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AN APPROXIMATION METHOD FOR THE ANALYSIS OF *GI/G/*1 QUEUES

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We study in this paper an approximation method for the calculation of various performance measures of a *GI/G/*1 queue. Instead of solving the waiting time directly, we analyze the idle-period distribution as the starting point. The result is then taken as input to many known results to get other performance measures. We show that the distribution of the *GI/G/*1 idle period satisfies a nonlinear integral equation. This equation directly leads to an accurate approximate solution of the idle-period distribution of the *GI/G/*1 queue where the interarrival times have a generalized hyperexponential distribution (*GH*). Since all distribution functions can be approximated by a *GH* distribution at any given accuracy (Botta and Harris 1986), the solution method developed in this paper serves as a unified basis for the analysis of *GI/G/*1 queues.

We study in this paper various performance measures of a *GI/G/*1 queue. Derivation of exact solutions of these measures typically requires solving Lindley's equation on the complex plane (Lindley 1952). Because of mathematical complications, closed-form solutions have been difficult to achieve. Consequently, approximations have been studied extensively. For example, one can approximate a *GI/G/*1 queue, as suggested in Neuts (1981) and (1989), by either a *GI/PH/*1 queue or a *PH/G/*1 queue, where *PH* denotes a phase-type distribution. Such approximations make possible the analysis of embedded Markov chains that appropriately monitor the resulting phase-type processes. Alternatively, a *GI/G/*1 queue can also be approximated by a *GH/G/*1 queue. By definition, a random variable A is said to have a *GH* distribution if

$$P(A \leqslant t) = \sum_{i=1}^{N} a_i(1 - e^{-\lambda_i t}), \tag{1}$$

where $\Sigma_{i=1}^{N} a_i = 1$, and $\lambda_i > 0$, $\Sigma_{i=1}^{N} a_i\lambda_i e^{-\lambda_i t} \geqslant 0$ for all $t > 0$. Thus, a *GH* distribution is a linear combination of exponential distributions with mixing parameters that sum to unity. Mathematically speaking, the class of *GH* distributions is a direct extension of the well-known class of hyperexponential distributions (*HE*), which are of the same form but with the additional requirement that $\{a_i\}$ be strictly positive. Because of this relaxation of the nonnegativity condition, $\{a_i\}$ in (1) can no longer be interpreted as probabilities. Hence, the traditional embedded-Markov-chain method for the analysis of the *HE/G/*1 and the *GI/HE/*1 queues does not readily apply to the *GH/G/*1 and the *GI/GH/*1 queues. As a result, the structural simplicity of the *GH* distribution remains largely unexploited so far in the literature.

In this paper, we introduce a new method for the analysis of the *GI/G/*1 queue that is based on *GH* approximations. We show that once a *GI/G/*1 queue is approximated by a *GH/G/*1 queue, it can be efficiently solved without resorting to the embedded-Markov-chain method. The main difference between our method and the classical ones is that we do not directly analyze the waiting time; instead, we focus on the idle period. The results for the idle period are then taken as input to the calculation of other performance measures of the system.

The close link between the waiting-time and the idle-period distributions has long been observed in the literature. For example, in solving Lindley's equation, Smith (1953) obtained the first explicit relationship between the waiting-time and the idle-period distributions of the *GI/G/*1 queue. Let A, S, W, and I be generic random variables that represent, respectively, the interarrival time, the service time, the waiting time, and the idle period of the *GI/G/*1 queue. The distributions of these variables are denoted by $A(x)$, $S(x)$, $W(x)$, and $I(x)$, and their Laplace-Stieltjes transforms by $\tilde{A}(x)$, $\tilde{S}(x)$, $\tilde{W}(x)$, and $\tilde{I}(x)$. Then Smith's result relating $W(x)$ to $I(x)$ is given in the Laplace-Stieltjes-transform from

$$\tilde{W}(s) = \frac{\alpha_0 \tilde{I}(s)}{\tilde{A}(-s)\tilde{S}(s) - 1} \tag{2}$$

(also see Equation (6.5) of Gross and Harris 1985), where α_0 is the probability that a randomly selected arrival finds the system empty, given by

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$$\alpha_0 = \frac{1}{E(I)}(E(A) - E(S)). \tag{3}$$

(See Equation (3) of Marshall 1968.) Relations (2) and (3) clearly indicate that the analysis of the waiting-time distribution of the $GI/G/1$ queue can be accomplished by that of the idle period of the system. When the latter is done, other performance measures can also be solved accordingly. For example, by letting $X = A - S$, the first two moments of the waiting time are given in Marshall (1968) as

$$E(W) = -\frac{E(X^2)}{2E(X)} - \frac{E(I^2)}{2E(I)}, \quad \text{and} \tag{4}$$

$$E(W^2) = \frac{E(I^3)}{3E(I)} + 2\left[\frac{E(X^2)}{2E(X)}\right]^2 + \frac{E(X^2)E(I^2)}{2E(X)E(I)} - \frac{E(X^3)}{3E(X)}. \tag{5}$$

Marshall's method also applies to all higher moments of W provided that these moments exist. In addition, with knowledge of $E(I)$, the expected duration of busy period can be obtained by the identity

$$E(B) = \frac{\rho}{1-\rho}E(I), \tag{6}$$

assuming $\rho = E(S)/E(A) < 1$. It has also been shown in the literature that given the idle-period distribution, not only the standard $GI/G/1$ queue can be solved, but a number of its variations can also be solved along the same line of analysis (Li 1989 and Li and Niu 1992).

The main difficulty in adopting this approach, however, is the lack of a "stand-alone" equation that characterizes the idle-period distribution of the $GI/G/1$ queue. The main relationship known in the literature for the idle-period distribution is the Wiener-Hopf integral equation which involves the waiting-time distribution as an indispensable factor (Feller 1971). For this reason, the close link between the idle-period distribution and other performance measures remains largely unexploited in the literature.

In this paper, we show that the idle-period distribution of the $GI/G/1$ queue by itself satisfies a nonlinear integral equation. Although this integral equation can be solved by the classical method of successive substitutions, it turns out that the equation is remarkably suited for approximating $GH/G/1$ queues. The critical observation in this regard is that the idle period of a $GH/G/1$ queue also has a GH distribution, i.e., if $A(x)$ follows the form in (1), then $I(x) = \sum_{i=1}^{N} r_i(1 - e^{-\lambda_i x})$, where $\sum_{i=1}^{N} r_i = 1$, with $\{r_i\}$ being functions of $\{a_i\}$ and $\{\lambda_i\}$. Hence, if $\{r_i\}$ are determined, the $GH/G/1$ queue can be solved by (2)–(6).

The remainder of the paper is organized as follows. In Section 1, we derive the nonlinear integral equation for $I(x)$, and develop a linear approximation of this integral equation. In Section 2, we apply the linear approximation to the $GH/G/1$ queue. Numerical examples and their comparison with simulation results are given in Section 3.

1. THE IDLE-PERIOD DISTRIBUTION

We consider the standard $GI/G/1$ queue. Since our main focus is on the idle periods, which are invariant with respect to all work-conserving service disciplines, we select the preemptive-resume-LIFO (last in first out) service discipline. In this discipline, if a customer experiences multiple interruptions, the duration of each of the interruptions and the time between two consecutive interruptions form two sequences of i.i.d. random variables (see Fakinos 1986 and Niu 1988). We now present the first result of this paper.

Theorem 1. *The probability distribution of idle periods in a standard $GI/G/1$ queue satisfies the following integral equation*:

$$I(x) = \int_0^\infty [A(t+x) - A(t)]\, dS(t) + \int_0^\infty \int_0^t B(x, t-u)\, dA(u)\, dS(t), \tag{7}$$

where $B(x, s)$ is the distribution of the forward recurrence time, as observed at time s, of a renewal process whose inter-event times are distributed as $I(x)$.

Proof. We assume, without loss of generality, that the first customer arrives at time 0, initiating a busy period. By conditioning on the time of the next arrival, the distribution of the idle period that follows this busy period can be written as

$$I(x) = P(I \leqslant x, A > S) + P(I \leqslant x, A \leqslant S). \tag{8}$$

Clearly, the first probability on the right-hand side of (8) is given by the first integral in (7). For the second probability, note that if $A \leqslant S$, then the first customer is interrupted by the next arrival, which initiates a new busy period, and at this next arrival epoch, the first customer has remaining service $R = S - A$. Now, under the preemptive-resume LIFO service discipline, the remaining service of R will be processed only during idle periods that follow subsequent busy periods generated by further arrivals. At the epoch when service to the first customer is completed, the excess associated with the idle period in progress is stochastically identical to the forward recurrence time at epoch R of a renewal process whose inter-event times are distributed as $I(\cdot)$. This establishes the second integral in (7), and the proof is complete.

It is well known that $B(x, s)$ is a function of $I(\cdot)$ alone, thus (7) represents a "stand-alone" integral equation for the idle-period distribution of the $GI/G/1$ queue. In this sense, it can be viewed as the counterpart of Lindley's equation.

The explicit relationship between $B(x, s)$ and $I(x)$ is given by

$$B(x, s) = I(s+x) - I(s) + \int_0^s [I(s-u+x) - I(s-u)]\, dm(u), \tag{9}$$

where $m(u)$ is the renewal function of $I(\cdot)$ (see, for example, Equation 5-45 of Heyman and Sobel 1982). Because $B(x, s)$ is the integral kernel of (7) and is nonlinear in $I(x)$, (7) is, by definition, a nonlinear integral equation of $I(x)$. It can be shown (see Li and Ou 1995) that (7) uniquely determines $I(x)$. However, practical solution methods exist only for linear integral equations. This situation motivates us to linearize (7) so that standard solution methods can be applied.

Theorem 2. *When the $GI/G/1$ queue is specialized to either the $M/G/1$ or the $GI/M/1$ queues,* (7) *reduces to a linear integral equation.*

Proof. The $M/G/1$ case is obvious. For the $GI/M/1$ case, recall that $P(I \leq x, A \leq S) = P(A \leq S) \cdot E[B(x, S - A)|A \leq S] = P(A \leq S)E[B(x, R)]$. By conditioning on whether or not the remaining service of the first customer is interrupted, $E[B(x, R)]$ can also be expressed in

$$E[B(x, R)] = E[B(x, R - I)|I < R]P(I < R)$$
$$+ E[B(x, R)|I \geq R]P(I \geq R) = E[B(x, R)]P(I < S)$$
$$+ \int_0^\infty [I(t + x) - I(t)]\, dS(t). \qquad (10)$$

The last equality holds because S and R are identically distributed exponential variables. Solving $E[B(x, R)]$ from (10), we get

$$E[B(x, R)] = \frac{1}{P(I \geq S)} \int_0^\infty [I(t + x) - I(t)]\, dS(t),$$

where the integral kernel is linear on $I(\cdot)$ while $1/P(I \geq S)$ can be treated as a constant, determined by normalization. Hence $E[B(x, R)]$ is a linear function of $I(\cdot)$. The proof is complete.

For $GI/G/1$ queues other than the $M/G/1$ and the $GI/M/1$ type, approximation is needed to linearize (7). From (9), we see that the nonlinearity of (7) is due to the renewal function $m(u)$. Observe that when $u \to \infty$,

$$\lim_{u \to \infty} dm(u) = \frac{1}{E(I)}\, du. \qquad (11)$$

Hence, if we replace $dm(u)$ in (9) by its limiting form in (11), then (7) will be converted into a linear integral equation. Formally, by substituting (11) into (9) and then (9) into (7), we have

$$I(x) = \int_0^\infty [A(t + x) - A(t)]\, dS(t)$$
$$+ \int_0^\infty \int_0^t \Big[I(t - u + x) - I(t - u)$$
$$+ \sigma \int_0^{t-u} [I(t - u - v + x)$$
$$- I(t - u - v)]\, dv \Big]\, dA(u)\, dS(t), \qquad (12)$$

where $\sigma = 1/E(I)$, which is to be determined by normalization. Thus, (12) is linear. In addition, it can be verified that (12) is equivalent to (i.e., has the same solution as) (7) in both the $M/G/1$ and the $GI/M/1$ cases.

From the theoretical view point, the quality of the above linearization scheme depends on the rate of convergence of the limit in (11) and its impact on the evaluation of the second term in (7). In the latter regard, we observe that the renewal function $m(u)$ for the $GI/M/1$ queue is not a linear function by itself, but (12) is still equivalent to (7). This, together with our numerical experiment on other service distributions, seems to suggest that $E[B(x, R)]$ is not very sensitive to the linear approximation in (12).

The quality of the approximation in (12) also depends on the relative weights of the two integrals in (7). Recall that these two integrals are $P(I \leq x, A > S)$ and $P(I \leq x, A \leq S)$ respectively. In light traffic, we have $P(A > S) \cong 1$ and hence $P(I \leq x, A > S) \cong I(x)$, suggesting that $I(x)$ is dominantly determined by the first term on the right-hand side of (7). On the other hand, even if traffic is heavy, it is not unreasonable to expect $P(A > S) > \frac{1}{2}$ since $\rho < 1$. Hence, the linearization in (11) can produce fairly accurate approximations in general.

It is well known in integral-equation theory that the solution to (12) is uniquely determined by the first term (i.e., the "free term") of (12) (see, for example, Mikhlin 1957, 8–12). This unique solution can be derived by the following iterative scheme:

$$I_0(x) = \int_0^\infty [A(t + x) - A(t)]\, dS(t), \quad \text{and} \qquad (13)$$

$$I_{n+1}(x) = \int_0^\infty [A(t + x) - A(t)]\, dS(t)$$
$$+ \int_0^\infty \int_0^t \Big[I_n(t - u + x) - I_n(t - u)$$
$$+ \sigma \int_0^{t-u} [I_n(t - u - v + x)$$
$$- I_n(t - u - v)]\, dv \Big]\, dA(u)\, dS(t). \qquad (14)$$

Because the second term in (14) is bounded from above by $P(A \leq S) < 1$, the convergence of the iterations is guaranteed (Mikhlin).

2. THE *GH/G/1* QUEUE

The iteration in (13) and (14) provides a method for the numerical solution of (12). However, we show in this section that (12) can be solved explicitly for a $GH/G/1$ queue.

Theorem 3. *The idle periods of a $GH/G/1$ queue also follow a GH distribution. More specifically, if $A(t)$ assumes the form in* (1), *then*

$$I(t) = \sum_{i=1}^{N} r_i (1 - e^{-\lambda_i t}), \quad \textit{where} \tag{15}$$

$$r_i = a_i \int_0^\infty \left[e^{-\lambda_i t} \Big/ \sum_{k=1}^{N} a_k e^{-\lambda_k t} \right] dF(t), \tag{16}$$

with $F(t)$ being the distribution of the age of the interarrival time observed at the end of a randomly selected busy period.

Proof. Since an idle period is distributed as the excess of the interarrival time observed at the end of a busy period, we have

$$P(I > x) = \int_0^\infty P(A - t > x / A > t)\, dF(t).$$

Substituting (1) into the above and simplifying yields (15), completing the proof.

Theorem 3 shows that if we can determine $\{r_i\}$ in (15), then (15) gives the analytical solution of $I(x)$ for the $GH/G/1$ queue.

Theorem 4. *The solution of* (12) *for a $GH/G/1$ queue is given in* (15), *with*

$$r_i = a_i \tilde{S}(\lambda_i)/\Delta_i, \quad \textit{where} \tag{17}$$

$$\Delta_i = 1 + a_i \lambda_i \tilde{S}'(\lambda_i) - \sigma a_i \left[\frac{1}{\lambda_i} [1 - \tilde{S}(\lambda_i)] + \tilde{S}'(\lambda_i) \right]$$
$$- \sum_{k \neq i}^{N} \frac{a_k \lambda_k}{\lambda_k - \lambda_i} [\tilde{S}(\lambda_i) - \tilde{S}(\lambda_k)] - \frac{\sigma}{\lambda_i} \sum_{k \neq i}^{N} a_k \lambda_k$$
$$\cdot \left[\frac{1}{\lambda_k} [1 - \tilde{S}(\lambda_k)] - \frac{1}{\lambda_k - \lambda_i} [\tilde{S}(\lambda_i) - \tilde{S}(\lambda_k)] \right], \tag{18}$$

with $\tilde{S}'$ denoting the derivative of $\tilde{S}(t)$.

Proof. The basic result follows from Theorem 3. To show (17) and (18), we rewrite (12) as

$$\bar{I}(x) = \int_0^\infty \bar{A}(t + x)\, dS(t) + \int_0^\infty \int_0^t \left[\bar{I}(t - u + x) + \sigma \int_0^{t-u} \bar{I}(t - u - v + x)\, dv \right] dA(u)\, dS(t), \tag{19}$$

where $\bar{I}(x)$ is the tail distribution of I. Substituting (15) into (19) yields

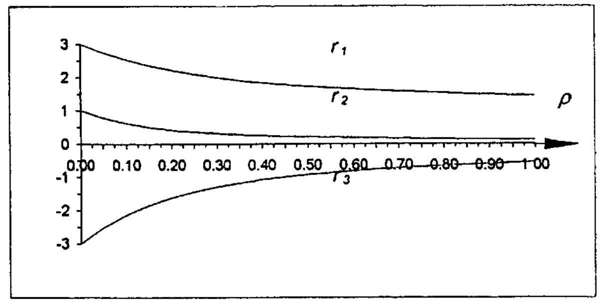

Figure 1. $\{r_i\}$ calculated from Equation (17).

$$\sum_{i=1}^{N} r_i e^{-\lambda_i x} = \int_0^\infty \sum_{i=1}^{N} a_i e^{-\lambda_i (t+x)}\, dS(t) + \int_0^\infty \int_0^t \left[\sum_{i=1}^{N} r_i e^{-\lambda_i (t+x-u)} + \sigma \int_0^{t-u} \sum_{i=1}^{N} r_i e^{-\lambda_i (t+x-u-v)}\, dv \right] \cdot \sum_{k=1}^{N} a_k \lambda_k e^{-\lambda_k u}\, du\, dS(t). \tag{20}$$

By working out the integral on the right-hand side of (20), we get

$$\sum_{i=1}^{N} r_i e^{-\lambda_i x} = \sum_{i=1}^{N} \Bigg\{ a_i \tilde{S}(\lambda_i) - r_i a_i \lambda_i \tilde{S}'(\lambda_i) + \sigma r_i a_i \left[\frac{1}{\lambda_i} [1 - \tilde{S}(\lambda_i)] + \tilde{S}'(\lambda_i) \right] + r_i \sum_{k \neq i}^{N} \frac{a_k \lambda_k}{\lambda_k - \lambda_i} [\tilde{S}(\lambda_i) - \tilde{S}(\lambda_k)] + \frac{\sigma r_i}{\lambda_i} \sum_{k \neq i}^{N} a_k \lambda_k \left[\frac{1}{\lambda_k} [1 - \tilde{S}(\lambda_k)] \times - \frac{1}{\lambda_k - \lambda_i} [\tilde{S}(\lambda_i) - \tilde{S}(\lambda_k)] \right] \Bigg\} e^{-\lambda_i x}. \tag{21}$$

The result follows by comparing the coefficients of $e^{-\lambda_i x}$ on both sides of (21) and subsequently solving for $\{r_i\}$. The proof is complete.

3. NUMERICAL RESULTS

In this section, we compare solutions obtained from (17) with simulation results. We use the typical GH distribution given by Botta and Harris as the interarrival distribution in our $GH/G/1$ model. That is,

$$A(x) = 3(1 - e^{-x}) + (-3)(1 - e^{-2x}) + (1 - e^{-3x}).$$

The service times are assumed to be uniform over $[0, b)$, with b varying from 0 to $2E(A)$. The solutions obtained from (17) under different traffic intensities are presented

Table I
$E(I)$ and $E(I^2)$ versus $E(I)_s$ and $E(I^2)_s$, where $R(f)$ is the relative error defined as $R(f) = 100 \times [E(f) - E(f_s)]/E(f)$

ρ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
$E(I)$	1.6635	1.5365	1.4461	1.3808	1.3323	1.2951	1.2658	1.2422	1.2227	1.2063
$E(I)_s$	1.6635	1.5365	1.4458	1.3810	1.3324	1.2954	1.2663	1.2427	1.2229	—
$E(I^2)$	4.1262	3.7021	3.4073	3.1972	3.0423	2.9243	2.8317	2.7571	2.6956	2.6441
$E(I^2)_s$	4.1259	3.7022	3.4064	3.1986	3.0417	2.9246	2.8335	2.7588	2.6964	—
$R(I)$	−0.0004	0.0032	0.0201	−0.0170	−0.0095	−0.0261	−0.0357	−0.0385	−0.0199	—
$R(I^2)$	0.0084	0.0040	0.0262	−0.0449	0.0190	−0.0115	−0.0649	−0.0643	−0.0291	

Table II
$E(W)$ and $E(W^2)$ versus $E(W)_s$ and $E(W^2)_s$

ρ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$E(W)$	0.0006441	0.007922	0.03316	0.09244	0.21176	0.44071	0.89294	1.90742	5.17647
$E(W)_s$	0.0006440	0.007913	0.03317	0.09247	0.21172	0.44024	0.89320	1.90786	5.16953
$E(W^2)$	0.0000835	0.002241	0.01593	0.06958	0.24355	0.78046	2.53728	9.61170	60.8129
$E(W^2)_s$	0.0000836	0.002240	0.01595	0.06965	0.24342	0.77912	2.54238	9.61167	60.8735
$R(W)$	0.027950	0.12080	−0.0149	−0.0312	0.0184	0.1075	−0.02893	−0.02288	−0.13398
$R(W^2)$	−0.148591	−0.05443	−0.1206	−0.1111	−0.0515	0.1710	−0.20102	−0.00039	−0.09949

in Figure 1. With $\{r_i\}$, moments of I and W are obtained. Tables I and II show the results derived via (17) and their simulated counterparts (subscripted by s). The simulations, coded in Pascal, were run on a 486-DX PC. Twenty million samples are collected for each simulation run.

Both Table I and Table II indicate that (17) gives very accurate approximations for the idle-period distribution even if the system is drastically different from both the $M/G/1$ and the $GI/M/1$ queues. The absolute values of both $R(I)$ and $R(I^2)$ are well below 0.1% and those of $R(W)$ and $R(W^2)$ below 0.3%. Similar accuracy is also observed in our computation experiments for other examples of GH distributions.

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