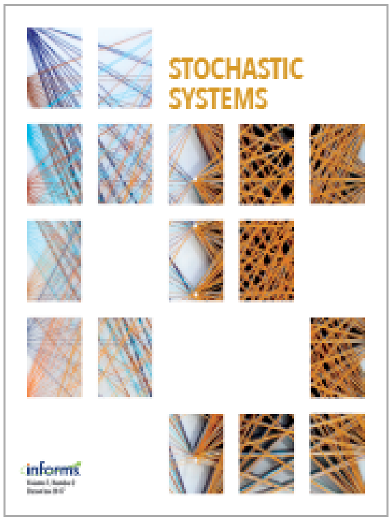

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Likelihood Ratio Gradient Estimation for Steady-State Parameters

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Abstract. We consider a discrete-time Markov chain $\mathbf{\Phi}$ on a general state-space X, whose transition probabilities are parameterized by a real-valued vector $\boldsymbol{\theta}$. Under the assumption that $\mathbf{\Phi}$ is geometrically ergodic with corresponding stationary distribution $\pi(\boldsymbol{\theta})$, we are interested in using Monte Carlo simulation for estimating the gradient $\nabla\alpha(\boldsymbol{\theta})$ of the steady-state expectation $\alpha(\boldsymbol{\theta}) = \pi(\boldsymbol{\theta}) f$. To this end, we first give sufficient conditions for the differentiability of $\alpha(\boldsymbol{\theta})$ and for the calculation of its gradient via a sequence of finite horizon expectations. We then propose two different likelihood ratio estimators and analyze their limiting behavior.

Keywords: simulation • gradient estimation • likelihood ratio

1. Introduction

Consider a discrete-time Markov chain $\mathbf{\Phi} = \{\Phi_k : k \geq 0\}$ on a general state space X whose transition kernel $P(\theta) = \{P(\theta, x, A) : x \in \mathsf{X},\ A \subseteq \mathsf{X}\}$ is parameterized by $\theta \in \Theta \subseteq \mathbb{R}$. We assume that Φ has a unique invariant distribution $\pi(\theta) = \{\pi(\theta, A) : A \subseteq \mathsf{X}\}$ and that we are interested in using Monte Carlo simulation to estimate the derivative of

$$\alpha(\theta) = \pi(\theta) f = \int_{\mathsf{X}} f(x)\pi(\theta, dx),$$

at a specific point $\theta_0 \in \theta$, for some function f such that $\pi(\theta)|f| < \infty$. Alternatively, we may be interested in estimating the derivative of the expected cumulative reward,

$$E_{\theta_0}\left[\sum_{j=0}^{n-1} f(\Phi_j)\right],$$

over the time horizon $[0, n)$, especially in situations when n is large.

Throughout the paper, we focus on the case when $\mathbf{\Phi}$ is geometrically ergodic, where the conditions for the existence of the derivative $\alpha'(\theta)$ are stated more concisely and are easier to verify (more general conditions for the existence of the gradient can be found in Rhee and Glynn 2016). However, rather than trying to exploit the regenerative structure that our assumptions provide, we focus on nonregenerative methods that require only simulating the chain's transitions under $P(\theta_0)$. This approach can be particularly useful in situations where estimators based on regenerative cycles are difficult to implement or the length of the cycles is too long to make them efficient.

We now give an informal description of the type of estimators that we study; the necessary assumptions will be made precise in the following section. From the strong law of large numbers, we can expect that when Φ evolves according to $P(\theta)$, then

$$\alpha_n = \frac{1}{n}\sum_{j=0}^{n-1} f(\Phi_j) \to \alpha(\theta) \qquad P(\theta) - \text{a.s.}$$

as $n \to \infty$ for any initial distribution. Moreover, if we assume that there exists a family of densities $\{p(\theta, x, y) : x, y \in \mathsf{X}\}$ such that the transition probabilities satisfy

$$P(\theta, x, dy) = p(\theta, x, y)P(\theta_0, x, dy),$$

then we can construct the likelihood ratio

$$L_n(\theta) = \prod_{j=1}^{n} p(\theta, \Phi_{j-1}, \Phi_j), \qquad n \geq 1$$

and use it to compute the expectation of α_n via the identity

$$E_\theta[\alpha_n] = E_{\theta_0}[\alpha_n L_n(\theta)]. \tag{1}$$

Here, $E_\theta[\,\cdot\,]$ denotes the expectation with respect to the transition probabilities $P(\theta)$ when the chain is started according to some fixed distribution μ. Details regarding this identity can be found in Glynn and L'Ecuyer (1995, theorem 1). Next, provided we have uniform integrability, we would have that

$$E_\theta[\alpha_n] \to \alpha(\theta)$$

as $n \to \infty$, and if we can further justify the exchange of derivative and expectation, then

$$\frac{d}{d\theta} E_\theta[\alpha_n] = E_{\theta_0}\left[\alpha_n L_n'(\theta)\right] \to \alpha'(\theta) \qquad n \to \infty. \tag{2}$$

If we are interested instead in estimating the derivative of the finite horizon expected cumulative reward, the corresponding estimator would be $n\alpha_n L_n'(\theta)$. For details on the estimation of gradients via likelihood ratios and other methods, as well as a variety of applications in finance, operations research, and engineering, we refer the reader to L'Ecuyer (1990), Glasserman (1991), and Fu (2006).

The observations made previously suggest that one could think of using $\alpha_n L_n'(\theta)$ with n sufficiently large as an estimator for $\alpha'(\theta)$. Unfortunately, $\alpha_n L_n'(\theta)$ turns out to be a poorly behaved estimator since it is of order $O_p(\sqrt{n})$. In fact, under some additional assumptions, $n^{-1/2}\alpha_n L_n'(\theta)$ converges in distribution as $n \to \infty$ to a normal random variable (see Proposition 1). Here and throughout the paper, we write $X_n = O_p(a_n)$ to mean that X_n/a_n is stochastically bounded. Note that the order of magnitude is important not only for computing $\alpha'(\theta)$ but also for computing finite horizon cumulative rewards whenever n is large, making the need for better estimators important to both cases. The first of our two proposed estimators, described in detail in Section 3, uses $L_n'(\theta_0)$ as a control variate to reduce the variance of $\alpha_n L_n'(\theta_0)$. The resulting estimator, after choosing the optimal control variate coefficient, is given by

$$(\alpha_n - \alpha(\theta_0))L_n'(\theta_0) \tag{3}$$

and is shown to be of order $O_p(1)$ in Proposition 2. A consistent estimator can then be obtained by computing the average of independent and identically distributed copies of the estimator in Equation (3). An estimator of this type has been shown in Hashemi et al. (2016) to be successful in practice, where it was used to compute the sensitivities in reaction networks.

Our second estimator, described in detail in Section 4, exploits the martingale structure of $L_n'(\theta_0)$ to obtain an alternative representation for $E_{\theta_0}[\alpha_n L_n'(\theta_0)]$ as the expectation of the discrete stochastic integral

$$\frac{1}{n}\sum_{k=1}^{n-1}\sum_{l=k}^{n-1} f(\Phi_l) D_k, \tag{4}$$

where the $\{D_k\}$ are the martingale differences in $\mathbb{R}$ satisfying $L_n'(\theta_0) = \sum_{k=1}^{n} D_k$. As is the case with $\alpha_n L_n'(\theta)$, this estimator turns out to be of order $O_p(\sqrt{n})$ (see Proposition 3), but can dramatically be improved by centering it with respect to $\alpha(\theta_0)$. The optimized estimator takes the form

$$\frac{1}{n}\sum_{k=1}^{n-1}\sum_{l=k}^{n-1}(f(\Phi_l) - \alpha(\theta_0)) D_k \tag{5}$$

and will be shown to be of order $O_p(1)$ as $n \to \infty$. Moreover, the analysis of the asymptotic variance of estimators (3) and (5), included in Section 5, suggests that (5) is a better estimator than (3).

It is worth mentioning that the two improved estimators (3) and (5) require the knowledge of $\alpha(\theta_0)$, which in practice may need to be approximated. However, this paper does not explore the impact that such an approximation could have on the behavior of the proposed estimators, and the assumption throughout is that $\alpha(\theta_0)$ is known.

The first part of the paper establishes sufficient conditions on the Markov chain Φ and the function f under which $\alpha'(\theta_0)$ exists and the following limit holds:

$$E_{\theta_0}[\alpha_n L_n'(\theta_0)] \to \alpha'(\theta_0) \qquad n \to \infty. \tag{6}$$

Conditions under which the exchange of derivative and expectation in Equation (2) is valid can be found in L'Ecuyer (1995), so we will not focus on this point. Once the convergence in Equation (6) is established in Section 2, we move on to the analysis of $\alpha_n L_n'(\theta_0)$ and the control variates estimator given in Equation (3); the corresponding limit theorems are stated in Section 3. The limit theorems for the the integral-type estimators given in Equations (4) and (5) are included in Section 4. To conclude the expository part of the paper, we compute in Section 5 the asymptotic variance of our two proposed estimators. Finally, Section 6 contains the majority of the proofs.

2. The Model

Throughout the paper, we consider a discrete-time Markov chain $\mathbf{\Phi} = \{\Phi_k : k \geq 0\}$ on a general state space X equipped with a countably generated σ-field $\mathscr{B}(\mathsf{X})$, and governed by the transition kernel $P(\boldsymbol{\theta}) = \{P(\boldsymbol{\theta}, x, A) : x \in \mathsf{X}, A \subseteq \mathsf{X}\}$ parameterized by $\boldsymbol{\theta} \in \mathbf{\Theta}$. Because the analysis of our estimators easily extends to the vector case, we assume from this point onward that $\mathbf{\Theta} \subseteq \mathbb{R}^d$ is a family of continuous parameters for $P(\boldsymbol{\theta})$, and adopt the boldface notation to denote vectors. Under the conditions given in this section, the Markov chain will possess a unique stationary distribution $\pi(\boldsymbol{\theta}) = \{\pi(\boldsymbol{\theta}, A) : A \subseteq \mathsf{X}\}$, and we are interested in obtaining a Monte Carlo estimator for either:

i. The gradient of the steady-state mean

$$\alpha(\boldsymbol{\theta}) = \pi(\boldsymbol{\theta}) f,$$

at some fixed point $\boldsymbol{\theta}_0 \in \mathbf{\Theta}$, for some function $f : \mathsf{X} \to \mathbb{R}$ such that $\pi(\boldsymbol{\theta})|f| < \infty$.

ii. Or, the gradient of the expected cumulative reward

$$E_{\boldsymbol{\theta}_0}\left[\sum_{j=0}^{n-1} f(\Phi_j)\right]$$

for large time horizons $[0, n)$.

In terms of notation, we use νf to denote the expectation of f with respect to measure ν, that is,

$$\nu f = \int_{\mathsf{X}} f(x)\nu(dx).$$

Similarly, for any Markov transition kernel P, we use

$$Pf(x) = \int_{\mathsf{X}} f(y) P(x, dy).$$

Whenever the context is clear, we denote the gradient of a function g at the point $\boldsymbol{\theta}_0$ by $\nabla g(\boldsymbol{\theta}_0)$, and when confusion may arise we will use the more precise notation $\nabla g(\boldsymbol{\theta})\big|_{\boldsymbol{\theta}=\boldsymbol{\theta}_0}$. The convention is to think of vectors as column vectors and to use $\mathbf{x}'$ to denote the transpose of $\mathbf{x}$.

Before giving the main set of assumptions for the Markov chain $\mathbf{\Phi}$, we include for completeness some basic norm definitions.

Definition 1. For $h : \mathsf{X} \to [1, \infty)$, let L_h^∞ denote the space of all measurable functions g on X such that $|g(x)|/h(x)$ is bounded in x, equipped with the norm

$$|g|_h = \sup_{x \in \mathsf{X}} \frac{|g(x)|}{h(x)}.$$

Definition 2. For $h : \mathsf{X} \to [1, \infty)$, define the *h-total variation norm* of any signed measure ν as

$$\|\nu\|_h = \sup_{g : |g| \leq h} |\nu g|.$$

Definition 3. For a positive function $V : \mathsf{X} \to [1,\infty)$ we define the *V-operator norm* distance between two Markov transition kernels P_1 and P_2 as

$$|||P_1 - P_2|||_V = \sup_{h \in L_V^\infty, |h|_V = 1} |(P_1 - P_2)h|_V.$$

Note: It can be shown that the h-operator norm distance can be written in terms of the h-total variation norm as

$$|||P_1 - P_2|||_V = \sup_{x \in \mathsf{X}} \frac{\|P_1(x,\cdot) - P_2(x,\cdot)\|_V}{V(x)}.$$

We can now state a set of sufficient conditions that will guarantee that $\nabla\alpha(\boldsymbol{\theta}_0)$ exists and that Equation (6) holds.

Assumption 1. *Let $\boldsymbol{\Phi} = \{\Phi_n : n \geq 0\}$ be a Markov chain taking values on X and having one-step transition probabilities $P(\boldsymbol{\theta}) = \{P(\boldsymbol{\theta}, x, dy) : x, y \in \mathsf{X}\}$, where $\boldsymbol{\theta} \in \Theta \subseteq \mathbb{R}^d$. Fix $\epsilon > 0$ and define $B_\epsilon(\boldsymbol{\theta}_0) = \{\boldsymbol{\theta} \in \Theta : \max_{1 \leq i \leq d} |\theta_i - \theta_{0,i}| < \epsilon\}$.*

i. *Suppose that $\boldsymbol{\Phi}$ is ψ-irreducible for all $\boldsymbol{\theta} \in B_\epsilon(\boldsymbol{\theta}_0)$.*

ii. *Suppose that for all $\boldsymbol{\theta} \in B_\epsilon(\boldsymbol{\theta}_0)$ there exist densities $p(\boldsymbol{\theta}, x, y)$, differentiable at $\boldsymbol{\theta}_0$ and such that*

$$P(\boldsymbol{\theta}, x, dy) = p(\boldsymbol{\theta}, x, y) P(\boldsymbol{\theta}_0, x, dy).$$

iii. *Suppose there exists a set $K \subseteq \mathscr{B}(\mathsf{X})$, $\delta > 0$, $m \in \mathbb{N}$, and a probability measure ν such that*

$$P^m(\boldsymbol{\theta}, x, dy) \geq \delta\nu(dy) \qquad \text{for all } x \in K,$$

for all $\boldsymbol{\theta} \in B_\epsilon(\boldsymbol{\theta}_0)$.

iv. *For the set K in* iii, *suppose there exists a function $V : \mathsf{X} \to [1,\infty)$ and constants $0 < \lambda < 1$, $b < \infty$, such that*

$$P(\boldsymbol{\theta})V(x) \leq \lambda V(x) + b1_K(x) \tag{7}$$

for all $\boldsymbol{\theta} \in B_\epsilon(\boldsymbol{\theta}_0)$.

v. *Let*

$$P^{(i)}(\boldsymbol{\theta}_0, x, dy) = \frac{\partial}{\partial\theta_i} p(\boldsymbol{\theta}, x, y)\bigg|_{\boldsymbol{\theta} = \boldsymbol{\theta}_0} P(\boldsymbol{\theta}_0, x, dy)$$

and $\mathbf{e}_i$ be the vector that has a 1 in the ith component and zeros elsewhere. Assume that for each $1 \leq i \leq d$, we have $|||P^{(i)}(\boldsymbol{\theta}_0)|||_V < \infty$ and

$$\lim_{h \to 0} \left|\left|\left| \frac{P(\boldsymbol{\theta}_0 + h\mathbf{e}_i) - P(\boldsymbol{\theta}_0)}{h} - P^{(i)}(\boldsymbol{\theta}_0) \right|\right|\right|_V = 0.$$

vi. *Suppose that $|g_{ii}|_V < \infty$ for each $1 \leq i \leq d$, where*

$$g_{ii}(x) = \int_{\mathsf{X}} \left(\frac{\partial}{\partial\theta_i} p(\boldsymbol{\theta}_0, x, y) \right)^2 P(\boldsymbol{\theta}_0, x, dy).$$

vii. *Suppose $|f|_{\sqrt{V}} < \infty$.*

Remark 1.

i. By iterating Equation (7), we obtain

$$E_{\boldsymbol{\theta},x}[V(\Phi_k)] = P^k(\boldsymbol{\theta})V(x) \leq \lambda^k V(x) + b\sum_{i=0}^{k-1} \lambda^i < \infty$$

for all $x \in \mathsf{X}$ and all $k \in \mathbb{N}$.

ii. A set K satisfying Assumption 1(iii) is said to be a *small set.*

The conditions in Assumption 1, which essentially impose geometric ergodicity (see e.g., Meyn and Tweedie 1993) of the chain $\boldsymbol{\Phi}$, are not necessary for the main convergence result of this section (Theorem 1), but have the advantage of allowing us to keep the arguments concise and focus on the estimators in the following sections. A similar set of conditions was used in Heidergott et al. (2006) (see Section 4.1). More general conditions ensuring the existence of the gradient outside of the geometric ergodicity setting can be found in Heidergott and Hordijk (2009), and, more recently, in Rhee and Glynn (2016).

We will now proceed to give some properties of $\mathbf{\Phi}$, for which we will need the following definition. Proofs not included immediately after the corresponding statement can be found in Section 6.

Definition 4. We say that the Markov chain $\mathbf{\Phi}$ is *h-ergodic* if $h : \mathsf{X} \to [1, \infty)$ and

i. The Markov chain $\mathbf{\Phi} = \{\Phi_k : k \geq 0\}$ is positive Harris recurrent with invariant probability π.
ii. The expectation πh is finite.
iii. For every initial condition $x \in \mathsf{X}$,

$$\lim_{k\to\infty} \|P^k(x,\cdot) - \pi\|_h = 0.$$

Lemma 1. *Under Assumption 1, the Markov chain $\mathbf{\Phi} = \{\Phi_k : k \geq 0\}$ is V-ergodic for each $\boldsymbol{\theta} \in B_\epsilon(\boldsymbol{\theta}_0)$. Furthermore, for all $1 \leq i \leq d$,*

$$\lim_{h\to 0} \|\pi(\boldsymbol{\theta}_0 + h\mathbf{e}_i) - \pi(\boldsymbol{\theta}_0)\|_V = 0.$$

Proof. Fix $\boldsymbol{\theta} \in B_\epsilon(\boldsymbol{\theta}_0)$. By Assumption 1(iv),

$$P(\boldsymbol{\theta})\hat{V}(x) \leq \hat{V}(x) - V(x) + \hat{b}1_K(x),$$

where $\hat{V} = (1-\lambda)^{-1}V$, $\hat{b} = (1-\lambda)^{-1}b$, and K is a small set. Then, by theorem 14.2.6 in Meyn and Tweedie (1993), $\mathbf{\Phi}$ is V-regular, which in turn implies, by theorem 14.3.3 in the same reference, that $\mathbf{\Phi}$ is V-ergodic. To establish the convergence in V-norm of the invariant probabilities, first note that Assumption 1(v) yields

$$\lim_{h\to 0} |||P(\boldsymbol{\theta}_0 + h\mathbf{e}_i) - P(\boldsymbol{\theta}_0)|||_V = 0,$$

from which it follows that $\|\pi(\boldsymbol{\theta}_0 + h\mathbf{e}_i) - \pi(\boldsymbol{\theta}_0)\|_V \to 0$ as $h \to 0$ (see Glynn and Meyn 1996, section 4.2). ∎

The main idea behind the analysis of the gradient of the likelihood ratio $L_n(\boldsymbol{\theta})$ is that under appropriate conditions each of its components is a square integrable martingale with respect to the family of filtrations generated by $\mathbf{\Phi}$. The next lemma makes this statement precise; its proof can be found in Section 6.

Lemma 2. *Suppose that Assumption 1 is satisfied. Define*

$$D_j^k = \frac{\partial}{\partial\theta_k}\log p(\boldsymbol{\theta}, \Phi_{j-1}, \Phi_j)\bigg|_{\boldsymbol{\theta}=\boldsymbol{\theta}_0} = \frac{\partial}{\partial\theta_k}p(\boldsymbol{\theta}_0, \Phi_{j-1}, \Phi_j), \qquad j = 1, 2, \ldots,$$

and $\mathbf{D}_j = (D_j^1, \ldots, D_j^d)$. Let $\mathcal{F}_j$ denote the σ-field generated by $\Phi_0, \Phi_1, \ldots, \Phi_j$. Then, under $P(\boldsymbol{\theta}_0)$,

$$\nabla L_n(\boldsymbol{\theta}_0) = \sum_{j=1}^n \mathbf{D}_j$$

is a square-integrable martingale in $\mathbb{R}^d$, that is, $\mathrm{M}_n^k = \sum_{j=1}^n \mathrm{D}_j^k$ ($\mathrm{M}_0^k \equiv 0$) is a square integrable-martingale adapted to $\mathcal{F}_k$ for each $k = 1, \ldots, d$.

The analysis of $\alpha_n \nabla L_n(\boldsymbol{\theta}_0)$ and of its expectation is based on a second martingale, one constructed via a solution $\hat{f}$ to Poisson's equation:

$$\hat{f} - P(\boldsymbol{\theta}_0)\hat{f} = f - \pi(\boldsymbol{\theta}_0)f. \tag{8}$$

Note that if this solution exists, then the centered estimator $\alpha_n - \alpha(\boldsymbol{\theta}_0)$ can be written as follows:

$$\begin{aligned} n(\alpha_n - \alpha(\boldsymbol{\theta}_0)) &= \sum_{k=1}^n (f(\Phi_{k-1}) - \pi(\boldsymbol{\theta}_0)f) \\ &= \sum_{k=1}^n \left(\hat{f}(\Phi_{k-1}) - P(\boldsymbol{\theta}_0)\hat{f}(\Phi_{k-1})\right) \\ &= \hat{f}(\Phi_0) - \hat{f}(\Phi_n) + \sum_{k=1}^n \left(\hat{f}(\Phi_k) - P(\boldsymbol{\theta}_0)\hat{f}(\Phi_{k-1})\right), \end{aligned} \tag{9}$$

where the terms $\hat{f}(\Phi_k) - P(\boldsymbol{\theta}_0)\hat{f}(\Phi_{k-1})$ can be shown to be martingale differences. It follows that provided $E_{\boldsymbol{\theta}_0}[|\hat{f}(\Phi_0) - \hat{f}(\Phi_n)|]/n \to 0$ as $n \to \infty$, we have that $E_{\boldsymbol{\theta}_0}[\alpha_n \nabla L_n(\boldsymbol{\theta}_0)] = E_{\boldsymbol{\theta}_0}[(\alpha_n - \alpha(\boldsymbol{\theta}_0))\nabla L_n(\boldsymbol{\theta}_0)]$ is the expectation of a product of two martingales. Lemma 3 gives precise properties of this second martingale.

Lemma 3. *Suppose that Assumption 1 is satisfied. Then, the expectations $\pi(\boldsymbol{\theta}_0)V$ and $\pi(\boldsymbol{\theta}_0)f^2$ are both finite, and there exists a solution $\hat{f}$ to Poisson's equation (Equation (8)) satisfying $|\hat{f}| \le c_1\sqrt{V}$ for some constant $c_1 < \infty$. Moreover, under $P(\boldsymbol{\theta}_0)$, $Z_n = \sum_{k=1}^n (\hat{f}(\Phi_k) - P(\boldsymbol{\theta}_0)\hat{f}(\Phi_{k-1}))$ is a square-integrable martingale adapted to $\mathscr{F}_k = \sigma(\Phi_0, \Phi_1, \ldots, \Phi_k)$.*

We are now ready to state our result for the convergence in Equation (6).

Theorem 1. *Under Assumption 1, $\alpha(\boldsymbol{\theta})$ is differentiable at $\boldsymbol{\theta}_0$ and*

$$E_{\boldsymbol{\theta}_0}[\alpha_n \nabla L_n(\boldsymbol{\theta}_0)] \to \nabla\alpha(\boldsymbol{\theta}_0) \qquad n \to \infty.$$

3. A First Likelihood Ratio Estimator

In view of Theorem 1, the remainder of the paper is devoted to the analysis of potential estimators for $E_{\boldsymbol{\theta}_0}[\alpha_n \nabla L_n(\boldsymbol{\theta}_0)]$. An obvious first choice would be to consider

$$\alpha_n \nabla L_n(\boldsymbol{\theta}_0) \tag{10}$$

itself. Unfortunately, as mentioned in the introduction, $\alpha_n \nabla L_n(\boldsymbol{\theta}_0)$ is poorly behaved for large values of n. In fact, under additional assumptions, $n^{-1/2}\alpha_n \nabla L_n(\boldsymbol{\theta}_0)$ converges in distribution to a multivariate normal random vector, which implies that $\alpha_n \nabla L_n(\boldsymbol{\theta}_0)$ is of order $O_p(\sqrt{n})$. This observation is a simple consequence of the following weak convergence result, which will also be helpful in the analysis of the estimators considered in Section 4.

Throughout the rest of the paper, let $D([0,1],\mathbb{R}^d)$ denote the space of right-continuous $\mathbb{R}^d$-valued functions on $[0,1]^d$ with left limits equipped with the standard Skorohod topology. We use $\Rightarrow$ to denote weak convergence. From this point forward, the Markov chain $\boldsymbol{\Phi}$ is always assumed to evolve according to $P(\boldsymbol{\theta}_0)$.

Theorem 2. *Suppose that Assumption 1 is satisfied and let $\hat{f}$ be the solution to Poisson's equation (Equation (8)) from Lemma 3. Define the functions g_{ij} according to*

$$g_{ij}(x) = \int_{\mathsf{X}} \frac{\partial}{\partial\theta_i} p(\boldsymbol{\theta}_0, x, y) \frac{\partial}{\partial\theta_j} p(\boldsymbol{\theta}_0, x, y) P(\boldsymbol{\theta}_0, x, dy), \qquad 1 \le i, j \le d,$$
$$g_{i0}(x) = g_{0i}(x) = \int_{\mathsf{X}} \hat{f}(y) \frac{\partial}{\partial\theta_i} p(\boldsymbol{\theta}_0, x, y) P(\boldsymbol{\theta}_0, x, dy), \qquad 1 \le i \le d,$$
$$g_{00}(x) = \int_{\mathsf{X}} \left(\hat{f}(y) - P(\boldsymbol{\theta}_0)\hat{f}(x)\right)^2 P(\boldsymbol{\theta}_0, x, dy) = P(\boldsymbol{\theta}_0)\hat{f}^2(x) - \left(P(\boldsymbol{\theta}_0)\hat{f}(x)\right)^2.$$

Let $G(x) \in \mathbb{R}^{(d+1)\times(d+1)}$ be the matrix whose (i,j)th element is $g_{ij}(x)$ for $0 \le i, j \le d$. Then,

$$\left(n^{-1/2}\lfloor n\cdot\rfloor(\alpha_{\lfloor n\cdot\rfloor} - \alpha(\boldsymbol{\theta}_0)),\ n^{-1/2}\nabla L_{\lfloor n\cdot\rfloor}(\boldsymbol{\theta}_0)'\right) \Rightarrow \mathbf{B}' \qquad n \to \infty,$$

in $D([0,1],\mathbb{R}^{d+1})$, where $\mathbf{B}(t) = (B_0(t), B_1(t), \ldots, B_d(t))'$ is a $(d+1)$-dimensional mean zero Brownian motion with covariance matrix $\pi(\boldsymbol{\theta}_0)G = (\pi(\boldsymbol{\theta}_0)g_{ij})$.

In view of this theorem, we have the following result for $n^{-1/2}\alpha_n \nabla L_n(\boldsymbol{\theta}_0)$.

Proposition 1. *Suppose that Assumption 1 is satisfied and define for $1 \le i, j \le d$ the functions g_{ij} according to Theorem 2. Then,*

$$n^{-1/2}\alpha_n \nabla L_n(\boldsymbol{\theta}_0) \Rightarrow \alpha(\boldsymbol{\theta}_0)\hat{\mathbf{B}}(1) \qquad n \to \infty,$$

where $\hat{\mathbf{B}}$ is a d-dimensional mean zero Brownian motion with covariance matrix $\Sigma = (\sigma_{ij})$, where $\sigma_{ij} = \pi(\boldsymbol{\theta}_0)g_{ij}$ for $1 \le i, j \le d$.

Proof. By Lemma 1 $\boldsymbol{\Phi}$ is V-ergodic, and since $|f|_{\sqrt{V}} < \infty$, we have $\pi(\boldsymbol{\theta}_0)|f| < \infty$. Then, by theorem 17.0.1 in Meyn and Tweedie (1993),

$$\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} f(\Phi_j) = \pi(\boldsymbol{\theta}_0)f = \alpha(\boldsymbol{\theta}_0) \qquad \text{a.s. } P(\boldsymbol{\theta}_0).$$

By Theorem 2, we have

$$n^{-1/2}\nabla L_n(\boldsymbol{\theta}_0) \Rightarrow \hat{\mathbf{B}}(1),$$

where $\hat{\mathbf{B}}(t) = (B_1(t), \dots, B_d(t))'$ is a d-dimensional mean zero Brownian motion with covariance matrix Σ. It follows by Slutsky's lemma that

$$n^{-1/2}\alpha_n \nabla L_n(\boldsymbol{\theta}_0) \Rightarrow \alpha(\boldsymbol{\theta}_0)\hat{\mathbf{B}}(1) \qquad n \to \infty. \quad \blacksquare$$

Proposition 1 shows that $\alpha_n \nabla L_n(\boldsymbol{\theta}_0)$ is of order $O_p(\sqrt{n})$, making it an inefficient estimator for large values of n. To reduce its variance, we propose using as a control variate $\nabla L_n(\boldsymbol{\theta}_0)$, that is, we seek an estimator of the form

$$\mathbf{Y}(C) \triangleq \alpha_n \nabla L_n(\boldsymbol{\theta}_0) + C\nabla L_n(\boldsymbol{\theta}_0),$$

where C is a $d \times d$ constant matrix. Let $\Sigma_{\mathbf{Y}(C)}$ be the covariance matrix of $\mathbf{Y}(C)$,

$$\Sigma_{\mathbf{Y}(C)} = E_{\boldsymbol{\theta}_0}[(\mathbf{Y}(C) - E_{\boldsymbol{\theta}_0}[\mathbf{Y}(C)])(\mathbf{Y}(C) - E_{\boldsymbol{\theta}_0}[\mathbf{Y}(C)])'].$$

Our goal is to minimize the so-called generalized variance of $\mathbf{Y}(C)$, defined as the determinant of $\Sigma_{\mathbf{Y}(C)}$. The optimal choice for C is given by

$$C_n^* = E_{\boldsymbol{\theta}_0}[(\alpha_n \nabla L_n(\boldsymbol{\theta}_0) - E_{\boldsymbol{\theta}_0}[\alpha_n \nabla L_n(\boldsymbol{\theta}_0)])(\nabla L_n(\boldsymbol{\theta}_0))'](E_{\boldsymbol{\theta}_0}[(\nabla L_n(\boldsymbol{\theta}_0))(\nabla L_n(\boldsymbol{\theta}_0))'])^{-1}$$

(see Rubinstein and Marcus 1985). In the notation of Lemma 2,

$$C_n^* = E_{\boldsymbol{\theta}_0}[(\alpha_n \mathbf{M}_n - E_{\boldsymbol{\theta}_0}[\alpha_n \mathbf{M}_n])\mathbf{M}_n'](E_{\boldsymbol{\theta}_0}[\mathbf{M}_n\mathbf{M}_n'])^{-1} = E_{\boldsymbol{\theta}_0}[\alpha_n \mathbf{M}_n \mathbf{M}_n'](E_{\boldsymbol{\theta}_0}[\mathbf{M}_n\mathbf{M}_n'])^{-1},$$

where we have used the observation that $E_{\boldsymbol{\theta}_0}[\mathbf{M}_n'] = 0$ since $\{\mathbf{M}_n : n \geq 0\}$ is a mean-zero martingale.

It can be shown (following the same arguments used in the proof of Theorem 2) that

$$\frac{1}{n}E_{\boldsymbol{\theta}_0}[\alpha_n \mathbf{M}_n \mathbf{M}_n'] \to \alpha(\boldsymbol{\theta}_0)\Sigma \qquad \text{and} \qquad \frac{1}{n}E_{\boldsymbol{\theta}_0}[\mathbf{M}_n\mathbf{M}_n'] \to \Sigma$$

as $n \to \infty$. Therefore, $C_n^* \to \alpha(\boldsymbol{\theta}_0)\Sigma\Sigma^{-1} = \alpha(\boldsymbol{\theta}_0)I$, where I is the identity matrix of $\mathbb{R}^{d\times d}$. We then have that $\alpha(\boldsymbol{\theta}_0)I$ is the asymptotically optimal choice for the control variate coefficient and our new suggested estimator is

$$\mathbf{Y}(C_n^*) = (\alpha_n - \alpha(\boldsymbol{\theta}_0))\nabla L_n(\boldsymbol{\theta}_0).$$

Using Theorem 2 again, we obtain the following convergence result.

Proposition 2. *Suppose that Assumption 1 is satisfied and let $\hat{f}$ be the solution to Poisson's equation (Equation (8)) from Lemma 3. Define the functions g_{ij} for $0 \leq i,j \leq d$ according to Theorem 2. Then,*

$$\mathbf{Y}(C_n^*) \Rightarrow B_0(1)\hat{\mathbf{B}}(1) \qquad n \to \infty,$$

where $\mathbf{B}(t) = (B_0(t), B_1(t), \dots, B_d(t))'$ is a $(d+1)$-dimensional mean zero Brownian motion with covariance matrix $\Sigma = (\sigma_{ij})$, where $\sigma_{ij} = \pi(\boldsymbol{\theta}_0)g_{ij}$ for $0 \leq i,j \leq d$, and $\hat{\mathbf{B}}(t) = (B_1(t), \dots, B_d(t))'$.

Proof. By Theorem 2, we have

$$(n^{1/2}(\alpha_n - \alpha(\boldsymbol{\theta}_0)),\, n^{-1/2}\nabla L_n(\boldsymbol{\theta}_0)') \Rightarrow \mathbf{B}(1)' \qquad n \to \infty,$$

where $\mathbf{B}(t) = (B_0(t), B_1(t), \dots, B_d(t))'$ is a $(d+1)$-dimensional mean zero Brownian motion with covariance matrix $\Sigma = \pi(\boldsymbol{\theta}_0)G$. Then, by the continuous mapping principle,

$$(\alpha_n - \alpha(\boldsymbol{\theta}_0))\nabla L_n(\boldsymbol{\theta}_0) \Rightarrow B_0(1)\hat{\mathbf{B}}(1) \qquad n \to \infty. \quad \blacksquare$$

We conclude that $\mathbf{Y}(C_n^*)$ is of order $O_p(1)$ as $n \to \infty$, and therefore is a more suitable estimator for $E_{\boldsymbol{\theta}_0}[\alpha_n \nabla L_n(\boldsymbol{\theta}_0)]$. In the next section, we consider other alternatives.

4. An Integral-Type Estimator

As mentioned in the introduction, our second proposed estimator is obtained by first deriving an alternative representation for $E_{\boldsymbol{\theta}_0}[\alpha_n \nabla L_n(\boldsymbol{\theta}_0)]$ in terms of a discrete stochastic integral. More precisely, we exploit the martingale properties of $\nabla L_n(\boldsymbol{\theta}_0)$ to obtain that

$$\begin{aligned} E_{\boldsymbol{\theta}_0}[\alpha_n \nabla L_n(\boldsymbol{\theta}_0)] &= E_{\boldsymbol{\theta}_0}\left[\frac{1}{n}\sum_{l=0}^{n-1} f(\Phi_l) \sum_{k=1}^{n} \mathbf{D}_k\right] \\ &= \frac{1}{n}\sum_{k=1}^{n}\sum_{l=0}^{k-1} E_{\boldsymbol{\theta}_0}[f(\Phi_l) E_{\boldsymbol{\theta}_0}[\mathbf{D}_k | \mathcal{F}_{k-1}]] + E_{\boldsymbol{\theta}_0}\left[\frac{1}{n}\sum_{k=1}^{n-1}\sum_{l=k}^{n-1} f(\Phi_l)\mathbf{D}_k\right] \\ &= E_{\boldsymbol{\theta}_0}\left[\frac{1}{n}\sum_{k=1}^{n-1}\sum_{l=k}^{n-1} f(\Phi_l)\mathbf{D}_k\right], \end{aligned}$$

where $D_k^i = (\partial/\partial\theta^i) p(\boldsymbol{\theta}_0, \Phi_{k-1}, \Phi_k)$ and $\mathbf{D}_k = (D_k^1, \ldots, D_k^d)'$. This suggests using

$$\mathbf{Y}_n \triangleq \frac{1}{n}\sum_{k=1}^{n-1}\sum_{l=k}^{n-1} f(\Phi_l)\mathbf{D}_k \tag{11}$$

as an estimator for $E_{\boldsymbol{\theta}_0}[\alpha_n \nabla L_n(\boldsymbol{\theta}_0)]$.

Unfortunately, just as the estimator $\alpha_n \nabla L_n(\boldsymbol{\theta}_0)$, the estimator $\mathbf{Y}_n$ defined in Equation (11) is of order $O_p(\sqrt{n})$ as $n \to \infty$, as the following result shows. Its proof is a straightforward consequence of Theorem 2 again.

Proposition 3. *Suppose that Assumption 1 is satisfied and define for $1 \le i, j \le d$ the functions g_{ij} according to Theorem 2. Then,*

$$n^{-1/2}\mathbf{Y}_n \Rightarrow \int_0^1 \alpha(\boldsymbol{\theta}_0)(1-s) I d\hat{\mathbf{B}}(s) \qquad n \to \infty,$$

where $\hat{\mathbf{B}}$ is a d-dimensional mean zero Brownian motion with covariance matrix $\Sigma = (\sigma_{ij})$, with $\sigma_{ij} = \pi(\boldsymbol{\theta}_0) g_{ij}$ for $1 \le i, j \le d$, and I is the identity matrix of $\mathbb{R}^{d\times d}$.

As before, we can try to solve the problem of the unboundedness of Y_n by using a centered estimator of the form

$$\mathbf{Y}_n^* \triangleq \frac{1}{n}\sum_{k=1}^{n-1}\sum_{l=k}^{n-1}(f(\Phi_l) - \alpha(\boldsymbol{\theta}_0))\mathbf{D}_k.$$

This modification turns out to be the right one, and we obtain the following convergence result for this new estimator.

Proposition 4. *Suppose that Assumption 1 is satisfied, and define for $0 \le i, j \le d$ the functions g_{ij} according to Theorem 2. Then,*

$$\mathbf{Y}_n^* \Rightarrow \int_0^1 (B_0(1) - B_0(s)) I \, d\hat{\mathbf{B}}(s) \qquad n \to \infty,$$

where $\mathbf{B}(t) = (B_0(t), B_1(t), \ldots, B_d(t))'$ is a $(d+1)$-dimensional mean zero Brownian motion with covariance matrix $\Sigma = (\sigma_{ij})$, where $\sigma_{ij} = \pi(\boldsymbol{\theta}_0) g_{ij}$ for $0 \le i, j \le d$, $\hat{\mathbf{B}}(t) = (B_1(t), \ldots, B_d(t))'$, and I is the identity matrix of $\mathbb{R}^{d\times d}$.

Proof. Let $S_n(t) = \lfloor nt \rfloor (\alpha_{\lfloor nt \rfloor} - \alpha(\boldsymbol{\theta}_0)) = \sum_{j=0}^{\lfloor nt \rfloor - 1}(f(\Phi_j) - \alpha(\boldsymbol{\theta}_0))$, and note that by Theorem 2 we have

$$n^{-1/2}(S_n, \nabla L_{\lfloor n\cdot \rfloor}(\boldsymbol{\theta}_0)') \Rightarrow \mathbf{B}' \qquad n \to \infty,$$

in $D([0,1], \mathbb{R}^{d+1})$, where $\mathbf{B}(t) = (B_0(t), B_1(t), \ldots, B_d(t))'$ is a $(d+1)$-dimensional mean zero Brownian motion with covariance matrix Σ. Now, define the process $\hat{W}_n(t) = \sum_{j=\lfloor nt \rfloor}^{n-1}(f(\Phi_j) - \alpha(\boldsymbol{\theta}_0))$ with the convention that $\hat{W}_n(1) \equiv 0$. It follows that $\hat{W}_n(t) = S_n(1) - S_n(t)$ and the continuous mapping theorem gives

$$n^{-1/2}\left(\hat{W}_n(\cdot), \nabla L_{\lfloor n\cdot \rfloor}(\boldsymbol{\theta}_0)'\right) \Rightarrow (B_0(1) - B_0(\cdot), B_1(\cdot), \ldots, B_d(\cdot)) \qquad n \to \infty \tag{12}$$

in $D([0,1], \mathbb{R}^{d+1})$.

Next, define the processes $X_n(t) = n^{-1/2}\hat{W}_n(t)I$, $X(t) = (B_0(1) - B_0(t))I$, $\mathbf{Z}_n(t) = n^{-1/2}\nabla L_{\lfloor nt\rfloor}(\boldsymbol{\theta}_0)$, and $\mathbf{Z}(t) = (B_1(t), \dots, B_d(t))'$. Let $\mathcal{G}_{n,t} = \mathcal{F}_{\lfloor nt\rfloor}$. Clearly, $\{X_n(t) : t \in [0,1]\}$ and $\{\mathbf{Z}_n(t) : t \in [0,1]\}$ are $\{\mathcal{G}_{n,t}\}$-adapted and $\mathbf{Z}_n(t)$ is a $\{\mathcal{G}_{n,t}\}$-martingale. Also, for $t_i = i/n$,

$$\mathbf{Y}_n^* = \frac{1}{n}\sum_{k=1}^{n-1}\sum_{l=k}^{n-1}(f(\Phi_l) - \alpha(\boldsymbol{\theta}_0))\mathbf{D}_k = \sum_{k=1}^{n-1} X_n(t_{k-1})(\mathbf{Z}_n(t_k) - \mathbf{Z}_n(t_{k-1})) = \int_0^{1-\frac{1}{n}} X_n(s-)\, d\mathbf{Z}_n(s).$$

The same steps used in the proof of Proposition 3 show that the conditions of theorem 2.7 of Kurtz and Protter (1991) are satisfied, and we obtain that

$$\left(X_n, \mathbf{Z}_n, \int_0^{1-\frac{1}{n}} X_n\, d\mathbf{Z}_n\right) \Rightarrow \left(X, \mathbf{Z}, \int_0^1 X\, d\mathbf{Z}\right) \qquad n \to \infty$$

in $D([0,1], \mathbb{R}^{d\times d} \times \mathbb{R}^d \times \mathbb{R}^d)$. ■

It follows that $\mathbf{Y}_n^*$ is a suitable estimator for $E_{\boldsymbol{\theta}_0}[\alpha_n \nabla L_n(\boldsymbol{\theta}_0)]$. It remains to compare $\mathbf{Y}_n^*$ to $\mathbf{Y}(C_n^*)$ from Section 3.

5. Computation of the Asymptotic Variance

Sections 3 and 4 provide details on two potential estimators for $E_{\boldsymbol{\theta}_0}[\alpha_n \nabla L_n(\boldsymbol{\theta}_0)]$, namely,

$$\mathbf{Y}(C_n^*) = (\alpha_n - \alpha(\boldsymbol{\theta}_0))\nabla L_n(\boldsymbol{\theta}_0)$$

and

$$\mathbf{Y}_n^* = \frac{1}{n}\sum_{k=1}^{n-1}\sum_{j=k}^{n-1}(f(\Phi_j) - \alpha(\boldsymbol{\theta}_0))\mathbf{D}_k,$$

where $\mathbf{D}_k = \nabla p(\boldsymbol{\theta}_0, \Phi_{k-1}, \Phi_k)$. Both of these estimators have the property, under Assumption 1, that their expectation converges to $\nabla\alpha(\boldsymbol{\theta}_0)$, that is,

$$E_{\boldsymbol{\theta}_0}[\mathbf{Y}(C_n^*)] \to \nabla\alpha(\boldsymbol{\theta}_0) \qquad \text{and} \qquad E_{\boldsymbol{\theta}_0}[\mathbf{Y}_n^*] \to \nabla\alpha(\boldsymbol{\theta}_0),$$

as $n \to \infty$ (Theorem 1), and, unlike the estimators given in Equations (10) and (11), they have order $O_p(1)$ (Propositions 2 and 4). For comparison purposes, in this section, we compute the variance of these limiting distributions.

First, by Proposition 2, we have $\mathbf{Y}(C_n^*) \Rightarrow Z_0\,\hat{\mathbf{Z}}$, where $\mathbf{Z} = (Z_0, Z_1, \dots, Z_d)'$ is a $(d+1)$-dimensional multivariate normal with covariance matrix Σ and $\hat{\mathbf{Z}} = (Z_1, \dots, Z_d)'$. Therefore, by Isserlis' theorem, the (i,j)th component, $1 \le i,j \le d$, of the limiting distribution's covariance matrix is given by

$$\begin{aligned}\mathrm{Cov}_{\boldsymbol{\theta}_0}(Z_0\,\hat{\mathbf{Z}})_{ij} &= E_{\boldsymbol{\theta}_0}[Z_0^2 Z_i Z_j] - E_{\boldsymbol{\theta}_0}[Z_0 Z_i]E_{\boldsymbol{\theta}_0}[Z_0 Z_j]\\ &= (\sigma_{00}\sigma_{ij} + 2\sigma_{0i}\sigma_{0j}) - \sigma_{0i}\sigma_{0j}\\ &= \sigma_{00}\sigma_{ij} + \sigma_{0i}\sigma_{0j}. \end{aligned} \tag{13}$$

Similarly, by Proposition 4, we have $\mathbf{Y}_n^* \Rightarrow \int_0^1 (B_0(1) - B_0(s))Id\hat{\mathbf{B}}(s)$, where $\mathbf{B}(t) = (B_0(t), B_1(t), \dots, B_d(t))'$ is a $(d+1)$-dimensional Brownian motion with covariance matrix Σ and $\hat{\mathbf{B}}(t) = (B_1(t), \dots, B_d(t))'$. Since the calculation of the covariance of the limiting distribution in this case is somewhat lengthier, we state the result in the following lemma and postpone its proof to Section 6.

Lemma 4. *Let $\mathbf{B}(t) = (B_0(t), B_1(t), \dots, B_d(t))'$ be a $(d+1)$-dimensional Brownian motion with covariance matrix Σ. Let $\mathcal{I} = \int_0^1 (B_0(1) - B_0(s))I\, d\hat{\mathbf{B}}(s)$, where I is the $\mathbb{R}^{d\times d}$ identity matrix and $\hat{\mathbf{B}}(t) = (B_1(t), \dots, B_d(t))'$. Then, the (i,j)th component of the covariance matrix of $\mathcal{I}$ is given by*

$$\mathrm{Cov}_{\boldsymbol{\theta}_0}(\mathcal{I})_{ij} = \frac{\sigma_{00}\sigma_{ij}}{2}. \tag{14}$$

To simplify the notation, let

$$A = \mathrm{Cov}_{\boldsymbol{\theta}_0}(Z_0\,\hat{\mathbf{Z}}) \qquad \text{and} \qquad B = \mathrm{Cov}_{\boldsymbol{\theta}_0}(\mathcal{I})$$

denote the asymptotic covariances of $\mathbf{Y}(C_n^*)$ and $\mathbf{Y}_n^*$, respectively. Next, define $\mathbf{v} = (\sigma_{01}, \sigma_{02}, \dots, \sigma_{0d})'$ and note that Equations (13) and (14) give

$$A = 2B + \mathbf{v}\mathbf{v}'.$$

We now compare the generalized variances of the two estimators, that is, the determinants of their covariance matrices. Provided B is positive definite, we obtain

$$\begin{aligned}
\det(A) &= \det(2B + \mathbf{v}\mathbf{v}') \\
&= \det(2B)\det\left(I + \frac{1}{2}B^{-1}\mathbf{v}\mathbf{v}'\right) \\
&= \det(2B)\left(1 + \frac{1}{2}\mathbf{v}'B^{-1}\mathbf{v}\right) \qquad \text{(by Sylvester's determinant theorem)} \\
&= 2^d \det(B)\left(1 + \frac{1}{2}\mathbf{v}'B^{-1}\mathbf{v}\right),
\end{aligned}$$

where I is the $\mathbb{R}^{d\times d}$ identity matrix. Since B is positive definite, so is B^{-1}, and therefore $\mathbf{v}'B^{-1}\mathbf{v} \geq 0$. We conclude that

$$\det(A) \geq 2^d \det(B),$$

which suggests that $\mathbf{Y}_n^*$ is a better estimator for $\nabla\alpha(\boldsymbol{\theta}_0)$ than $\mathbf{Y}(C_n^*)$.

6. Proofs

This last section of the paper contains all the proofs that were not given in the prior sections. The first one corresponds to the martingale properties of $\nabla L_n(\boldsymbol{\theta}_0)$.

Proof of Lemma 2. We start by noting that for any $\boldsymbol{\theta} \in \Theta$ and $1 \leq i \leq d$,

$$\begin{aligned}
\frac{\partial}{\partial\theta_i} L_n(\boldsymbol{\theta}) &= \frac{\partial}{\partial\theta_i}\prod_{j=1}^n p(\boldsymbol{\theta}, \Phi_{j-1}, \Phi_j) = \sum_{j=1}^n \frac{L_n(\boldsymbol{\theta})}{p(\boldsymbol{\theta}, \Phi_{j-1}, \Phi_j)} \cdot \frac{\partial}{\partial\theta_i} p(\boldsymbol{\theta}, \Phi_{j-1}, \Phi_j) \\
&= L_n(\boldsymbol{\theta}) \sum_{j=1}^n \frac{\partial}{\partial\theta_i} \log p(\boldsymbol{\theta}, \Phi_{j-1}, \Phi_j).
\end{aligned}$$

Since $L_n(\boldsymbol{\theta}_0) \equiv 1$, it follows that

$$\nabla L_n(\boldsymbol{\theta}_0) = \sum_{j=1}^n \mathbf{D}_j.$$

Next, note that for any fixed $x \in \mathsf{X}$, we have

$$\begin{aligned}
&\left| \frac{\partial}{\partial\theta_i}\int_{\mathsf{X}} p(\boldsymbol{\theta}, x, y) P(\boldsymbol{\theta}_0, x, dy)\right|_{\boldsymbol{\theta}=\boldsymbol{\theta}_0} - \int_{\mathsf{X}} \frac{\partial}{\partial\theta_i} p(\boldsymbol{\theta}_0, x, y) P(\boldsymbol{\theta}_0, x, dy) \Bigg| \\
&\quad= \lim_{h\to 0}\left| \int_{\mathsf{X}} \left(\frac{p(\boldsymbol{\theta}_0 + h\mathbf{e}_i, x, y) - p(\boldsymbol{\theta}_0, x, y)}{h} - \frac{\partial}{\partial\theta_i} p(\boldsymbol{\theta}_0, x, y)\right) P(\boldsymbol{\theta}_0, x, dy)\right| \\
&\quad\leq \lim_{h\to 0} \sup_{g:|g|\leq V} \left| \int_{\mathsf{X}} g(y) \left(\frac{p(\boldsymbol{\theta}_0 + h\mathbf{e}_i, x, y) - p(\boldsymbol{\theta}_0, x, y)}{h} - \frac{\partial}{\partial\theta_i} p(\boldsymbol{\theta}_0, x, y)\right) P(\boldsymbol{\theta}_0, x, dy)\right| \\
&\quad\leq V(x) \lim_{h\to 0} \left|\left|\left| \frac{P(\boldsymbol{\theta}_0 + h\mathbf{e}_i) - P(\boldsymbol{\theta}_0)}{h} - P^{(i)}(\boldsymbol{\theta}_0) \right|\right|\right|_V.
\end{aligned}$$

Therefore, by Assumption 1(v),

$$\frac{\partial}{\partial\theta_i}\int_{\mathsf{X}} p(\boldsymbol{\theta}, x, y) P(\boldsymbol{\theta}_0, x, dy)\bigg|_{\boldsymbol{\theta}=\boldsymbol{\theta}_0} = \int_{\mathsf{X}} \frac{\partial}{\partial\theta_i} p(\boldsymbol{\theta}_0, x, y) P(\boldsymbol{\theta}_0, x, dy)$$

for all $x \in \mathsf{X}$. It follows that

$$\begin{aligned} E_{\boldsymbol{\theta}_0}[D_k^i|\mathcal{F}_{k-1}] &= E_{\boldsymbol{\theta}_0}\left[\frac{\partial}{\partial\theta_i}p(\boldsymbol{\theta}_0,\Phi_{k-1},\Phi_k)\middle|\Phi_{k-1}\right] \\ &= \int_{\mathsf{X}}\frac{\partial}{\partial\theta_i}p(\boldsymbol{\theta}_0,\Phi_{k-1},y)P(\boldsymbol{\theta}_0,\Phi_{k-1},dy) \\ &= \frac{\partial}{\partial\theta_i}\int_{\mathsf{X}}p(\boldsymbol{\theta},\Phi_{k-1},y)P(\boldsymbol{\theta}_0,\Phi_{k-1},dy)\bigg|_{\boldsymbol{\theta}=\boldsymbol{\theta}_0} \\ &= 0 \qquad\qquad \text{(since the integral is equal to one for all } \boldsymbol{\theta}), \end{aligned}$$

which establishes that $\mathbf{M}_n \triangleq \nabla L_n(\boldsymbol{\theta}_0)$ is a martingale. To see that it is square integrable, let $\mathbf{M}_n = (M_n^1, \dots, M_n^d)'$ and note that $E_{\boldsymbol{\theta}_0,x}[(M_n^i)^2] = \sum_{k=1}^n E_{\boldsymbol{\theta}_0}[(D_k^i)^2]$ and

$$\begin{aligned} E_{\boldsymbol{\theta}_0}[(D_k^i)^2] &= E_{\boldsymbol{\theta}_0}[E_{\boldsymbol{\theta}_0}[(D_k^i)^2|\mathcal{F}_{k-1}]] \\ &= E_{\boldsymbol{\theta}_0}\left[\int_{\mathsf{X}}\left(\frac{\partial}{\partial\theta_i}p(\boldsymbol{\theta}_0,\Phi_{k-1},y)\right)^2 P(\boldsymbol{\theta}_0,\Phi_{k-1},dy)\right] \\ &= E_{\boldsymbol{\theta}_0}[g_{ii}(\Phi_{k-1})], \end{aligned}$$

which is finite since $|g_{ii}|_V < \infty$ by Assumption 1(vi) and $E_{\boldsymbol{\theta}_0}[V(\Phi_{k-1})] < \infty$. ■

The next proof corresponds to the martingale constructed using the solution to Poisson's equation.

Proof of Lemma 3. We start by pointing out that by Lemma 1, the chain $\boldsymbol{\Phi}$ is V-ergodic for each $\boldsymbol{\theta} \in B_\epsilon(\boldsymbol{\theta}_0)$, and therefore, $\pi(\boldsymbol{\theta}_0)V < \infty$. Also, by Assumption 1(vii), we have that $\pi(\boldsymbol{\theta}_0)f^2 < \infty$.

We now proceed to show the existence of a solution $\hat{f}$ to Poisson's equation. To this end, note that by Jensen's inequality and Assumption 1(iv) we have

$$P(\boldsymbol{\theta}_0)\sqrt{V(x)} \le \sqrt{P(\boldsymbol{\theta}_0)V(x)} \le \sqrt{\lambda V(x) + b1_K(x)} \le \sqrt{\lambda V(x)} + \sqrt{b}1_K(x). \tag{15}$$

Next, define $\tilde{V}(x) = (1-\sqrt{\lambda})^{-1}(1\vee\kappa)\sqrt{V(x)}$, where $\kappa = |f|_{\sqrt{V}}$ and $x \vee y = \max\{x, y\}$. Using Equation (15) we obtain

$$\begin{aligned} P(\boldsymbol{\theta}_0)\tilde{V}(x) &= (1-\sqrt{\lambda})^{-1}(1\vee\kappa)P(\boldsymbol{\theta}_0)\sqrt{V(x)} \\ &\le (1-\sqrt{\lambda})^{-1}(1\vee\kappa)\left(\sqrt{\lambda V(x)} + \sqrt{b}1_K(x)\right) \\ &= \sqrt{\lambda}\tilde{V}(x) + (1-\sqrt{\lambda})^{-1}(1\vee\kappa)\sqrt{b}1_K(x) \\ &= \tilde{V}(x) - (1\vee\kappa)\sqrt{V(x)} + (1-\sqrt{\lambda})^{-1}(1\vee\kappa)\sqrt{b}1_K(x). \end{aligned}$$

It follows that condition (V3) in Meyn and Tweedie (1993) (see equation (14.16) in Meyn and Tweedie 1993 or equation (8) in Glynn and Meyn 1996) is satisfied with $\tilde{V}$ everywhere finite, $(1\vee\kappa)\sqrt{V} \ge 1$, and K a small set (hence K petite). Moreover, by Jensen's inequality,

$$\pi(\boldsymbol{\theta}_0)\tilde{V} = (1-\sqrt{\lambda})^{-1}(1\vee\kappa)\pi(\boldsymbol{\theta}_0)\sqrt{V} \le (1-\sqrt{\lambda})^{-1}(1\vee\kappa)\sqrt{\pi(\boldsymbol{\theta}_0)V} < \infty.$$

Then, since $|f| \le (1\vee\kappa)\sqrt{V}$, theorem 2.3 in Glynn and Meyn (1996) (theorem 17.4.2 in Meyn and Tweedie 1993) ensures that there exists a solution $\hat{f}$ to Poisson's equation satisfying $|\hat{f}| \le c_0(\tilde{V}+1)$ for some constant $c_0 < \infty$. This last inequality also implies that $\pi(\boldsymbol{\theta}_0)\hat{f}^2 < \infty$. Choose $c_1 = 2c_0(1-\sqrt{\lambda})^{-1}(1\vee\kappa)$ to obtain the statement of the lemma.

It remains to show that Z_n is a square-integrable martingale. Clearly,

$$E_{\boldsymbol{\theta}_0}\left[\hat{f}(\Phi_k) - P(\boldsymbol{\theta}_0)\hat{f}(\Phi_{k-1})\right] = E_{\boldsymbol{\theta}_0}\left[E_{\boldsymbol{\theta}_0}\left[\hat{f}(\Phi_k)\middle|\mathcal{F}_{k-1}\right] - P(\boldsymbol{\theta}_0)\hat{f}(\Phi_{k-1})\right] = 0,$$

so Z_n is a martingale. To see that it is square-integrable, note that

$$\begin{aligned} E_{\boldsymbol{\theta}_0}\left[\left(\hat{f}(\Phi_k) - P(\boldsymbol{\theta}_0)\hat{f}(\Phi_{k-1})\right)^2\right] &= E_{\boldsymbol{\theta}_0}\left[\left(f(\Phi_k) - \pi(\boldsymbol{\theta}_0)f\right)^2\right] \\ &\le E_{\boldsymbol{\theta}_0}\left[f(\Phi_k)^2\right] + E_{\boldsymbol{\theta}_0}\left[|f(\Phi_k)|\right]\pi(\boldsymbol{\theta}_0)f + (\pi(\boldsymbol{\theta}_0)f)^2. \end{aligned}$$

Since $|f|_{\sqrt{V}} < \infty$ and both $E_{\boldsymbol{\theta}_0}[V(\Phi_k)] < \infty$ and $\pi(\boldsymbol{\theta}_0)V < \infty$, then the above expression is finite, which completes the proof. ■

Next, we give the proof of Theorem 1, which states that under Assumption 1 the expectation of $\alpha_n \nabla L_n(\boldsymbol{\theta}_0)$ converges to $\nabla\alpha(\boldsymbol{\theta}_0)$.

Proof of Theorem 1. Define $M_n^i = \sum_{j=1}^n D_j^i$, $1 \le i \le d$ as in Lemma 2, and

$$Z_n = \sum_{k=1}^n \Big(\hat{f}(\Phi_k) - P(\boldsymbol{\theta}_0)\hat{f}(\Phi_{k-1})\Big), \qquad \xi_k = Z_k - Z_{k-1},$$

as in Lemma 3. By those same lemmas we have that M_n^i and Z_n are square-integrable martingales.

Next, note that

$$\begin{aligned}
E_{\boldsymbol{\theta}_0}[\alpha_n \nabla L_n(\boldsymbol{\theta}_0)]_i &= E_{\boldsymbol{\theta}_0,x}[\alpha_n M_n^i] \\
&= E_{\boldsymbol{\theta}_0}[(\alpha_n - \alpha(\boldsymbol{\theta}_0))M_n^i] \\
&= \frac{1}{n} E_{\boldsymbol{\theta}_0}[\hat{f}(\Phi_0)M_n^i] - \frac{1}{n} E_{\boldsymbol{\theta}_0}[\hat{f}(\Phi_n)M_n^i] + \frac{1}{n}\sum_{k=1}^n E_{\boldsymbol{\theta}_0}[\xi_k M_n^i] \\
&= \frac{1}{n}\sum_{j=1}^n E_{\boldsymbol{\theta}_0}[\hat{f}(\Phi_0)D_j^i] - \frac{1}{n} E_{\boldsymbol{\theta}_0}[\hat{f}(\Phi_n)M_n^i] + \frac{1}{n}\sum_{k=1}^n\sum_{j=1}^n E_{\boldsymbol{\theta}_0}[\xi_k D_j^i] \\
&= -\frac{1}{n} E_{\boldsymbol{\theta}_0}[\hat{f}(\Phi_n)M_n^i] + \frac{1}{n}\sum_{k=1}^n\sum_{j=1}^k E_{\boldsymbol{\theta}_0}[\xi_k D_j^i] \\
&= -\frac{1}{n} E_{\boldsymbol{\theta}_0}[\hat{f}(\Phi_n)M_n^i] + \frac{1}{n}\sum_{k=1}^n E_{\boldsymbol{\theta}_0}[\xi_k D_k^i],
\end{aligned}$$

where in the fifth equality we used the fact that the $\{D_j^i\}$ are martingale differences to obtain that $E_{\boldsymbol{\theta}_0}\big[\hat{f}(\Phi_0)D_j^i\big] = E_{\boldsymbol{\theta}_0}\big[\hat{f}(\Phi_0)E_{\boldsymbol{\theta}_0}\big[D_j^i|\mathscr{F}_0\big]\big] = 0$ for $j > 0$ and $E_{\boldsymbol{\theta}_0}\big[\xi_k D_j^i\big] = E_{\boldsymbol{\theta}_0}\big[\xi_k E_{\boldsymbol{\theta}_0}\big[D_j^i|\mathscr{F}_k\big]\big] = 0$ for $j > k$, and in the sixth equality we used the observation that the $\{\xi_k\}$ are also martingale differences to get $E_{\boldsymbol{\theta}_0}\big[\xi_k D_j^i\big] = E_{\boldsymbol{\theta}_0}\big[D_j^i E_{\boldsymbol{\theta}_0}[\xi_k|\mathscr{F}_j]\big] = 0$ for $j < k$.

To show that $(1/n)\Big|E_{\boldsymbol{\theta}_0}[\hat{f}(\Phi_n)M_n^i]\Big| \to 0$ as $n \to \infty$, note that by the Cauchy–Schwarz inequality

$$\begin{aligned}
\frac{1}{n}\Big|E_{\boldsymbol{\theta}_0}[\hat{f}(\Phi_n)M_n^i]\Big| &\le \frac{1}{n}\Big(E_{\boldsymbol{\theta}_0}[\hat{f}(\Phi_n)^2]\Big)^{1/2}\big(E_{\boldsymbol{\theta}_0}[(M_n^i)^2]\big)^{1/2} \\
&= \left(\frac{1}{n}E_{\boldsymbol{\theta}_0}[\hat{f}(\Phi_n)^2]\right)^{1/2}\left(\frac{1}{n}\sum_{j=1}^n E_{\boldsymbol{\theta}_0}[(D_j^i)^2]\right)^{1/2}.
\end{aligned}$$

Also, by Lemma 3 we have $\hat{f}^2 \le c_1 V$, and since by Lemma 1 $\boldsymbol{\Phi}$ is V-ergodic, we obtain that $E_{\boldsymbol{\theta}_0}[\hat{f}(\Phi_n)^2] \to \pi(\boldsymbol{\theta}_0)\hat{f}^2 < \infty$ as $n \to \infty$. This in turn implies that $1/n\, E_{\boldsymbol{\theta}_0}[\hat{f}(\Phi_n)^2] \to 0$ as $n \to \infty$. For the other term we have by Assumption 1 (vi) that $|g_{ii}|_V < \infty$, and therefore $E_{\boldsymbol{\theta}_0}[g_{ii}(\Phi_n)] \to \pi(\boldsymbol{\theta}_0)g_{ii} < \infty$. Hence,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^n E_{\boldsymbol{\theta}_0}[(D_j^i)^2] = \lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^n E_{\boldsymbol{\theta}_0}[g_{ii}(\Phi_{j-1})] = \pi(\boldsymbol{\theta}_0)g_{ii}.$$

We conclude that $1/n\Big|E_{\boldsymbol{\theta}_0}[\hat{f}(\Phi_n)M_n^i]\Big| \to 0$ as $n \to \infty$.

To show that $1/n\sum_{k=1}^n E_{\boldsymbol{\theta}_0}[\xi_k D_k^i] \to \frac{\partial}{\partial\theta_i}\alpha(\boldsymbol{\theta}_0)$, note that

$$\begin{aligned}
E_{\boldsymbol{\theta}_0}[\xi_k D_k^i] &= E_{\boldsymbol{\theta}_0}[E_{\boldsymbol{\theta}_0}[\xi_k D_k^i|\mathscr{F}_{k-1}]] \\
&= E_{\boldsymbol{\theta}_0}\left[\int_{\mathsf{X}}\Big(\hat{f}(y) - P(\boldsymbol{\theta}_0)\hat{f}(\Phi_{k-1})\Big)\frac{\partial}{\partial\theta_i}p(\boldsymbol{\theta}_0,\Phi_{k-1},y)P(\boldsymbol{\theta}_0,\Phi_{k-1},dy)\right] \\
&= E_{\boldsymbol{\theta}_0}\left[\int_{\mathsf{X}}\hat{f}(y)\frac{\partial}{\partial\theta_i}p(\boldsymbol{\theta}_0,\Phi_{k-1},y)P(\boldsymbol{\theta}_0,\Phi_{k-1},dy)\right] \\
&\quad - E_{\boldsymbol{\theta}_0}\left[P(\boldsymbol{\theta}_0)\hat{f}(\Phi_{k-1})\int_{\mathsf{X}}\frac{\partial}{\partial\theta_i}p(\boldsymbol{\theta}_0,\Phi_{k-1},y)P(\boldsymbol{\theta}_0,\Phi_{k-1},dy)\right] \\
&= E_{\boldsymbol{\theta}_0}\left[\int_{\mathsf{X}}\hat{f}(y)\frac{\partial}{\partial\theta_i}p(\boldsymbol{\theta}_0,\Phi_{k-1},y)P(\boldsymbol{\theta}_0,\Phi_{k-1},dy)\right],
\end{aligned}$$

where we have used the observation that $\int_{\mathsf{X}}(\partial/\partial\theta_i)p(\boldsymbol{\theta}_0,x,y)P(\boldsymbol{\theta}_0,x,dy) = (\partial/\partial\theta_i)\int_{\mathsf{X}} p(\boldsymbol{\theta}_0,x,y)P(\boldsymbol{\theta}_0,x,dy) = 0$ for any $x \in \mathsf{X}$. Let $h_i(x) = \int_{\mathsf{X}} \hat{f}(y)(\partial/\partial\theta_i)p(\boldsymbol{\theta}_0,x,y)P(\boldsymbol{\theta}_0,x,dy)$ and note that by the Cauchy–Schwarz inequality

$$
\begin{aligned}
|h_i(x)| &\le \left(\int_{\mathsf{X}} \hat{f}^2(y)P(\boldsymbol{\theta}_0,x,dy)\right)^{1/2}\left(\int_{\mathsf{X}}\left(\frac{\partial}{\partial\theta_i}p(\boldsymbol{\theta}_0,x,y)\right)^2 P(\boldsymbol{\theta}_0,x,dy)\right)^{1/2} \\
&= \left(P(\boldsymbol{\theta}_0)\hat{f}^2(x)\right)^{1/2}(g_{ii}(x))^{1/2} \\
&\le c_1(P(\boldsymbol{\theta}_0)V(x))^{1/2}(\|g_{ii}\|_V V(x))^{1/2} \\
&\le c_1(\|g_{ii}\|_V V(x)(\lambda V(x)+b))^{1/2} \\
&\le c_1(\|g_{ii}\|_V(\lambda+b))^{1/2}V(x),
\end{aligned}
$$

and therefore $\|h_i\|_V < \infty$. It follows from the same arguments used above that

$$
\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} E_{\boldsymbol{\theta}_0}[\xi_k D_k^i] = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} E_{\boldsymbol{\theta}_0}[h_i(\Phi_{k-1})] = \pi(\boldsymbol{\theta}_0)h_i,
$$

with the limit $\pi(\boldsymbol{\theta}_0)h_i$ well-defined and finite. It only remains to show that $\pi(\boldsymbol{\theta}_0)h_i = (\partial/\partial\theta_i)\alpha(\boldsymbol{\theta}_0)$. To do this first note that

$$
\begin{aligned}
\frac{\partial}{\partial\theta_i}\alpha(\boldsymbol{\theta}_0) &= \frac{\partial}{\partial\theta_i}\int_{\mathsf{X}} f(x)\pi(\boldsymbol{\theta},dx)\bigg|_{\boldsymbol{\theta}=\boldsymbol{\theta}_0} \\
&= \frac{\partial}{\partial\theta_i}\int_{\mathsf{X}}\left(\hat{f}(x) - P(\boldsymbol{\theta}_0)\hat{f}(x) + \alpha(\boldsymbol{\theta}_0)\right)\pi(\boldsymbol{\theta},dx)\bigg|_{\boldsymbol{\theta}=\boldsymbol{\theta}_0} \\
&= \frac{\partial}{\partial\theta_i}\int_{\mathsf{X}}\left(\hat{f}(x) - P(\boldsymbol{\theta}_0)\hat{f}(x)\right)\pi(\boldsymbol{\theta},dx)\bigg|_{\boldsymbol{\theta}=\boldsymbol{\theta}_0} \\
&= \lim_{h\to 0}\int_{\mathsf{X}}\left(\hat{f}(x) - P(\boldsymbol{\theta}_0)\hat{f}(x)\right)\frac{\pi(\boldsymbol{\theta}_0+h\mathbf{e}_i,dx)-\pi(\boldsymbol{\theta}_0,dx)}{h} \\
&= \lim_{h\to 0}\int_{\mathsf{X}}\left(\hat{f}(x) - P(\boldsymbol{\theta}_0)\hat{f}(x)\right)\frac{\pi(\boldsymbol{\theta}_0+h\mathbf{e}_i,dx)}{h} \\
&= \lim_{h\to 0}\int_{\mathsf{X}}\left(\hat{f}(x) - P(\boldsymbol{\theta}_0+h\mathbf{e}_i)\hat{f}(x) + P(\boldsymbol{\theta}_0+h\mathbf{e}_i)\hat{f}(x) - P(\boldsymbol{\theta}_0)\hat{f}(x)\right)\frac{\pi(\boldsymbol{\theta}_0+h\mathbf{e}_i,dx)}{h} \\
&= \lim_{h\to 0}\int_{\mathsf{X}}\left(\frac{P(\boldsymbol{\theta}_0+h\mathbf{e}_i)\hat{f}(x) - P(\boldsymbol{\theta}_0)\hat{f}(x)}{h}\right)\pi(\boldsymbol{\theta}_0+h\mathbf{e}_i,dx),
\end{aligned}
$$

where in the fifth and seventh steps we used the identity $\pi(\boldsymbol{\theta})P(\boldsymbol{\theta}) = \pi(\boldsymbol{\theta})$ for all $\boldsymbol{\theta} \in B_\epsilon(\boldsymbol{\theta}_0)$. Next, note that $h_i(x) = P^{(i)}(\boldsymbol{\theta}_0)\hat{f}(x)$, from where it follows that

$$
\begin{aligned}
&\left|\pi(\boldsymbol{\theta}_0)h_i - \frac{\partial}{\partial\theta_i}\alpha(\boldsymbol{\theta}_0)\right| \\
&= \left|\int_{\mathsf{X}} P^{(i)}(\boldsymbol{\theta}_0)\hat{f}(x)\pi(\boldsymbol{\theta}_0,dx) - \lim_{h\to 0}\int_{\mathsf{X}}\left(\frac{P(\boldsymbol{\theta}_0+h\mathbf{e}_i)\hat{f}(x) - P(\boldsymbol{\theta}_0)\hat{f}(x)}{h}\right)\pi(\boldsymbol{\theta}_0+h\mathbf{e}_i,dx)\right| \\
&\le \lim_{h\to 0}\left|\int_{\mathsf{X}} P^{(i)}(\boldsymbol{\theta}_0)\hat{f}(x)(\pi(\boldsymbol{\theta}_0,dx) - \pi(\boldsymbol{\theta}_0+h\mathbf{e}_i,dx))\right| \qquad (16) \\
&\quad + \lim_{h\to 0}\left|\int_{\mathsf{X}}\left(P^{(i)}(\boldsymbol{\theta}_0)\hat{f}(x) - \frac{P(\boldsymbol{\theta}_0+h\mathbf{e}_i)\hat{f}(x) - P(\boldsymbol{\theta}_0)\hat{f}(x)}{h}\right)\pi(\boldsymbol{\theta}_0+h\mathbf{e}_i,dx)\right|. \qquad (17)
\end{aligned}
$$

It remains to show that the last two limits are zero. To analyze Equation (16) recall that $\|h_i\|_V < \infty$, from where it follows that Equation (16) is bounded by

$$
\lim_{h\to 0}\|\pi(\boldsymbol{\theta}_0) - \pi(\boldsymbol{\theta}_0+h\mathbf{e}_i)\|_V = 0 \qquad \text{(by Lemma 1).}
$$

To show that Equation (17) is zero as well, note that

$$\begin{aligned}
&\left| P^{(i)}(\boldsymbol{\theta}_0)\hat{f}(x) - \frac{P(\boldsymbol{\theta}_0 + h\mathbf{e}_i)\hat{f}(x) - P(\boldsymbol{\theta}_0)\hat{f}(x)}{h} \right| \\
&\le |\hat{f}|_V \left\| P^{(i)}(\boldsymbol{\theta}_0, x, \cdot) - \frac{P(\boldsymbol{\theta}_0 + h\mathbf{e}_i, x, \cdot) - P(\boldsymbol{\theta}_0, x, \cdot)}{h} \right\|_V \\
&\le |\hat{f}|_V V(x) \left\|\left| P^{(i)}(\boldsymbol{\theta}_0) - \frac{P(\boldsymbol{\theta}_0 + h\mathbf{e}_i) - P(\boldsymbol{\theta}_0)}{h} \right\|\right|_V,
\end{aligned}$$

which, combined with $\pi(\boldsymbol{\theta}_0)V < \infty$, gives that Equation (17) is bounded by

$$\begin{aligned}
&\lim_{h\to 0} \int_{\mathsf{X}} |\hat{f}|_V V(x) \left\|\left| P^{(i)}(\boldsymbol{\theta}_0) - \frac{P(\boldsymbol{\theta}_0 + h\mathbf{e}_i) - P(\boldsymbol{\theta}_0)}{h} \right\|\right|_V \pi(\boldsymbol{\theta}_0 + h\mathbf{e}_i, dx) \\
&= |\hat{f}|_V \lim_{h\to 0} \left\|\left| P^{(i)}(\boldsymbol{\theta}_0) - \frac{P(\boldsymbol{\theta}_0 + h\mathbf{e}_i) - P(\boldsymbol{\theta}_0)}{h} \right\|\right|_V \pi(\boldsymbol{\theta}_0 + h\mathbf{e}_i)V \\
&\le |\hat{f}|_V \lim_{h\to 0} \left\|\left| P^{(i)}(\boldsymbol{\theta}_0) - \frac{P(\boldsymbol{\theta}_0 + h\mathbf{e}_i) - P(\boldsymbol{\theta}_0)}{h} \right\|\right|_V (\pi(\boldsymbol{\theta}_0)V + \|\pi(\boldsymbol{\theta}_0 + h\mathbf{e}_i) - \pi(\boldsymbol{\theta}_0)\|_V) = 0.
\end{aligned}$$

This completes the proof. ∎

The following is the proof of the main weak convergence theorem that is used to describe the behavior of all four estimators considered in Sections 3 and 4. It is essentially an application of the functional central limit theorem for multivariate martingales found in Whitt (2007) (see also theorems 1.4 and 1.2 in chapter 7 of Ethier and Kurtz 1986).

Proof of Theorem 2. For $m \in \mathbb{N}$, let $Z_m = \sum_{k=1}^m (\hat{f}(\Phi_k) - P(\boldsymbol{\theta}_0)\hat{f}(\Phi_{k-1}))$ and $\mathbf{M}_m = \nabla L_m(\boldsymbol{\theta}_0)$. Next, define the process $\mathbf{X}_n(t) = (X_n^0(t), X_n^1(t), \dots, X_n^d(t))' = n^{-1/2}(Z_{\lfloor nt\rfloor}, \mathbf{M}'_{\lfloor nt\rfloor})'$ and the filtrations $\mathscr{G}_{n,t} = \mathscr{F}_{\lfloor nt\rfloor} = \sigma(\Phi_0, \dots, \Phi_{\lfloor nt\rfloor})$. Note that by Lemmas 2 and 3 $\{X_n(t) : t \in [0,1]\}$ is a square integrable martingale with respect to $\mathscr{G}_{n,t}$. Moreover, by Equation (9), we have

$$(n^{-1/2}\lfloor nt\rfloor(\alpha_{\lfloor nt\rfloor} - \alpha(\boldsymbol{\theta}_0)),\ n^{-1/2}\nabla L_{\lfloor nt\rfloor}(\boldsymbol{\theta}_0)') = \left(n^{-1/2}(\hat{f}(\Phi_0) - \hat{f}(\Phi_{\lfloor nt\rfloor})),\ \mathbf{0}'\right) + X_n(t),$$

where $\mathbf{0}$ is the zero vector in $\mathbb{R}^d$. Note that

$$\sup_{0\le t\le 1} n^{-1/2}|\hat{f}(\Phi_0) - \hat{f}(\Phi_{\lfloor nt\rfloor}))| \le 2\max_{0\le k\le n} \frac{|\hat{f}(\boldsymbol{\Phi}_k)|}{n^{1/2}}.$$

Since $\boldsymbol{\Phi}$ is V-ergodic and $|\hat{f}^2|_V < \infty$, theorem 17.3.3 in Meyn and Tweedie (1993) gives

$$\max_{1\le k\le n} \frac{(\hat{f}(\boldsymbol{\Phi}_k))^2}{n} \to 0 \qquad \text{a.s. } P(\boldsymbol{\theta}),$$

which in turn implies that $(n^{-1/2}(\hat{f}(\Phi_0) - \hat{f}(\Phi_{\lfloor nt\rfloor})), \mathbf{0}') \Rightarrow \mathbf{0}'$ in $D([0,1], \mathbb{R}^{d+1})$. It follows by Slutsky's lemma that it suffices to show that $\mathbf{X}_n \Rightarrow \mathbf{B}$ in $D([0,1], \mathbb{R}^{d+1})$. We will do so by showing that $\mathbf{X}_n$ satisfies condition (ii) of theorem 2.1 in Whitt (2007).

Let $\boldsymbol{\xi}_k^n = \mathbf{X}_n(k/n) - \mathbf{X}_n((k-1)/n)$ and consider the matrix $A_n = A_n(t) \in \mathbb{R}^{(d+1)\times(d+1)}$ whose (i,j)th component is given by

$$A_n^{ij}(t) = \sum_{k=1}^{\lfloor nt\rfloor} E_{\boldsymbol{\theta}_0}[(\boldsymbol{\xi}_k^n)_i(\boldsymbol{\xi}_k^n)_j | \mathscr{F}_{k-1}].$$

Then $X_n^i(t)X_n^j(t) - A_n^{ij}(t)$ is a martingale adapted to $\mathscr{G}_{n,t}$ for each $0 \le i,j \le d$, and therefore, the $A_n^{ij} = \langle X_n^i, X_n^j\rangle$ are the predictable quadratic-covariation processes of $\mathbf{X}_n$. Also, for $1 \le i,j \le d$, we have

$$\begin{aligned}
A_n^{ij}(t) &= \frac{1}{n}\sum_{k=1}^{\lfloor nt\rfloor} E_{\boldsymbol{\theta}_0}\left[\frac{\partial}{\partial\theta_i} p(\boldsymbol{\theta}_0, \Phi_{k-1}, \Phi_k)\, \frac{\partial}{\partial\theta_j} p(\boldsymbol{\theta}_0, \Phi_{k-1}, \Phi_k) \middle| \mathscr{F}_{k-1} \right] \\
&= \frac{1}{n}\sum_{k=1}^{\lfloor nt\rfloor} \int_{\mathsf{X}} \frac{\partial}{\partial\theta_i} p(\boldsymbol{\theta}_0, \Phi_{k-1}, y)\, \frac{\partial}{\partial\theta_j} p(\boldsymbol{\theta}_0, \Phi_{k-1}, y) P(\boldsymbol{\theta}_0, \Phi_{k-1}, dy) \\
&= \frac{1}{n}\sum_{k=1}^{\lfloor nt\rfloor} g_{ij}(\Phi_{k-1}),
\end{aligned}$$

and for $1 \le i \le d$,

$$A_n^{0i}(t) = A_n^{i0}(t) = \frac{1}{n}\sum_{k=1}^{\lfloor nt\rfloor} E_{\boldsymbol{\theta}_0}\left[\frac{\partial}{\partial\theta_i} p(\boldsymbol{\theta}_0, \Phi_{k-1}, \Phi_k)\left(\hat{f}(\Phi_k) - P(\boldsymbol{\theta}_0)\hat{f}(\Phi_{k-1})\right)\middle|\mathscr{F}_{k-1}\right]$$
$$= \frac{1}{n}\sum_{k=1}^{\lfloor nt\rfloor}\int_{\mathsf{X}} \frac{\partial}{\partial\theta_i} p(\boldsymbol{\theta}_0, \Phi_{k-1}, y)\hat{f}(y)P(\boldsymbol{\theta}_0, \Phi_{k-1}, dy)$$
$$= \frac{1}{n}\sum_{k=1}^{\lfloor nt\rfloor} g_{0i}(\Phi_{k-1}).$$

Similarly,

$$A_n^{00}(t) = \frac{1}{n}\sum_{k=1}^{\lfloor nt\rfloor} g_{00}(\Phi_{k-1}).$$

Since $\pi(\boldsymbol{\theta}_0)|g_{ij}| \le \pi(\boldsymbol{\theta}_0)(g_{ii}g_{jj})^{1/2} \le (\pi(\boldsymbol{\theta}_0)g_{ii})^{1/2}(\pi(\boldsymbol{\theta}_0)g_{jj})^{1/2} < \infty$ for each $0 \le i, j \le d$, we have by theorem 17.0.1 in Meyn and Tweedie (1993) that

$$\frac{\lfloor nt\rfloor}{n}\cdot\frac{1}{\lfloor nt\rfloor}\sum_{k=1}^{\lfloor nt\rfloor} g_{ij}(\Phi_{k-1}) \to t\pi(\boldsymbol{\theta}_0)g_{ij} \qquad \text{a.s. } P(\boldsymbol{\theta}).$$

Also, for each $0 \le i, j \le d$,

$$\lim_{n\to\infty} E_{\boldsymbol{\theta}_0}\left[\sup_{t\in[0,1]}\left|A_n^{ij}(t) - A_n^{ij}(t-)\right|\right] = \lim_{n\to\infty} E_{\boldsymbol{\theta}_0}\left[\max_{1\le k\le n}\left|E_{\boldsymbol{\theta}_0}[(\boldsymbol{\xi}_k^n)_i(\boldsymbol{\xi}_k^n)_j|\mathscr{F}_{k-1}]\right|\right]$$
$$= \lim_{n\to\infty}\frac{1}{n}E_{\boldsymbol{\theta}_0}\left[\max_{1\le k\le n}\left|g_{ij}(\Phi_{k-1})\right|\right]$$
$$\le \|g_{ij}\|_V \lim_{n\to\infty}\frac{1}{n}E_{\boldsymbol{\theta}_0}\left[\max_{1\le k\le n} V(\Phi_{k-1})\right]$$
$$\le \|g_{ij}\|_V \lim_{n\to\infty}\frac{1}{n}E_{\boldsymbol{\theta}_0}\left[\max_{1\le k\le n}\left(\sqrt{n} + V(\Phi_{k-1})1(V(\Phi_{k-1}) > \sqrt{n})\right)\right]$$
$$\le \|g_{ij}\|_V \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} E_{\boldsymbol{\theta}_0}\left[V(\Phi_{k-1})1(V(\Phi_{k-1}) > \sqrt{n})\right].$$

To see that the last expression converges to zero, let $h_m(x) = V(x)1(V(x) > m)$, and note that $E_{\boldsymbol{\theta}_0}[h_m(\Phi_k)] \to \pi(\boldsymbol{\theta}_0)h_m < \infty$ as $k \to \infty$, and monotone convergence gives $\pi(\boldsymbol{\theta}_0)h_m \to 0$ as $m \to \infty$; therefore, we can choose $N \in \mathbb{N}$ large enough so that $\pi(\boldsymbol{\theta}_0)h_N < \delta$. It follows that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} E_{\boldsymbol{\theta}_0}\left[h_{\sqrt{n}}(\Phi_{k-1})\right] \le \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} E_{\boldsymbol{\theta}_0}[h_N(\Phi_{k-1})] = \pi(\boldsymbol{\theta}_0)h_N < \delta,$$

and since $\delta > 0$ was arbitrary, the limit is zero.

For a vector $\mathbf{x} = (x_0, x_1, \ldots, x_d)' \in \mathbb{R}^{d+1}$, let $|\mathbf{x}| = \left(\sum_{i=0}^{d} x_i^2\right)^{1/2}$. Then, by similar arguments as those used earlier,

$$\lim_{n\to\infty} E_{\boldsymbol{\theta}_0}\left[\sup_{t\in[0,1]}|\mathbf{X}_n(t) - \mathbf{X}_n(t-)|^2\right] = \lim_{n\to\infty} E_{\boldsymbol{\theta}_0}\left[\max_{1\le k\le n}|\boldsymbol{\xi}_k^n|^2\right]$$
$$= \sum_{i=0}^{d}\lim_{n\to\infty} E_{\boldsymbol{\theta}_0}\left[\max_{1\le k\le n}(\boldsymbol{\xi}_k^n)_i^2\right]$$
$$\le \sum_{i=0}^{d}\lim_{n\to\infty}\sum_{k=1}^{n} E_{\boldsymbol{\theta}_0}\left[(\boldsymbol{\xi}_k^n)_i^2 1((\boldsymbol{\xi}_k^n)_i^2 > n^{-1/2})\right]$$
$$= \sum_{i=0}^{d}\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} E_{\boldsymbol{\theta}_0}\left[\hat{h}_{i,\sqrt{n}}(\Phi_{k-1})\right],$$

where

$$\hat{h}_{i,m}(x) = \int_X \left(\frac{\partial}{\partial\theta_i} p(\boldsymbol{\theta}_0, y, x)\right)^2 1\left(\left(\frac{\partial}{\partial\theta_i} p(\boldsymbol{\theta}_0, y, x)\right)^2 > m\right) P(\boldsymbol{\theta}_0, x, dy), \qquad 1 \le i \le d,$$

$$\hat{h}_{0,m}(x) = \int_X \left(\hat{f}(y) - P(\boldsymbol{\theta}_0)\hat{f}(x)\right)^2 1\left(\left(\hat{f}(y) - P(\boldsymbol{\theta}_0)\hat{f}(x)\right)^2 > m\right) P(\boldsymbol{\theta}_0, x, dy).$$

Since we have that $\hat{h}_{i,m} \le |g_{ii}|_V V$ for all $0 \le i \le d$, then $E_{\boldsymbol{\theta}_0}[\hat{h}_{i,m}(\Phi_{k-1})] \to \pi(\boldsymbol{\theta}_0)\hat{h}_{i,m} < \infty$ for each fixed $m \in \mathbb{N}$ and the same arguments used before give

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} E_{\boldsymbol{\theta}_0}\left[\hat{h}_{i,\sqrt{n}}(\Phi_{k-1})\right] = 0.$$

It now follows from theorem 2.1 in Whitt (2007) that $\mathbf{X}_n \Rightarrow \mathbf{B}$ in $D([0,1], \mathbb{R}^{d+1})$. ■

The next proof corresponds to Proposition 3, which shows the lack of convergence of the discrete integral

$$\frac{1}{n}\sum_{k=1}^{n-1}\sum_{l=k}^{n-1} f(\boldsymbol{\Phi}_l)\mathbf{D}_k.$$

Proof of Proposition 3. First note that the V-ergodicity of $\boldsymbol{\Phi}$, the observation that $\pi(\boldsymbol{\theta}_0)|f| < \infty$, and theorem 17.0.1 in Meyn and Tweedie (1993), gives for any $t \in [0,1]$,

$$Q_n(t) = \frac{1}{n}\sum_{l=0}^{n-1} f(\boldsymbol{\Phi}_l) - \frac{\lfloor nt\rfloor + 1}{n} \cdot \frac{1}{\lfloor nt\rfloor + 1}\sum_{l=0}^{\lfloor nt\rfloor} f(\boldsymbol{\Phi}_l) \to \alpha(\boldsymbol{\theta}_0)(1-t) = Q(t) \qquad \text{a.s. } P(\boldsymbol{\theta}_0),$$

as $n \to \infty$. Moreover,

$$\begin{aligned} n(Q_n(t) - Q(t)) &= \sum_{l=\lfloor nt\rfloor+1}^{n-1} (f(\boldsymbol{\Phi}_l) - \alpha(\boldsymbol{\theta}_0)) + (nt - \lfloor nt\rfloor - 1)\alpha(\boldsymbol{\theta}_0) \\ &= S_n(1) - S_n(t) + (nt - \lfloor nt\rfloor - 1)\alpha(\boldsymbol{\theta}_0), \end{aligned}$$

where $S_n(t) = \sum_{j=0}^{\lfloor nt\rfloor - 1}(f(\boldsymbol{\Phi}_j) - \alpha(\boldsymbol{\theta}_0))$ and $n^{-1/2}S_n \Rightarrow B_0$ in $D([0,1],\mathbb{R})$ by Theorem 2, with B_0 a mean zero Brownian motion. It follows that $Q_n(t) - Q(t) \Rightarrow 0$ in $D([0,1],\mathbb{R})$. Also, by Theorem 2, again we have that $n^{-1/2}\nabla L_{\lfloor n\cdot\rfloor}(\boldsymbol{\theta}_0) \Rightarrow \hat{\mathbf{B}}$ in $D([0,1],\mathbb{R}^d)$, where $\hat{\mathbf{B}}$ is a zero-mean d-dimensional Brownian motion with covariance matrix Σ.

It follows that since Q is a nonrandom element of $D([0,1],\mathbb{R})$,

$$(Q_n,\, n^{-1/2}\nabla L_{\lfloor n\cdot\rfloor}(\boldsymbol{\theta}_0)') \Rightarrow (Q, \hat{\mathbf{B}}') \qquad n \to \infty \tag{18}$$

in $D([0,1],\mathbb{R}^{d+1})$.

Next, define the processes $X_n(t) = Q_n(t)I$, $X(t) = Q(t)I$, $\mathbf{Z}_n(t) = n^{-1/2}\nabla L_{\lfloor nt\rfloor}(\boldsymbol{\theta}_0)$, and $\mathbf{Z}(t) = \hat{\mathbf{B}}(t)$, where I is the identity matrix of $\mathbb{R}^{d\times d}$. Define $\mathcal{G}_{n,t} = \mathcal{F}_{\lfloor nt\rfloor}$. Clearly, X_n and $\mathbf{Z}_n$ are $\{\mathcal{G}_{n,t}\}$-adapted and $\mathbf{Z}_n$ is a $\{\mathcal{G}_{n,t}\}$-martingale. Also, for $t_i = i/n$,

$$n^{-1/2}\mathbf{Y}_n = \frac{1}{n^{1/2}}\frac{1}{n}\sum_{k=1}^{n-1}\sum_{l=k}^{n-1} f(\Phi_l)\mathbf{D}_k = \sum_{k=1}^{n-1} X_n(t_{k-1})(\mathbf{Z}_n(t_k) - \mathbf{Z}_n(t_{k-1})) = \int_0^{1-\frac{1}{n}} X_n(s-)\, d\mathbf{Z}_n(s).$$

Consider now the process

$$[\mathbf{Z}_n]_t = \frac{1}{n}\sum_{k=1}^{\lfloor nt\rfloor} \mathbf{D}_k\mathbf{D}_k'$$

and note that the (i,j)th element of $|E_{\boldsymbol{\theta}_0}[[\mathbf{Z}_n]_t]|$ $(1 \le i,j \le d)$ is

$$\begin{aligned} \left|\frac{1}{n}\sum_{k=1}^{\lfloor nt\rfloor} E_{\boldsymbol{\theta}_0}[D_k^i D_k^j]\right| &= \left|\frac{1}{n}\sum_{k=1}^{\lfloor nt\rfloor} E_{\boldsymbol{\theta}_0}\left[E_{\boldsymbol{\theta}_0}[D_k^i D_k^j | \mathcal{F}_{k-1}]\right]\right| = \left|E_{\theta_0}\left[\frac{1}{n}\sum_{k=1}^{\lfloor nt\rfloor} g_{ij}(\Phi_{k-1})\right]\right| \\ &\le t\sup_n E_{\boldsymbol{\theta}_0}\left[\left|\frac{1}{n}\sum_{k=1}^{n} g_{ij}(\Phi_{k-1})\right|\right]. \end{aligned}$$

For each $\alpha > 0$, let $\tau_n^\alpha = 2\alpha$, and note that $P_{\boldsymbol{\theta}_0}(\tau_n^\alpha \le \alpha) = 0 \le 1/\alpha$ and

$$\sup_n E_{\boldsymbol{\theta}_0}[[\mathbf{Z}_n]_{t\wedge\tau_n^\alpha}] \le \sup_n E_{\boldsymbol{\theta}_0}[[\mathbf{Z}_n]_{2\alpha}] \le 2\alpha \max_{1\le i,j\le d} \sup_n E_{\boldsymbol{\theta}_0}\left[\left|\frac{1}{n}\sum_{k=1}^n g_{ij}(\Phi_{k-1})\right|\right] < \infty.$$

Finally, by Equation (18), we have $(X_n, \mathbf{Z}_n) \Rightarrow (X, \mathbf{Z})$ in $D([0,1], \mathbb{R}^{d\times d} \times \mathbb{R}^d)$. Therefore, the conditions of theorem 2.7 of Kurtz and Protter (1991) are satisfied and we have

$$\left(X_n, \mathbf{Z}_n, \int_0^{1-\frac{1}{n}} X_n\, d\mathbf{Z}_n\right) \Rightarrow \left(X, \mathbf{Z}, \int_0^1 X\, d\mathbf{Z}\right) \qquad n \to \infty$$

in $D([0,1], \mathbb{R}^{d\times d} \times \mathbb{R}^d \times \mathbb{R}^d)$. ■

The last proof in the paper corresponds to the calculation of the variance of $\int_0^1 (B_0(1) - B_0(s)) I\, d\hat{\mathbf{B}}(s)$.

Proof of Lemma 4. Note that we can write the limit $\mathcal{I}$ as $B_0(1)\hat{\mathbf{B}}(1) - \int_0^1 B_0(s) I\, d\hat{\mathbf{B}}(s)$. Let $\mathbf{U} = B_0(1)\hat{\mathbf{B}}(1)$ and $\mathbf{V} = \int_0^1 B_0(s) I\, d\hat{\mathbf{B}}(s)$. Then, since $E_{\boldsymbol{\theta}_0}[\mathbf{V}] = 0$, the covariance matrix of the limiting distribution is given by

$$\begin{aligned}
\mathrm{Cov}_{\boldsymbol{\theta}_0}(\mathcal{I}) &= E_{\boldsymbol{\theta}_0}[(\mathbf{U} - \mathbf{V} - E_{\boldsymbol{\theta}_0}[\mathbf{U}])(\mathbf{U} - \mathbf{V} - E_{\boldsymbol{\theta}_0}[\mathbf{U}])'] \\
&= E_{\boldsymbol{\theta}_0}[\mathbf{U}\mathbf{U}'] - E_{\boldsymbol{\theta}_0}[\mathbf{U}\mathbf{V}'] - E_{\boldsymbol{\theta}_0}[\mathbf{U}]E_{\boldsymbol{\theta}_0}[\mathbf{U}]' - E_{\boldsymbol{\theta}_0}[\mathbf{V}\mathbf{U}'] + E_{\boldsymbol{\theta}_0}[\mathbf{V}\mathbf{V}'] \\
&= \mathrm{Cov}_{\boldsymbol{\theta}_0}(\mathbf{U}) - E_{\boldsymbol{\theta}_0}[\mathbf{U}\mathbf{V}'] - (E_{\boldsymbol{\theta}_0}[\mathbf{U}\mathbf{V}'])' + E_{\boldsymbol{\theta}_0}[\mathbf{V}\mathbf{V}'].
\end{aligned}$$

Note that $\mathbf{U} \overset{\mathcal{D}}{=} Z_0\hat{\mathbf{Z}}$, that is, the limiting distribution of $\mathbf{Y}(C_n^*)$ (see Equation (13)), so the (i,j)th component of $\mathrm{Cov}_{\boldsymbol{\theta}_0}(\mathbf{U})$ is $\sigma_{00}\sigma_{ij} + \sigma_{0i}\sigma_{0j}$. To compute the remaining expectations, let $\mathbf{W}(t) = (W_0(t), W_1(t), \ldots, W_d(t))'$ be a standard $(d+1)$-dimensional Brownian motion and write $\Sigma = AA'$, where $A = (a_{ij}) \in \mathbb{R}^{(d+1)\times(d+1)}$. Then, we can rewrite

$$\mathbf{V} = \int_0^1 B_0(s) A d\mathbf{W}(s), \qquad \mathbf{B}(t) = \int_0^t A d\mathbf{W}(s),$$

and

$$\begin{aligned}
(E_{\boldsymbol{\theta}_0}[\mathbf{V}\mathbf{V}'])_{ij} = E_{\boldsymbol{\theta}_0}[V_i V_j] &= E_{\boldsymbol{\theta}_0}\left[\int_0^1 B_0(s) \sum_{k=0}^d a_{ik} dW_k(s) \int_0^1 B_0(s) \sum_{l=0}^d a_{jl} dW_l(s)\right] \\
&= \sum_{k=0}^d a_{ik} \sum_{l=0}^d a_{jl}\, E_{\boldsymbol{\theta}_0}\left[\int_0^1 B_0(s) dW_k(s) \int_0^1 B_0(s) dW_l(s)\right] \\
&= \sum_{k=0}^d a_{ik} a_{jk}\, E_{\boldsymbol{\theta}_0}\left[\left(\int_0^1 B_0(s) dW_k(s)\right)^2\right] \\
&= \sigma_{ij} \int_0^1 E_{\boldsymbol{\theta}_0}\left[(B_0(s))^2\right] ds \\
&= \sigma_{ij} \int_0^1 \sigma_{00} s\, ds = \frac{\sigma_{00}\sigma_{ij}}{2}.
\end{aligned}$$

To compute $E_{\boldsymbol{\theta}_0}[\mathbf{U}\mathbf{V}']$, first note that we can write it as

$$\begin{aligned}
(E_{\boldsymbol{\theta}_0}[\mathbf{U}\mathbf{V}'])_{ij} = E_{\boldsymbol{\theta}_0}[U_i V_j] &= E_{\boldsymbol{\theta}_0}\left[B_0(1) B_i(1) \int_0^1 B_0(s) \sum_{k=0}^d c_{jk} dW_k(s)\right] \\
&= E_{\boldsymbol{\theta}_0}\left[\sum_{m=0}^d c_{0m} W_m(1) \sum_{l=0}^d c_{il} W_l(1) \int_0^1 \sum_{n=0}^d c_{0n} W_n(s) \sum_{k=0}^d c_{jk} dW_k(s)\right] \\
&= \sum_{n=0}^d c_{0n} \sum_{k=0}^d c_{jk} \sum_{m=0}^d c_{0m} \sum_{l=0}^d c_{il}\, E_{\boldsymbol{\theta}_0}\left[W_m(1) W_l(1) \int_0^1 W_n(s) dW_k(s)\right].
\end{aligned}$$

Now, for each of the remaining expectations, use the product rule $W_m(1)W_l(1) = \int_0^1 W_m(s)dW_l(s) + \int_0^1 W_l(s)dW_m(s) + 1(m = l)$ to obtain

$$\begin{aligned}
E_{\boldsymbol{\theta}_0}\left[W_m(1)W_l(1)\int_0^1 W_n(s)dW_k(s)\right] &= E_{\boldsymbol{\theta}_0}\left[\int_0^1 W_m(s)dW_l(s)\int_0^1 W_n(s)dW_k(s)\right] \\
&\quad + E_{\boldsymbol{\theta}_0}\left[\int_0^1 W_l(s)dW_m(s)\int_0^1 W_n(s)dW_k(s)\right] \\
&= \int_0^1 E_{\boldsymbol{\theta}_0}[W_m(s)W_n(s)]ds\,1(l = k) \\
&\quad + \int_0^1 E_{\boldsymbol{\theta}_0}[W_l(s)W_n(s)]ds\,1(m = k) \\
&= \int_0^1 s1(m = n)\,ds\,1(l = k) + \int_0^1 s1(l = n)\,ds\,1(m = k) \\
&= \frac{1}{2}1(m = n)1(l = k) + \frac{1}{2}1(l = n)1(m = k).
\end{aligned}$$

Substituting in the expression for $E_{\boldsymbol{\theta}_0}[\mathbf{UV}']$, we obtain

$$\begin{aligned}
\left(E_{\boldsymbol{\theta}_0}[\mathbf{UV}']\right)_{ij} &= \frac{1}{2}\sum_{n=0}^{d} c_{0n}\sum_{k=0}^{d} c_{jk}c_{0n}c_{ik} + \frac{1}{2}\sum_{n=0}^{d} c_{0n}\sum_{k=0}^{d} c_{jk}c_{0k}c_{in} \\
&= \frac{1}{2}\sigma_{00}\sigma_{ij} + \frac{1}{2}\sigma_{0i}\sigma_{0j} = (E_{\boldsymbol{\theta}_0}[\mathbf{UV}'])_{ji}.
\end{aligned}$$

Therefore, the (i, j)th component, $1 \leq i, j \leq d$, of the limiting distribution's covariance matrix is given by

$$\mathrm{Cov}_{\boldsymbol{\theta}_0}(\mathcal{I})_{ij} = \sigma_{00}\sigma_{ij} + \sigma_{0i}\sigma_{0j} - (\sigma_{00}\sigma_{ij} + \sigma_{0i}\sigma_{0j}) + \frac{\sigma_{00}\sigma_{ij}}{2} = \frac{\sigma_{00}\sigma_{ij}}{2}. \quad \blacksquare$$

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