions for strictly increasing and decreasing returns to scale for production technologies with multiple outputs are rare in the literature. One exception is Panzar and Willig (1977), but their analyses were mainly local in nature. Another exception is Boussemart et al. (2009, definition 2.1, p. 334). They defined strictly increasing and decreasing returns to scale using the weakly efficient subset. Both definitions involves sets. In contrast, the strict concepts of increasing returns to scale and decreasing returns to scale can be handled easily by Definition 1 using the ray production function $g(s;\boldsymbol{v},\boldsymbol{u})$. In summary, adopting Definition 1 enables us to use functions to discuss all concepts of returns to scale in multioutput, multi-input production technologies. Note that the ray production function projects outputs proportionally to the production frontier. So, Definition 1 indicates that the concept of returns to scale is about the responses of outputs to proportional changes of inputs on the production frontier. It turns out that these three types of returns to scale are closely related to "average output" on the frontier, as shown in the following proposition. In this paper, a function $f(\boldsymbol{x})$ is *weakly increasing* if $\boldsymbol{a} \geq \boldsymbol{b} \Longrightarrow f(\boldsymbol{a}) \geqq f(\boldsymbol{b})$. It is *strictly increasing* if $\boldsymbol{a} > \boldsymbol{b} \Longrightarrow f(\boldsymbol{a}) > f(\boldsymbol{b})$.

Proposition 1. *Let* $(\boldsymbol{v}^0,\boldsymbol{u}^0) \in \boldsymbol{I}^x \times \boldsymbol{I}^y$ *be any feasible inout unit mix and* $s>0$ *be any productive input factor for* $(\boldsymbol{v}^0,\boldsymbol{u}^0)$.

a. $g(\theta s;\boldsymbol{v}^0,\boldsymbol{u}^0) \geqq (>) \theta g(s;\boldsymbol{v}^0,\boldsymbol{u}^0)$ *for* $\theta>1$ *if and only if* $g(\lambda s;\boldsymbol{v}^0,\boldsymbol{u}^0) \leqq (<) \lambda g(s;\boldsymbol{v}^0,\boldsymbol{u}^0)$ *for* $0<\lambda<1$ *if and only if* $g(s;\boldsymbol{v}^0,\boldsymbol{u}^0)/s$ *is weakly (strictly) increasing.*

b. $g(\theta s;\boldsymbol{v}^0,\boldsymbol{u}^0) = \theta g(s;\boldsymbol{v}^0,\boldsymbol{u}^0)$ *for* $\theta>0$ *if and only if* $g(\theta s;\boldsymbol{v}^0,\boldsymbol{u}^0)/\theta s = g(s;\boldsymbol{v}^0,\boldsymbol{u}^0)/s$.

c. $g(\theta;\boldsymbol{v}^0,\boldsymbol{u}^0) \leqq (<) \theta g(s;\boldsymbol{v}^0,\boldsymbol{u}^0)$ *for* $\theta>1$ *if and only if* $g(\lambda s;\boldsymbol{v}^0,\boldsymbol{u}^0) \geqq (>) \lambda g(s;\boldsymbol{v}^0,\boldsymbol{u}^0)$ *for* $0<\lambda<1$ *if and only if* $g(s;\boldsymbol{v}^0,\boldsymbol{u}^0)/s$ *is weakly (strictly) decreasing.*

Proof. See Appendix A. □

We have noted previously that the ray production function is closely related to the Farrell output measure of technical efficiency. Boussemart et al. (2009, propositions 3.1 and 3.2, pp. 334–335) characterized SIRS and SDRS using efficiency measures. For example, SIRS is equivalent to $E_o(\boldsymbol{x}^0,\boldsymbol{q}^0) > [E_i(\boldsymbol{x}^0,\boldsymbol{q}^0)]^{-1}$. Boussemart et al. (2009) also discuss another relevant concept, α-returns to scale, which assumes homogeneity of the production technology. Both discussions are useful to understand the relationships between output and input measures of technical efficiency under various types of returns to scale. However, such characterizations are not convenient for discussing RVRS, and we do not pursue them further.

Conventional definitions of NDRS, NIRS, and CRS are defined in terms of the technology set. The technology set exhibits NDRS, NIRS, and CRS if $\theta\Im \subseteq \Im$ for $\theta>1$, $\Im \subseteq \theta\Im$ for $\theta>1$, and $\theta\Im = \Im$ for $\theta>1$, respectively. See Färe et al. (1994, section 2.2) for an example. Proposition 2 states that Definition 1 is consistent with them.

Proposition 2. *Given Definition 1, the following are true.*

a. *The production technology exhibits nondecreasing returns to scale if and only if* $\theta\Im \subseteq \Im$ *for* $\theta>1$.

b. *The production technology exhibits nonincreasing returns to scale if and only if* $\Im \subseteq \theta\Im$ *for* $\theta>1$.

c. *The production technology exhibits constant returns to scale if and only if* $\theta\Im = \Im$ *for* $\theta>0$.

Proof. See Appendix A. □

From our knowledge, the concept of the "Regular Ultra-Passum Law" has not been formally extended to multioutput production. The term "RVRS" is vigorously defined below by extending the content of the Regular Ultra-Passum Law to multiple outputs without differentiability.

Definition 2. Let $(\boldsymbol{v}^0, \boldsymbol{u}^0)$ be a productive inout unit mix in $\Im$. This production technology exhibits *regular variable returns to scale* with respect to $(\boldsymbol{v}^0, \boldsymbol{u}^0)$ if there exists a productive input factor s^* such that the following conditions are satisfied.

a. For $0 < s \leqq s^*$ and $0 < \theta < 1$, $g(\theta s; \boldsymbol{v}^0, \boldsymbol{u}^0) \leqq \theta g(s; \boldsymbol{v}^0, \boldsymbol{u}^0)$ and $g(\theta s^*; \boldsymbol{v}^0, \boldsymbol{u}^0) < \theta g(s^*; \boldsymbol{v}^0, \boldsymbol{u}^0)$. Furthermore, the weak inequality is strict for some value of s and some value of θ.

b. For $s > s^*$ and $\theta > 1$, $g(\theta s; \boldsymbol{x}^0, \boldsymbol{q}^0) \leqq \theta g(s; \boldsymbol{x}^0, \boldsymbol{q}^0)$, and the strict inequality holds for some $\theta' > 1$.

When a technology exhibits RVRS, the set $\{\boldsymbol{x} = s\boldsymbol{v}^0 : 0 \leqq s < s^*\}$ in the above definition is called the increasing returns to scale region of inputs with respect to $(\boldsymbol{v}^0, \boldsymbol{u}^0)$. The set $\left\{\boldsymbol{x} = s\boldsymbol{v}^0 : s > s^*, \frac{g(s;\boldsymbol{v}^0,\boldsymbol{u}^0)}{s} < \frac{g(s^*;\boldsymbol{v}^0,\boldsymbol{u}^0)}{s^*}\right\}$ is called the decreasing returns to scale region of inputs with respect to $(\boldsymbol{v}^0, \boldsymbol{u}^0)$. Finally, the set $\left\{\boldsymbol{x} = s\boldsymbol{v}^0 : s \geqq s^*, \frac{g(s;\boldsymbol{v}^0,\boldsymbol{u}^0)}{s} = \frac{g(s^*;\boldsymbol{v}^0,\boldsymbol{u}^0)}{s^*}\right\}$ is called the constant returns to scale region. Thus, along any ray from the origin, the order of appearance of these three regions will be IRS, CRS, and DRS. We say that the production technology exhibits RVRS if Definition 2 holds for all feasible inout unit mixes $(\boldsymbol{v}, \boldsymbol{u})$. When all NDRS, CRS, and NIRS do not hold but the production technology does not exhibit RVRS, we call it having irregular variable returns to scale (IVRS).

Note that the term "*variable returns to scale*" in the existing literature (such as in Briec et al. 2004, p. 163, and Sickles and Zelenyuk 2019, p. 48) covers both RVRS and IVRS. A typical example of IVRS production technology is the FDH technology set. Consider the ray function of any production activity with positive inputs and outputs. When the data set is sufficiently large, local SIRS and SDRS in general show up alternately several times.

Proposition 1 has stated the close relations between returns to scale and "average output." We can define RVRS using "average output."

Proposition 3. *The technology set $\Im$ is characterized by RVRS if and only if for any productive inout unit mix $(\boldsymbol{v}^0, \boldsymbol{u}^0)$ and the corresponding productive input factor s^* in Definition 2, the following are true.*

a. *For $0 < s < s^*$, $g(s; \boldsymbol{v}^0, \boldsymbol{u}^0)/s$ is weakly increasing in s and $g(s; \boldsymbol{v}^0, \boldsymbol{u}^0)/s < g(s^*; \boldsymbol{x}^0, \boldsymbol{q}^0)/s^*$. Furthermore, there exist $s^a, s^b > 0$ such that $s^a < s^b < s^*$ and $g(s^a; \boldsymbol{v}^0, \boldsymbol{u}^0)/s^a < g(s^b; \boldsymbol{v}^0, \boldsymbol{u}^0)/s^b$.*

b. *For $s > s^*$, $g(s; \boldsymbol{x}^0, \boldsymbol{q}^0)/s$ is weakly decreasing in s and $g(s'; \boldsymbol{x}^0, \boldsymbol{q}^0)/s' < g(s; \boldsymbol{x}^0, \boldsymbol{q}^0)/s$ for some $s' > s$.*

Proof. Apply Proposition 1 to Definition 2. □

When RVRS appears, the technology set can be convex or not convex. The following diagrams illustrate this in a one-input, one-output case. In Figure 1, (a) and (b), the technology set satisfies RVRS. The technology set in Figure 1(a) is not convex, whereas the technology set in Figure 1(b) is convex for $q > 0$. In contrast, panels (c) and (d) of Figure 1 exhibit irregular variable returns to scale.

Figure 1(b) is a common type of production technology adopted in empirical studies in DEA. However, Figure 1(a) has rarely been explored. Several of these rare examples are in Kerstens and Vanden Eeckaut (1999), Post (2001b), Olesen and Petersen (2013), and Olesen and Ruggiero (2014). This paper explores measuring technical efficiency in the presence of RVRS without convexity.

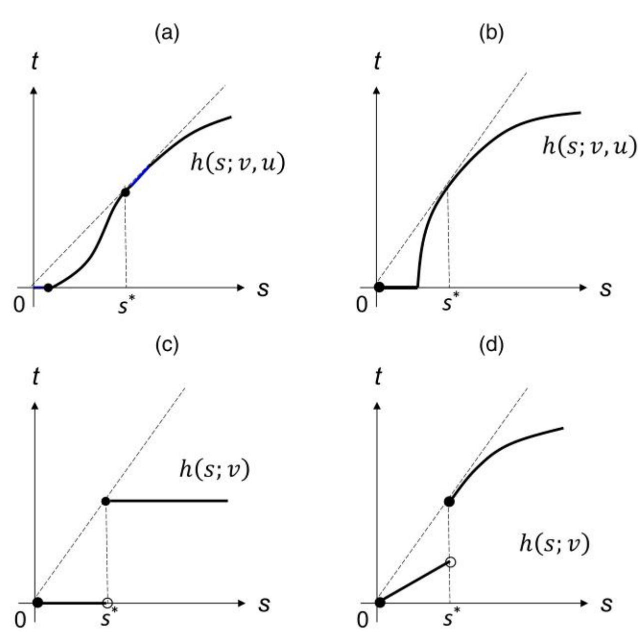
When there is only one output (i.e., $M = 1$), the production technology can be represented by the production function defined by $f(\boldsymbol{x}) = \max_u \{u : (\boldsymbol{x}, u) \in \Im\} = g(s; \boldsymbol{v}, 1) = h(s, \boldsymbol{v})$, where $\boldsymbol{v} = \boldsymbol{x}/\|\boldsymbol{x}\|$ and $s = \|\boldsymbol{x}\|$. In terms of the production function, Definition 2 is rewritten below.

Definition 2'. Let $f(\boldsymbol{x})$ be any production function. This production function exhibits *regular variable returns to scale* if for any productive input unit mix $\boldsymbol{v}^0$, there exists a productive input factor s^* such that

a. for $0 < s \leqq s^*$ and $0 < \theta < 1$, $h(\theta s, \boldsymbol{v}^0) \leqq \theta h(s, \boldsymbol{v}^0)$ and $h(\theta s^*, \boldsymbol{v}^0) < \theta h(s^*, \boldsymbol{v}^0)$. Furthermore, the weak inequality is strict for some value of s and some value of θ.

b. for $s > s^*$ and $\theta > 1$, $h(\theta s, \boldsymbol{v}^0) \leqq \theta h(s, \boldsymbol{v}^0)$, and the inequality is strict for at least one value of θ.

Figure 1. Convex and Nonconvex Production Frontiers



Notes. (a) RVRS without convexity. (b) RVRS with convexity. (c) IRVRS. (d) IRVRS.

Definition 2′ requires the weak inequality "$h(\theta s, \boldsymbol{v}^0) \leqq \theta h(s, \boldsymbol{v}^0)$" to be strict under some conditions. This plugs a loophole in the definition of "regular variable returns to scale" (the word "regular" is added for consistency) provided by Li (2019, definition vrs, p. 162). See the explanation provided by Li et al. (2024).

In the following discussions, a production technology exhibiting regular variable returns to scale is called an *RVRS production technology*, and the corresponding $\Im$ is called an *RVRS technology set*.

3. Extensions of the Production Technology

The difficulty of modeling RVRS comes from the different behaviors of increasing returns to scale and decreasing returns to scale in one production technology. For example, a small proportional expansion of a feasible inout vector on the production frontier is feasible in a region of strictly increasing returns to scale but is infeasible in a region of strictly decreasing returns to scale. To overcome the difficulties in the coexistence of increasing and decreasing returns to scale, this section separates these two types of scale by using extensions of the production technology.

With reference to Briec et al. (2000) for convex technologies, we define extensions of any general technology as follows. For any technology set $\Im$, define its NDRS extension by $\Im^{ndrs} := \{(\boldsymbol{x}, \boldsymbol{q}) : (\boldsymbol{x}, \boldsymbol{q}) = (\alpha \boldsymbol{x}', \alpha \boldsymbol{q}')$ for some $(\boldsymbol{x}', \boldsymbol{q}') \in \Im,\ \alpha \geqq 1\} = \cup_{\alpha \geqq 1} \alpha \Im$. Similarly, the NIRS extension and the CRS extension of $\Im$ are $\Im^{nirs} := \{(\boldsymbol{x}, \boldsymbol{q}) : (\boldsymbol{x}, \boldsymbol{q}) = (\alpha \boldsymbol{x}', \alpha \boldsymbol{q}')$ for some $(\boldsymbol{x}', \boldsymbol{q}') \in \Im,\ \alpha \in [0, 1]\} = \cup_{0 < \alpha \leqq 1} \alpha \Im$ and $\Im^{crs} := \{(\boldsymbol{x}, \boldsymbol{q}) : (\boldsymbol{x}, \boldsymbol{q}) = (\alpha \boldsymbol{x}', \alpha \boldsymbol{q}')$ for some $(\boldsymbol{x}', \boldsymbol{q}') \in \Im,\ \alpha > 0\} = \cup_{\alpha > 0} \alpha \Im$, respectively.

The properties of these extensions are stated in the following lemma. The proof is simple and thus, not included here.

Lemma 2. *Let $\Im$ be any technology set with extensions $\Im^{ndrs}$, $\Im^{nirs}$, and $\Im^{crs}$. For rts = ndrs, nirs, crs, the following properties of these extensions hold.*

a. *The technology set $\Im^{rts}$ exhibits the returns to scale of rts.*
b. *$\Im \subseteq \Im^{ndrs} \subseteq \Im^{crs}$ and $\Im \subseteq \Im^{nirs} \subseteq \Im^{crs}$.*
c. *If $\Im$ exhibits returns to scale rts, then $\Im^{rts} = \Im$.*
d. *If $\Im$ has any one of the following properties, then $\Im^{rts}$ has that property too:*
  i. *strongly disposable inputs and outputs,*
  ii. *convex input requirement set,*
  iii. *convex output possibilities set, or*
  iv. *convex technology set.*

There are close relations among these extensions. One is that the CRS extension of any technology is always equal to the union of the other two extensions.

Proposition 4. *For any technology set $\Im$, $\Im^{crs} = \Im^{ndrs} \cup \Im^{nirs}$.*

Proof. See Appendix B. □

Proposition 4 has been observed by Briec et al. (2000, propositions 1a and 5a) for some nonparametric production technologies in DEA. The above proposition states that such relations on the union of $\Im^{ndrs}$ and $\Im^{nirs}$ hold for the extensions of any technology, including both convex and nonconvex technologies. Although Briec et al. (2000, propositions 1b and 5b) also noted that the intersection of $\Im^{ndrs}$ and $\Im^{nirs}$ for convex technology is a "variable returns to scale" technology when CRS, NIRS, and NDRS do not hold, our focus is to find out whether such a relationship also holds for nonconvex technologies or in what situation it may hold.

An important relation can be obtained when certain patterns of returns of scale appear. If a production technology exhibits one of the four patterns of returns to scale stated above, then we can recover it from its NDRS and NIRS extensions. It is stated in Proposition 5. The following lemma is an immediate consequence of Definition 2. It is useful for the proof of Proposition 5.

Lemma 3. *Let $(\boldsymbol{v}^0, \boldsymbol{u}^0)$ be a productive inout unit mix. Suppose the conditions of Definition 2 are satisfied for the number $s^* > 0$. Let $s > 0$ be a productive input factor.*

a. *For $0 < s < s^*$ and $1 < \lambda \leqq s^*/s$, $g(\lambda s; \boldsymbol{v}^0, \boldsymbol{u}^0) \geqq \lambda g(s; \boldsymbol{v}^0, \boldsymbol{u}^0)$.*
b. *For $s > s^*$ and $s^*/s \leqq \lambda < 1$, $g(\lambda s; \boldsymbol{v}^0, \boldsymbol{u}^0) \geqq \lambda g(s; \boldsymbol{v}^0, \boldsymbol{u}^0)$.*

Proof. Apply Proposition 1 to the conditions of Definition 2. □

Proposition 5. *Let $\Im$ be a technology set exhibiting RVRS with extensions $\Im^{ndrs}$ and $\Im^{nirs}$ as defined before. Then, $\Im = \Im^{ndrs} \cap \Im^{nirs}$. The equality also holds when $\Im$ shows instead any of the following returns to scale: CRS, NDRS, NIRS, SIRS, and SDRS.*

Proof. See Appendix B. □

Briec et al. (2000, propositions 1b and 5b) have shown that the result of Proposition 5 holds for convex technology sets. Here, we point out that the pattern of returns to scale is the crucial property of this result. This equality holds even if the technology set is not convex.

Consider any $rts = crs, ndrs, nirs$ and any productive inout unit mix $(\boldsymbol{v}^0, \boldsymbol{u}^0)$. If $(s\boldsymbol{v}^0, t\boldsymbol{u}^0) \in \Im^{rts}$ for some $s > 0$ and $t > 0$, then the corresponding ray production function is $g^{rts}(s; \boldsymbol{v}^0, \boldsymbol{u}^0) := \max_{t > 0} \{t : (s\boldsymbol{v}^0, t\boldsymbol{u}^0) \in \Im^{rts}\}$. The corollary below immediately follows from the previous two propositions.

Corollary 1 (Corollary to Propositions 4 and 5). *Consider any technology set $\Im$ and $g(s; \boldsymbol{v}^0, \boldsymbol{u}^0)$ its ray production function with respect to $(\boldsymbol{v}^0, \boldsymbol{u}^0) \in \boldsymbol{I}^x \times \boldsymbol{I}^y$ for $s > 0$. Let $\Im^{rts}$ be its rts extension and $g^{rts}(s; \boldsymbol{v}^0, \boldsymbol{u}^0)$ be its rts ray production function with respect to $(\boldsymbol{v}^0, \boldsymbol{u}^0)$, where rts = crs, ndrs, nirs and $s > 0$.*

a. $g^{crs}(s;\boldsymbol{v}^0,\boldsymbol{u}^0) = \max\{g^{ndrs}(s;\boldsymbol{v}^0,\boldsymbol{u}^0), g^{nirs}(s;\boldsymbol{v}^0,\boldsymbol{u}^0)\}$.

b. *If $\Im$ exhibits CRS, NDRS, NIRS, or RVRS, then*

$$g(s;\boldsymbol{v}^0,\boldsymbol{u}^0) = \min\{g^{ndrs}(s;\boldsymbol{v}^0,\boldsymbol{u}^0), g^{nirs}(s;\boldsymbol{v}^0,\boldsymbol{u}^0)\}.$$

Proposition 4 holds for any arbitrary production technology. On the other hand, the technology set $\Im$ is known to exhibit one of some specific returns to scale in Proposition 5. In empirical studies, sometimes we know that the two empirical technology sets are NDRS and NIRS. The next question is whether an RVRS technology set can be constructed from the intersection of arbitrary NDRS and NIRS technology sets.

4. Constructing an RVRS Production Technology from Intersection

Suppose there are $k = 1,\dots,K$ observations of input and output vectors $(\boldsymbol{x}^k,\boldsymbol{q}^k) \in \mathbb{R}_+^N \times \mathbb{R}_+^M$. We assume that $\boldsymbol{x}^k \geq \boldsymbol{0}_N$ and $\boldsymbol{q}^k \geq \boldsymbol{0}_M$ for all k. We use $T^0 := \{(\boldsymbol{x}^k,\boldsymbol{q}^k) : k = 1,\dots,K\}$ to denote the data set. We call a technology set that contains T^0 an *empirical technology set*. This section fills the gap in the literature on modeling RVRS by providing a general approach to construct an empirical RVRS production technology.

The following notations will be used. Let Ψ^{ndrs} and Ψ^{nirs} be any technology sets that exhibit nondecreasing returns to scale and nonincreasing returns to scale, respectively. They satisfy the maintained assumptions, specifically strongly disposable inputs and outputs and the convex technology set. Further, they contain the data set T^0. Briec et al. (2004, remark 1, p. 166) suggested using the intersection of Ψ^{ndrs} and Ψ^{nirs} to form a technology with "variable returns to scale." However, they have not shown that such an intersection does exhibit RVRS. To proceed, define a technology by the intersection of these two sets: $\Psi^{int} := \Psi^{ndrs} \cap \Psi^{nirs}$. The major issue is that given the hypotheses in this paragraph, will Ψ^{int} exhibit RVRS? If the answer is yes, then we can construct an RVRS production technology with two technology sets with different returns to scale. We explore this below.

The following two lemmas are immediate from the properties of Ψ^{ndrs} and Ψ^{nirs}.

Lemma 4. *Given Ψ^{ndrs}, Ψ^{nirs}, and Ψ^{int} as defined before, Ψ^{int} is an empirical technology set with the following properties.*

a. *Inputs and outputs are strongly disposable in Ψ^{int}.*

b. *The input set $\mathcal{L}^{int}(\boldsymbol{q}) := \{\boldsymbol{x} : (\boldsymbol{x},\boldsymbol{q}) \in \Psi^{int}\}$ is closed for any output vector $\boldsymbol{q}$.*

Proof. See Appendix C. □

Lemma 5. *For any productive inout unit mix $(\boldsymbol{v}^0,\boldsymbol{u}^0)$ in Ψ^{rts}, define $g^{rts}(s;\boldsymbol{v}^0,\boldsymbol{u}^0) = \max_{t>0}\{t : (s\boldsymbol{v}^0, t\boldsymbol{u}^0) \in \Psi^{rts}\}$ for $s > 0$, $rts = ndrs, nirs, int$.*

a. *The function $g^{rts}(s;\cdot)$ is weakly increasing in s.*

b. *For $s > 0$, the function $g^{rts}(s;\cdot)/s$ is continuous from above in s.*

Proof. See Appendix C. □

The main result of this section assumes the following condition with respect to a unit mix of the relevant technology:

Condition 1 (Intersection Condition). *Let $(\boldsymbol{v}^0,\boldsymbol{u}^0)$ be any productive inout unit mix in Ψ^{rts} and the ray production function with respect to $(\boldsymbol{v}^0,\boldsymbol{u}^0)$ be $g^{rts}(s;\boldsymbol{v}^0,\boldsymbol{u}^0) = \max_{t>0}\{t : (s\boldsymbol{v}^0, t\boldsymbol{u}^0) \in \Psi^{rts}\}$ for $s > 0$, $rts = ndrs, nirs$. The following conditions are satisfied.*

i. *There exist $0 < s_1 < s_2$ such that $0 < g^{ndrs}(s_1;.) < g^{nirs}(s_1;.)$ and $g^{ndrs}(s_2;.) > g^{nirs}(s_2;.) > 0$.*

ii. *For every productive input factor $s > 0$, there exists $\tau > 1$ such that $g^{nirs}(\tau s;.) < \tau g^{nirs}(s;.)$.*

When the intersection condition (Condition 1) holds with respect to all productive inpout unit mixes, we say that the production technology satisfies the intersection condition (Condition 1). The intersection condition (Condition 1(i)) makes sure that $g^{ndrs}(s;.)$ and $g^{nirs}(s;.)$ do not coincide before decreasing returns to scale and after increasing returns to scale set in. Condition 1(ii) requires that strictly decreasing returns to scale finally appear for the NIRS extension.

Lemma 6. *Let $(\boldsymbol{v}^0,\boldsymbol{u}^0)$ be any productive inout unit mix in Ψ^{int} and the function $g^{int}(s;\boldsymbol{v}^0,\boldsymbol{u}^0)$ as defined before. If the intersection condition (Condition 1(i)) holds, the solution set of the following problem is either a singleton or a closed interval $[s_{min}, s_{max}]$, where $0 < s_{min} < s_{max}$:*

$$\max_s \left\{\frac{g^{int}(s;\boldsymbol{v}^0,\boldsymbol{u}^0)}{s}\right\}.$$

Proof. See Appendix C. □

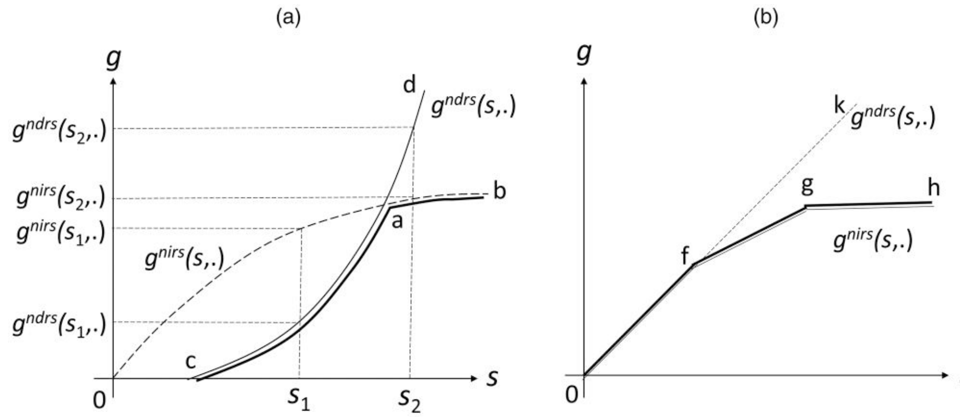
Note that $[s_{min}, s_{max}]$ is the set of most productive scale sizes and that the production there exhibits constant returns to scale. Lemma 6 states that the production in the constant returns to scale region for any given productive inout unit mix is a line segment including the end points. Lemmas 4–6 are useful in the proof of one major result of this paper as stated in the following proposition.

Proposition 6. *If the intersection condition (Condition 1) holds with respect to all productive input unit mixes, then $\Psi^{int} = \Psi^{ndrs} \cap \Psi^{nirs}$ has strongly disposable inputs and outputs, with closed technology, and exhibits RVRS.*

Proof. See Appendix C. □

Here, we briefly discuss the above proposition. Suppose we construct Ψ^{ndrs} and Ψ^{nirs} from two data sets. The corresponding $g^{ndrs}(s;.)$ and $g^{nirs}(s;.)$ can be derived accordingly. In Figure 2, the NDRS function $g^{ndrs}(s;.)$ is shown with the thin curves, and the NIRS function $g^{nirs}(s;.)$ is shown with the dashed curves. The ray function of $\Psi^{ndrs} \cap \Psi^{nirs}$ is the thick curve.

Figure 2. The Intersection of NDRS and NIRS Functions



Notes. (a) Satisfying Condition 1. (b) Violating Condition 1(i).

In panel (a) of Figure 2, the NDRS function $g^{ndrs}(s;.)$ is *cad*, and the NIRS function $g^{nirs}(s;.)$ is *0ab*. It is obvious that (i) and (ii) in Condition 1 are satisfied, and the intersection exhibits strictly increasing returns to scale before point a and strictly decreasing returns to scale after point a. In panel (b) of Figure 2, the NDRS function *0fk* exhibits constant returns to scale and has an overlap section *0f* with the NIRS function *0fgh*. This violates Condition 1(i). Although Condition 1(ii) is satisfied, their intersection is an NIRS function, not RVRS.

In summary, Condition 1(i) ensures that strictly increasing (decreasing) returns to scale appears before (after) some point of a ray. Condition 1(ii) avoids the case in Figure 3 as explained in Li et al. (2024).

The difficulty of proving Proposition 6 comes from three sources. (i) The two ray production functions are not restricted to continuous functions. (ii) The characteristics of these two functions are different in the sense that the average value of the function (i.e., $g^{rts}(s;.)/s$) is weakly increasing in the NDRS function and weakly decreasing in the NIRS function. (iii) The possibility of changing convexity makes the proof more complicated. One specific example is that the NDRS function is strictly convex and that the NIRS function is strictly concave, as in Figure 1(a). The proof of this proposition is very technical and lengthy. It is placed in Appendix C.

When two reference technology sets contain T^0 and satisfy certain conditions, Proposition 6 states that we can construct an RVRS technology set from them. This is an important result. The crucial step is to find such reference technology sets. One suggestion is provided in the following section.

5. Measuring Technical Efficiency Under RVRS Without Convexity

The target of this section is to find an appropriate production frontier for measuring technical efficiency with respect to the data set T^0 and the properties of strong disposability and RVRS, without assuming convexity. The first task is to find the minimal extrapolation of T^0. The second task is to show that measuring efficiency with this frontier is better than existing methods.

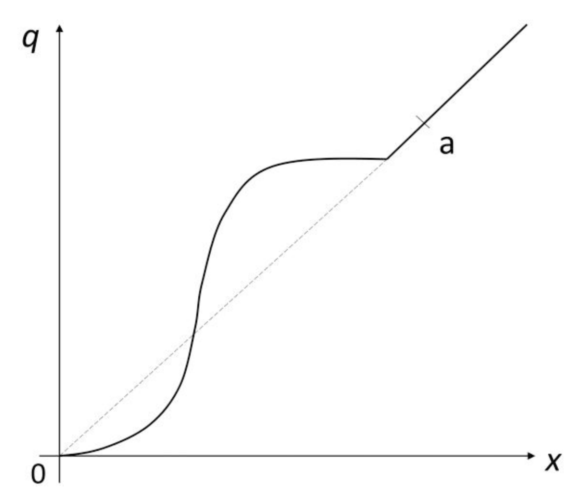
Before we proceed, it is more convenient to clarify some concepts of nonconvexity in production analysis. The meaning of the convex technology set is well known. When we say the technology set is not convex, it may take many possible forms. Specifically, there are always subsets of the technology set that are locally convex. The following types of convexity are classified in this paper.

i. The technology is fully convex. For $(\boldsymbol{x}^a, \boldsymbol{q}^a), (\boldsymbol{x}^b, \boldsymbol{q}^b) \in \Im$ and $0 < \theta < 1$, $(\theta \boldsymbol{x}^a + (1-\theta)\boldsymbol{x}^a, \theta \boldsymbol{q}^a + (1-\theta)\boldsymbol{q}^a) \in \Im$.

ii. The technology is partially convex in input sets. For $(\boldsymbol{x}^a, \boldsymbol{q}), (\boldsymbol{x}^b, \boldsymbol{q}) \in \Im$ and $0 < \theta < 1$, $(\theta \boldsymbol{x}^a + (1-\theta)\boldsymbol{x}^a, \boldsymbol{q}) \in \Im$, and (i) does not hold.

iii. The technology is partially convex in output sets. For $(\boldsymbol{x}, \boldsymbol{q}^a), (\boldsymbol{x}, \boldsymbol{q}^b) \in \Im$, and $0 < \theta < 1$, $(\boldsymbol{x}, \theta \boldsymbol{q}^a + (1-\theta)\boldsymbol{q}^a) \in \Im$, and (i) does not hold.

Figure 3. A Special Case of the Definition of Li (2019)



iv. The technology is *partially convex along rays*. For any productive inout unit mix $(\boldsymbol{v}, \boldsymbol{u})$ in $\Im$, the ray function $g(s; \boldsymbol{v}, \boldsymbol{u})$ is concave in s for all $s > 0$ such that $g(s; \boldsymbol{v}, \boldsymbol{u}) > 0$, and (i) does not hold.

When at least one of (ii)–(iv) is true but the technology set is not fully convex, we say that the technology is *partially convex*. If it is desirable to point out the specific types of convexity, we say the technology is partially convex in the relevant subsets. When all of (ii)–(iv) are not true, we say that the technology is *completely not convex*. Note the difference between partial convexity and quasiconvexity. A production function that is, for example, partially convex in input sets is a quasiconcave function, but it is not a concave function. Conversely, a quasiconcave production function does not rule out the possibility of "fully convex."

Following Banker (1984, p. 39) and Banker et al. (1984, p. 1081), the property of minimal extrapolation is defined as follows.

Definition 3. Let A be a set of the regularity condition. An empirical technology set T^* is the *minimal extrapolation* of T^0 for A if the following conditions are satisfied. (i) $T^0 \subseteq T^*$. (ii) T^* satisfies A. (iii) If T is any technology set that contains T^0 and satisfies A, then $T^* \subseteq T$.

A common empirical technology set in the DEA literature is

$$T^{cv-rvrs} = \left\{ (\boldsymbol{x}, \boldsymbol{q}) : \sum_{k=1}^{K} z_k \boldsymbol{x}^k \leqq \boldsymbol{x}, \sum_{k=1}^{K} z_k \boldsymbol{q}^k \geqq \boldsymbol{q}, \sum_{k=1}^{K} z_k = 1,\ z_k \geqq 0, k = 1, \ldots, K \right\}. \quad (2)$$

It is well known that this technology set is fully convex with strongly disposable inputs and outputs. Its extensions are $T^{cv-crs} = \{(\boldsymbol{x}, \boldsymbol{q}) : \sum_{k=1}^K z_k \boldsymbol{x}^k \leqq \boldsymbol{x}, \sum_{k=1}^K z_k \boldsymbol{q}^k \geqq \boldsymbol{q},\ z_k \geqq 0, k = 1, \ldots, K\}$, $T^{cv-ndrs} = \{(\boldsymbol{x}, \boldsymbol{q}) : \sum_{k=1}^K z_k \boldsymbol{x}^k \leqq \boldsymbol{x}, \sum_{k=1}^K z_k \boldsymbol{q}^k \geqq \boldsymbol{q},\ \sum_{k=1}^K z_k \geqq 1,\ z_k \geqq 0, k = 1, \ldots, K\}$, and $T^{cv-nirs} = \{(\boldsymbol{x}, \boldsymbol{q}) : \sum_{k=1}^K z_k \boldsymbol{x}^k \leqq \boldsymbol{x}, \sum_{k=1}^K z_k \boldsymbol{q}^k \geqq \boldsymbol{q},\ \sum_{k=1}^K z_k \leqq 1,\ z_k \geqq 0, k = 1, \ldots, K\}$. As observed by Kersten and Vanden Eeckaut (1999), $T^{cv-crs} = T^{cv-ndrs} \cup T^{cv-nirs}$ and $T^{cv-rvrs} = T^{cv-ndrs} \cap T^{cv-nirs}$. The first equality verifies Proposition 4. The second equality confirms that according to Proposition 6 and subject to the intersection condition (Condition 1), the technology set in (2) exhibits RVRS according to our definition.

On the other hand, the free disposal hull empirical technology set is completely not convex:

$$\begin{aligned} T^{fdh} &= \{(\boldsymbol{x}, \boldsymbol{q}) : \boldsymbol{x}^k \leqq \boldsymbol{x}, \boldsymbol{q}^k \geqq \boldsymbol{q} \text{ for some } k = 1, \ldots, K\} \\ &= \left\{ (\boldsymbol{x}, \boldsymbol{q}) : \sum_{k=1}^{K} z_k \boldsymbol{x}^k \leqq \boldsymbol{x}, \sum_{k=1}^{K} z_k \boldsymbol{q}^k \geqq \boldsymbol{q},\ \sum_{k=1}^{K} z_k = 1,\ z_k \in \{0, 1\}, k = 1, \cdots, K \right\}. \end{aligned} \quad (3)$$

Following Kersten and Vanden Eeckaut (1999), the extensions of T^{fdh} are $T^{fdh-crs} := \{(\boldsymbol{x}, \boldsymbol{q}) : \alpha \boldsymbol{x}^k \leqq \boldsymbol{x}, \alpha \boldsymbol{q}^k \geqq \boldsymbol{q}$ for some $k = 1, \ldots, K$ and $\alpha \geqq 0\}$, $T^{fdh-ndrs} := \{(\boldsymbol{x}, \boldsymbol{q}) : \alpha \boldsymbol{x}^k \leqq \boldsymbol{x}, \alpha \boldsymbol{q}^k \geqq \boldsymbol{q}$ for some $k = 1, \ldots, K$ and $\alpha \geqq 1\}$, and $T^{fdh-nirs} := \{(\boldsymbol{x}, \boldsymbol{q}) : \alpha \boldsymbol{x}^k \leqq \boldsymbol{x}, \alpha \boldsymbol{q}^k \geqq \boldsymbol{q}$ for some $k = 1, \ldots, K$ and $0 \leqq \alpha \leqq 1\}$. These extensions exhibit constant returns to scale, nondecreasing returns to scale, and nonincreasing returns to scale, respectively. Using the terms introduced at the beginning of this section, $T^{fdh}, T^{fdh-ndrs}$, and $T^{fdh-nirs}$ are completely not convex, whereas $T^{fdh-crs}$ is partially convex in rays. The following results are useful.

Lemma 7 (Briec et al. 2004, Proposition 1, p. 166). *For rts = ndrs, nirs, the technology set $T^{fdh-rts}$ is the minimal extrapolation for the properties of strong disposability of inputs and outputs and returns to scale rts.*

One can verify that $T^{fdh-crs} = T^{fdh-ndrs} \cup T^{fdh-nirs}$, which is a special case of Proposition 4. Unlike $T^{cv-rvrs}$, Kersten and Vanden Eeckaut (1999) pointed out that $T^{fdh} \neq T^{fdh-ndrs} \cap T^{fdh-nirs}$. From our previous analyses, the two sides are not equal because there is no specific pattern of returns to scale in the construction of T^{fdh}. It turns out that the empirical technology constructed from this intersection as in Destefanis (2002) and Destefanis and Storti (2002) is suitable for DEA as stated in the following proposition.

Proposition 7. *Construct $T^{fdh-ndrs}$ and $T^{fdh-nirs}$ for the data set T^0 as defined before. Let $T^{int} = T^{fdh-ndrs} \cap T^{fdh-nirs}$. If Condition 1(i) is satisfied, then T^{int} is the minimal extrapolation for the conditions of strongly disposable inputs and outputs and RVRS for T^0.*

Proof. See Appendix D. □

From now on, the intersection T^{int} is denoted by the notation $T^{fdh-rvrs}$. The input-oriented measures of technical efficiency with respect to the FDH-NDRS and FDH-NIRS frontiers for firm j are

$$\begin{aligned} te_i^{fdh-ndrs}(\boldsymbol{x}^j, \boldsymbol{q}^j) &= \min_{\lambda} \{\lambda : (\lambda \boldsymbol{x}^j, \boldsymbol{q}^j) \in T^{fdh-ndrs}\} \\ &= \min_{\lambda} \{\lambda : \alpha \boldsymbol{x}^k \leqq \lambda \boldsymbol{x}^j, \alpha \boldsymbol{q}^k \geqq \boldsymbol{q}^j \\ &\qquad \text{for some } k = 1, \ldots, K \text{ and } \alpha \geqq 1\}. \end{aligned} \quad (4)$$

$$\begin{aligned} te_i^{fdh-nirs}(\boldsymbol{x}^j, \boldsymbol{q}^j) &= \min_{\lambda} \{\lambda : (\lambda \boldsymbol{x}^j, \boldsymbol{q}^j) \in T^{fdh-nis}\} \\ &= \min_{\lambda} \{\lambda : \alpha \boldsymbol{x}^k \leqq \lambda \boldsymbol{x}^j, \alpha \boldsymbol{q}^k \geqq \boldsymbol{q}^j \\ &\qquad \text{for some } k = 1, \ldots, K \text{ and } \alpha \geqq 1\}. \end{aligned} \quad (5)$$

The input-oriented measure of technical efficiency with respect to $T^{fdh-rvrs}$ is then $te_i^{fdh-rvrs}(\boldsymbol{x}^j, \boldsymbol{q}^j) = \max\{te_i^{fdh-ndrs}(\boldsymbol{x}^j, \boldsymbol{q}^j), te_i^{fdh-nirs}(\boldsymbol{x}^j, \boldsymbol{q}^j)\}$.

Podinovski (2004) has shown that $te_i^{fdh-ndrs}(\boldsymbol{x}^j, \boldsymbol{q}^j)$ and $te_i^{fdh-nirs}(\boldsymbol{x}^j, \boldsymbol{q}^j)$ can be found by solving equivalent mixed integer linear programs. We find that Formulae (4) and (5) are easier without using linear programming.

Pseudocodes are included in Appendix E for reference. The output-oriented measures of technical efficiency with respect to $T^{fdh-rvrs}$ are similar and thus, are not shown.

If there is a "true" production technology behind an observed data set, we would like to have the values of our efficiency measure as close to the "true" values as possible. Further, it is desirable that the technically efficient production activities identified by the efficiency measure are feasible. The following proposition shows the superiority of $te_i^{fdh-rvrs}$ over other existing popular measures.

Proposition 8. *Suppose the data of K firms are observed, $(\boldsymbol{x}^k, \boldsymbol{q}^k), k = 1, \ldots, K$, and these firms are facing the same technology set $\Im^*$. Construct the empirical frontiers T^{fdh}, $T^{fdh-rvrs}$, and $T^{cv-rvrs}$ as defined previously. The input-oriented technical efficiency of firm j with respect to these technology sets is $te_i^s(\boldsymbol{x}^j, \boldsymbol{q}^j) = \min\{\lambda : (\lambda \boldsymbol{x}^j, \boldsymbol{q}^j) \in T^s\}, s = *, fdh, fdh-rvrs, cv-rvrs$. If inputs and outputs are strongly disposable and $\Im^*$ exhibits RVRS, then*

$$te_i^*(\boldsymbol{x}^j, \boldsymbol{q}^j) \leqq te_i^{fdh-rvrs}(\boldsymbol{x}^j, \boldsymbol{q}^j) \leqq te_i^{fdh}(\boldsymbol{x}^j, \boldsymbol{q}^j). \tag{6}$$

Proof. See Appendix D. □

Hence, under the hypotheses of Proposition 8, $te_i^{fdh}(\boldsymbol{x}^j, \boldsymbol{q}^j)$ underestimates technical inefficiency of unit j. This conclusion holds whether the production technology is convex or nonconvex.

On the other hand, the conventional empirical production set $T^{cv-rvrs}$ is convex. Thus, it cannot capture properties of nonconvexity. One type of nonconvexity draws our attention: the S-shaped production. This pattern has been observed in and applied to many fields, as mentioned in Section 1. It has been defined in the one-output production (Li 2019, definition (ss-curve), p. 162). The multioutput version is defined below.

Definition 4. Let $(\boldsymbol{v}^0, \boldsymbol{u}^0)$ be a productive inout unit mix. A technology is called *strictly ray S shaped* with respect to $(\boldsymbol{v}^0, \boldsymbol{u}^0)$ if there exist numbers $0 \leqq s^0 < s_{min} \leqq s_{max}$ such that (i) $g(0; \boldsymbol{v}^0, \boldsymbol{u}^0) = 0$ and if $s^0 > 0$, then $g(s; \boldsymbol{v}^0, \boldsymbol{u}^0) = 0$ for $0 \leqq s < s^0$; (ii) $g(s; \boldsymbol{v}^0, \boldsymbol{u}^0)$ is strictly convex in s for $s^0 < s < s_{min}$; and (iii) $g(s; \boldsymbol{v}^0, \boldsymbol{u}^0)$ is concave in s for $s > s_{min}$ and strictly concave in s for $s > s_{max}$. The technology is called *strictly S shaped* if $g(s; \boldsymbol{v}, \boldsymbol{u})$ is strictly ray S shaped with respect to all productive inout unit mixes $(\boldsymbol{v}, \boldsymbol{u})$.

For a strictly S-shaped production technology, the conventional VRS technology has a big deficiency as stated in the following proposition.

Proposition 9. *Let T^* be a strictly S-shaped production set. Then, it is possible that*

$$te_i^*(\boldsymbol{x}^j, \boldsymbol{q}^j) > te_i^{cv-rvrs}(\boldsymbol{x}^j, \boldsymbol{q}^j). \tag{7}$$

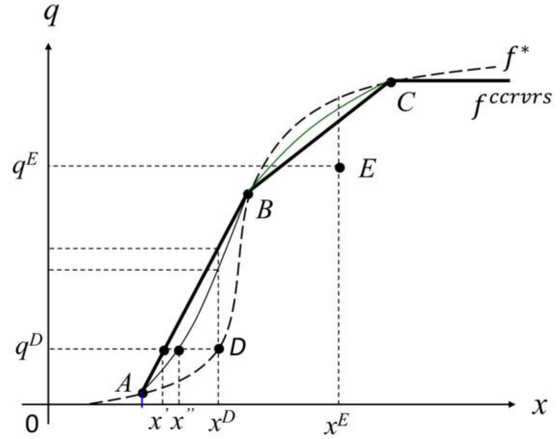
Proof. See Appendix D. □

Thus, if the technology is not ray concave, Proposition 9 points out that the conventional DEA technology $te_i^{cv-rvrs}(\boldsymbol{x}^j, \boldsymbol{q}^j)$ with full convexity may identify an infeasible production activity as a firm's target to achieve. Rather than proving Proposition 9, we explain its logic using Figure 4, which is similar to Dekker and Post (2001, figure 2, p. 299). The thick solid frontier in Figure 4 is the conventional VRS frontier. The magnitude of the error depends on many factors. If the "true" production frontier is the thin curve in Figure 4, then the IRS region is closer to line segment AB, and the magnitude of the error will be smaller. If the "true" frontier is the dashed curve in Figure 4, then the IRS region is highly curved, and there will be a large error. The production frontier between A and B depends on these two points only. Thus, the error in the IRS region will not become smaller even if more observed points are collected. In contrast, if the theoretical technology is convex, the errors in nonconvex technologies may overestimate firm performance in small data sets. When the data set is getting larger, the difference between the empirical values and the theoretical values will be getting smaller. In conclusion, $te_i^{fdh-rvrs}(\boldsymbol{x}^j, \boldsymbol{q}^j)$ is a better choice for the efficiency measurement when there is no convexity in the production technology, but RVRS is true.

A final remark is on the piecewise log-linear frontier (Banker and Maindiratta 1986), which is able to capture the properties of RVRS in multioutput production. Let $\hat{q} = (\ln(q_1), \ldots, \ln(q_M))$ and $\hat{x} = (\ln(x_1), \ldots, \ln(x_M))$. Their technology set is

$$T^{bm} = \left\{ (x, q) : q_m \leqq \prod_{k=1}^{K} q_{km}^{z_k}, m = 1, \ldots, M; x_n \geqq \prod_{k=1}^{K} x_{kn}^{z_k}, n = 1, \ldots, N; \sum_{k=1}^{K} z_k = 1;\ z_k \geqq 0\ \forall k \right\}. \tag{8}$$

Figure 4. (Color online) Technical Efficiency for the Not Convex Technology Set



This technology set can model the S-shaped pattern in multiple outputs. Post (2001b) generalized the logarithm of inputs to a transformation function. Although his method is an improvement of the piecewise log-linear frontier, it is limited to one output only. Furthermore, both methods share similar weaknesses. We explain this using the piecewise log-linear frontier. Consider the one-input, one-output example in Figure 4. The slightly curved line in Figure 4 is the production frontier implied in the IRS region implied by (8). Along any ray of input, any nonextreme point must be the convex combination of two points (after taking the logarithm) on the ray. Thus, for any interior point of the slightly curved line in in Figure 4, $(\boldsymbol{x}, q)$, $ln(q) = a + \varepsilon \ln(x)$, where a and ε are constants. Thus,

$$\frac{d\ln(q)}{d\ln(x)} = \frac{dq}{dx}\frac{x}{q} = \varepsilon$$

is the elasticity of scale. Then,

$$\frac{d\varepsilon}{dx} = \frac{d^2q}{dx^2}\frac{x}{q} + \frac{dq}{dx}\left(\frac{q - x\frac{dq}{dx}}{q^2}\right) = 0.$$

It follows that

$$\frac{d^2q}{dx^2} = \frac{1}{x}\frac{dq}{dx}(\varepsilon - 1).$$

In Figure 4, point B is the most productive scale size. Between A and B in Figure 4, increasing returns to scale prevails. We have $\varepsilon > 1$, and hence, q is a strictly convex function of x. Between B and C in Figure 4, decreasing returns to scale prevails. We have $\varepsilon < 1$, and hence, q is a strictly concave function of x. The reader can verify that this result also holds for multiple inputs. Therefore, the log-linear function captures the properties of an S-shaped function well. However, there are two caveats. (i) When the true technology is highly convex in the IRS region, the corresponding solid curve in Figure 4 depends on points A and B only. The information of point D in Figure 4 is not considered. The efficient input level x'' identified in the IRS region is infeasible to produce q^D. (ii) The function $f^{bm}(\boldsymbol{x})$ is piecewise linear in the log input and log output space, and it is piecewise nonlinear in the input and output space. From our previous discussion, the function is piecewise strictly convex in the IRS region and piecewise strictly concave in the DRS region. If the theoretical function is concave, some parts of the function are forced to be convex in the empirical function. Therefore, although the piecewise log-linear frontier can capture some properties of S-shaped production, it has weaknesses, especially when the IRS region is highly curved.

In summary, the empirical production technology $T^{fdh-rvrs}$ is the minimal extrapolation of multioutput production data under the assumptions of strong disposability and RVRS. It is an appropriate tool for efficiency measurement with one output and multioutput production data.

6. An Illustration: The Technical Efficiency of Listed Firms in China

This section applies the production frontiers discussed in this paper to investigate the input-oriented technical efficiency of listed companies in China. Special attention is paid to the relationship between technical efficiency and company size. We collect the data of the nonmetallic mineral products industry (code: C27) and the pharmaceutical industry (code: C30) in the year 2019 from the China Stock Market & Accounting Research Database. The data set contains 241 listed companies (182 in C27 and 59 in C30). The basic unit of Chinese currency is RMB. Inputs are the number of employees (x_1) and the value of fixed assets in million RMB (x_2). Output is the total revenue in million RMB (q_1). The descriptive statistics are reported in Table 1.

Three production technologies have been discussed in previous sections: the nonconvex technology that exhibits RVRS and strong disposability $T^{fdh-rvrs}$, the conventional convex VRS technology $T^{cv-rvrs}$, and the free disposal hull technology T^{fdh}. They are used to measure the technical efficiency of companies in the two industries. The frontiers of these technologies are constructed for each industry separately. The corresponding input-oriented measures of technical efficiency are $te_i^{fdh-rvrs}$, $te_i^{cv-rvrs}$, and te_i^{fdh}, respectively. The geometric means of the values of these measures are listed in Table 2.

Considering the values of efficiency measures for the full sample. The efficiency measures te_i^{fdh}, $te_i^{fdh-rvrs}$, and $te_i^{cv-rvrs}$ give geometric means of 0.654, 0.558, and 0.382, respectively, for the industry C27. The corresponding geometric means for industry C30 are 0.802, 0.713, and 0.544, respectively. These values indicate that both industries are technically inefficient, and the pharmaceutical industry (C30) is more efficient than the nonmetallic mineral products industry (C27). The results are robust to the empirical production frontier adopted. The values for te_i^{fdh} and $te_i^{fdh-rvrs}$ also verify the second inequality of (6) in Proposition 8. Indeed, under our maintained assumptions of RVRS and strong disposability, Proposition 8 also points out that $te_i^{fdh-rvrs}$ improves upon te_i^{fdh} because the value of $te_i^{fdh-rvrs}$ is closer to the "true value" of technical efficiency than te_i^{fdh}. So, we focus on discussing the differences between $te_i^{fdh-rvrs}$ and $te_i^{cv-rvrs}$.

The last column of Table 2 shows the percentage difference between the FDH-RVRS and CV-RVRS efficiency values. It is adopted to indicate the effect of relaxing convexity. The inverse of the ratio in the formula is similar to the CRTE$_i$ ratio in Briec et al. (2004,

Table 1. Descriptive Statistics of Input-Output Data

Variable	Observations	Mean	Standard deviation	Min	Max
C27: Nonmetallic mineral products sector					
q_1: Total revenue	182	2,861.4	4,121.3	0.7	28,585.2
x_1: Employees	182	2,824.3	3,888.6	241	31,370
x_2: Fixed asset	182	2,761.2	5,092.8	40.3	55,716.3
C30: Pharmaceutical sector					
q_1: Total revenue	59	3,509.8	5,614.8	31.7	32,897.5
x_1: Employees	59	3,247.3	5,515.1	91	32,873
x_2: Fixed asset	59	3,086.5	4,529.7	16.5	21,937.5

Note. q_1 and x_2 are measured in million RMB yuan.

definition 7, p. 181), which is used as a nonparametric goodness-of-fit test for convexity. For the full sample, the geometric mean of $te_i^{fdh-rvrs}$ is 45.8% in C27 and 31.1% in C30 over that of $te_i^{cv-rvrs}$. This means that if the production technology is not convex, the conventional CV-RVRS frontier would overestimate technical inefficiency by at least 30%. To identify the source of such large differences, production units are classified according to their returns to scale in Table 2. We will say that companies in the IRS and DRS regions are IRS and DRS companies, respectively. Investigating the geometric means of technical efficiency of companies in the IRS and DRS regions, we find the following.

1. The source of percentage difference mainly comes from the difference in the IRS region. It is obvious that the difference between the geometric means of $te_i^{fdh-rvrs}$ and $te_i^{cv-rvrs}$ is large in the IRS region (they are 50.5% in C27 and 37.4% in C30). In contrast, the geometric means of these two measures are relatively close in the DRS region, with a difference of 10.2% in C27 and 14.5% in C30. This indicates that the differences between the FDH-RVRS and CV-RVRS frontiers identified in the full sample are mainly because of the efficiency values in the IRS region. Such large differences suggest that the production is partially not convex on rays in the IRS region. As explained in the proof of Proposition 8, some potential gains identified by the CV-RVRS frontier may not be achievable.

2. The ranking of companies may change. Consider the conventional production frontier $T^{cv-rvrs}$ of industry C30. The geometric mean of $te_i^{cv-rvrs}$ is 0.511 in the IRS region and 0.600 in the DRS region. Thus, DRS companies look on average more technically efficient than IRS companies. For the nonconvex RVRS frontier $T^{fdh-rvrs}$, the corresponding numbers are 0.703 in the IRS region and 0.687 in the DRS region. That means that DRS companies are less technically efficient. This may lead to misleading policy implications.

Table 2. Input-Oriented Technical Efficiency by Returns to Scale

Sample	n	te_i^{fdh}	$te_i^{fdh-rvrs}$	$te_i^{cv-rvrs}$	$\left(\frac{te_i^{fdh-rvrs}}{te_i^{cv-rvrs}}-1\right)\times 100\%$
C27: Nonmetallic mineral products industry					
Full sample	182	0.654	0.558	0.382	45.8
IRS	165	0.643	0.550	0.365	50.5
DRS	13	0.747	0.556	0.504	10.2
C30: Pharmaceutical industry					
Full sample	59	0.802	0.713	0.544	31.1
IRS	46	0.774	0.703	0.511	37.4
DRS	10	0.883	0.687	0.600	14.5

Note. The CRS region is omitted because only four in C27 and three in C30 are in CRS.

3. The relation between firm size and technical efficiency. The finding of company ranking in (2) raises another issue: the relation between firm size and technical efficiency. In the agricultural sector, the inverse farm size productivity hypothesis has attracted researchers' attention since Sen (1962). This hypothesis states that small farms are more productive than large farms. A recent discussion of this hypothesis can be found in Helfand and Taylor (2021). Studies adopting data envelopment analysis (Jha et al. 2000, Nowak et al. 2015) and stochastic frontier analysis (Latruffe et al. 2004, de Freitas et al. 2019) to explore the impacts of firm size on technical efficiency usually have convex production technologies. To illustrate the importance of appropriate efficiency measures, fixed assets are used to measure firm size. The Pearson correlation coefficient between the full sample geometric mean of technical efficiency and the fixed assets (x_2) is reported in Table 3 below.

In both industries, the value of the correlation coefficient is positive for the convex CV-RVRS frontier, whereas it is negative for the FDH-RVRS frontier. When firm size is adopted as a determinant to explain technical efficiency, the efficiency scores derived from our new production frontier may lead to a very different conclusion. We note that the absolute values of the correlation coefficient are very small. However, as an illustration, the pattern of the

Table 3. Correlation Coefficient Between Technical Efficiency and Firm Size

	$te_i^{fdh-rvrs}$ and x_2	$te_i^{cv-rvrs}$ and x_2
C27	−0.083	0.007
C30	−0.041	0.049

values of the correlation coefficient shown in Table 3 should be sufficient to alert researchers to the importance of appropriate modeling of the IRS region.

The conclusions arising from measuring technical inefficiency derived from the new FDH-RVRS frontier may be very different from those of using the conventional production frontier in DEA. Such differences between the efficiency values computed from these two frontiers reflect the effects of relaxing the assumption of convexity under RVRS. In summary, (i) when there are many IRS firms, such differences mainly happen in the IRS region, and (ii) the ranks of production units may be very different with the new production frontier so that the relation between technical efficiency and its determinants may undergo drastic changes. Although only firm size is discussed here, we conjecture that the same conclusion applies to other determinants of technical efficiency.

7. Conclusion

In the literature on efficiency measurement, the empirical technology set is usually convex. This may result in overestimating the technical inefficiency of small firms when there are increasing returns to scale without convexity. Although the convex VRS production technology for multiple outputs was introduced to the literature by Banker et al. (1984), no satisfactory solution has been found for nonconvex RVRS. When researchers want to impose RVRS and strong disposability to the production technology without convexity, they find no tools in their toolbox.

This paper fills in the gap in the literature on modeling nonconvex production technology with RVRS. By focusing on pure scale changes of inputs and outputs, a vigorous definition of RVRS has been suggested. We have found that the FDH-RVRS frontier suggested by Briec et al. (2004, remark 1, p. 166) is the minimal extrapolation for strong disposability and RVRS. This is a suitable choice for measuring technical efficiency. It also provides a tool for researchers to test the existence of the nonconvex IRS region in future research.

Applying the new frontier to two Chinese industries, we have found that the input-oriented efficiency scores using the conventional CV-RVRS frontier are, in general, much smaller than those using the FDH-RVRS frontier, especially for firms in the increasing returns to scale regions. If the researcher adopts the assumption of regular variable returns to scale and is willing to abandon convexity completely, then Proposition 8 implies that the conventional RVRS frontier may overestimate the inefficiency of production units relative to the "true" frontier for some firms. Some implied potential improvements from $te_i^{cv-rvrs}$ may be infeasible. Furthermore, without imposing RVRS, efficiency values based on the FDH frontier are farther from the theoretical value of technical efficiency than $te_i^{fdh-rvrs}$.

The results in this paper call for serious thought on modeling nonconvexity when there are many production units in the increasing returns to scale region. If a researcher is willing to assume RVRS and suspects that convexity may not hold, then we recommend the new empirical frontier ($T^{fdh-rvrs}$) introduced in this paper.

Finally, as proved in Li et al. (2024), the empirical technology introduced by Li (2019) is the minimal extrapolation under the assumption of strong disposability, convex input sets, and RVRS in one-output production. Extreme difficulties appear when we want to extend the technology of Li (2019) to multiple outputs by adding partial convexity to $T^{fdh-rvrs}$. This will be left for future research.

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Appendix A. Proofs for Section 2

Proof of Lemma 1.

a. By definition, $g(s^0;\boldsymbol{v}^0,\boldsymbol{u}^0)=\max_{t>0}\{t:(s^0\boldsymbol{v}^0,t\boldsymbol{u}^0)\in\Im\}\geqq t'$, where $(s^0\boldsymbol{v}^0,t'\boldsymbol{u}^0)\in\Im$. Specifically, $g(s^0;\boldsymbol{v}^0,\boldsymbol{u}^0)\geqq t^0$.

b. Let $0<s^a<s^b$ and $t^*=g(s^a;\boldsymbol{v}^0,\boldsymbol{u}^0)$. Then, $(s^a\boldsymbol{v}^0;t^*\boldsymbol{u}^0)\in\Im$. By strong disposability of inputs, $s^a<s^b$ implies $(s^b\boldsymbol{v}^0;t^*\boldsymbol{u}^0)\in\Im$. Therefore, $g(s^b;\boldsymbol{v}^0,\boldsymbol{u}^0)=\max_{t>0}\{t:(s^b\boldsymbol{v}^0,t\boldsymbol{u}^0)\in\Im\}\geqq t^*=g(s^a;\boldsymbol{v}^0,\boldsymbol{u}^0)$. Thus, the ray production function is weakly increasing. Because all monotonic univariate functions are quasiconcave, the ray production function is quasiconcave in s.

c. The input set of $\Im$ for output vector $\boldsymbol{q}=t\boldsymbol{u}^0$ is $\mathcal{L}(\boldsymbol{q})=\{\boldsymbol{x}:(\boldsymbol{x},\boldsymbol{q})\in\Im\}$. When inputs are restricted to be along the ray passing through $\boldsymbol{v}^0$, we have $\boldsymbol{x}=s\boldsymbol{v}^0$ for some $s>0$. Then, the restricted input set is

$$\mathcal{L}(\boldsymbol{q})=\{\boldsymbol{x}=s\boldsymbol{v}^0:(s\boldsymbol{v}^0,t\boldsymbol{u}^0)\in\Im,s>0\}=\{s>0:(s\boldsymbol{v}^0,t\boldsymbol{u}^0)\in\Im\}\boldsymbol{v}^0. \tag{A.1}$$

If $g(s;\boldsymbol{v}^0,\boldsymbol{u}^0)\geqq t$, then $(s\boldsymbol{v}^0;g(s;\boldsymbol{v}^0,\boldsymbol{u}^0)\boldsymbol{u}^0)\in\Im$, and hence, $(s\boldsymbol{v}^0;t\boldsymbol{u}^0)\in\Im$ by the strong disposability of outputs. Together with (a), $g(s;\boldsymbol{v}^0,\boldsymbol{u}^0)\geqq t$ if and only if $(s\boldsymbol{v}^0,t\boldsymbol{u}^0)\in\Im$. The upper contour set for the ray production function is then $V(t)=\{s>0:g(s;\boldsymbol{v}^0,\boldsymbol{u}^0)\geqq t\}=\{s>0:(s\boldsymbol{v}^0;t\boldsymbol{u}^0)\in\Im\}$. From (A.1), if $\mathcal{L}(\boldsymbol{q})$ is a closed set, then $\{s>0:(s\boldsymbol{v}^0,t\boldsymbol{u}^0)\in\Im\}$ is a closed interval. The continuity of the upper contour set ensures that the ray production function is continuous from above.

d. Let $0<s^a<s^b$ be two positive numbers, and $t^i=g(s^i;\boldsymbol{v}^0,\boldsymbol{u}^0)$, $i=a,b$. Then, $(s^i\boldsymbol{v}^0;t^i\boldsymbol{u}^0)\in\Im$, $i=a,b$. For $0<\theta<1$, let $s^\theta=\theta s^a+(1-\theta)s^b$ and $t^\theta=\theta t^a+(1-\theta)t^b$. Because $\Im$ is a convex set, we have $(\theta s^a\boldsymbol{v}^0+(1-\theta)s^b\boldsymbol{v}^0;\theta t^a\boldsymbol{u}^0+(1-\theta)t^b\boldsymbol{u}^0)=(s^\theta\boldsymbol{v}^0;t^\theta\boldsymbol{u}^0)\in\Im$. By (a) and the definitions of $t^i=g(s^i;\boldsymbol{v}^0,\boldsymbol{u}^0)$, $i=a,b,\theta$,

$$\begin{aligned}g(s^\theta;\boldsymbol{v}^0,\boldsymbol{u}^0)&=\max_{t>0}\{t:(s^\theta\boldsymbol{v}^0,t\boldsymbol{u}^0)\in\Im\}\geqq t^\theta=\theta t^a+(1-\theta)t^b\\&=\theta g(s^a;\boldsymbol{v}^0,\boldsymbol{u}^0)+(1-\theta)g(s^b;\boldsymbol{v}^0,\boldsymbol{u}^0).\end{aligned}$$

Hence, $g(s;\boldsymbol{v}^0,\boldsymbol{u}^0)$ is a concave function. □

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Proof of Proposition 1.

a. The proof of the first equivalence is identical to the proof of a similar relation for a production function exhibiting increasing returns to scale, which can be found in advanced microeconomics textbooks. For the second equivalence, consider three positive numbers λ, s^a, and s^b such that $0<\lambda<1$ and $s^a=\lambda s^b<s^b$. Hence,

$$g(\lambda s^b;\cdot) \leqq (<) \lambda g(s^b;\cdot) \Leftrightarrow \frac{g(\lambda s^b;\cdot)}{\lambda s^b} \leqq (<) \frac{g(s^b;\cdot)}{s^b} \Leftrightarrow \frac{g(s^a;\cdot)}{s^a} \leqq (<) \frac{g(s^b;\cdot)}{s^b}$$

$\Leftrightarrow g(s;\cdot)/s$ is weakly (strictly) increasing in s.

The proofs for the rest of the proposition are similar. □

The following lemma contains standard results that are useful for the proof of Proposition 2. They are stated without proofs.

Lemma A.1. *Let* $\theta>1$. (i) $\theta\mathfrak{I} \subseteq \mathfrak{I} \Leftrightarrow \mathfrak{I} \subseteq \lambda\mathfrak{I}$ *for* $0<\lambda<1$. (ii) $\mathfrak{I} \subseteq \theta\mathfrak{I} \Leftrightarrow \lambda\mathfrak{I} \subseteq \mathfrak{I}$ *for* $0<\lambda<1$.

Proof of Proposition 2.

a. (if) Suppose $\theta\mathfrak{I} \subseteq \mathfrak{I}$ for $\theta>1$. Let $(\boldsymbol{v}^0,\boldsymbol{u}^0)$ be any productive inout unit mix in $\mathfrak{I}$. We want to show that $g(\theta s;\boldsymbol{v}^0,\boldsymbol{u}^0) \geqq \theta g(s;\boldsymbol{v}^0,\boldsymbol{u}^0)$ for any productive input factor s. Multiplying s by θ in (1′) and manipulating terms,

$$\begin{aligned} g(\theta s;\boldsymbol{v}^0,\boldsymbol{u}^0) &= \max_{t/\theta}\left\{\theta\frac{t}{\theta} : \left(s\boldsymbol{v}^0, \frac{t}{\theta}\boldsymbol{u}^0\right) \in \frac{1}{\theta}\mathfrak{I}\right\} \\ &= \theta \max_{\gamma}\left\{\gamma : (s\boldsymbol{v}^0, \gamma\boldsymbol{u}^0) \in \frac{1}{\theta}\mathfrak{I}\right\}, \end{aligned} \tag{A.2}$$

where $\gamma=t/\theta$. Because $\theta>1$, we have $0<1/\theta<1$. By the hypothesis of (if) and Lemma A.1(i), $\mathfrak{I} \subseteq \frac{1}{\theta}\mathfrak{I}$. Hence,

$$\max_{\gamma}\left\{\gamma : (s\boldsymbol{v}^0, \gamma\boldsymbol{u}^0) \in \frac{1}{\theta}\mathfrak{I}\right\} \geqq \max_{\gamma}\{\gamma : (s\boldsymbol{v}^0, \gamma\boldsymbol{u}^0) \in \mathfrak{I}\} = g(s;\boldsymbol{v}^0,\boldsymbol{u}^0). \tag{A.3}$$

Combining (A.2) and (A.3),

$$g(\theta s;\boldsymbol{v}^0,\boldsymbol{u}^0) \geqq \theta g(s;\boldsymbol{v}^0,\boldsymbol{u}^0).$$

(only if) Conversely, suppose for any given productive inout unit mix $(\boldsymbol{v}^0,\boldsymbol{u}^0)$ in $\mathfrak{I}$, we have $g(\theta s;\boldsymbol{v}^0,\boldsymbol{u}^0) \geqq \theta g(s;\boldsymbol{v}^0,\boldsymbol{u}^0)$ for $\theta>1$, where s is any feasible input factor. We want to show that $\theta\mathfrak{I} \subseteq \mathfrak{I}$.

Let $(\boldsymbol{x}',\boldsymbol{q}') \in \theta\mathfrak{I}$ for some $\theta>1$. Then, there exists $(\boldsymbol{x},\boldsymbol{q}) \in \mathfrak{I}$ such that $(\boldsymbol{x}',\boldsymbol{q}')=(\theta\boldsymbol{x},\theta\boldsymbol{q})$. This means $(\boldsymbol{x},\boldsymbol{q})=(\boldsymbol{x}'/\theta,\boldsymbol{q}'/\theta)$. Let $(\boldsymbol{x},\boldsymbol{q})=(s^0\boldsymbol{v}^0,t^0\boldsymbol{u}^0)$ for some feasible input and output factors. Let $s=s^0$ in the hypothesis of (only if). Then,

$$g(\theta s^0;\boldsymbol{v}^0,\boldsymbol{u}^0) \geqq \theta g(s^0;\boldsymbol{v}^0,\boldsymbol{u}^0). \tag{A.4}$$

By Lemma 1(a), $g(s^0;\boldsymbol{v}^0,\boldsymbol{u}^0) \geqq t^0$. Let $t^*=g(\theta s^0;\boldsymbol{v}^0,\boldsymbol{u}^0)$. Then, (A.4) becomes

$$\frac{t^*}{\theta} = \frac{g(\theta s^0;\boldsymbol{v}^0,\boldsymbol{u}^0)}{\theta} \geqq g(s^0;\boldsymbol{v}^0,\boldsymbol{u}^0) \geqq t^0. \tag{A.5}$$

By definition, $t^*=g(\theta s^0;\boldsymbol{v}^0,\boldsymbol{u}^0)$ means $(\theta s^0\boldsymbol{v}^0,t^*\boldsymbol{u}^0)=\left(\theta\boldsymbol{x},\frac{t^*}{t^0}\boldsymbol{q}\right)$ is in $\mathfrak{I}$. Hence,

$$\left(\theta\boldsymbol{x},\frac{t^*}{t^0}\boldsymbol{q}\right) \in \mathfrak{I} \Leftrightarrow \left(\theta\frac{\boldsymbol{x}'}{\theta},\frac{t^*}{t^0}\frac{\boldsymbol{q}'}{\theta}\right) \in \mathfrak{I} \Leftrightarrow \left(\boldsymbol{x}',\frac{t^*}{\theta t^0}\boldsymbol{q}'\right) \in \mathfrak{I}. \tag{A.6}$$

By (A.5), $\frac{t^*}{\theta t^0}\boldsymbol{q}' \geqq \boldsymbol{q}'$. By strong disposability of outputs, the right-hand side of (A.6) implies $(\boldsymbol{x}',\boldsymbol{q}') \in \mathfrak{I}$. Therefore, $\theta\mathfrak{I} \subseteq \mathfrak{I}$.

The proofs for the rest of the proposition are similar. □

Appendix B. Proofs for Section 3

Proof of Proposition 4. From the definition of $\mathfrak{I}^s, s=crs, ndrs, nirs$,

$$\mathfrak{I}^{crs} = \bigcup_{\alpha>0}\alpha\mathfrak{I} = \bigcup_{0<\alpha\leqq 1}\alpha\mathfrak{I} \cup \bigcup_{\alpha\geqq 1}\alpha\mathfrak{I} = \mathfrak{I}^{ndrs} \cup \mathfrak{I}^{nirs}. \quad \square$$

Proof of Proposition 5. By Lemma 2(b), $\mathfrak{I} \subseteq \mathfrak{I}^{ndrs} \cap \mathfrak{I}^{nirs}$. We just need to show that $\mathfrak{I}^{ndrs} \cap \mathfrak{I}^{nirs} \subseteq \mathfrak{I}$.

Let $(\boldsymbol{x},\boldsymbol{q}) \in \mathfrak{I}^{ndrs} \cap \mathfrak{I}^{nirs}$, where $\boldsymbol{x}$ and $\boldsymbol{q}$ are nonzero vectors. By the definition of extensions, there exist $\alpha \geqq 1$ and $\alpha' \in [0,1]$ such that (i) $(\boldsymbol{x},\boldsymbol{q})=(\alpha\boldsymbol{v},\alpha\boldsymbol{u})$ for some $(\boldsymbol{v},\boldsymbol{u}) \in \mathfrak{I}$ and (ii) $(\boldsymbol{x},\boldsymbol{q})=(\alpha'\boldsymbol{v}',\alpha'\boldsymbol{u}')$ for some $(\boldsymbol{v}',\boldsymbol{u}') \in \mathfrak{I}$. If either $\alpha=1$ or $\alpha'=1$, then $(\boldsymbol{x},\boldsymbol{q}) \in \mathfrak{I}$, and we are done. So, suppose $\alpha>1$ and $\alpha'<1$.

Let $(\boldsymbol{x},\boldsymbol{q})=(s^0\boldsymbol{v}^0,t^0\boldsymbol{u}^0)=(\alpha\boldsymbol{v},\alpha\boldsymbol{u})$, where $(\boldsymbol{v}^0,\boldsymbol{u}^0)$ is a feasible inout mix in $\mathfrak{I}$ and $s^0>0$. Consider the ray production function $g(s;\boldsymbol{v}^0,\boldsymbol{u}^0) := \max_{t>0}\{t : (s\boldsymbol{v}^0,t\boldsymbol{u}^0) \in \mathfrak{I}\}$. Under RVRS, there exist $s^*>0$ such that the conditions in Definition 2 are satisfied with respect to $(\boldsymbol{v}^0,\boldsymbol{u}^0)$. There are two exhaustive and mutually exclusive cases.

Case B.1. $s^0 \leqq s^*$.

Choose $s=s^0/\alpha<s^0$ and $\lambda=\alpha=s^0/s>1$. Under the hypothesis of $s^0 \leqq s^*$, $0<s<s^*$ and $1<\lambda \leqq s^*/s$. Lemma 3(a) is applicable. Thus, $g(s^0;\boldsymbol{v}^0,\boldsymbol{u}^0)=g(\lambda s;\boldsymbol{v}^0,\boldsymbol{u}^0) \geqq \lambda g(s;\boldsymbol{v}^0,\boldsymbol{u}^0)=\alpha g(s;\boldsymbol{v}^0,\boldsymbol{u}^0)$. Using the relations between $(\boldsymbol{v}^0,\boldsymbol{u}^0)$ and $(\boldsymbol{v},\boldsymbol{u})$, we have $g(s;\boldsymbol{v}^0,\boldsymbol{u}^0)=\max_{t>0}\{t : \left(\frac{s}{s^0}s^0\boldsymbol{v}^0,\frac{t}{t^0}t^0\boldsymbol{u}^0\right) \in \mathfrak{I}\}=\max_{\alpha t/t^0>0}\{\frac{1}{\alpha}t^0\alpha\frac{t}{t^0} : \left(\frac{s}{s^0}\alpha\boldsymbol{v},\frac{t}{t^0}\alpha\boldsymbol{u}\right) \in \mathfrak{I}\}=\frac{s}{s^0}t^0\max_{t'>0}\{t' : (\boldsymbol{v},t'\boldsymbol{u}) \in \mathfrak{I}\} \geqq \frac{s}{s^0}t^0$, where $t'=\alpha t/t^0$. The last inequality comes from the fact that $\max_{t'>0}\{t' : (\boldsymbol{v},t'\boldsymbol{u}) \in \mathfrak{I}\} \geqq 1$ because $(\boldsymbol{v},\boldsymbol{u}) \in \mathfrak{I}$. Therefore, $g(s^0;\boldsymbol{v}^0,\boldsymbol{u}^0) \geqq \lambda g(s;\boldsymbol{v}^0,\boldsymbol{u}^0) \geqq \alpha\frac{s}{s^0}t^0=t^0$. By Lemma 1(a), $(s^0\boldsymbol{v}^0, g(s^0;\boldsymbol{v}^0,\boldsymbol{u}^0)\boldsymbol{u}^0)=(\boldsymbol{x},g(s^0;\boldsymbol{v}^0,\boldsymbol{u}^0)\boldsymbol{u}^0) \in \mathfrak{I}$. Because $g(s^0;\boldsymbol{v}^0,\boldsymbol{u}^0)\boldsymbol{u}^0 \geqq t^0\boldsymbol{u}^0 = \boldsymbol{y}$, we have $(\boldsymbol{x},\boldsymbol{y}) \in \mathfrak{I}$ by strong disposability of outputs.

Case B.2. $s^*<s^0$.

Let $s=s^0/\alpha'>s^0$ and $\lambda=\alpha'=s^0/s<1$. Then, $s>s^*$ and $s^*/s<\lambda<1$. Lemma 3(b) is applicable. The remaining steps of the proof are similar to those in Case B.1. Again, $(\boldsymbol{x},\boldsymbol{q}) \in \mathfrak{I}$. Hence, Cases B.1 and B.2 imply that $\mathfrak{I}^{ndrs} \cap \mathfrak{I}^{nirs} \subseteq \mathfrak{I}$.

Combining the above results, we get $\mathfrak{I}=\mathfrak{I}^{ndrs} \cap \mathfrak{I}^{nirs}$. The proofs for other returns to scale are essentially the same and are skipped here. □

Appendix C. Proofs for Section 4

Proof of Lemma 4. Because both Ψ^{ndrs} and Ψ^{nirs} contain T^0, their intersection must contain T^0 too. Hence, Ψ^{int} is an empirical technology set.

a. Let $(\boldsymbol{x},\boldsymbol{q}) \in \Psi^{int}$ and $(-\boldsymbol{x}',\boldsymbol{q}') \leqq (-\boldsymbol{x},\boldsymbol{q})$. Because $(\boldsymbol{x},\boldsymbol{q})$ is in both Ψ^{ndrs} and Ψ^{nirs}, $(\boldsymbol{x}',\boldsymbol{q}')$ is also in these two sets by the assumption of strong disposability. It follows that $(\boldsymbol{x}',\boldsymbol{q}') \in \Psi^{int}$.

b. The input set for any output vector $\boldsymbol{q}$ in Ψ^{int} can be written as $\mathcal{L}^{int}(\boldsymbol{q})=\mathcal{L}^{ndrs}(\boldsymbol{q}) \cap \mathcal{L}^{nirs}(\boldsymbol{q})$. Under the maintained assumption, $\mathcal{L}^{rts}(\boldsymbol{q})$ is closed, $rts = ndrs, nirs$. Thus, $\mathcal{L}^{int}(\boldsymbol{q})$, which is the intersection of two closed sets, is closed. □

Proof of Lemma 5.

a. By Lemma 4(a), inputs are strongly disposable in all three technologies. By Lemma 1(a), the ray function $g^{int}(s;\cdot)$ is weakly increasing, $rts = ndrs, nirs, int$.

b. By assumptions and Lemma 4(b), the input sets of Ψ^{rts} for all rts are closed. Lemma 1(d) states that the corresponding ray production functions are continuous from above. By the properties of continuous functions, $g^{int}(s;\cdot)/s$ is also continuous from above in $s > 0$. □

Given that Ψ^{ndrs} and Ψ^{nirs} exhibit nondecreasing and nonincreasing returns to scale, Definition 1 and Proposition 1 imply for $t > 0$, $0 < \lambda < 1$, and $\delta > 1$, the following inequalities.

$$g^{ndrs}(\lambda t;(\boldsymbol{x}^0,\boldsymbol{q}^0)) \leqq \lambda g^{ndrs}(t;(\boldsymbol{x}^0,\boldsymbol{q}^0)). \tag{C.1}$$

$$g^{nirs}(\lambda t;(\boldsymbol{x}^0,\boldsymbol{q}^0)) \geqq \lambda g^{nirs}(t;(\boldsymbol{x}^0,\boldsymbol{q}^0)). \tag{C.2}$$

$$g^{ndrs}(\delta t;(\boldsymbol{x}^0,\boldsymbol{q}^0)) \geqq \delta g^{ndrs}(t;(\boldsymbol{x}^0,\boldsymbol{q}^0)). \tag{C.3}$$

$$g^{nirs}(\delta t;(\boldsymbol{x}^0,\boldsymbol{q}^0)) \leqq \delta g^{nirs}(t;(\boldsymbol{x}^0,\boldsymbol{q}^0)). \tag{C.4}$$

These inequalities are useful in this appendix.

Proof of Lemma 6. Let $s_{min} = \inf A$ and $s_{max} = \sup A$, where A is the solution set of the maximization problem in Lemma 6. We show that A is a closed convex set.

To simplify notations, let $r^{rts}(s;\boldsymbol{x}^0,\boldsymbol{q}^0) = g^{rts}(s;\boldsymbol{x}^0,\boldsymbol{q}^0)/s$, $rts = ndrs, nirs, int$. Thus, the maximization problem becomes $\max_s\{r^{rts}(s;\boldsymbol{x}^0,\boldsymbol{q}^0)\}$.

1. The solution set is not empty on the interval $[s_1, s_2]$.

By Lemma 5(b), $r^{int}(s;.)$ is continuous from above. The interval $[s_1, s_2]$ is a compact set. According to a version of the Weierstrass theorem (see Ali El-Hodiri 1991, p. 5 for an example), the ratio $r^{int}(s;.)$ attains a maximum at $s_0 \in [s_1, s_2]$.

From the intersection condition (Condition 1), $g^{int}(s_1;.) = g^{ndrs}(s_1;.) \leqq g^{nirs}(s_1;.)$. Combining with (C.1) and (C.2), we have $g^{ndrs}(\theta s_1;.) \leqq \theta g^{ndrs}(s_1;.) \leqq \theta g^{nirs}(s_1;.) \leqq g^{nirs}(\theta s_1;.)$ for $0 < \theta < 1$. Note that $0 < \theta s_1 < s_1$. We can conclude that $g^{int}(s;.) = g^{ndrs}(s;.)$ for $0 < s \leqq s_1$. Together with Proposition 1(a), $r^{int}(s;.) = g^{ndrs}(s;.)/s$ is weakly increasing. It follows that $r^{int}(s;.) \leqq r^{rts}(s_1;.)$ for $0 < s \leqq s_1$. Similarly, $r^{int}(s;.) \leqq r^{int}(s_2;.)$ for $s \geqq s_2$. In summary, $\max_{s\in\{s_1,s_2\}}\left\{\frac{g^{int}(s;.)}{s}\right\} = \max_{s>0}\left\{\frac{g^{int}(s;.)}{s}\right\}$. Hence, $s_0 \in A$.

2. The solution set is convex.

If A is a singleton, then $s_0 = s_{min} = s_{max}$, and A is a convex set by definition.

Suppose there are two different values in A: $0 < s' < s''$. Because both are solutions to the maximization problem,

$$r^{int}(s;.) \leqq r^{int}(s';.) = r^{int}(s'';.) \text{ for } s > 0. \tag{C.5}$$

Let $s^\theta = \theta s' + (1-\theta)s''$ for $0 \leqq \theta \leqq 1$. Then, $s_{min} \leqq s' \leqq s^\theta \leqq s'' \leqq s_{max}$. By Proposition 1, (a) and (c), $r^{ndrs}(s;.)$ is weakly increasing, and $r^{nirs}(s;.)$ is weakly decreasing. Thus,

$$r^{ndrs}(s_{min};.) \leqq r^{ndrs}(s';.) \leqq r^{ndrs}(s^\theta;.) \leqq r^{ndrs}(s'';.) \leqq r^{ndrs}(s_{max};.). \tag{C.6}$$

$$r^{nirs}(s_{min};.) \geqq r^{nirs}(s';.) \geqq r^{nirs}(s^\theta;.) \geqq r^{nirs}(s'';.) \geqq r^{nirs}(s_{max};.). \tag{C.7}$$

There are three possibilities.

i. $g^{int}(s';\boldsymbol{x}^0,\boldsymbol{q}^0) = g^{ndrs}(s';\boldsymbol{x}^0,\boldsymbol{q}^0)$ and $g^{int}(s'';\boldsymbol{x}^0,\boldsymbol{q}^0) = g^{ndrs}(s'';\boldsymbol{x}^0,\boldsymbol{q}^0)$.

Using (C.6) and the hypotheses of (i),

$$r^{int}(s';.) \leqq r^{ndrs}(s^\theta;.) \leqq r^{int}(s'';.). \tag{C.8}$$

From the monotonic property of $g^{rts}(s;.)/s$, $rts = ndrs, nirs$,

$$r^{nirs}(s^\theta;.) \geqq r^{nirs}(s'';.) \geqq r^{ndrs}(s'';.) \geqq r^{ndrs}(s^\theta;.). \tag{C.9}$$

Using (C.8) and recalling that $s', s'' \in A$, the inequalities in (C.8) are not strict. Expression (C.9) implies that $g^{nirs}(s^\theta;.) \geqq g^{ndrs}(s^\theta;.)$ after cancelling out the denominators of the first and last terms. Thus, $g^{int}(s^\theta;.) = g^{ndrs}(s^\theta;.)$, and hence,

$$r^{int}(s';.) = \frac{g^{int}(s';.)}{s'} = \frac{g^{int}(s^\theta;.)}{s^\theta} = \frac{g^{int}(s'';.)}{s''} = r^{int}(s'';.).$$

Therefore, $s^\theta \in A$. This shows that A is a convex set.

ii. $g^{int}(s';\boldsymbol{x}^0,\boldsymbol{q}^0) = g^{ndrs}(s';\boldsymbol{x}^0,\boldsymbol{q}^0)$ and $g^{int}(s'';\boldsymbol{x}^0,\boldsymbol{q}^0) = g^{nirs}(s'';\boldsymbol{x}^0,\boldsymbol{q}^0)$.

Combine (C.5), (C.6), and (C.7).

From (C.6), $r^{ndrs}(s^\theta;.) \geqq r^{ndrs}(s';.) = r^{int}(s';.)$. From (C.7), $r^{int}(s'';.) = r^{nirs}(s'';.) \leqq r^{nirs}(s^\theta;.)$. From (C.5), $r^{int}(s';.) = r^{int}(s'';.)$. Combining these three expressions, we get

$$r^{ndrs}(s^\theta;.) \geqq r^{int}(s';.) \leqq r^{nirs}(s^\theta;.). \tag{C.10}$$

By the definition of $g^{int}(s^\theta;\boldsymbol{x},\boldsymbol{q})$ and the fact that $s' \in A$,

$$r^{int}(s^\theta;.) = \frac{g^{int}(s^\theta;.)}{s^\theta} = \frac{\min\{g^{ndrs}(s^\theta;.), g^{nirs}(s^\theta;.)\}}{s^\theta} \leqq \frac{g^{int}(s';.)}{s'}. \tag{C.11}$$

Inequalities (C.10) and (C.11) together imply $s^\theta \in A$.

iii. $g^{int}(s';\boldsymbol{x}^0,\boldsymbol{q}^0) = g^{nirs}(s';\boldsymbol{x}^0,\boldsymbol{q}^0)$ and $g^{int}(s'';\boldsymbol{x}^0,\boldsymbol{q}^0) = g^{nirs}(s'';\boldsymbol{x}^0,\boldsymbol{q}^0)$.

Using the strategy of proving (i), we can derive that $s^\theta \in A$.

Hence, $s^\theta \in A$ for $0 \leqq \theta \leqq 1$. This means that A is a convex set and hence, is an interval if A is not a singleton.

3. The solution set is closed.

From previous discussion, $g^{int}(s;\boldsymbol{x}^0,\boldsymbol{q}^0)/s$ is continuous from above for $s > 0$, and the solution set is convex. This means that when s is sufficiently close to s_{min} from the right, this ratio is the maximum. Let $r_{max} = g^{int}(s;\boldsymbol{x}^0,\boldsymbol{q}^0)/s$ for $s \in A$. If $g^{int}(s_{min};\boldsymbol{x}^0,\boldsymbol{q}^0)/s_{min} = r_{max}$, then $s_{min} \in A$, and we are done. Now, suppose $g^{int}(s_{min};.)/s_{min} < r_{max}$. Let $\tau = r_{max} - g^{int}(s_{min};.)\ /s_{min} > 0$. For $s > s_{min}$,

$$r_{max} + s_{min} - s < \frac{g^{int}(s;.)}{s} < r_{max} + s - s_{min}$$

Taking limits,

$$\lim_{\substack{s\to s_{min}^+\\ s\in A}} (r_{max} + s - s_{min}) \leqq \lim_{\substack{s\to s_{min}^+\\ s\in A}} \frac{g^{int}(s;.)}{s} \leqq \lim_{\substack{s\to s_{min}^+\\ s\in A}} (r_{max} + s - s_{min}).$$

$$r_{max} \leqq \lim_{\substack{s\to s_{min}^+\\ s\in A}} \frac{g^{int}(s;.)}{s} \leqq r_{max}.$$

By the squeeze theorem, $\lim_{\substack{s\to s_{min}^+\\ s\in A}} \frac{g^{int}(s;.)}{s} = r_{max}$. From previous discussions, $g^{int}(s;.)/s$ is continuous from above for $s > 0$. So,

$\lim_{\substack{s\to s_{min}^+ \\ s\in A}} \frac{g^{int}(s;.)}{s} = \frac{g^{int}(s_{min};.)}{s_{min}}$. Hence,

$$\frac{g^{int}(s_{min};.)}{s_{min}} = r_{max}.$$

That is, $s_{min} \in A$.

Next, we show that $s_{max} \in A$. Suppose not. Then, $g^{int}(s_{max};.)/s_{max} < g^{int}(s_{min};.)/s_{min}$. Because the ray function is weakly increasing in s, then $g^{int}(s;.) \leqq g^{int}(s_{max};.)$ for $0 < s < s_{max}$. It follows that $\lim_{s\to s_{max}^-} g^{int}(s;.) \leqq g^{int}(s_{max};.)$. Using a strategy like that in the previous paragraph, one can derive that

$$\lim_{s\to s_{max}^-} \frac{g^{int}(s;.)}{s} = \frac{g^{int}(s_{min};.)}{s_{min}} \leqq \frac{g^{int}(s_{max};.)}{s_{max}}.$$

Suppose the inequality is strict. Then,

$$\frac{g^{int}(s_{min};\boldsymbol{x},\boldsymbol{q})}{s_{min}} < \frac{g^{int}(s_{max};\boldsymbol{x},\boldsymbol{q})}{s_{max}}.$$

This contradicts the assumption that $s_{min} \in A$. Thus,

$$\frac{g^{int}(s_{min};\boldsymbol{x},\boldsymbol{q})}{s_{min}} = \frac{g^{int}(s_{max};\boldsymbol{x},\boldsymbol{q})}{s_{max}}.$$

Therefore, $s_{max} \in A$. In conclusion, the solution set is the interval $[s_{min}, s_{max}] \subseteq [s', s'']$. □

Proof of Proposition 6. To show the existence of RVRS, we need to find the number $s^* > 0$ such that Definition 2 is satisfied.

Let $\boldsymbol{x}^0 \in \mathbb{R}_+^N$ and $\boldsymbol{q}^0 \in \mathbb{R}_+^M$ be any arbitrary nonzero input and output vectors. From the intersection condition (Condition 1(i)) and the definition of Ψ^{int}, we have $g^{int}(s_1;\boldsymbol{x}^0,\boldsymbol{q}^0) = g^{ndrs}(s_1;\boldsymbol{x}^0,\boldsymbol{q}^0)$ and $g^{int}(s_2;\boldsymbol{x}^0,\boldsymbol{q}^0) = g^{nirs}(s_2;\boldsymbol{x}^0,\boldsymbol{q}^0)$. Further, there exist $\boldsymbol{v} \geq \boldsymbol{0}_N$ and $\boldsymbol{u} \geq \boldsymbol{0}_M$ such that $(\boldsymbol{v},\boldsymbol{u}) \in \Psi^{int}$; otherwise, (i) cannot be true. By Lemma 6, the maximum of $r^{int}(s;\boldsymbol{x}^0,\boldsymbol{q}^0) = g^{int}(s;\boldsymbol{x}^0,\boldsymbol{q}^0)/s$ exists with a closed convex solution set: $[s_{min}, s_{max}]$.

The following relations immediately come from the definitions of s_{min} and s_{max}.

$$\frac{g^{int}(s_{min};.)}{s_{min}} = \frac{g^{int}(s;.)}{s} \geqq \frac{g^{int}(t;.)}{t} \text{ for } t>0 \text{ and } s_{min} \leqq s \leqq s_{max}. \quad (C.12)$$

$$\frac{g^{int}(s_{min};.)}{s_{min}} > \frac{g^{int}(\lambda s_{min};.)}{\lambda s_{min}} \text{ for } 0<\lambda<1. \quad (C.13)$$

$$\frac{g^{int}(s_{max};.)}{s_{max}} > \frac{g^{int}(\theta s_{max};.)}{\theta s_{max}} \text{ for } \theta>1. \quad (C.14)$$

Choose $s^* = s_{min}$. Now, we are ready to check the conditions of Definition 2.

Part 1. We show that the condition of Definition 2(a) is satisfied.

First, consider $0 < s < s^*$. Let $\lambda = s/s^* < 1$. If $g^{ndrs}(s^*;.) \leqq g^{nirs}(s^*;.)$, then $g^{int}(s^*;.) = g^{ndrs}(s^*;.)$. Using (C.1) and (C.2),

$$g^{ndrs}(\lambda s^*;.) \leqq \lambda g^{ndrs}(s^*;.) \leqq \lambda g^{nirs}(s^*;.) \leqq g^{nirs}(\lambda s^*;.).$$

Recall $\lambda s^* = s$. This means $g^{int}(s;.) = g^{ndrs}(s;.)$ for $0<s<s^*$.

On the other hand, suppose $g^{ndrs}(s^*;.) > g^{nirs}(s^*;.)$. Combining with (C.2), $\lambda g^{int}(s^*;.) = \lambda g^{nirs}(s^*;.) \leqq g^{nirs}(\lambda s^*;.)$. Dividing by λs^*, $\frac{g^{int}(s^*;.)}{s^*} \leqq \frac{g^{nirs}(\lambda s^*;.)}{\lambda s^*}$. If $g^{int}(\lambda s^*;.) = g^{nirs}(\lambda s^*;.)$, then $s^* = s_{min}$ cannot be an infimum in Lemma 6, a contradiction. Thus, $g^{int}(s;.) = g^{ndrs}(s;.)$.

In summary, $g^{int}(s^*;.) = g^{ndrs}(s^*;.)$ or $g^{nirs}(s^*;.)$. However, $g^{int}(s;.) = g^{ndrs}(s;.)$ for $0 < s < s^*$. Expression (C.1) implies that $g^{int}(\lambda s;.) \leqq \lambda g^{int}(s;.)$ for $0 < \lambda < 1$. The first condition of Definition 2(a) is satisfied.

Next, by the definition of s^* and (C.13),

$$r^{int}(\lambda s^*;.) = \frac{g^{int}(\lambda s^*;.)}{\lambda s^*} < \frac{g^{int}(s^*;.)}{s^*} = r^{int}(s^*;.).$$

This implies that $g^{int}(\lambda s^*;.) < \lambda g^{int}(s^*;.)$. The second condition of Definition 2(a) is satisfied.

Therefore, all conditions of Definition 2(a) are satisfied.

Part 2. We show that the conditions of Definition 2(b) are satisfied.

Consider $s > s^*$. Let $\theta = s/s^* > 1$. When $s = \theta s^* \leqq s_{max}$,

$$\frac{g^{int}(s^*;.)}{s^*} = \frac{g^{int}(s;.)}{s}$$

by (C.12).

When $s = \theta s^* > s_{max}$, the proof of the first part of Definition 2(b) is similar to the proof of the first part of Definition 2(a).

To prove the second part of Definition 2(b), recall from the intersection condition (Condition 1(ii)) that $g^{nirs}(\tau s;.) < \tau g^{nirs}(s;.)$ for some $\tau > 1$. Similar to the proof in Part 1, it can be shown that $g^{int}(s;.) = g^{nirs}(s;.)$. Therefore, for all $s > s^*$,

$$g^{int}(\tau s;.) < \tau g^{int}(s;.) \text{ for some } \tau > 1.$$

The second condition of Definition 2(b) is satisfied.

In conclusion, all conditions of Definition 2 are satisfied. The technology set Ψ^{int} exhibits RVRS. □

Appendix D. Proofs for Section 5

Proof of Proposition 7. It can be verified that Condition 1(ii) is always satisfied by T^{int}. Because Condition 1(i) is assumed in Proposition 7, both properties of Condition 1 are satisfied. Hence, T^{int} exhibits RVRS with strongly disposable inputs and outputs according to Proposition 6. Let T^* be the required minimal extrapolation. By definition of minimal extrapolation, $T^* \subseteq T^{int}$. By construction, $T^0 \subseteq T^{int}$. It remains to show that $T^{int} \subseteq T^*$.

Define the NDRS and NIRS extensions of T^*: $T^{*ndrs} := \{(x,q) : (x,q) = (\theta x', \theta q'),\ (x',q') \in T^*,\ \theta \geqq 1\}$ and $T^{*nirs} := \{(x,q) : (x,q) = (\theta x', \theta q'),\ (x',q') \in T^*,\ \theta \in [0,1]\}$. By Proposition 5, $T^* = T^{*ndrs} \cap T^{*nirs}$. By Lemma 4, for $rts = ndrs, nirs$, $T^{fdh-rts}$ is the minimal extrapolation for the relevant set of regularity conditions. Thus, $T^{fdh-rts} \subseteq T^{*rts}$. Hence, $T^{int} = T^{fdh-ndrs} \cap T^{fdh-nirs} \subseteq T^{*ndrs} \cap T^{*nirs} = T^*$. Therefore, $T^{int} \subseteq T^*$. Combining the above discussions, we have $T^{int} = T^*$. □

Proof of Proposition 8. The first inequality of (6) follows from Proposition 7 that $T^{fdh-rvrs}$ is the minimum extrapolation of the observed data set under strong disposability and RVRS. Observing (3) and the definitions of $T^{fdh-ndrs}$ and $T^{fdh-nirs}$, we have $T^{fdh} \subseteq T^{fdh-ndrs}$ and $T^{fdh} \subseteq T^{fdh-nirs}$. Thus, $T^{fdh} \subseteq T^{fdh-ndrs} \cap T^{fdh-nirs} = T^{fdh-rvrs}$. The second inequality of (6) follows immediately. □

Proof of Proposition 9. Figure 4 illustrates one possibility of (7) in which the production frontier is S shaped as in Figure 1(a). Five firms (A, B, C, D, and E) are observed. From this diagram, $te_i^*(\boldsymbol{x}^D, \boldsymbol{q}^D) = 1$, $te_i^{cv-rvrs}(\boldsymbol{x}^D, \boldsymbol{q}^D) < 1$. Thus, $te_i^*(\boldsymbol{x}^D, \boldsymbol{q}^D) > te_i^{cv-rvrs}(\boldsymbol{x}^D, \boldsymbol{q}^D)$. □

Appendix E. Pseudocodes for Computing Technical Efficiency Without Convexity

Pseudocodes for computing input-oriented technical efficiency in the FDH-NDRS frontier are

$$te_i^{fdh-ndrs}(x^j, q^j) = \min_{\lambda}\{\lambda : \delta q^k \geqq q^j, \delta x^k \leqq \lambda x^j, k = 1, \ldots, K,\ \delta \geqq 1\}.$$

For computation, start with $k = 0$.

i. $k = k + 1$.

ii. If $\max_m\{q_{mj}/q_{mk}\} \geqq 1$, then $\delta = \max_m\{q_{mj}/q_{mk}\}$. Otherwise, $\delta = 1$.

$$\lambda^k = \max_n\{\delta x_{nk}/x_{nj}\}.$$

iii. If $k < K$, go back to (i). Otherwise, go to the next step.

iv. The technical efficiency score for firm j is

$$te_i^{fdh-ndrs}(x^j, q^j) = \min_{k=1,\ldots,K}\{\lambda^k\}.$$

Pseudocodes for computing input-oriented technical efficiency in FDH-NIRS frontier are

$$te_i^{fdh-nirs}(x^j, q^j) = \min_{\lambda}\{\lambda : \delta q^k \geqq q^j, \delta x^k \leqq \lambda x^j, k = 1, \ldots, K,$$
$$0 \leqq \delta \leqq 1\}.$$

For computation, start with $k = 0$.

i. $k = k + 1$.

ii. If $\max_m\{q_{mj}/q_{mk}\} \leqq 1$, then $\delta = \max_m\{q_{mj}/q_{mk}\}$; else, $\delta = 999$.

iii. If $\delta \leqq 1$, $\lambda^k = \max_n\{\delta x_{nk}/x_{nj}\}$. Otherwise, $\lambda^k = 999$.

iv. If $k < K$, go back to (i). Otherwise, go to the next step.

v. The technical efficiency score for firm j is

$$te_i^{fdh-ndrs}(x^j, q^j) = \min_{k=1,\ldots,K}\{\lambda^k\}.$$

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