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Letter to the Editor—Note on Klein's "Direct Use of Extremal Principles in Solving Certain Problems Involving Inequalities"

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The reviewer's criticism of the study on the grounds of the simplicity of its mathematics seems astonishing Can it be that we sometimes advocate complexity for complexity's sake?

NOTE ON KLEIN'S "DIRECT USE OF EXTREMAL PRINCIPLES IN SOLVING CERTAIN PROBLEMS INVOLVING INEQUALITIES"*

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B. KLEIN in the May 1955 issue of this JOURNAL[1] proposed that ordinary methods of the calculus be used to minimize a function z of n variables, $x_1, x_2, \ldots, x_n$, where the latter are subject to *inequality* constraints instead of usual equality constraints As a first step he replaces all inequalities such as

$$f_i(x_1, x_2, \ldots, x_n) \leq 0 \qquad (i=1, 2, \ldots, m) \quad (1)$$

by equalities by the introduction of new variables u_i satisfying

$$f_i(x_1, x_2, \ldots, x_n)+u_i^2=0 \quad (2)$$

After this the problem can be solved (in theory) by the use of Lagrange multipliers For the linear-programming case, this method will yield the information that the solution is at a vertex where some of the constraints are equalities and others are not For the non-linear case it will state that the solution is inside or on the boundary Our purpose will be to discuss whether this method can be used constructively to determine the optimum for large m and n

Applied to the standard linear-programming problem

$$\sum_{j=1}^{n} a_{ij} x_j = b_i, \qquad (i=1, 2, \cdot\ \ldots, m) \quad (3)$$

$$x_j \geqq 0, \qquad (j=1, 2, \ldots, n) \quad (4)$$

$$\sum_{j=1}^{n} c_j x_j = z, \qquad (z \text{ to be minimum}) \quad (5)$$

the inequality conditions (4) are replaced by

$$-x_j+u_j^2=0 \qquad (j=1, 2, \ldots, n) \quad (4')$$

Let the Lagrange multipliers associated with the ith equation of (3) be π_i, and those associated with the jth equation of (4') be δ_j, then the main problem is reduced to finding the unconditional minimum of the form

$$z' = \sum_{j=1}^{n} c_j x_j - \sum_{i=1}^{m} \pi_i \left(\sum_{j=1}^{n} a_{ij} x_j - b_i \right) + \sum_{j=1}^{n} \delta_j (-x_j+u_j^2), \quad (5)$$

* Prepared at the request of the Editor of OPERATIONS RESEARCH

where the constants π_i and δ_j are selected so that x_j and u_j at the minimum satisfy (3) and (4) Setting $\partial z'/\partial x_j=0$ and $\partial z'/\partial u_j=0$ yields *necessary* conditions for a minimum

$$c_j-\sum_{i=1}^{m}\pi_i\, a_{ij}-\delta_j=0, \qquad (j=1, 2, \cdots, n) \quad (6)$$

$$\delta_j\, u_j=0, \qquad (j=1, 2, \cdots, n) \quad (7)$$

or, solving for δ_j in (6) and substituting in (7),

$$\left(c_j-\sum_{i=1}^{m}\pi_i\, a_{ij}\right) u_j=0, \qquad (j=1, 2, \cdots, n) \quad (8)$$

which are well known relations which must exist between optimal solutions of the primal and dual linear-programming problems

The essence of Klein's proposal is that (8) implies for each j that either the first or the second term must be zero This gives rise to 2^n possible ways that (8) can be satisfied—say that for $j=\nu_1, \nu_2, \cdots, \nu_k, \cdots, \nu_r$ the first term of the product vanishes, while for all other j the second vanishes The $n-r$ conditions $u_j=0$ (i e , $x_j=0$) for $j\neq\nu_k$, when substituted into (3) give rise to m equations (9) in r variables x_j where $j=\nu_k$, the r conditions $\delta_j=0$ for $j=\nu_k$ give rise to r equations (10) in m variables π_i

$$\sum_{k=1}^{r} a_{i\nu_k}\, x_{\nu_k}=b_i, \qquad (i=1, 2, \cdots, m) \quad (9)$$

$$\sum_{i=1}^{m} \pi_i\, a_{i\nu_k}=c_{\nu_k} \qquad (k=1, 2, \cdots, r) \quad (10)$$

If $r<m$, (9) will contain fewer equations than variables, if $r>m$, the second system (10), whose matrix is the transpose of the first, will have this property Thus in practice it may be expected that a large part of the 2^n combinations will result in an inconsistent system For example, if one could assume that the rank of any $m\times m$ submatrix of the coefficients of (1) is m, then all cases $r>m$ or $r<m$ can be dropped, so that instead of 2^n combinations, one could deal with the $\binom{n}{m}$ combinations where $r=m$ To avoid the practical difficulty of testing the assumption on rank, one could introduce 'artificial' variables as is usually done in linear programming and show that it is only necessary to consider submatrices whose rank is m

For cases where x_j exist satisfying (3) *and* (4) *and where there is a finite lower bound for the value of z, the procedure as outlined is completely valid One simply selects from among the combinations, solutions in which all $x_j\geq 0$ and finds one which minimizes the value of z*

Most people who have considered problems involving inequalities have gone over the same ground as Klein The practical difficulty begins only if the solution of $\binom{n}{m}$ systems, each $m\times m$, is too much work Thus if $n=7$, $m=2$ as in the illustrative example presented by Klein, all combinations can be enumerated quickly George Stigler[2] in trying to solve the nutrition problem made a search among the combinations of an ($n=77$, $m=9$) linear-programming nutrition problem Actually, he was able to eliminate many foods and needed to consider an

($n=23$, $m=9$) case, even here $\binom{23}{9}$ is very large and he had to resort to further devices to cut down the number of combinations to about 512 Since this was done in the days of hand methods he was forced to try a few of the combinations and take for his 'optimum' the best among them

The Simplex method[3] in fact begins where Klein leaves off The first part of the procedure consists of a method of selecting from among the $\binom{n}{m}$ combinations one in which all $x_j \geq 0$—called a *basic feasible solution* (the m variables x_{j1}, x_{j2}, , x_{jm} are called *basic variables*) With such a starting solution, the values of π_i and δ_j are determined next by (10) and (6) It can be shown that all δ_j should satisfy $\delta_j \geq 0$ for the solution to be optimum If not, there is a $\delta_s < 0$ The corresponding non-basic variable $x_s = 0$ is allowed to increase while all other non-basic variables remain at their zero value At a value $x_s = x_s^*$ sufficiently large, one of the values of the basic variables becomes $x_{i_0} = x_{i_0}^* = 0$ while all other $x_{jk} \geq 0$ This gives rise to a *second combination* in which x_s replaces x_{i_0} as basic variable The new value of z can be shown to be less than that for the previous combination The procedure is iterated until a solution is obtained satisfying (3) and (4) in which all $\delta_j \geq 0$ In practice, approximately m combinations are usually examined The shift from one combination to the next can be arranged so that $n \times m$ new multiplications are needed Thus each new combination does not mean solving from scratch a new $m \times m$ system of equations

The foregoing may be summarized as follows First, Klein's method leads to a set of equations which, *in the linear case*, have been well known for some years Second, the direct solution of these equations *in the case of a large number of variables*, had previously been thoroughly explored and shown to be prohibitively laborious Third, the Simplex method, applied to the same problem but following a different procedure, is capable of solving this type of problem with an acceptable amount of labor

However, it does not follow that Klein's work contains nothing of value, for at least two reasons First, the squared-variable transformation makes it possible to replace all inequality constraints by equality constraints, that is, to reduce non-classical to classical problems † Second, Klein's method has applicability to inequality-constrained *non*-linear *few*-variables problems, as indicated in his article itself

REFERENCES

1 B Klein, *Opns Res* **3,** 168, 548 (1955) See also A Charnes and W W Cooper, *ibid* **3,** 345 (1955)

2 George J Stigler, "The Cost of Subsistence," *J Farm Economics* **27,** No 2 (May 1945)

3 George B Dantzig, "Maximization of a Linear Function of Variables Subject to Linear Inequalities," *Activity Analysis of Production and Allocation*, T C Koopmans, ed , Wiley, New York, 1951

4 Glen D Camp, "Inequality-Constrained Stationary-Value Problems," *Opns Res* **3,** 548 (1955)

† For example, G Camp in reference 4 points out that this approach may make it possible effectively to attack *both* linear and non-linear problems by classical direct and semi-direct methods, and to develop useful special-purpose analog computers, and that it may lead to useful theoretical developments