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Distribution of the Mean Queue Size for the Time-Dependent Queue

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This paper derives the Laplace-Stieltjes transform of the mean queue size for the time-dependent $GI/G/1$ queue, with an arrival at $t=0$. It also obtains the probability of the queue being empty at time t. These results are based on the departure process for this queue.

WE ASSUME that customers $C_0, C_1, \cdots$ arrive at times $T_0=0, T_1, \cdots$, respectively. Suppose that $t_n=T_{n+1}-T_n$ is the interarrival interval between C_n and C_{n+1}, so that $T_n=\sum_{j=0}^{n-1} t_j$. We assume further that customers are served in the order of arrival, that is, with the first-come, first-served discipline. Let the durations of service of $C_0, C_1, \cdots$, be $s_0, s_1, \cdots$, respectively. We also make the usual independence assumptions for the $GI/G/1$ queue.

Write $P(t_1\leqq x)=A(x)$ and $P(s_1\leqq x)=B(x)$, with Laplace-Stieltjes transforms denoted by

$$\alpha(\theta)=E(e^{-\theta t_1})=\int_0^\infty e^{-\theta t}\,dA(t)$$

and

$$\beta(\theta)=E(e^{-\theta s_1})=\int_0^\infty e^{-\theta t}\,dB(t).$$

These integrals converge at least for $\mathrm{Re}(\theta)\geqq 0$.

THE EMPTINESS OF THE QUEUE

ASSUME THAT $m(t)$ is the number of departures in $(0, t]$ and write $U(t)=Em(t)$. Denote by $G(t)$ and $F(t)$ the distributions of the busy period and busy cycle, respectively. Let the Laplace-Stieltjes transforms of $U(t)$, $G(t)$, and $F(t)$ be denoted by $u(\theta)$, $g(\theta)$, and $f(\theta)$, respectively.

It is shown in equation (3.7) of ALI[1] that

$$u(\theta)=\{\beta(\theta)/[1-\beta(\theta)]\}\{[1-g(\theta)]/[1-f(\theta)]\}. \tag{1}$$

We can interpret (1) formally as follows. If the queue is such that the server is never idle in $(0, t]$, then all the interdeparture intervals are just the service times, and hence $u(\theta)$ is given by $h(\theta)=\beta(\theta)/[1-\beta(\theta)]$. Thus, we may say that the other factor $\Phi(\theta)=[1-g(\theta)]/[1-f(\theta)]$ allows, in some way, for the possibility of the queue being empty in the interval $(0, t]$. Therefore, it is not surprising that $\Phi(\theta)$ can be expressed in terms of the emptiness of the queue.

Suppose $E_0=T_0=0, E_1, E_2, \cdots$, are the epochs of commencement of successive busy periods. It is well known that these epochs $(E_0, E_1, \cdots)$ form a renewal process, for which $e_r=E_{r+1}-E_r$ $(r=0, 1, \cdots)$ are the busy cycles. Let $X(t)$ be the expected number of renewals in $(0, t]$. Then $X(t)=\sum_{n=1}^{n=\infty} P(E_n\leqq t)$ and, by renewal-theory arguments,

$$\int_0^\infty e^{-\theta t}\, dX(t)=f(\theta)/[1-f(\theta)].$$

Let $P_0(t)$ be the probability that the queue is empty at time t, $P_0(0)=0$. Hence, $1-P_0(t)$ is the probability that the server is busy at time t, and this event can happen in the following mutually exclusive ways: (i) the initial busy period is still in progress at time t, (ii) the last customer to find the server idle is C_n $(n=1, 2, \cdots)$, who arrives at the instant u, where $0<u<t$, and the subsequent busy period, which starts at time u, extends beyond time t. By our assumptions, the initial busy period has the same distribution as that of the subsequent ones. Hence, by the theorem of total probability,

$$\begin{aligned}1-P_0(t)&=1-G(t)+\sum_{n=1}^{\infty}\int_0^t dP(E_n\leqq u)[1-G(t-u)]\\&=1-G(t)+\int_0^t[1-G(t-u)]\, dX(u).\end{aligned}$$

If we now take ordinary Laplace transforms, we get

$$\int_0^\infty e^{-\theta t}[1-P_0(t)]\, dt=\{[1-g(\theta)]/\theta\}\{1+f(\theta)/[1-f(\theta)]\}=(1/\theta)[1-g(\theta)]/[1-f(\theta)]$$
$$=\Phi(\theta)/\theta.$$

Hence $\Phi(\theta)=1-\theta r(\theta)$, where

$$r(\theta)=\int_0^\infty e^{-\theta t}P_0(t)\, dt. \tag{2}$$

We can finally state the following theorem.

Theorem 1. *In the $GI/G/1$ queue,*

$$u(\theta)=\{\beta(\theta)/[1-\beta(\theta)]\}[1-\theta r(\theta)] \tag{3}$$

where $r(\theta)$ is defined in (2).

Define $H(t)$ as the renewal function of an ordinary renewal process in which the distribution of lifetimes is that of the service time. Then $h(\theta)$ is the Laplace-Stieltjes transform of $H(t)$. Hence, we can invert (3) to get

$$U(t)=\int_0^t[1-P_0(t-y)]\, dH(y). \tag{4}$$

As a simple application of (4), we consider the queue $GI/M/1$, where, in the usual notation, $dH(y)=\mu\, dy$. Hence

$$U(t)=\mu\int_0^t[1-P_0(y)]\, dy=\mu E\{\text{total occupation time in } (0, t]\},$$

which is a known result.

Again for this queue, $h(\theta)=\mu/\theta$ and from (3) $u(\theta)=\mu[1-\theta r(\theta)]/\theta$, so that $r(\theta)=1/\theta-u(\theta)/\mu$. For this queue, $u(\theta)$ is given in Theorem 3 of Ali,[1] so that

$$r(\theta)=1/\theta-\{1/[1-\alpha(\theta)]\}\{[1-\delta(\theta)]/[\theta+\mu(1-\delta(\theta))]\},$$

where $\delta(\theta)$ is the unique root in $|v|<1$ of $v=\alpha[\theta+\mu(1-v)]$. This last result is given in CONOLLY.[2]

THE MEAN QUEUE SIZE

WE DEFINE THE queue size $q(t)$ as the number of customers waiting or being served at time t. In the $M/M/1$ queue, $q(t)$ is Markov and can be handled straightforwardly. For more general queues, this Markov property does not hold and special techniques, such as the inclusion of supplementary variables and the imbedded Markov chain, have to be employed.

We indicate how the mean queue size for general queues may be obtained easily from our results. Suppose $n(t)$ is the number of arrivals in $[0\ t]$ (we include the initial customer who arrives at $T_0=0$). Then it is clear that $q(t)=n(t)-m(t)$. If we write $Q(t)=Eq(t)$, $N(t)=En(t)$; then, taking expectations, we have

$$Q(t)=N(t)-U(t). \tag{5}$$

Taking Laplace-Stieltjes transforms of (5), we get

$$\int_0^\infty e^{-\theta t}\, dQ(t)=\int_0^\infty e^{-\theta t}\, dN(t)-u(\theta).$$

By renewal-theory arguments, we have $N(t)=\sum_{n=0}^{n=\infty} A_n(t)$, where $A_n(t)$ is the n-fold convolution of $A(t)$ with itself. Hence

$$\int_0^\infty e^{-\theta t}\, dQ(t)=1/[1-\alpha(\theta)]-u(\theta).$$

Two representations for $u(\theta)$ are given in Ali.[1]

These results are collected in the following statement.

THEOREM 2. *In the $GI/G/1$ queue*

$$\int_0^\infty e^{-\theta t}\, dQ(t)=1/[1-\alpha(\theta)]-u(\theta).$$

For the $GI/M/1$ queue, using the formula for $u(\theta)$ given in Ali,[1] we have

$$\int_0^\infty e^{-\theta t}\, dQ(t)=\frac{1}{1-\alpha(\theta)}\,\frac{\theta}{\theta+\mu[1-\delta(\theta)]},$$

where $\delta(\theta)$ is defined as above.

As stated in Ali,[1] $u(\theta)$ can always be evaluated for either arrival or service-time distributions with a rational Laplace-Stieltjes transform. Consequently, a similar remark can be made for the mean queue size.

REFERENCES

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