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A Note on the $M/M/1$ Queue with $\lambda = \mu$

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The simplest Markovian queue where the arrival rate equals the service rate ($\lambda = \mu$) has been successfully treated using a diffusion approximation that assumes a continuous state space in continuous time. Here it is shown that an intuitively appealing approach using a simple restricted random walk with a discrete state space in discrete time yields the same results.

A GENERAL TREATMENT of the single channel queue using a diffusion approximation that assumes a continuous state space in continuous time may be found in Newell [1971]. Results for the cases $M/M/1$ and $M/G/1$ (the latter due to Gaver [1968]) under heavy traffic ($\rho = \lambda/\mu = 1 - \epsilon$) are presented in Gross and Harris [1974]. A successful diffusion approximation to the $G1/G/1$ queue under heavy traffic is given in Heyman [1975] which also includes the $M/M/1$ queue as a special case. Here the $M/M/1$ queue will be considered and it will be shown that results analogous to the ones previously obtained follow directly from the embedded chain of the $M/M/1$ queue without resorting to the diffusion equation.

The time dependent solution for the diffusion approximation of the $M/M/1$ queue when the arrival rate equals the service rate ($\lambda = \mu$) is given in Newell. Let $Q(t)$ denote the number of customers in the queue at time t. The expected number in the queue by time t for a process that starts in state 0 is given by $E[Q(t)] = (2t\Delta/\pi)^{1/2}$ where Δ, the variance of queue change per unit time, is of magnitude 2μ when $\lambda = \mu$; the parameter Δ can be made equal to 1 with a suitable choice of the time scale. To obtain this expression the number in the queue is assumed to be a continuum and the laws governing the evolution of queue distributions are modeled by the diffusion equation. Essential to the solution technique is the use of the reflection principle, a good treatment of which may be found in either Karlin and Taylor [1975], or Feller [1968].

Here the total number of customers in the system will be modeled as the position of a particle in a symmetric random walk with a reflecting

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barrier at the origin. The barrier allows the process to reach state 0 but the particle will bounce back to state 1 at the next epoch with probability one. Consider a simple random walk $\{Z_n\colon n = 0, 1, \cdots\}$ where Z_n is the position of the particle at epoch n. Let $Z_n = Z_{n-1} + Y_n$ where Y_n is a random variable such that $P[Y_n = 1] = p$ and $P[Y_n = -1] = q = 1 - p$. In this case $\Delta = 4pq\,(\Delta = 1$ for $p = q = ½)$. A symmetric random walk requires $p = q = ½$ and placing a reflecting barrier at the origin implies the condition $P[Z_{n+1} = 1/Z_n = 0] = 1$. Let $p^j_{n,k}$ denote the probability that the random walk with a reflecting barrier at the origin is at position k by epoch n given that it started at position $j\,(Z_0 = j)$; let $F^j_{n,k} = P[Z_n \leq k/Z_0 = j]$ and let $cF^j_{n,k}$ denote the probability complement of $F^j_{n,k}$. Establishing the equations of balance for the probability mass $P^j_{n,k}$ and summing $\sum_{r=0}^{k} P^j_{n,r}$ yields the equation

$$F^j_{n,k} = pF^j_{n-1,k-1} + qF^j_{n-1,k+1} \qquad k \geq 1 \tag{1}$$

with boundary conditions $F^j_{n,n+j} = 1$ and $F^j_{n,0} = P^j_{n,0}$. Let $\bar{F}^j_{n,k}$ and $\bar{P}^j_{n,k}$ denote the same as $F^j_{n,k}$ and $P^j_{n,k}$ for an unrestricted random walk, and consider the symmetric random walk $(p = q = ½)$. From the reflection principle, $F^j_{n,k} = \bar{F}^j_{n,k} - c\bar{F}^{-j}_{n,k}$, so

$$F^j_{n,k} = (½)^n \textstyle\sum_{i=-(k+j)}^{k-j} \displaystyle\binom{n}{(n+i)/2}. \tag{2}$$

Equation (2) is a solution to (1) for $p = q = ½$. It is implicitly assumed in (2) and in anywhere else that a binomial coefficient such as $\binom{n}{m}$ is used, that if m is negative, fractional or greater than n the coefficient is 0. A well-known identity given in Feller is the following:

$$\binom{x+1}{r} = \binom{x}{r} + \binom{x}{r-1}. \tag{3}$$

Substituting (2) in both sides of (1) and using (3) verifies that the equation is satisfied. From $p^j_{n,k} = F^j_{n,k} - F^j_{n,k-1}$ for $k \geq 1$ and $p^j_{n,0} = F^j_{n,0}$ the distribution $\{P^j_{n,k}\}_k$ is easily obtained which for $j = 0$ further simplifies to

$$P^0_{n,k} = \begin{cases} (½)^{n-1}\displaystyle\binom{n}{(n+k)/2} & k \geq 1 \\ (½)^n \displaystyle\binom{n}{n/2} & k = 0. \end{cases}$$

At any epoch n, the restricted random walk (starting at the origin) is at position $Z_n = \sum_{j=1}^{n} Y_j$. Each jump Y_j is state dependent with the Markov property holding. If j is even this means Z_{j-1} cannot be zero; thus $E[Y_j] = 0$. For j odd $E[Y_j/Z_{j-1} = 0] = 1$ and $E[Y_j/Z_{j-1} > 0] = 0$; therefore $E[Y_1] = 1$ and $E[Y_j] = P^0_{j-1,0}$ for $j = 3, 5, \cdots$. Thus, for $n \geq 1$ and odd the expected number of customers in the system is given by $E[Z_n] = 1 + \sum_{j=1}^{(n-1)/2} P_{2j,0}^{0}$. Stirling's formula provides an adequate approximation which is easier to compute.

$$E[Z_n] = 1 + (1/\sqrt{\pi}) \sum_{j=1}^{(n-1)/2} (1/\sqrt{j}). \qquad (4)$$

Finally, replacing the summation in (4) by an integral and integrating from 0 to $(n/2)$ yields $E[Z_n] = 1 + (2n/\pi)^{1/2}$, a result identical to that given in Newell. Let $N(t)$ denote the number of transitions by time t for the $M/M/1$ queue. In heavy traffic ($\lambda = \mu$) the intertransition times become i.i.d. exponential random variables with mean $\frac{1}{2}\mu$. As $t \to \infty$, $t/N(t) \to \frac{1}{2}\mu$ w.p. 1.; hence the relation $(t/n) \doteq \frac{1}{2}\mu$ is well established.

At any epoch $n = 2i + 1$ the increment in the expected position of the particle is exactly the mass assigned to the probability of being at position 0 at the epoch $2i$. This probability is being recovered or it is weighing the jump of magnitude 1 given the particle is at state 0. The total displacement by epoch n is the accumulated effect of bouncing off the barrier. Using indicator random variables it is easily shown that the expected number of returns to the origin is the same as the total displacement with both going to infinity since the series $\sum_{j=1}^{\infty} (1/j)^{1/2}$ diverges. Let $m(n)$ denote the mean return time (in number of steps) to state 0 by epoch n where $m(n)$ is the inverse of the rate of returning to state 0. Using the same argument given to derive (4) yields $E[Z_n] = \sum_{j=1}^{n} P[Z_{i-1} = 0]$ from which the following expression is obtained.

$$\begin{aligned}(1/m(n)) &= (1/n) \textstyle\sum_{i=1}^{n} P[Z_i = 0]\\ &= (1/n)[\textstyle\sum_{i=1}^{n+1} P[Z_{i-1} = 0] - P[Z_0 = 0]]\\ &= (1/n)[E(Z_{n+1}) - 1]\\ &= (1/n)(2n/\pi)^{1/2}.\end{aligned}$$

It follows that $m(n) = (\pi n/2)^{1/2}$. It is finite for all finite n and increasing on n without bounds.

A simple approach which highlights the null recurrent character of the process being considered has led to the resolution of a difficult problem.

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REFERENCES

FELLER, W. 1968. *An Introduction to Probability Theory and Its Applications, Vol. 1.* John Wiley & Sons, New York.

GAVER, D. P. 1968. Diffusion Approximations and Models for Certain Congestion Problems. *J. Appl. Prob.* **5**, 607–623.

Gross, D., and C. M. Harris. 1974. *Fundamentals of Queueing Theory*. John Wiley & Sons, New York.

Heyman, D. P. 1975. A Diffusion Model Approximation for the $GI/G/1$ Queue in Heavy Traffic. *Bell System Tech. J.* **54**, 1637-1646.

Karlin, S., and H. M. Taylor. 1975. *A First Course in Stochastic Processes*. Academic Press, New York.

Newell, G. F. 1971. *Applications of Queueing Theory*. Chapman & Hall Ltd., London.

Analysis of a Preference Order Traveling Salesman Problem

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Application is made to the preference order dynamic programming solution procedure proposed by Kao for a stochastic traveling salesman problem. Although the procedure is flawed from the myopic interpretation of the monotonicity condition, it may be used as a convenient heuristic tool for solving stochastic problems.

THE OBJECTIVE of this note is to report on certain difficulties the author encountered in trying to apply the preference order dynamic programming solution procedure proposed by Kao [1978] for a stochastic traveling salesman problem. The objective of Kao's procedure is to maximize the probability that the total travel time is equal to or smaller than a prespecified critical value. The analysis was motivated by the counterexample reported by Sniedovich [1979] showing that the procedure may yield nonoptimal solutions. This counterexample consists of $n = 4$ cities, the origin point 0, the critical completion time $C = 70$ and normal travel time variates whose means and variances are

$$\{\mu_{ij}\} = \begin{array}{c|ccccc} & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & — & 99 & 99 & 99 & 13 \\ 1 & 99 & — & 10 & 15 & 99 \\ 2 & 8 & 99 & — & 20 & 99 \\ 3 & 14 & 99 & 24 & — & 99 \\ 4 & 99 & 8 & 99 & 99 & — \end{array} \qquad \{\sigma_{ij}^2\} = \begin{array}{c|ccccc} & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & — & 0 & 0 & 0 & 0 \\ 1 & 0 & — & 6 & 2 & 0 \\ 2 & 2 & 0 & — & 12 & 0 \\ 3 & 9 & 0 & 2 & — & 11 \\ 4 & 0 & 1 & 0 & 0 & — \end{array}$$

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