Operations Research

Publication details, including instructions for authors and subscription information:
http://pubsonline.informs.org

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Probability of Success in the Search for a Moving Target

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(Received April 1980; revised April 1981; accepted July 1981)

The probability of detection in the search for a randomly moving target is calculated for the case of a target whose motion is a diffusion process and known searcher path. The probability of detection can be calculated by solving a backward diffusion equation. Corwin [1980] gives a solution of the backward equation for a special case. In general, exact solutions do not exist and other methods are needed. In this paper, the backward equation is solved approximately by using a formal asymptotic method, valid when the intensity of the random motion is small. The general solution is illustrated for the case of spatially homogeneous drift and diffusion coefficients. In this case, the asymptotic solution can be evaluated analytically.

ALTHOUGH PROBLEMS of search for moving targets have received considerable attention in recent years, some apparently simple problems remained unsolved. These problems are actually not simple, and it is the motion of the target which makes them hard to solve. One of these problems is the calculation of the probability of detection in a search when the search path is specified. In this case, one does not try to find an "optimal" path, but gives a search path and then calculates the probability of detection at the end of the search. We shall find this probability by solving a backward diffusion equation. This procedure is not as removed from optimal search as it seems. First, once the probability of detection is known, the optimal path can be obtained by a nonlinear programming procedure (e.g. Ciervo [1976]). Second, it turns out that when studying optimal search problems, in order to solve the conditions giving an optimal path, one needs to solve the equation treated in this

Subject classification: 452 probability of success in search, 563 search for diffusing targets.

Operations Research
Vol. 30, No. 1, January–February 1982

0030-364X/82/3001-0216 $01.25

paper (Lukka [1977], Stone [1979]). Recently, Corwin studied the appropriate diffusion equation and derived an exact infinite series representation for the solution in the special case of drift vectors linear in spatial variables and covariance matrix independent of spatial variables. Although exact, his work requires knowledge of the cumulants of the detection function, which may be hard to find. The present method, although approximate, will work for arbitrary drift and diffusion coefficients and requires no knowledge of cumulants.

There are basically two ways to calculate the probability of detection in search for a moving target. To illustrate them, consider targets moving in the plane and let $\rho(x, y)$ be the initial density for the location of the target. The first approach to calculating the probability of detection by time T, $P(T)$, uses the joint density for target location and unsuccessful search, $f(x, y, t, Z)$. Define

$$f(x, y, t, Z)dx\,dy = \text{Prob}\{\text{target} \in (x, x + dx; y, y + dy) \text{ and search along } Z(\tau),\ 0 \le \tau \le t \text{ was not successful}\} \tag{1}$$

so that

$$P(T) = 1 - \int\int f(x, y, t, Z)dx\,dy. \tag{2}$$

Mangel [1981] gives an approximate method (the ray method) for the calculation of $f(x, y, t, Z)$ when the target's motion is a diffusion process.

A second way to find the probability of detection by time T uses the probability of nondetection to time t, conditioned on starting at (x, y). If

$$u(x, y, t, s, Z) = \text{Prob}\{\text{nondetection in } (s, t)\,|\,\text{target starts at } (x, y) \text{ and the search is along } Z(\tau),\ s \le \tau \le t\} \tag{3}$$

then

$$P(T) = 1 - \int\int u(x, y, T, 0, Z)\rho(x, y)dx\,dy. \tag{4}$$

In this paper, $u(x, y, t, s, Z)$ is calculated by approximately solving the equation that it satisfies.

The difference between $f(x, y, t, Z)$ and $u(x, y, t, s, Z)$ is that in $f(x, y, t, Z)$ the independent variables are *future* time and position, while in $u(x, y, t, s, Z)$ the independent variables are past time and position. Thus $f(x, y, t, Z)$ satisfies a forward equation and $u(x, y, t, s, Z)$ satisfies a backward equation (see, e.g. Ludwig [1975]).

In Section 1, the detection problem is formulated and the backward equation for $u(x, y, t, s, Z)$ is given. In Section 2, the ray method is used to construct an approximate solution of the backward equation. Implementation of the solution is illustrated in Section 3 by considering the case of spatially homogeneous target motion and linear search paths.

1. TARGET MOTION MODEL, SEARCH MODEL AND BACKWARD EQUATION

In this section, the backward equation for $u(x, y, t, s, Z)$ is given. It is assumed that all variables are scaled (see, e.g. Cohen and Lewis [1967], Mangel [1981]).

The target is assumed to move as a diffusion process with drift vector $(b_x(x, y, t), b_y(x, y, t))$ and covariance matrix $\epsilon \begin{bmatrix} a_x(x, y, t) & 0 \\ 0 & a_y(x, y, t) \end{bmatrix}$. A diagonal covariance matrix is assumed without loss of generality. The variable ϵ arises in the scaling: $\epsilon = \alpha/(A_c^2/T_c)$; here α is a characteristic value for the diffusion coefficient of the target, A_c is a characteristic value for the size of the region to be searched, and T_c is a characteristic time of the search (see Mangel [1981]). It is assumed that $\epsilon \ll 1$. The position of the target at time t is $(X(t), Y(t), 0)$ and of the searcher is $Z(t) = (Z_x(t), Z_y(t), h(t))$.

Detection is characterized by the instantaneous detection rate, $\psi(x, y, t, Z)$ defined so that

$$\begin{aligned} \psi(x, y, t, Z)\Delta t = \text{Prob}\{&\text{detection in } (t, t + \Delta t) \mid \\ &X(t) = x,\ Y(t) = y, \\ &Z(t) = Z\} + o(\Delta t). \end{aligned} \tag{5}$$

It can be shown (see, e.g., Karlin and Taylor [1981]) that with the above assumptions $u(x, y, t, s, Z)$ satisfies the equation

$$\begin{aligned} \partial u/\partial s + \frac{\epsilon}{2}[a_x \cdot (\partial^2 u/\partial_x^2) + a_y \cdot (\partial^2 u/\partial y^2)] \\ + b_x \cdot (\partial u/\partial x) + b_y \cdot (\partial u/\partial y) - \psi(x, y, s, Z)u = 0. \end{aligned} \tag{6}$$

with the end condition $u(x, y, t, s, Z) \uparrow 1$ as $s \uparrow t$; boundary conditions are needed if the target moves in a finite region.

2. ASYMPTOTIC SOLUTION OF THE BACKWARD EQUATION

An approximate solution of Equation 6, valid for small ϵ, can be obtained by using the ray method (Ludwig, Mangel [1981]). We seek a solution of (6) in the form

$$u(x, y, t, s, Z) = \textstyle\sum_{k=0} \epsilon^k g_k(x, y, t, s, Z)\exp(-\phi(x, y, t, s, Z)/\epsilon) \tag{7}$$

where $\phi(x, y, t, s, Z)$ and the functions $g_k(x, y, t, s, Z)$ are to be determined.

After evaluating derivatives and substituting into Equation 6, terms are collected according to powers of ϵ. Asymptotically (for small ϵ), $\phi(x, y, t, s, Z)$ and $g_0(x, y, t, s, Z)$ will satisfy the eiconal and transport equations (Cohen and Lewis, Mangel [1981])

$$(\partial\phi/\partial_s) + b_x \cdot (\partial\phi/\partial_x) + b_y \cdot (\partial\phi/\partial y) - (a_x/2)(\partial\phi/\partial x)^2 - (a_y/2)(\partial\phi/\partial y)^2 = 0 \tag{8}$$

$$(\partial/\partial s)g_0 + (\partial/\partial x)g_0(b_x - a_x(\partial\phi/\partial x)) + (\partial/\partial y)g_0(b_y - a_y(\partial\phi/\partial y)) = (\psi(x, y, s, Z) + (a_x/2)(\partial^2\phi/\partial x^2) + (a_y/2)(\partial^2\phi/\partial y^2))g_0. \tag{9}$$

From the form of the ansatz (7) and the end condition for $u(x, y, t, s, Z)$ we require

$$\phi(x, y, t, t, Z) = 0$$
$$g_0(x, y, t, t, Z) = 1. \tag{10}$$

Equation 8 is a first order partial differential equation and can be solved by the method of characteristics (Courant and Hilbert (1962)). Set $p = \partial\phi/\partial x$, $q = \partial\phi/\partial y$ and $H = b_x p + b_y q - a_x p^2/2 - a_y q^2/2$. Pick a point (x_t, y_t) in the search domain at time t. The characteristic or ray equations corresponding to (8) are

$$\begin{aligned}
&ds/d\tau = 1 \\
&dx/d\tau = b_x(x, y, s) - a_x(x, y, s)p, && x(t) = x_t \\
&dy/d\tau = b_y(x, y, s) - a_y(x, y, s)q, && y(t) = y_t \\
&d\phi/d\tau = -\tfrac{1}{2}[a_x(x, y, s)p^2 + a_y(x, y, s)q^2], && \phi(t) = 0 \\
&dp/d\tau = -\partial H/\partial x \\
&dq/d\tau = -\partial H/\partial y.
\end{aligned} \tag{11}$$

The values of $p(t)$ and $q(t)$ are determined by requiring that the solution of (11) goes through the point (x, y) when $\tau = s$. In $(x(\tau), y(\tau))$ space, the solution of (11) is a curve or ray joining $(x(s), y(s)) = (x, y)$ and $(x(t), y(t)) = (x_t, y_t)$. The solution of (11) will be denoted by $x(\tau, s, t, x_t, y_t)$, $y(\tau, s, t, s_t, y_t)$, $\phi(x(\tau, s, t, x_t, y_t), y(\tau, s, t, x_t, y_t))$, with the convenient short-hand $x(\tau)$, $y(\tau)$, $\phi(\tau)$ if there is no chance of confusion.

The solution $u(x, y, t, s, Z)$ is the sum of contributions over the rays covering the (x_t, y_t) plane.

Along the ray $(x(\tau), y(\tau))$, Equation 9 becomes

$$(dg_0/d\tau) = [\psi(x(\tau), y(\tau), \tau, Z(\tau)) + (a_x/2)(\partial^2\phi/\partial x^2) + (a_y/2)(\partial^2\phi/\partial y^2)]g_0. \tag{12}$$

Once $\partial^2\phi/\partial x^2$ and $\partial^2\phi/\partial y^2$ are expressed in terms of τ (by using the solution of (8)) g_0 is obtained by a single quadrature.

It is possible that the rays calculated from (11) intersect or have an envelope. In those cases the simple ansatz given by (7) breaks down and a more complicated (uniform) one is needed (see, e.g., Mangel [1981] or Cohen and Lewis).

The product $g_0 e^{-\phi/\epsilon}$ is, to $0(\epsilon)$, the probability that the target starts at (x, y) at time s, is not detected by time t, and is in a neighborhood of (x_t, y_t) at time t. Combining all the above results gives

$$P(t) = 1 - \int\int\int\int \rho\,(x, y)\exp\Big[-\int_s^t [(a_x/2)(\partial^2\phi/\partial x^2) + (a_y/2)(\partial^2\phi/\partial y^2) + \psi(x(\tau), y(\tau), \tau, Z(\tau))]d\tau\Big] \cdot \exp[-(1/\epsilon)\phi(x(s, t, x_t, y_t), y(s, t, x_t, y_t), s)]dx_t dy_t dx\, dy. \quad (13)$$

3. SPATIALLY HOMOGENEOUS TARGET MOTION AND LINEAR SEARCH

To illustrate the use of the ray equations, we now consider an important, and common, special case of (13), in which the drift parameters depend only upon time, the diffusion parameters are constant, and the search path is piecewise linear. The ray equations become

$$\begin{aligned} dx/d\tau &= b_x(\tau) - a_x p, & x(t) &= x_t \\ dy/d\tau &= b_y(\tau) - a_y q, & y(t) &= y_t \\ d\phi/d\tau &= -\tfrac{1}{2}[a_x p^2 + a_y q^2], & \phi(t) &= 0 \\ dp/d\tau &= dq/d\tau = 0. \end{aligned} \quad (14)$$

These equations can be integrated immediately to give

$$\begin{aligned} x(\tau) &= x_t - a_x p(\tau - t) + \int_t^\tau b_x(s)ds \\ y(\tau) &= y_t - a_y q(\tau - t) + \int_t^\tau b_y(s)ds \\ \phi(\tau) &= -\tfrac{1}{2}[a_x p^2 + a_y q^2](\tau - t). \end{aligned} \quad (15)$$

The values of p and q are chosen so that the solution (15) passes through (x, y) at $\tau = s$; we find

$$\begin{aligned} p &= \left(x_t + \int_t^s b_x(s')ds' - x\right)\Big/ a_x(s - t) \\ q &= \left(y_t + \int_t^s b_y(s')ds' - y\right)\Big/ a_y(s - t). \end{aligned} \quad (16)$$

Using (16) in the third equation in (14) gives

$$\phi(s) = \tfrac{1}{2}\left(x_t - \int_s^t b_x(s')ds' - x\right)^2 \Big/ a_x(t-s) + \left(y_t - \int_s^t b_y(s')ds' - y\right)^2 \Big/ a_y(t-s) \tag{17}$$

and
$$(a_x/2)(\partial^2\phi/\partial x^2) + (a_y/2)(\partial^2\phi/\partial y^2) = 1/(t-s). \tag{18}$$

From (17) and (18), we see that u will have the form of a Gaussian function times a function due to search.

If the search path is piecewise linear, corresponding to constant velocities between course changes, so that

$$\begin{aligned} Z_x(t) &= Z_x(t_{j-\iota}) + V_{xj}(t - t_{j-1}) \\ Z_y(t) &= Z_y(t_{j-\iota}) + V_{yj}(t - t_{j-1}) \\ h(t) &= h(t_{j-\iota}) + V_h(t - t_{j-1}) \end{aligned} \tag{19}$$

the integral of the detection function can often be evaluated explicitly. For example, in visual detection (Koopman [1980]), the detection rate is

$$\psi = kh/(h^2 + (Z_x - x)^2 + (Z_y - y)^2)^{3/2} \tag{20}$$

and the integral giving g_0 can be evaluated exactly in terms of elementary functions. The integral of ψ is evaluated along the ray and in this case can be found exactly (Mangel [1981]). Use of this method for search planning is reported in Mangel [1979].

ACKNOWLEDGMENTS

I thank Dr. Lawrence Stone for discussions about using the backward equation in search problems and Dr. Eugene Hamilton for a number of discussions about the Feynman-Kac formula. This work was started at the Operations Evaluation Group, Center for Naval Analyses, and partially supported at the University of California by Office of Naval Research Contract N00014-81-K-0030.

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