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Technical Note—Capacity Expansion in Convex Cost Networks with Uncertain Demand

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2. The PNET code referred to in Reference 3 is approximately 10 percent slower than the PNET-1 code benchmarked in Table I.

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Capacity Expansion in Convex Cost Networks with Uncertain Demand

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The purpose of this investigation is to suggest a simple algorithm for solving capacity expansion problems in networks with uncertain demand. ***We assume that the cost of expanding the capacity of*** **each arc is a convex function and that there is a concave salvage value**

each arc is a convex function and that there is a concave salvage value associated with excess capacity. We adopt the two-stage programming approach but with the essential difference that our independent variable is an assumed first-stage demand. We show that the objective function of the equivalent convex program is a convex function of the assumed first-stage demand and use this fact to propose an algorithm.

IN ANALYZING certain transportation or communication networks for which an increase in the demand for flow of passengers, goods or information is anticipated, it may be necessary to find an optimal method for expanding capacity even though the anticipated demand is not known deterministically.

Consider a directed network (N, A), where N is the set of nodes and A represents the set of arcs. Let s be the source node and t be the sink node. The capacity expansion problem for a convex cost network where we seek to minimize cost while satisfying a demand, v, on the flow from source to sink can be stated as follows:

$$\begin{aligned} \text{minimize} \quad & c(x), \\ Ef = & \{\underline{v}\}, \\ f - x \leqq & b, \\ x \geqq & 0, \end{aligned} \qquad (1)$$

where E is the node-arc matrix of the directed network, x is a vector variable representing increase in arc capacities, b is a known vector representing present arc capacities, f is the flow vector, $c(x)$ is the convex objective function to be minimized, and $\{\underline{v}\}$ represents the vector $\{v, 0, 0, \cdots, 0, -v\}^T$ where v is a random flow demand. We assume that $c(0) \equiv 0$ and that v has a known probability distribution.

Problem (1) is a nonlinear stochastic programming problem for which we need to determine the optimal value of the vector x. We shall adopt the two-stage approach for solving (1). The method is based on the concept that we expand the capacity of each arc (i, j) by an amount x_{ij} (to satisfy an assumed first stage demand, v^1) in the first stage when the demand is unknown; then when the demand v becomes known deterministically, we take corrective action by solving a second-stage problem either to increase capacity by a further amount y_{ij}^+ or reduce capacity by the amount y_{ij}^-. We shall assume that $g_{ij}(x_{ij}, y_{ij}^+)$, the cost of increasing capacity of arc (i, j) in the second stage, is a convex function of both the first-stage expansion, x_{ij}, and the second-stage expansion y_{ij}^+. Also assume that the salvage value associated with excess capacity of arc (i, j) in the second stage,

$h_{ij}(x_{ij}, y_{ij}^-)$, is a concave function of both x_{ij} and y_{ij}^-. g and h are assumed to depend on the first-stage variable x so that we may avoid the possibility of reducing capacity and then increasing capacity of the same arc in the second stage.

We place no bounds on the amount by which the capacity of any arc may be increased. Such bounds can be adequately taken care of by proper choice of the cost functions; e.g., if K_{ij} is the maximum amount by which capacity of arc (i, j) may be increased, assume c_{ij} and g_{ij} are such that $c_{ij}(x_{ij})=\infty$ for $x_{ij}>K_{ij}$ and $g_{ij}(x_{ij}, y_{ij}^+)=\infty$ for $x_{ij}\leqq K_{ij}<x_{ij}+y_{ij}^+$. Also $g(x, y)-h(x, y)>0$ for any x, y.

For simplicity, we modify problem (1) to leave out all existing capacities $(b=0)$ and consider only increases in demand above the present level. Then the two-stage program takes the form:

$$\begin{gathered} \text{minimize} \quad c(x)+E_v\{g(x, y^+)-h(x, y^-)\}, \\ Ef=\{\underline{v}\}, \\ f-x-y^+ + y^- \leqq 0, \\ x, y^+, y^- \geqq 0. \end{gathered} \tag{2}$$

We now break problem (2) into two stages, assuming for the purpose of the first stage that there is a deterministic flow demand v^1. The first-stage problem is

$$\begin{gathered} \text{minimize} \quad c(x), \\ Ef=\{\underline{v}^1\}, \\ f-x\leqq 0, \\ x\geqq 0. \end{gathered} \tag{3}$$

If problem (3) has a feasible solution, then it has an optimal solution $\hat{x}$. The variable x clearly depends on the chosen value of v^1, i.e , $x=x(v^1)$

The second-stage problem is:

$$\begin{gathered} Q\{x(v^1), v\}=[\text{minimize } g\{x(v^1), y^+\}-h\{x(v^1), y^-\}], \\ Ef=\{\underline{v}\}, \\ f-y^+ + y^- \leqq x(v^1), \\ y^+, y^- \geqq 0. \end{gathered} \tag{4}$$

Problem (4) has an optimal feasible solution. The objective function of problem (2) can now be written as $z\{x(v^1)\}=c\{x(v^1)\}+E_v\{Q[x(v^1), v]\}$. Since the value of $E_v\{Q[x(v^1), v]\}$ depends only on $x(v^1)$ and therefore on v^1, it is now obvious that $z\{x(v^1)\}$ is a function of v^1, the assumed first-stage demand.

THE CONVEX PROGRAM

We shall now prove a few results that will be used in our solution algorithm for problem (2).

PROPOSITION 1. *Consider the convex programming problem*

$$\text{minimize} \quad g(x, y^+) - h(x, y^-),$$
$$Ef = \{\underline{v}\},$$
$$f - y^+ + y^- \leqq 0,$$
$$y^+, y^- \geqq 0.$$

Let $g-h>0$ *for each* x *and* y $(y^+=y^-=y)$, *and let the above problem have a unique optimal solution* $\{\hat{y}^+, \hat{y}^-\}$. *Then* $\hat{y}^+$ *is nondecreasing with increasing* v.

Proof. We shall obtain a proof that exploits the network structure of the problem. Let P_1 and P_2 be the least-cost paths for sending the first and second units of flow, respectively, through the directed network. Let the cost of sending one unit of flow $(v=1)$ be $c(P_1)$ and of sending two units of flow $(v=2)$ be $c(P_1)+c(P_2)$. We have $\hat{y}_{ij}^+(v=1)=1$, for arcs $(i, j)\epsilon P_1$, $\hat{y}_{ij}^+(v=1)=0$, otherwise.

We must show $\hat{y}_{ij}^+(v=2)\geqq\hat{y}_{ij}^+(v=1)$ for $(i, j)\epsilon P_1$. Suppose not. Then there are optimal paths P_1^1, P_2^1 such that for at least one arc $(k, l)\epsilon P_1$, we have $(k, l)\notin P_1^1$ or P_2^1. But by optimality of P_1 and uniqueness of $\hat{y}^+$, $c(P_1)<c(P_1^1)$. Also, the set of arcs from which we can choose P_2^1, $S(P_2^1)$, is smaller than $S(P_2)$, the set of arcs from which we can choose P_2. Therefore $c(P_2)\leqq c(P_2^1)$, i.e., $c(P_1)+c(P_2)<c(P_1^1)+c(P_2^1)$, which means P_1^1 and P_2^1 are not optimal and, therefore, $\hat{y}_{ij}(v=2)\geqq\hat{y}_{ij}(v=1)$. The process may be continued for $v>2$.

COROLLARY 1. $\hat{y}^-$ *is nonincreasing with increasing* v.

COROLLARY 2. *For the general network problem where* $\{\hat{y}^+, \hat{y}^-\}$ *are nonunique, we can always find an optimal sequence* $\{\hat{y}^+(v), \hat{y}^-(v)\}$ *that satisfies Proposition 1 and Corollary 1.*

PROPOSITION 2. *Consider the first stage problem (3). If* $\hat{x}=\hat{x}(v^1)$ *is the optimal solution vector, then* $c\{\hat{x}(v^1)\}$ *is a convex function of* v^1.

Proof. Follows from Theorem 1 of the appendix.

The results of Proposition 1 and Corollaries 1 and 2 can be used to divide problem (4) into two subproblems, only one of which needs to be solved for any particular value of the actual demand v. For $v>v^1$, problem (4) is equivalent to

$$\text{minimize} \quad g\{x(v^1), y^+\},$$
$$Ef=\{\underline{v}\},$$
$$f-y^+\leqq x(v^1), \tag{5}$$
$$y^+\geqq 0.$$

If we let $v^+=v-v^1$, then problem (5) is equivalent to

$$\text{minimize} \quad g\{x(v^1), y^+\},$$
$$Ef=\{\underline{v}^+\},$$
$$f-y^+\leqq 0, \tag{6}$$
$$y^+\geqq 0.$$

We have left out the y^- variable in (6) since, by Corollaries 1 and 2, $y^-=0$ for $v>v^1$.

For $v<v^1$, we need to solve

$$\text{minimize} \quad [-h\{x(v^1), y^-\}],$$
$$Ef=\{\underline{v}^-\},$$
$$f-y^-\geqq 0, \tag{7}$$
$$y^-\geqq 0,$$

where $$v^-=v^1-v.$$

PROPOSITION 3. $Q\{x(v^1), v\}$, *the optimal solution to the second-stage problem (4), is a convex function of* v^1 *for any* v.

Proof. Follows from Theorem 1 of the appendix and the convexity in x of $g(x, y^+)$ and $\{-h(x, y^-)\}$.

PROPOSITION 4. $E_v[Q\{x(v^1), v\}]$ *is a convex function of* v^1.

Proof. Follows from the fact that the expected value is a nonnegative combination of a number of functions, each of which is convex.

We therefore observe that $z\{x(v^1)\}=c\{x(v^1)\}+E_v\{Q[x(v^1), v]\}$ is a convex function of v^1. We use this fact to propose our algorithm.

SOLUTION ALGORITHM

For the purpose of the following algorithm, we shall assume that the demand v is a discrete random variable. For a network problem such as ours, we can assume that v takes only integer values. However, a sampling procedure can easily take care of a continuous distribution of v. Let the bounds on v be v^0 and v^f. If the bounds are unknown, assume $v^0=0$ and $v^f=M$, an arbitrarily chosen large number.

The Algorithm

First Stage:
Step 1. Solve

$$\begin{aligned} \text{minimize} \quad & c(x^k), \\ Ef &= \{\underline{v}^k\}, \\ f - x^k &\leqq 0, \\ x^k &\geqq 0. \end{aligned}$$

Initially, $v^k = v^0$. Step 1 can be solved by various algorithms for convex programming problems. Hu[2] has provided an efficient network algorithm. Let $\hat{x}^k$, $c(\hat{x}^k)$ be the optimal values.

Second Stage:
Step 2. For each $v^i > v^k$, $v^i \epsilon [v^0, v^f]$, solve

$$\begin{aligned} \text{minimize} \quad g\{\hat{x}^k, y^+\} &= Q^+(\hat{x}^k, v^i), \\ Ef &= \{\underline{v}^i - \underline{v}^k\} \\ f - y^+ &\leqq 0, \\ y^+ &\geqq 0. \end{aligned}$$

A solution may be obtained by the same procedure as in Step 1.
Step 3. Find $\sum_{\{i: v^i > v^k\}} p^i Q^+(\hat{x}^k, v^i) = Q^+(\hat{x}^k)$,
where $p^i =$ probability $\{v = v^i\}$.
Step 4. For each $v^i < v^k$, $v^i \epsilon [v^0, v^f]$, solve

$$\begin{aligned} \text{minimize} \quad [-h\{\hat{x}^k, y^-\}] &= Q^-(\hat{x}^k, v^i), \\ Ef &= \{\underline{v}^k - \underline{v}^i\}, \\ f - y^- &\geqq 0, \\ y^- &\geqq 0. \end{aligned}$$

Step 5. Find $\sum_{\{i: v^i < v^k\}} p^i Q^-(\hat{x}^k, v^i) = Q^-(\hat{x}^k)$.
Step 6. If $v^k \geqq v^f$, stop. $\hat{x}^k$ is the optimal decision vector and $c(\hat{x}^k)$ is the optimal value of the objective function. For $v^k < v^f$, find $z(\hat{x}^k) = c(\hat{x}^k) + Q^+(\hat{x}^k) + Q^-(\hat{x}^k)$. If $z(\hat{x}^k) \geqq z(x^{k-1})$, stop. $\hat{x}^{k-1}$ is the optional decision vector and $z(\hat{x}^{k-1})$ is the optimal value of the objective function.
Step 7. Let $v^{k+1} = v^k + 1$ and go to Step 1, replacing v^k by v^{k+1} and x^k by x^{k+1}.

Comment

In case the number of discrete values that the demand v can take is large, it would be more efficient to use a search technique such as the Fibonacci search for finding new values of v in Step 7. This procedure requires an appropriate alteration in the stopping criterion of Step 6.

APPENDIX

Def. 1. A set $\Gamma \subset R^n$ is a convex set if for any $\lambda \epsilon R$, $0 \leqq \lambda \leqq 1$, and any x^1, $x^2 \epsilon \Gamma$, $(1-\lambda)x^1 + x^2 \epsilon \Gamma$.

Def. 2. A numerical function θ defined on a convex set Γ is convex on Γ if and only if for any x^1, $x^2 \epsilon \Gamma$ and $0 \leqq \lambda \leqq 1$,

$$(1-\lambda)\theta(x^1) + \lambda\theta(x^2) \geqq \theta[(1-\lambda)x^1 + \lambda x^2].$$

THEOREM 1. *Let $b \epsilon \bar{\Gamma}$, a convex set, and let θ be a convex function of x on R^n. Let $\sigma(b) = \{\min \theta(x) | Ax = b, x \geqq 0\}$. Then $\sigma(b)$ is a convex function of b on $\bar{\Gamma}$.*

Proof. Let $x(b^1)$, $x(b^2)$ be the optimal vectors and $\sigma(b^1)$, $\sigma(b^2)$ be the optimal values corresponding to any two points b^1, b^2 in $\bar{\Gamma}$. Let $b_\lambda = (1-\lambda)b^1 + b^2$ and $x_\lambda = (1-\lambda)x(b^1) + \lambda x(b^2)$, $0 \leqq \lambda \leqq 1$. Let $\sigma(b_\lambda) = \{\min \theta(x) | Ax = b_\lambda, x \geqq 0\}$ with $x(b_\lambda)$ the optimal vector corresponding to b_λ. Since $x(b_\lambda)$ may or may not be equal to x_λ, we have

$$\sigma(b_\lambda) = \theta\{x(b_\lambda)\} \leqq \theta(x_\lambda) \leqq (1-\lambda)\theta\{x(b^1)\} + \lambda\theta\{x(b^2)\} = (1-\lambda)\sigma(b^1) + \lambda\sigma(b^2),$$

which proves the convexity of $\sigma(b)$ on $\bar{\Gamma}$.

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