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Technical Note—On a Paper by Simon

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is found to be greater than the current best solution, a new solution is generated from the exclusive branch. If the new solution is not better than the current best, the branch is simply cut. Bounds on the exclusive branches created during the generation of a 'new solution' are disregarded so that the principles of branch and bound are violated.

Source of the Better Feasible Solution

Research carried out by the writer under the direction of JOHN F. MUTH of the Indiana University Graduate School of Business, *Optimizing Reliability with Backup Components* (D.B.A. dissertation, 1972), provides computer times for a revised version of the Ghare-Taylor algorithm.

The better feasible solution is, in fact, optimal. It was obtained by a version of the Ghare-Taylor algorithm that adhered strictly to branch-and-bound principles.

On a Paper by Simon

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In a paper on "Stationary Properties of a Two-Echelon Inventory Model for Low-Demand Items" [*Opns. Res.* 19, 761–773 (1971)], R. M. Simon presents an argument for finding the stationary distribution of inventory position that is not always valid. This note gives a corrected version of Simon's method.

THIS NOTE assumes the reader is familiar with R. M. SIMON's paper on "Stationary Properties of a Two-Echelon Inventory Model for Low-Demand Items." [1] A counterexample shows that Simon's derivation of the probability distribution of the base inventory position is not completely valid; however, it is correct for the two special cases in which nonbase-reparable failures are either all depot-reparable, or all nondepot-reparable.

Building on Simon's analysis, we derive more complex expression that is always valid. However, to obtain a solution using this expression, a large amount of enumeration is required. No attempt is made to assess the degree to which Simon's result could serve as a useful approximation to this new expression. Numerical approximations are possible.

Notation

We use Simon's basic definitions and notational scheme, which are repeated here with some additions for the reader's convenience.

r_j = the probability that an item failed at base j will be repaired at base j ($j=1, \cdots, J$).

ρ = the probability that a failed carcass that is not base reparable will be depot reparable.

$D_j^B(t)$, $t \geqq 0$ = the stochastic process of units repaired at base j.

$D_j^D(t)$, $t \geqq 0$ = the stochastic process of units sent to the depot from base j for repair.

$D_j^C(t)$, $t \geqq 0$ = the stochastic process of units condemned at base j. All condemnation occurs at bases, never at depots.

$D_j(t) = D_j^C(t) + D_j^D(t)$, $t \geqq 0$.

$D_0^D(t) = \sum_{j=1}^{j=J} D_j^D(t)$, $t \geqq 0$.

$D_0^C(t) = \sum_{j=1}^{j=J} D_j^C(t)$, $t \geqq 0$.

$D_0(t) = D_0^C(t) + D_0^D(t)$, $t \geqq 0$.

$X(t)$ = the depot asset position at time t.

$B_j(t)$ = the number of backorders at base j at time t.

R_j = the deterministic repair time at facility j ($j=0, 1, \cdots, J$; $j=0$ denotes the depot).

t_j = the deterministic delivery time from depot to base j. With no loss in generality, this is assumed to be zero.

t_0 = the deterministic procurement lead time from the exogenous source to the depot. We assume, as does Simon, $R_0 \leqq t_0$.

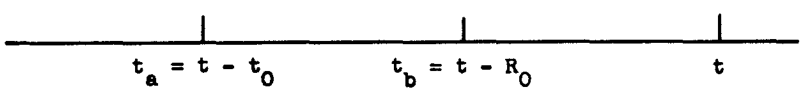

Figure 1

λ_j = total demand rate on facility j, $j=0, 1, \cdots, J$.

$U_j(t)$ = the units in the base j repair cycle plus the units on order from the depot by base j at time t.

$(s_j - 1, s_j)$ = replenishment policy used at base.

s_0, S = replenishment policy used at depot.

$Q = S - s_0$.

$N(t', t) = N(t) - N(t'^{+})$ for any stochastic process, $t \geqq 0$.

$t_a = t - t_0$.

$t_b = t - R_0$.

Lower-case letters are used to denote a particular realization of a random variable. Also, *see* Fig. 1.

COUNTEREXAMPLE

SIMON'S ARGUMENT [reference 1, page 768] for ignoring $D_j^D(t-t_0-t_j, t-R_r-t_j)$ and $D_0^D(t-t_0-t_j, t-R_0-t_j)$ is not strictly correct. While the values of these variables have no effect on the total quantity due out at the depot at time t, they can affect to whom a due out is due. As such, they cannot be ignored in Simon's equation (2) except as an approximation.

For a simple counterexample, suppose

$$X(t-t_0-t_j)=s_0+k, \qquad D_0^C(t-t_0-t_j,\, t-R_0-t_j-t')=s_0+k+1,$$
$$D_0^C(t-R_0-t_j-t',\, t-R_0-t_j)=0, \qquad D_0^D(t-t_0-t_j,\, t-R_0-t_j-t')=0,$$
$$D_0^D(t-R_0-t_j-t',\, t-R_0-t_j)=1.$$

If the last condemnable demand is from base 1, and the depot reparable demand from base 2, a FIFO priority dictates that the due out will be to base 2 and not to base 1, as the model would claim.

A corrected method for obtaining $\mathrm{pr}[B_j(t)=b]$ is presented in the next section.

Basic Approach

We obtain an expression for $P_r[B_j(t)=b]$ that can be evaluated in the limit (as $t\to\infty$). Our approach uses a convenient dichotomy. Noting that total depot assets available to the bases by time t is $x(t_a)+d_0^D(t_a,t_b)$, i.e., depot assets at time t_a plus those returned for repair prior to t_b, then, if $d_0^C(t_a,t_b)\leqq x(t_a)$, demands on the depot in $[t_b, t]$ drawn against an amount $x(t_a)-d_0^C(t_a,t_b)$ determine which are unsatisfied by t. On the other hand, if $d_0^C(t_a,t_b)>x(t_a)$, there is no guarantee that all the $d_0^D(t_a,t_b)$ demands will be satisfied and, hence, total demands on the depot in $[t_a, t]$ drawn against an amount $x(t_a)+d_0^D(t_a,t_b)$ need to be considered.

Analysis

As a consequence of the s_j, s_j-1 base policy, $B_j(t)=b$ if and only if the units in base repair at time t plus the units on order from the depot by base j at time t equal s_j+b. Thus,

$$P_r[B_j(t)=b]=P_r[U_j(t)=s_j+b]. \tag{1}$$

We investigate

$$P_r[U_j(t)=s_j+b \mid X(t_a)=x(t_a),\, D_0^C(t_a,t_b)=d_0^C(t_a,t_b)] \tag{2}$$

for the two mutually exclusive events

$$D_0^C(t_a,t_b)\leqq x(t_a), \tag{2A}$$
$$D_0^C(t_a,t_b)> x(t_a). \tag{2B}$$

In the limit, $x(t_a)$ is uniformly distributed between the depot's reorder point, and reorder up to point.[3] Hence, if (2) can be evaluated, $\lim_{t\to\infty} P_r[U_j(t)=s_j+b]$ can be calculated by enumeration over $D_0^C(t_a,t_b)$.

Case 2A. All demands on the depot occurring in (t_a,t_b) are satisfied by time t, since by assumption $d_0(t_a,t_b)<x(t_a)+d_0^D(t_a,t_b)$. Only the demands on the depot occurring in (t_b^+, t) could possibly be unfilled by t. Also, only the base-reparable demands occurring in $(t-R_j, t)$ are in the base repair cycle at time t.

Let $U_j(t)=U_j^1(t)+U_j^2(t)$, where $U_j^1(t)=$ units in base j repair cycle at time t, and $U_j^2(t)=$ units ordered from the depot by base j in (t_b^+, t) that are unsatisfied by time t. Since $U_j^1(t)$ and $U_j^2(t)$ are independent, $P_r[U_j(t)=s_j+b]$ can be obtained by a convolution of $U_j^1(t)$ and $U_j^2(t)$. $U_j^1(t)$ is a simple Poisson variable with parameter $r_j\lambda_jR_j$, so a convolution can be taken if $P_r[U_j^2(t)=d]$ can be calculated.

Let us concentrate on $U_j^2(t)$. At time t_b^+ there are exactly $x(t_a)-d_0^C(t_a, t_b)>0$ depot assets that are available or will become available to the bases by time t. Suppose we are given $D_0(t_b, t)=d_0(t_b, t)$ and $D_j(t_b, t)=d_j(t_b, t)$. Then $U_j^2(t)=d'$ if and only if $d_j(t_b, t)-d'$ of base j demands can be filled by time t; i.e., $d_j(t_b, t)-d'$ of the first $x(t_a)-d_0^C(t_a, t_b)$ depot demands after time t_b must come from base j. Using Simon's theorem (*see* Appendix A of reference 2) to calculate the probability of this event, we obtain

$$P_r[U_j^2(t)=d'|X(t_a)=x(t_a),\, D_0^C(t_a, t_b)=d_0^C(t_a, t_b),$$
$$D_0(t_b, t)=d_0(t_b, t),\, D_j(t_b, t)=d_j(t_b, t)] \tag{3}$$
$$=\binom{d_j(t_b, t)}{d_j(t_b, t)-d'}\binom{d_0(t_b, t)-d_j(t_b, t)}{x(t_a)-d_0^C(t_a, t_b)-d_j(t_b, t)+d'}\bigg/\binom{d_0(t_b, t)}{x(t_a)-d_0^C(t_a, t_b)},$$

where the probability is calculated as 0 if any denominators are negative.

Now, $x(t_a)$, $D_0^C(t_a, t_b)$, and $D_0(t_b, t)$ are all independent. $D_0^C(t_a, t_b)$ is Poisson with parameter $\mu_0=\sum_{j=1}^{j=J}(1-\rho)(1-r_j)\lambda_j(t_a-t_b)$. $D_j(t_b, t)$ is conditional on $D_0(t_b, t)$ with probability function (*see* reference 2)

$$P_r[D_j(t_b, t)=d_j(t_b, t)|D_0(t_b, t)=d_0(t_b, t)]$$
$$=\binom{d_0(t_b, t)}{d_j(t_b, t)}[(1-r_j)\lambda_j/\lambda_0]^{d_j(t_b, t)}[1-(1-r_j)\lambda_j/\lambda_0]^{d_0(t_b, t)-d_j(t_b, t)},$$

which follows from the Poisson properties.

To find equation (2A), first compute $P_r[U_j^2(t)=d'|X(t_a)=x(t_a), D_0^C(t_a, t_b)=d_0^C(t_a, t_b)]$ from (3) by enumerating over the ranges of $D_0(t_b, t)$ and $D_j(t_b, t)$ such that all values within the combinatorial expressions of (3) are nonnegative. Then find (2A) by convoluting the probability expressions for $U_j^1(t)$ and $U_j^2(t)$.

Case 2B. At time t_b^+, there are no depot assets that can be used to satisfy base demands on the depot after t_b by time t, since $d_0(t_a, t_b)\geqq x(t_a)+d_0^D(t_a, t_b)$. Thus, all depot demands from base j in (t_b, t), as well as base j reparable demands in $t-R_j$, t, are still on order and in the base repair cycle, respectively, at time t.

Let $U_j^1(t)$ = units in base j repair cycle at time t plus units ordered from the depot in (t_b, t) by base j, and $U_j^2(t)$ = depot demands from base j in (t_a, t_b) that are unsatisfied by time t.

With these new definitions of $U_j^1(t)$ and $U_j^2(t)$, $P_r[U_j(t)=s_j+b]$ can still be obtained through convolution. $U_j^1(t)$ is again a Poisson variable, now with parameter $r_j\lambda_jR_j+(1-r_j)\lambda_jR_0$. If we are given $D_0^D(t_a, t_b)=d_0^D(t_a, t_b)$ and $D_j(t_a, t_b)=d_j(t_a, t_b)$, then $U_j^2(t)=d'$ if and only if exactly $d_j(t_a, t_b)-d'$ of base j demands can be filled by time t; i.e., $d_j(t_a, t_b)-d'$ of the first $x(t_a)+d_0^D(t_a, t_b)$ depot demands in (t_a, t_b) must come from base j.

As before,

$$P_r[U_j^2(t)=d'|X(t_a)=x(t_a),\, D_0^C(t_a, t_b)=d_0^C(t_a, t_b),$$
$$D_0^D(t_a, t_b)=d_0^D(t_a, t_b),\, D_j(t_a, t_b)=d_j(t_a, t_b)]$$
$$=\binom{d_j(t_a, t_b)}{d_j(t_a, t_b)-d'}\binom{d_0^D(t_a, t_b)+d_0^C(t_a, t_b)-d_j(t_a, t_b)}{x(t_a)+d_0^D(t_a, t_b)-d_j(t_a, t_b)+d'}\bigg/ \tag{4}$$
$$\binom{d_0^D(t_a, t_b)+d_0^C(t_a, t_b)}{x(t_a)+d_0^D(t_a, t_b)}.$$

Since $P_r[D_0^D(t_a, t_b) = d_0^D(t_a, t_b)]$ is Poisson with parameter $\mu_0 = \sum_{j=1}^{j=J} (1-r_j)\rho\lambda_j(t_b - t_a)$, and, since the conditional probability

$$P_r[D_j(t_a, t_b) = d_j(t_a, t_b) | D_0^C(t_a, t_b) = d_0^C(t_a, t_b),\ D_0^D(t_a, t_b) = d_0^D(t_a, t_b)]$$

is expressed as

$$\binom{d_0^C(t_a, t_b) + d_0^D(t_a, t_b)}{d_j(t_a, t_b)} \cdot [(1-r_j)\lambda_j/\lambda_0]^{d_j(t_a,t_b)}[1-(1-r_j)\lambda_j/\lambda_0],^{d_0^C(t_a,t_b)+d_0^D(t_a,t_b)-d_j(t_a,t_b)}$$

equation 2B can be found by means similar to those used for 2A, ensuring that $D_0^D(t_a, t_b)$ and $D_j^D(t_a, t_b)$ take on only values such that the values within the combinatorial expressions of (4) are nonnegative.

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1. R. M. SIMON, "Stationary Properties of a Two-Echelon Inventory Model for Low-Demand Items," *Opns. Res.* **19**, 761–773 (1971).
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Corrections to a Paper by Hershaft on Decoy Effectiveness

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THE FOLLOWING corrections should be made in a paper by A. HERSHAFT on "The Effectiveness of Imperfect Decoys" [*Opns. Res.* **16**, 10–17(1968)]:

1. In equation (9′) the E in the denominator should be replaced by a D, so that the second line reads

$$\bar{S} \cong R - p_k W p_r R/(p_r R + p_d D). \qquad (9')$$

2. Equation (20), which is derived by transposing (9′), should be

$$R = \bar{S} + p_k W R/[R + (p_d/p_r)D] = \bar{S} + p_k W/[1 + (p_d/p_r)(D/\mathrm{R})]. \qquad (20)$$

3. The error in equation (20) is propagated in (21), (22), (23), and (24). Wherever p_d appears, it should be replaced by p_d/p_r. Thus, the final expression, when corrected, is

$$D/R = (p_r/p_d)\{[(p_k W/\bar{S}c)(p_d/p_r - c)]^{1/2} - 1\}. \qquad (24)$$