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The Impact of a Network Split on Cascading Failure Processes

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Abstract. Cascading failure models are typically used to capture the phenomenon where failures possibly trigger further failures in succession, causing knock-on effects. In many networks, this ultimately leads to a disintegrated network where the failure propagation continues independently across the various components. In order to gain insight into the impact of network splitting on cascading failure processes, we extend a well-established cascading failure model for which the number of failures obeys a power-law distribution. We assume that a single line failure immediately splits the network in two components and examine its effect on the power-law exponent. The results provide valuable qualitative insights that are crucial first steps toward understanding more complex network splitting scenarios.

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Keywords: cascading failures • power-law • blackout • rare-event

1. Introduction

Cascading failure models are used to describe systems of interconnected components where initial failures can trigger subsequent failures of other components. Despite the deceptively simple appearance, these models capture an extraordinary richness of different behaviors and have proven to be crucial in many application areas, such as material science, computer networks, traffic networks, earthquake dynamics, and power-transmission systems. It is therefore not surprising that cascading failures have received a lot of attention throughout the years (Pradhan et al. 2010).

In particular, the application domain of power systems has received increasing attention over the last 15 years (Albert et al. 2004, Kinney et al. 2005, Bienstock 2015, Rohden et al. 2016) and also provides the main inspiration for our research. Power grids have grown significantly in size and complexity. Moreover, various recent advances, such as the rise of renewable sources, have considerably increased the volatility in these systems. The occurrences of severe blackouts have increased rapidly around the world, in a time where society relies on a reliable power grid more and more. Notorious examples include the Northeast Blackout of 2003, the India Blackout of 2012, and the South Australia Blackouts of 2016 and 2017. Such catastrophic events cause significant economic and social disruption, and the analysis of severe blackouts has therefore become a crucial part of transmission-grid planning and operations (Newman et al. 2011, Wang et al. 2015).

Blackouts often occur through a cascade of failures that accelerate and outstrip control capabilities (Bienstock 2015). The failure mechanism causing a power outage entails long and complex sequences of failures, making the analysis of the failure propagation extremely difficult. Simulation techniques are typical approaches in order to obtain a better understanding of the cascading failure process. However, standard Monte Carlo simulation may become computationally intractable due to the low probability of a blackout event and the huge size of the network (curse of dimensionality) (Guo et al. 2017). Nevertheless, rare-event simulation, such as importance sampling and splitting (Kim et al. 2013, Shortle 2013), can be used to overcome these issues and analyze fairly complex cascading failure models.

Although advanced simulation techniques and scenario-testing approaches have proven indispensable, they provide little physical insight into the mechanism leading to a severe blackout. In contrast, macroscopic models, such as Motter and Lai (2002), Watts (2002), Dobson et al. (2004), Crucitti et al. (2004), and Blanchet et al. (2017),

focus on a few essential characteristics to obtain more qualitative insights. Such insights help in gaining a deeper understanding of the failure propagation. In particular, Dobson et al. (2004, 2005) construct a simple cascading failure model that captures four salient features of large blackouts: the large number of components, the initial disturbance stressing the network, the component failure when its capacity is exceeded, and the additional loading of other components when a component failure has occurred. This results in a tractable model that allows for a rigorous derivation of the number of component failures.

In this paper, we extend these models, allowing for another distinctive feature observed in occurrences of large blackouts: network splitting. Successive line failures may cause the network to disintegrate in disjoint components. Once a network split has occurred, the failure propagation continues independently among the various components. Network splitting is also known as islanding and is sometimes used as a tool in power systems to prevent blackouts to cascade to large-scale proportions (Bienstock 2015). Our results show the impact of islanding on the power-law exponent.

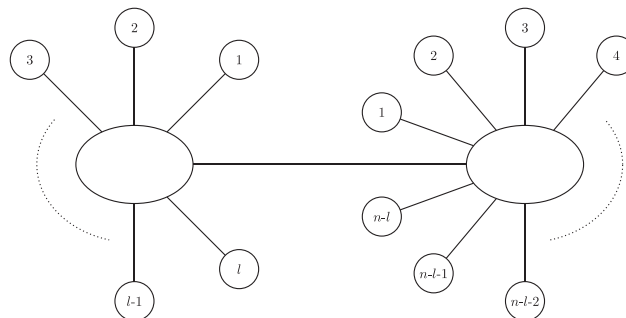
Specifically, we consider a network consisting of two star components connected by a single line; see Figure 1. Each line has an initial load that is exceeded by the capacity by a random margin. The cascading failure process is triggered by the failure of the line bridging the two components, causing all lines to be additionally loaded. When this load surge causes the capacity to be exceeded on a line, it fails. Every consecutive line failure causes all surviving lines connected to it to receive another supplementary load increase, possibly triggering massive knock-on effects. We emphasize that due to the network structure, no network splitting occurs after the failure of the bridging line. Therefore, the cascading failure propagation continues independently in the two components until the capacities at the surviving lines in both components are sufficient to meet the load surges. A detailed description of the model is given in Section 2.

We measure the reliability of the network by the probability that the total number of line failures exceeds a certain threshold, which we refer to as the *exceedance probability*. This objective is well understood in the case of a single star network under certain assumptions (Sloothaak et al. 2018a). That is, there is an initial disruption causing all lines to be additionally loaded, and every consecutive line failure causes subsequent load surges to all surviving lines. Under a particular condition, the exceedance probability obeys a power-law distribution with exponent $-1/2$. This heavy-tailed behavior reflects a relatively high risk of having severe blackouts and is of strong interest, as it appears in empirical analyses of historic blackout data (Carreras et al. 2000, Talukdar et al. 2003, Carreras et al. 2004, Newman et al. 2011).

The objective of this paper is to examine the impact of a single immediate network split on the exceedance probability. It turns out that the power-law property, which appeared in case of a star topology, mostly prevails. However, the splitting feature may possibly change the prefactor and the exponent depending on the threshold and component sizes. The results can intuitively be interpreted as follows. When the threshold is sufficiently smaller than the size of the smaller component, the threshold is most likely exceeded in just one of the components alone. If the threshold is approximately between the size of the smaller and the larger component, the threshold is most likely exceeded in the bigger component alone. In both cases, this property will imply that the power-law exponent is $-1/2$, as is the case in a single star network. For larger threshold values, both components need to have a significant number of line failures. Consequently, it is much less likely for the threshold to be exceeded, which causes a phase transition: The power-law exponent is reduced to -1 . This provides a possible explanation of why also other power-law exponents appear in empirical data analyses (Hines et al. 2009).

Our methodology uses an asymptotic analysis for the sum of two independent quasibinomially distributed random variables. We distinguish between different cases: the balanced case, where the sizes of both components are of the same order of magnitude; and the disparate case, where one is of a smaller order.

Figure 1. Visual Representation of the Network



Preliminary results appeared in a conference paper (Sloothaak et al. 2017), without proofs, where we focused on an approximation scheme for the exceedance probability and compared this to simulation results. In the present paper, we rigorously prove the asymptotic behavior. In the analysis, many subtleties need to be accounted for, which are most apparent when the threshold is close to the size of the larger component. These obstacles cannot be handled with existing techniques from the area of heavy-tailed distributions (Foss et al. 2011, Rhee et al. 2016). Our analysis in Section 3 aims to provide physical insights in these subtleties.

The paper is organized as follows. In Section 2, we describe our model in more detail and review some known results for the single star network. We state our main results in Section 3 and provide a high-level interpretation. The proofs of the main results are covered in Section 4. We conclude the paper with a discussion of some future research directions in Section 5.

2. Model Description and Preliminaries

We consider a network with $n + 2$ nodes, where n is large. The network consists of two components connected by a single line. The smaller component consists of $l := l_n \leq n/2$ lines, whereas the other component has $n - l$ lines, and hence $l \leq n - l$. Each line has a limited capacity for the amount of load it can carry before it fails. We assume that the network is initially stable in the sense that every line has enough capacity to carry its load. The difference between the initial load and capacity is called the *surplus capacity*, and we assume it to be independent and standard uniformly distributed at each of the n lines. A visual representation of the model is given in Figure 1.

The cascading failure process is initiated by the failure of the single line connecting the two components. This event creates two disjoint components and causes the load at all other lines to increase by θ/n for a certain constant $\theta > 0$. If this load increase exceeds the capacity of one or more lines, those lines will fail. Every subsequent failure again results in a load increase at the surviving lines, and we call such an increase the *load surge*. This cascading failure process continues until the surplus capacity for every surviving line exceeds its load. We assume that the load surge caused by each consecutive line failure in the smaller component is $1/l$, and in the larger component $1/(n - l)$, and that both components remain connected after every consecutive line failure. In other words, the cascading failure processes behave independently between the two components, and no further splitting will occur.

The vulnerability of the network is measured by the probability that the blackout size—that is, the number of failed lines, after the cascading failure propagation has stopped—exceeds a certain threshold $k := k_n$ as n grows large. The asymptotic behavior shows how the *exceedance probability* decays with respect to the threshold and furnishes valuable qualitative insights. We consider all thresholds k that are *growing* with n —that is, both $k \rightarrow \infty$ and $n - k \rightarrow \infty$ as $n \rightarrow \infty$.

Naturally, the behavior of the exceedance probability depends heavily on the sizes of the two components. We will consider the balanced case where both components have a size of order n , as well as the disparate case where the smaller component is of a size smaller than order n .

Next, we introduce some notation that will be used throughout the paper. Let A_n be the random variable representing the total number of line failures and $A_{l,n}$ the number of line failures in a component of size l disconnected from a component of size $n - l$. We assume both $\alpha := \lim_{n \rightarrow \infty} k/n$ and $\beta := \lim_{n \rightarrow \infty} l/n$ exist. We write $a_n = o(b_n)$ if $\lim_{n \rightarrow \infty} a_n/b_n = 0$ and $a_n = O(b_n)$ if $\limsup_{n \rightarrow \infty} a_n/b_n < \infty$. Similarly, we write $a_n = \omega(b_n)$ if $\lim_{n \rightarrow \infty} b_n/a_n = 0$ and $a_n = \Omega(b_n)$ if $\limsup_{n \rightarrow \infty} b_n/a_n < \infty$. Finally, we denote $a_n \sim b_n$ if $\lim_{n \rightarrow \infty} a_n/b_n = 1$ and $a_n \asymp b_n$ if $\lim_{n \rightarrow \infty} a_n/b_n \in (0, \infty)$.

The case of a single star network, where each line failure causes a single node to become isolated, has been studied rigorously in Sloothaak et al. (2018a). Specifically, this case involves a star network consisting of $n + 1$ nodes, n lines with uniformly distributed surplus capacities, an initial load surge of θ/n at all lines, and subsequent load surges of $1/n$ at all surviving lines. In that case, the following result holds.

Theorem 1. Let $k_\star := k_{n_\star}$ and $k^\star := k_{n^\star}$ be growing sequences of n with $k_\star \leq k^\star$ —that is, both $k_\star \rightarrow \infty$ and $n - k^\star \rightarrow \infty$ as $n \rightarrow \infty$. Then,

$$\lim_{n \rightarrow \infty} \sup_{k \in [k_\star, k^\star]} \left| k^{3/2} \sqrt{\frac{n-k}{n}} \mathbb{P}(A_n = k) - \frac{\theta}{\sqrt{2\pi}} \right| = 0, \quad (1)$$

and

$$\lim_{n \rightarrow \infty} \sup_{k \in [k_\star, k^\star]} \left| k^{1/2} \sqrt{\frac{n}{n-k}} \mathbb{P}(A_n \geq k) - \frac{2\theta}{\sqrt{2\pi}} \right| = 0. \quad (2)$$

Theorem 1 thus states that uniformly for all $k \in [k_*, k^*]$,

$$\mathbb{P}(A_n = k) \sim \frac{\theta}{\sqrt{2\pi}} \sqrt{\frac{n}{n-k}} k^{-3/2},$$

and

$$\mathbb{P}(A_n \geq k) \sim \frac{2\theta}{\sqrt{2\pi}} \sqrt{\frac{n-k}{n}} k^{-1/2}.$$

Equation (1) is theorem 1 of Sloothaak et al. (2018a). The proof of (2) can be found in the Appendix: It follows the lines of the proof of theorem 2 in Sloothaak et al. (2018a), but it is adapted to hold uniformly.

3. Main Results

The exceedance probability naturally depends on the threshold and the component sizes. In essence, we derive the tail distribution of $A_n = A_{l,n} + A_{n-l,n}$, where $A_{l,n}$ and $A_{n-l,n}$ are independent random variables. Note that $A_{l,n}$ involves the number of line failures in a single star network with initial load surge $\theta/n = \theta/l \cdot l/n$ and consecutive load surges $1/l$. Therefore, $A_{l,n}$ obeys a quasibinomial distribution (Dobson et al. 2005), where the asymptotic behavior is given by Theorem 1 (with θ replaced by $\theta \cdot l/n$). We point out that $A_{l,n}$ is thus heavy-tailed for all values that are not too close to l . We derive the asymptotic behavior of the probability that the sum of two quasibinomial distributed random variables exceeds a network-size-dependent threshold k . To see how accurate these approximations are against the actual exceedance probability for finite (large) n , we refer the reader to our conference paper (Sloothaak et al. 2017).

As mentioned earlier, we distinguish between two cases: the balanced case where $\beta = \lim_{n \rightarrow \infty} l/n > 0$ and the disparate case where $l = o(n)$. Henceforth, recall that $\alpha = \lim_{n \rightarrow \infty} k/n$.

3.1. Balanced Component Sizes

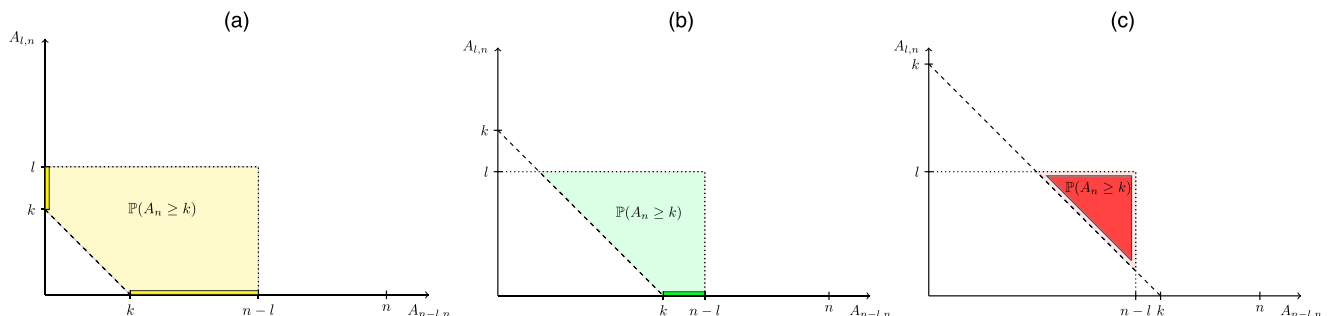
In this section, we consider the case where the two component sizes are of the same order and derive the tail of A_n . This tail behavior reflects the most likely scenarios for the number of line failures to exceed threshold k . Recall that A_n is essentially the sum of two heavy-tailed random variables (when $l \rightarrow \infty$ as $n \rightarrow \infty$). Moreover, the tail of both random variables typically obeys a power-law distribution with exponent $-1/2$ in the balanced case.

- (a) Case $0 \leq \alpha < \beta$.
- (b) Case $\beta \leq \alpha < 1 - \beta$.
- (c) Case $1 - \beta \leq \alpha < 1$.

This observation yields intuition for the asymptotic behavior of the exceedance probability. Figure 2 illustrates this intuition, where the bolder areas reflect which scenarios asymptotically contribute to the exceedance probability. When the threshold is significantly smaller than both component sizes ($\alpha < \beta$), the most likely scenario to exceed k is when it is exceeded in one of the components alone. In other words, the event where both $A_{l,n}$ and $A_{n-l,n}$ attain large values is much less likely to occur. Similarly, if the threshold is only significantly smaller than the larger component size ($\beta \leq \alpha < 1 - \beta$), the most likely scenario for A_n to exceed k is when it is exceeded in the larger component, whereas the smaller component only has very few line failures. We observe that in both cases the tail of A_n therefore obeys a power-law distribution with exponent $-1/2$.

If $1 - \beta < \alpha < 1$, both components must have many line failures. The threshold is then most likely to be exceeded if in both components a nonnegligible fraction of the lines have failed. This causes the power-law exponent to decrease to $-1/2 - 1/2 = -1$ —that is, a phase transition appears at $\alpha = 1 - \beta$. These notions lead to the following theorem.

Figure 2. Asymptotic Contributions to the Exceedance Probability in Theorem 2



Notes. (a) $0 \leq \alpha < \beta$. (b) $\beta \leq \alpha < 1 - \beta$. (c) $1 - \beta \leq \alpha < 1$.

Theorem 2. Suppose $\beta \in (0, 1/2]$ and $\alpha \neq 1 - \beta$. As $n \rightarrow \infty$, A_n asymptotically behaves as follows. If $0 \leq \alpha < \beta$, then

$$\mathbb{P}(A_n \geq k) \sim \frac{2\beta\theta}{\sqrt{2\pi}} \sqrt{1 - \frac{k}{l}} k^{-1/2} + \frac{2(1-\beta)\theta}{\sqrt{2\pi}} \sqrt{1 - \frac{k}{n-l}} k^{-1/2}. \tag{3}$$

If $\beta \leq \alpha < 1 - \beta$, then

$$\mathbb{P}(A_n \geq k) \sim \frac{2(1-\beta)\theta}{\sqrt{2\pi}} \sqrt{1 - \frac{k}{n-l}} k^{-1/2}. \tag{4}$$

Lastly, if $1 - \beta < \alpha < 1$, then

$$\mathbb{P}(A_n \geq k) \sim \frac{\alpha \sqrt{\beta(1-\beta)} \theta^2}{\pi} c(\alpha, \beta) k^{-1},$$

where

$$c(\alpha, \beta) = \int_{x=\frac{\alpha(1-\beta)}{\beta}}^1 \frac{x^{-3/2}}{\sqrt{1-x}} \sqrt{\frac{s(x)}{1-s(x)}} dx, \quad s(x) = \frac{\beta x - (\alpha - (1-\beta))}{1-\beta}.$$

To prove Theorem 2, we partition the event of exceeding the threshold in three terms:

$$\{A_n \geq k\} = \{A_n \geq k; A_{l,n} \leq s_\star\} \cup \{A_n \geq k; s_\star < A_{l,n} < s^\star\} \cup \{A_n \geq k; A_{l,n} \geq s^\star\}, \tag{5}$$

where s_\star is chosen appropriately small and s^\star appropriately large. Table 1 illustrates which term will yield the dominant behavior in each of the cases in Theorem 2.

The reasoning turns more subtle at the boundary where the threshold is either close to the larger component size or when it is close to n itself. In view of Theorem 2, the first case ($\alpha = 1 - \beta$) corresponds to the interval of threshold values where we move from a power-law distribution with exponent $-1/2$ to one with exponent -1 . When the larger component remains significantly larger than the smaller one ($0 < \beta < 1 - \beta < 1$), this phase transition occurs as follows. As long as threshold k is sufficiently smaller than l , the most likely scenario to exceed k remains when it is already exceeded in the larger component alone. However, the closer k is to l , the smaller this probability is, and it is in fact zero when $l > k$. From some specific point, the scenario where the number of line failures in the larger component is close to k , yet not exceeding it, becomes the most likely one. If $\alpha = \beta = 1 - \beta = 1/2$, a similar likely event can also occur for the smaller component. Figure 3 reflects this intuition of Theorem 3. Again, the bold areas indicate which scenarios possibly asymptotically contribute to the exceedance probability.

(a) Case $\beta \in (0, 1/2)$. (b) Case $\beta = 1 - \beta = 1/2$.

Finally, if the threshold is close to the network size itself ($\alpha = 1$), almost all lines in both of the components need to have failed. Visually, this case is comparable to the one in Figure 2(c), where the triangle is minuscule.

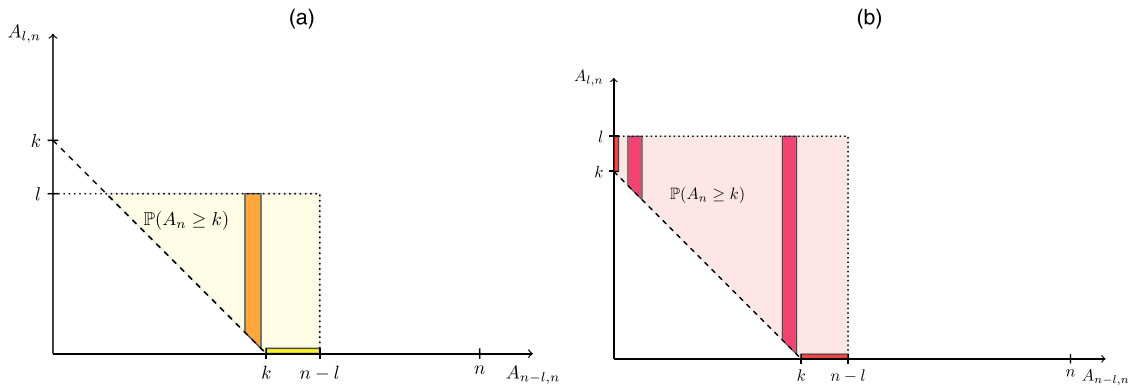
Theorem 3. Suppose $\alpha = 1 - \beta$ and $\beta \in (0, 1/2]$, and write $r := r_n = k - (n - l)$ and $t := t_n = k - l$. Then, as $n \rightarrow \infty$, A_n asymptotically behaves as follows. If $\beta \in (0, 1/2)$, then

$$\mathbb{P}(A_n \geq k) \sim \begin{cases} \frac{2(1-\beta)\theta}{\sqrt{2\pi}} \frac{\sqrt{-r}}{k} & \text{if } r < 0, -r = \omega((\log k)^2), \\ \frac{2(1-\beta)\theta}{\sqrt{2\pi}} \frac{\sqrt{-r}}{k} + \frac{\beta(1-\beta)\theta^2}{\pi} \frac{\log k}{k} & \text{if } r < 0, -r \propto (\log k)^2, \\ \frac{\beta(1-\beta)\theta^2}{\pi} \frac{\log k}{k} & \text{if } r < 0, -r = o(\log k)^2, \\ \frac{\beta(1-\beta)\theta^2}{\pi} \frac{\log(k/r)}{k} & \text{otherwise.} \end{cases}$$

Table 1. Road Map for Proof of Theorem 2

| Case | $\mathbb{P}(A_n \geq k; A_{l,n} \leq s_\star)$ | $\mathbb{P}(A_n \geq k; A_{l,n} \in (s_\star, s^\star))$ | $\mathbb{P}(A_n \geq k; A_{l,n} \geq s^\star)$ |
|---------------------------------|--|--|--|
| $\alpha \in [0, \beta)$ | $\mathbb{P}(A_{n-l,n} \geq k)$ | Negligible | $\mathbb{P}(A_{l,n} \geq k)$ |
| $\alpha \in [\beta, 1 - \beta)$ | $\mathbb{P}(A_{n-l,n} \geq k)$ | Negligible | 0 or negligible |
| $\alpha \in (1 - \beta, 1)$ | 0 | Dominant | 0 |

Figure 3. Asymptotic Contributions to $\mathbb{P}(A_n \geq k)$ in Theorem 3 if $\alpha = 1 - \beta$



Notes. (a) $\beta \in (0, 1/2)$. (b) $\beta = 1 - \beta = 1/2$.

If $\beta = 1/2$, write $\eta := \lim_{n \rightarrow \infty} t/r$. Then,

$$\mathbb{P}(A_n \geq k) \sim \begin{cases} \frac{\theta}{\sqrt{2\pi}} \frac{\sqrt{-r} + \sqrt{-t}}{k} & \text{if } -r = \omega((\log k)^2), \eta > 0, \\ \frac{\theta}{\sqrt{2\pi}} \frac{\sqrt{-r}}{k} & \text{if } -r = \omega((\log k)^2), \eta \leq 0, \\ \frac{\theta}{\sqrt{2\pi}} \frac{\sqrt{-r} + \sqrt{-t}}{k} + \frac{\theta^2 \log k}{2\pi k} & \text{if } r \propto -(\log k)^2, \eta > 0, \\ \frac{\theta}{\sqrt{2\pi}} \frac{\sqrt{-r}}{k} + \frac{\theta^2 \log k + \log(k/(|t| + 1))}{4\pi k} & \text{if } r \propto -(\log k)^2, \eta \leq 0, |t| \neq k^{1-o(1)}, \\ \frac{\theta}{\sqrt{2\pi}} \frac{\sqrt{-r}}{k} + \frac{\theta^2 \log k}{4\pi k} & \text{if } r \propto -(\log k)^2, t = k^{1-o(1)}, \\ \frac{\theta^2 \log(k/(|r| + 1)) + \log(k/(|t| + 1))}{4\pi k} & \text{if } r \text{ else, } \frac{k}{|r|+1} \neq \left(\frac{k}{|r|+1}\right)^{o(1)}, \\ \frac{\theta^2 \log(k/(|r| + 1))}{4\pi k} & \text{otherwise.} \end{cases}$$

If $\alpha = 1$, then as $n \rightarrow \infty$,

$$\mathbb{P}(A_n \geq k) \sim \frac{\theta^2}{2} (n - k)k^{-2}. \tag{6}$$

Because there is a sharp transition from a power law with exponent $-1/2$ to one with exponent -1 when $\alpha = 1 - \beta$, it is natural to consider the number of failures in the bigger component in more detail. In the proof of Theorem 3, we partition the event of exceeding the threshold with respect to the number of line failures in the bigger component.

When $\alpha = 1 - \beta$ with $\beta \in (0, 1/2)$, we use the identity

$$\begin{aligned} \mathbb{P}(A_n \geq k) &= \mathbb{P}(A_{n-l,n} \geq k) + \mathbb{P}(A_n \geq k; A_{n-l,n} \in [k - s_\star, k]) + \mathbb{P}(A_n \geq k; A_{n-l,n} \in (k - s^\star, k - s_\star)) \\ &\quad + \mathbb{P}(A_n \geq k; A_{n-l,n} \in [k - l, k - s^\star]), \end{aligned} \tag{7}$$

where s_\star and s^\star are chosen in a specific way. By labeling each term I, II, III, and IV, respectively, the asymptotic behavior of each term can be evaluated separately, which yields the result as in Table 2.

The result then follows by determining the dominant terms of (7) for the various cases of the threshold. It turns out there is a transition in dominant behavior when $-r \propto (\log k)^2$. For a smaller threshold, the threshold remains most likely to be exceeded in the larger component alone. Otherwise, it is most likely exceeded due to almost all lines failing in the larger component in conjunction with a growing number of line failures in the smaller component. This is summarized by Table 3.

Table 2. Asymptotic Behavior of Terms in (7)

| Term | Probability | Asymptotic behavior |
|------|--|--|
| I | $\mathbb{P}(A_{n-l,n} \geq k)$ | $\frac{2(1-\beta)\theta}{\sqrt{2\pi}} \frac{\sqrt{-r}}{k} \mathbb{1}_{\{-r > 0 \text{ growing}\}} + O(k^{-1}) \cdot \mathbb{1}_{\{r \leq 0 \text{ fixed}\}}$ |
| II | $\mathbb{P}(A_n \geq k; A_{n-l,n} \in [k - s_\star, k])$ | $o\left(\frac{\log(l/(r +1))}{k}\right)$ |
| III | $\mathbb{P}(A_n \geq k; A_{n-l,n} \in (k - s^\star, k - s_\star))$ | $\frac{\beta(1-\beta)\theta^2}{\pi} \frac{\log(k/(r +1))}{k}$ |
| IV | $\mathbb{P}(A_n \geq k; A_{n-l,n} \in [k - l, k - s^\star])$ | $o\left(\frac{\log(l/(r +1))}{k}\right)$ |

The situation turns even more subtle when $\alpha = \beta = 1/2$. In the most extreme case, we may have $l = n - l = n/2$, and one cannot distinguish between a smaller and larger component. The cascading process in the component of size l can therefore become more significant, leading to more possible scenarios likely to have occurred if the threshold is exceeded. To capture these scenarios, we need to refine the partitioning of events in (7) to

$$\begin{aligned} \mathbb{P}(A_n \geq k) &= \mathbb{P}(A_{n-l,n} \geq k) + \mathbb{P}(A_n \geq k; A_{n-l,n} \in [k - s_\star, k]) + \mathbb{P}(A_n \geq k; A_{n-l,n} \in (k - s^\star, k - s_\star)) \\ &\quad + \mathbb{P}(A_n \geq k; A_{n-l,n} \in (k - q_\star, k - s^\star]) + \mathbb{P}(A_n \geq k; A_{n-l,n} \in [k - q^\star, k - q_\star]) \\ &\quad + \mathbb{P}(A_n \geq k; A_{n-l,n} \in [k - l, k - q^\star]). \end{aligned} \tag{8}$$

In other words, we partition the event $\{A_n \geq k; A_{n-l,n} \in [k - l, k - s^\star]\}$ in (7) in three disjoint events in this case. In the proof, we determine the asymptotic behavior of the various terms in the identity (8), which leads to the result given in Table 4. Table 5 illustrates which terms contribute to the asymptotic tail behavior of A_n .

The final case of Theorem 3 involves the case where the threshold is close to the network size—that is, $\alpha = 1$. Both component sizes are therefore significantly smaller than the threshold. In this case, we partition the event of exceeding the threshold in only three disjoint events:

$$\begin{aligned} \mathbb{P}(A_n \geq k) &= \mathbb{P}(A_n \geq k; A_{n-l,n} \in (k - s_\star, n - l]) + \mathbb{P}(A_n \geq k; A_{n-l,n} \in [k - s^\star, k - s_\star]) \\ &\quad + \mathbb{P}(A_n \geq k; A_{n-l,n} \in (k - l, k - s^\star]). \end{aligned} \tag{9}$$

For appropriate choices of s_\star and s^\star , we will show that the second term is dominant and yields the result in Theorem 3.

3.2. Disparate Component Sizes

Next, we turn to the case $l = o(n)$. The smaller component is hence of a size that is (almost) negligible compared with the larger component. Essentially, this results in a framework that, for most thresholds ($0 < \alpha < 1$), no matter what occurs in the smaller component, the only likely manner to exceed the threshold is when it is exceeded in the larger component alone. This intuition remains true for $\alpha = 0$: The initial disturbance $\theta/n = \theta/l \cdot l/n$ is relatively minor in the smaller component and unlikely to cause the cascading failure process to propagate further.

When $\alpha = 1 - \beta = 1$, other scenarios to exceed k may become relevant. In particular, when $k > n - l$, the number of line failures in the larger component alone cannot exceed k . The partitioning of the event of exceeding threshold k needs to be done carefully, resulting in many phase transitions.

Theorem 4. *Suppose $\beta = 0$ and $r = k - (n - l)$. If $\alpha < 1$, or $\alpha = 1$ with $-r = \Omega(l)$, then as $n \rightarrow \infty$,*

$$\mathbb{P}(A_n \geq k) \sim \frac{2\theta}{\sqrt{2\pi}} \sqrt{1 - \frac{k}{n-l}} k^{-1/2}. \tag{10}$$

Table 3. Road Map for Proof of Theorem 3 with $\alpha = 1 - \beta$ with $\beta \in (0, 1/2)$

| Case | I | II | III | IV |
|---------------------------|--|------------|--|------------|
| $-r = \omega((\log k)^2)$ | $\frac{2(1-\beta)\theta}{\sqrt{2\pi}} \frac{\sqrt{-r}}{k}$ | Negligible | Negligible | Negligible |
| $-r \propto (\log k)^2$ | $\frac{2(1-\beta)\theta}{\sqrt{2\pi}} \frac{\sqrt{-r}}{k}$ | Negligible | $\frac{\beta(1-\beta)\theta^2}{\pi} \frac{\log k}{k}$ | Negligible |
| $ r = o(\log k)^2$ | 0 or negligible | Negligible | $\frac{\beta(1-\beta)\theta^2}{\pi} \frac{\log k}{k}$ | Negligible |
| $r > 0$ growing | 0 | Negligible | $\frac{\beta(1-\beta)\theta^2}{\pi} \frac{\log(k/r)}{k}$ | Negligible |

Table 4. Asymptotic Behavior of Terms in (8)

| Term | Probability | Asymptotic behavior |
|------|--|--|
| I | $\mathbb{P}(A_{n-l,n} \geq k)$ | $\frac{2(1-\beta)\theta}{\sqrt{2\pi}} \frac{\sqrt{-r}}{k} \mathbb{1}_{\{-r>0 \text{ growing}\}} + O(k^{-1}) \cdot \mathbb{1}_{\{r \leq 0 \text{ fixed}\}}$ |
| II | $\mathbb{P}(A_n \geq k; A_{n-l,n} \in [k - s_*, k])$ | $o\left(\frac{\log(l/ r +1)}{k}\right)$ |
| III | $\mathbb{P}(A_n \geq k; A_{n-l,n} \in (k - s_*, k - s_*))$ | $\frac{\theta^2 \log k}{4\pi} \frac{1}{k}$ |
| IV | $\mathbb{P}(A_n \geq k; A_{n-l,n} \in [k - q_*, k - s_*])$ | $o\left(\frac{\log(k/(r +1)}{k} + \frac{\log(k/(l +1)}{k}\right)$ |
| V | $\mathbb{P}(A_n \geq k; A_{n-l,n} \in [k - q^*, k - q_*])$ | $\frac{\theta^2 \log(k/(l +1)}{4\pi} \frac{1}{k}$ |
| VI | $\mathbb{P}(A_n \geq k; A_{n-l,n} \in [k - l, k - q^*])$ | $\frac{\theta}{\sqrt{2\pi}} \frac{\sqrt{-l}}{k} \mathbb{1}_{\{-l>0 \text{ growing}\}} + o\left(\frac{\log(l/(l +1)}{k}\right)$ |

If $k \leq n - l$, $-r = o(l)$ is growing with l , then as $n \rightarrow \infty$,

$$\mathbb{P}(A_n \geq k) \sim \begin{cases} \frac{2\theta}{\sqrt{2\pi}} \frac{\sqrt{-r}}{k}, & \text{if } l = o\left(\frac{n\sqrt{-r}}{\log n}\right), \\ \frac{2\theta}{\sqrt{2\pi}} \frac{\sqrt{-r}}{k} + \frac{\theta^2 l \log l}{\pi} k^{-2}, & \text{if } l \propto \frac{n\sqrt{-r}}{\log n}, \\ \frac{\theta^2 l \log l}{\pi} k^{-2}, & \text{if } l = \omega\left(\frac{n\sqrt{-r}}{\log n}\right). \end{cases} \quad (11)$$

If $r \leq 0$ is fixed, then as $n \rightarrow \infty$,

$$\mathbb{P}(A_n \geq k) \sim \begin{cases} \sum_{m=0}^{\max\{-r, \lceil \theta \rceil - 1\}} \frac{\theta(m - \theta)^m}{m!} e^{-(m-\theta)} k^{-1}, & \text{if } l = o\left(\frac{n}{\log n}\right), \\ \frac{\theta^2 l \log l}{\pi} k^{-2} + \sum_{m=0}^{\max\{-r, \lceil \theta \rceil\}} \frac{\theta(m - \theta)^m}{m!} e^{-(m-\theta)} k^{-1}, & \text{if } l \propto \frac{n}{\log n}, \\ \frac{\theta^2 l \log l}{\pi} k^{-2}, & \text{if } l = \omega\left(\frac{n}{\log n}\right). \end{cases} \quad (12)$$

If $k > n - l$ and $r = o(l)$ is growing with l , then as $n \rightarrow \infty$,

$$\mathbb{P}(A_n \geq k) \sim \frac{\theta^2 l \log(l/r)}{\pi} k^{-2}. \quad (13)$$

If $k > n - l$ and $\gamma := \lim_{n \rightarrow \infty} r/l \in (0, 1)$, then as $n \rightarrow \infty$,

$$\mathbb{P}(A_n \geq k) \sim \frac{\theta^2}{\pi} c(y) \frac{\sqrt{l}}{k^2}, \quad (14)$$

where

$$c(y) = \int_{y=\gamma}^1 \sqrt{\frac{1-y}{(y-\gamma)y}} dy.$$

Table 5. Road Map for Proof of Theorem 3 with $\alpha = 1 - \beta = 1/2$

| Case | I | II | III | IV | V | VI |
|--|--|------------|---|------------|---|--|
| $-r = \omega((\log k)^2)$, $\lim_{n \rightarrow \infty} t/r > 0$ | $\frac{\theta}{\sqrt{2\pi}} \frac{\sqrt{-r}}{k}$ | Negligible | Negligible | Negligible | Negligible | $\frac{\theta}{\sqrt{2\pi}} \frac{\sqrt{-l}}{k}$ |
| $-r = \omega((\log k)^2)$, $\lim_{n \rightarrow \infty} t/r \leq 0$ | $\frac{\theta}{\sqrt{2\pi}} \frac{\sqrt{-r}}{k}$ | Negligible | Negligible | Negligible | Negligible | Negligible |
| $-r \propto (\log k)^2$, $\lim_{n \rightarrow \infty} t/r > 0$ | $\frac{\theta}{\sqrt{2\pi}} \frac{\sqrt{-r}}{k}$ | Negligible | $\frac{\theta^2 \log k}{4\pi} \frac{1}{k}$ | Negligible | $\frac{\theta^2 \log k}{4\pi} \frac{1}{k}$ | $\frac{\theta}{\sqrt{2\pi}} \frac{\sqrt{-l}}{k}$ |
| $-r \propto (\log k)^2$, $ t \neq k^{1-o(1)}$, $\lim_{n \rightarrow \infty} t/r \leq 0$ | $\frac{\theta}{\sqrt{2\pi}} \frac{\sqrt{-r}}{k}$ | Negligible | $\frac{\theta^2 \log k}{4\pi} \frac{1}{k}$ | Negligible | $\frac{\theta^2 \log(\frac{k}{ r +1})}{4\pi} \frac{1}{k}$ | Negligible |
| $-r \propto (\log k)^2$, $t = k^{1-o(1)}$ | $\frac{\theta}{\sqrt{2\pi}} \frac{\sqrt{-r}}{k}$ | Negligible | $\frac{\theta^2 \log k}{4\pi} \frac{1}{k}$ | Negligible | Negligible | Negligible |
| r otherwise, $\frac{k}{ r +1} \neq \left(\frac{k}{ r +1}\right)^{o(1)}$ | Negligible | Negligible | $\frac{\theta^2 \log(\frac{k}{ r +1})}{4\pi} \frac{1}{k}$ | Negligible | $\frac{\theta^2 \log(\frac{k}{ r +1})}{4\pi} \frac{1}{k}$ | Negligible |
| Otherwise | Negligible | Negligible | $\frac{\theta^2 \log(\frac{k}{ r +1})}{4\pi} \frac{1}{k}$ | Negligible | Negligible | Negligible |

Finally, if $k > n - l$ and $r = l - o(l)$, then as $n \rightarrow \infty$,

$$\mathbb{P}(A_n \geq k) \sim \frac{\theta^2}{2}(n - k)k^{-2}. \quad (15)$$

The proof of Theorem 4 is analogous to the proofs of Theorems 2 and 3 (excluding the case $\alpha = \beta = 1/2$). Yet, we need to account for the disparity in component sizes, changing the points where the phase transitions occur.

Remark 5. We would like to stress that the main results in this paper can be extended immediately to more general surplus capacity distributions. The reason why is described in Sloothaak et al. (2018a) in case of the star topology. That is, suppose that the surplus capacities are independent and identically distributed with a distribution function $F(\cdot)$ that is continuous and has a strictly positive density in zero. Write $C_{(i)}^n, i = 1, \dots, n$ for the order statistics corresponding to the surplus capacities and, similarly, $U_{(i)}^n, i = 1, \dots, n$ for the standard uniformly distributed order statistics. Write $l^n(\cdot)$ for the load surge function. It holds that

$$\begin{aligned} \mathbb{P}(A_n \geq k) &= \mathbb{P}\left(C_{(i)}^n \leq l^n(i), \quad i = 1, \dots, k\right) \\ &= \mathbb{P}\left(F(C_{(i)}^n) \leq F(l^n(i)), \quad i = 1, \dots, k\right) \\ &= \mathbb{P}\left(U_{(i)}^n \leq F(l^n(i)), \quad i = 1, \dots, k\right). \end{aligned}$$

In other words, we require that

$$F(l^n(i)) \approx \frac{\theta + i - 1}{n}, \quad (16)$$

for the power-law behavior to prevail, where the meaning of the approximation operator is specified in Sloothaak et al. (2018a). Using this observation, it is immediate to extend the results in this paper to more general surplus capacities: We require $F(l^l(i)) \approx (\theta + i - 1)/l$ for the component with l edges and $F(l^{n-l}(i)) \approx (\theta + i - 1)/(n - l)$ for the other component. We refrained from doing so in this paper for the purpose of readability.

4. Proofs of Main Results

In order to prove Theorems 2–4, we use an asymptotic analysis for the sum of two independent heavy-tailed random variables. Specifically, we determine the asymptotic tail of $A_n = A_{l,n} + A_{n-l,n}$, where $A_{l,n}$ and $A_{n-l,n}$ are independent. In fact, the distribution of $A_{l,n}$ and $A_{n-l,n}$ is well understood due to the following observation. The first line failure disconnects the network in two separate star networks of sizes l and $n - l$. For the first component, this causes an initial disturbance of $\theta/n = (\theta/l) \cdot (l/n)$, and every consecutive line failure causes an additional load surge of $1/l$. Hence, it falls in the framework of the model studied in Sloothaak et al. (2018a), where l is the network size and initial disturbance constant $\theta \cdot l/n$. It implies that for every growing k ,

$$\mathbb{P}(A_{l,n} \geq k) \begin{cases} = 0 & \text{if } k > l, \\ \sim \chi(l) \cdot l^{-1} & \text{if } l - k \geq 0 \text{ fixed,} \\ \sim \frac{2\theta \cdot l/n}{\sqrt{2\pi}} \sqrt{\frac{l-k}{kl}} & \text{otherwise,} \end{cases} \quad (17)$$

where the latter holds uniformly in accordance with Theorem 1, and using equation (A.1) in Sloothaak et al. (2018a), we derive

$$\chi(l) = \sum_{m=0}^{\max\{l-k, \lceil \theta \cdot l/n \rceil - 1\}} \frac{\theta \cdot l/n (m - \theta \cdot l/n)^m}{m!} e^{-(m - \theta \cdot l/n)}.$$

Similarly, (17) holds for $A_{n-l,n}$, with l replaced by $n-l$. We see that it implies a certain order of magnitude, that is

$$\mathbb{P}(A_{l,n} \geq k) = \begin{cases} 0 & \text{if } k > l, \\ O\left(\frac{l}{n}l^{-1}\right) & \text{if } l-k \geq 0 \text{ fixed,} \\ O\left(\frac{l}{n}\frac{\sqrt{l-k}}{l}\right) & \text{if } l-k = o(l) \text{ growing,} \\ O\left(\frac{l}{n}\sqrt{\frac{l-k}{kl}}\right) & \text{if } \lim_{n \rightarrow \infty} k/l \in (0,1), \\ O\left(\frac{l}{n}k^{-1/2}\right) & \text{if } k = o(l), \end{cases}$$

and again, similarly for $A_{n-l,n}$ with l replaced by $n-l$. This will be used extensively throughout the proofs.

To derive the main results stated in Section 3, we determine which scenarios are most likely to cause $A_n = A_{l,n} + A_{n-l,n}$ to exceed the threshold.

4.1. Very Few or Many Failures in One Component

The strategy in all our proofs involves an appropriate partition of the event of exceeding the threshold. In this section, we state results on the asymptotic behavior of such joint events where there are very few or many line failures in one component.

4.1.1. Smaller Component. The proof of Theorem 2 partitions the event of exceeding threshold k in joint events where the number of failures in the smaller component is in a certain interval. The next two lemmas quantify the probability of $\{A_n \geq k; A_{l,n} \leq s_\star\}$ with s_\star very small, $\{A_n \geq k; A_{l,n} \geq s^\star\}$ with s^\star very large. The proofs are given in Appendix C.

Lemma 6. *Let $s_\star = o(\min\{k, l\})$ be growing. Then, as $n \rightarrow \infty$,*

$$\mathbb{P}(A_n \geq k; A_{l,n} \leq s_\star) \begin{cases} \sim \mathbb{P}(A_{n-l,n} \geq k) & \text{if } \alpha < 1 - \beta, \\ = 0 & \text{if } 1 - \beta < \alpha < 1. \end{cases}$$

Lemma 7. *Let s^\star be growing defined by $s^\star = k - o(k)$ if $k < l$ and $s^\star = l - o(l)$ otherwise. Then, as $n \rightarrow \infty$,*

$$\mathbb{P}(A_n \geq k; A_{l,n} \geq s^\star) \begin{cases} \sim \mathbb{P}(A_{l,n} \geq k) & \text{if } \alpha < \beta, \\ = o(k^{-1/2}) & \text{if } \beta \leq \alpha < 1 - \beta, \\ = o(k^{-1}) & \text{if } 1 - \beta < \alpha < 1. \end{cases}$$

4.1.2. Larger Component: Balanced Case. The proof of Theorem 3 partitions the event of exceeding threshold k in joint events where the number of failures in the larger component is in a certain interval. The next lemma shows the asymptotic behavior where almost all lines in the larger component have failed. Henceforth, recall that $r = k - (n - l)$.

Lemma 8. *Suppose $\alpha = 1 - \beta$ with $\beta \neq 0$. If $r < 0$ and $-r$ is growing with n or $|r|$ fixed, let $s_\star = o(l)$ be growing. Then,*

$$\mathbb{P}(A_n \geq k, A_{n-l,n} \in [k - s_\star, \min\{k - 1, n - l\}]) = O\left(\frac{s_\star}{n - l}\right). \tag{18}$$

If $\alpha = 1 - \beta$ and $r > 0$ is growing, let $s_\star = r + o(l)$ be growing such that $s_\star - r$ is growing. Then,

$$\mathbb{P}(A_n \geq k, A_{n-l,n} \geq k - s_\star) = o\left(\frac{\sqrt{s_\star - r}}{n - l}\right). \tag{19}$$

Finally, if $\alpha = 1$ and $\beta \neq 0$, let $s_\star = k - (n - l) + o(n - k) = r + o(n - k)$ such that $s_\star - r$ is growing. Then,

$$\mathbb{P}(A_n \geq k, A_{n-l,n} \geq k - s_\star) = o\left(\frac{n - k}{k^2}\right). \tag{20}$$

The second lemma in this section yields the asymptotic behavior where at least a significant number of lines in the smaller component have failed for $\alpha = 1 - \beta$ or $\alpha = 1$.

Lemma 9. *If $\beta \in (0, 1/2)$ and $\alpha = 1 - \beta$, suppose $s^* = o(l)$ such that $s^* - r > 0$ is growing. Then,*

$$\mathbb{P}(A_n \geq k, A_{n-l,n} \in [k-l, k-s^*]) = O(s^{*-1/2} k^{-1/2}). \quad (21)$$

If $\alpha = \beta = 1 - \beta = 1/2$ and $l - k$ is not growing, set $s^ = l - o(l)$ such that $l - s^*$ is growing. Then,*

$$\mathbb{P}(A_n \geq k, A_{n-l,n} \in [k-l, k-s^*]) = O\left(\frac{\sqrt{l-s^*}}{k}\right). \quad (22)$$

If $\alpha = \beta = 1 - \beta = 1/2$ and $l - k > 0$ is growing, set $s^ = k - o(l - k)$ such that $k - s^*$ is growing. Then, as $n \rightarrow \infty$,*

$$\mathbb{P}(A_n \geq k, A_{n-l,n} \in [k-l, k-s^*]) \sim \frac{\theta}{\sqrt{2\pi}} \frac{\sqrt{l-k}}{k}. \quad (23)$$

Finally, if $\alpha = 1$ and $\beta \in (0, 1/2]$, set $s^ = l - o(n - k)$ such that $l - s^* = \omega(1)$. Then,*

$$\mathbb{P}(A_n \geq k, A_{n-l,n} \in [k-l, k-s^*]) = o\left(\frac{n-k}{k^2}\right). \quad (24)$$

4.1.3. Larger Component: Disparate Case. In this section, we consider only the case $\alpha = 1 - \beta = 1$ with $\beta = 0$, where $r = o(l)$.

Lemma 10. *Suppose $|r| = o(l)$. If $-r > 0$ is growing, let $s_\star = o(l)$ be growing with l such that it satisfies $s_\star = o(\log l)$. Otherwise, let $s_\star = r + v$, where v is growing and satisfying $v = o(\log(l/|r| + 1))$. Then,*

$$\mathbb{P}(A_n \geq k; A_{n-l,n} \in [k-s_\star, k-1]) = o\left(\mathbb{1}_{\{r \leq 0\}} \frac{l \log l}{k^2} + \mathbb{1}_{\{r > 0\}} \frac{l \log(l/r)}{k^2}\right).$$

Lemma 11. *Suppose $|r| = o(l)$, and let $s^* = o(l)$ be growing with l such that it satisfies $s^* = \omega(l/(\log l)^2)$ when $r \leq 0$, and $s^* = \omega(l/(\log(l/r))^2)$ when $r > 0$. Then,*

$$\mathbb{P}(A_n \geq k; A_{n-l,n} \in [k-l, k-s^*]) = o\left(\mathbb{1}_{\{r \leq 0\}} \left(\frac{l \log l}{k^2}\right) + \mathbb{1}_{\{r > 0\}} \frac{l \log(l/r)}{k^2}\right).$$

4.2. Proof of Theorem 2

Next, we prove Theorem 2 using the approach outlined in Table 1.

Proof of Theorem 2. If $\alpha < \beta$, set $s_\star = o(k)$ and $s^* = k - \omega(k/s_\star)$ such that both are growing large with n . Then,

$$\mathbb{P}(A_n \geq k; s_\star < A_{l,n} < s^*) \leq \underbrace{\mathbb{P}(A_{l,n} \geq s_\star)}_{=O(s_\star^{-1/2})} \underbrace{\mathbb{P}(A_{n-l,n} \geq k-s^*)}_{=o((k/s_\star)^{-1/2})} = o(k^{-1/2}).$$

Applying Lemmas 6 and 7, together with (17), yields

$$\begin{aligned} \mathbb{P}(A_n \geq k) &\sim \mathbb{P}(A_{n-l,n} \geq k) + \mathbb{P}(A_{l,n} \geq k) \\ &\sim \frac{2\beta\theta}{\sqrt{2\pi}} \sqrt{1 - \frac{k}{l}} k^{-1/2} + \frac{2(1-\beta)\theta}{\sqrt{2\pi}} \sqrt{1 - \frac{k}{n-l}} k^{-1/2}, \end{aligned}$$

as $n \rightarrow \infty$.

If $\beta \leq \alpha < 1 - \beta$, set $s_\star = o(\min\{k, l\})$ and $s^* = \min\{k, l\} - o(\min\{k, l\})$ such that both are growing with n . Lemmas 6 and 7 imply that as $n \rightarrow \infty$,

$$\mathbb{P}(A_{l,n} \geq k) + \mathbb{P}(A_{n-l,n} \geq k) \sim \mathbb{P}(A_{n-l,n} \geq k) \sim \frac{2(1-\beta)\theta}{\sqrt{2\pi}} \sqrt{1 - \frac{k}{n-l}} k^{-1/2}.$$

The result follows in this case as well because

$$\mathbb{P}(A_n \geq k; s_\star < A_{l,n} < s^\star) \leq \underbrace{\mathbb{P}(A_{l,n} \geq s_\star)}_{=o(1)} \underbrace{\mathbb{P}(A_{n-l,n} \geq k-l)}_{=O(k^{-1/2})} = o(k^{-1/2}).$$

Finally, we consider the case $1 - \beta < \alpha < 1$. For every $s_\star = o(l)$, it holds that $\mathbb{P}(A_n \geq k; A_{l,n} < s_\star) = 0$. Lemma 7 implies for $s^\star = l - o(l)$,

$$\mathbb{P}(A_n \geq k; A_{l,n} \geq s^\star) = o(k^{-1}).$$

In addition, we have for $q^\star = n - l - o(n - l)$

$$\mathbb{P}(A_n \geq k; A_{n-l,n} \geq q^\star) \leq \underbrace{\mathbb{P}(A_{l,n} \geq k - (n - l))}_{=O(k^{-1/2})} \underbrace{\mathbb{P}(A_{n-l,n} \geq q^\star)}_{=O(q^{\star-1/2})} = o(k^{-1}).$$

Therefore, it remains to be shown that for these choices of q^\star and s^\star ,

$$\mathbb{P}(A_n \geq k; A_{l,n} < s^\star; A_{n-l,n} < q^\star) \sim \frac{\alpha \sqrt{\beta(1-\beta)} \theta^2}{\pi} c(\alpha, \beta) k^{-1}, \tag{25}$$

as $n \rightarrow \infty$, where $c(\alpha, \beta)$ is defined as in the theorem.

Remark 12. We note that $c(\alpha, \beta)$ is a positive finite constant. That is, the function within the integral is nonnegative and has a positive mass over the interval we integrate, and hence it is positive. Moreover, because $x/(1-x)$ is a nonnegative increasing function for all $x \in [0, 1)$ and $s(\cdot)$ is a linearly increasing function,

$$\int_{\frac{\alpha-(1-\beta)}{\beta}}^1 (1-x)^{-1/2} x^{-3/2} \sqrt{\frac{s(x)}{1-s(x)}} dx \leq 2 \sqrt{\frac{1-\alpha}{\alpha-(1-\beta)}} \sqrt{\frac{s(1)}{1-s(1)}} < \infty.$$

Indeed, the integral expression is a positive finite constant.

Theorem 1 yields as $n \rightarrow \infty$,

$$\mathbb{P}(A_n \geq k; A_{l,n} < s^\star; A_{n-l,n} < q^\star) \sim \frac{\beta(1-\beta)\theta^2}{2\pi} \sum_{j=k-q^\star+1}^{s^\star-1} \sum_{m=(n-l)-q^\star+1}^{j-(k-(n-l))} \sqrt{\frac{l}{(l-j)j^3}} \sqrt{\frac{n-l}{m(n-l-m)^3}}.$$

An upper bound for the summation term is given by

$$\begin{aligned} \sum_{j=k-q^\star+1}^{s^\star-1} \sum_{m=(n-l)-q^\star+1}^{j-(k-(n-l))} \sqrt{\frac{l}{(l-j)j^3}} \sqrt{\frac{n-l}{m(n-l-m)^3}} &\leq \int_{j=k-q^\star}^{s^\star} \int_{m=(n-l)-q^\star}^{j-(k-(n-l))} \sqrt{\frac{l \cdot (n-l)}{(l-j)j^3 m(n-l-m)^3}} dm dj \\ &\leq \frac{1}{\sqrt{l(n-l)}} \int_{x=\frac{k-(n-l)}{l}}^1 \int_{y=0}^{\frac{lx-(k-(n-l))}{n-l}} (1-x)^{-1/2} x^{-3/2} \cdot y^{-1/2} (1-y)^{-3/2} dy dx \\ &= \frac{2}{\sqrt{l(n-l)}} \int_{x=\frac{k-(n-l)}{l}}^1 (1-x)^{-1/2} x^{-3/2} \sqrt{\frac{(lx-(k-(n-l)))/(n-l)}{1-(lx-(k-(n-l)))/(n-l)}} dx. \end{aligned}$$

Similarly, a lower bound is given by

$$\sum_{j=k-q^\star+1}^{s^\star-1} \sum_{m=(n-l)-q^\star+1}^{j-(k-(n-l))} \sqrt{\frac{l}{(l-j)j^3}} \sqrt{\frac{n-l}{m(n-l-m)^3}} \geq \frac{1}{\sqrt{l(n-l)}} \int_{x=\frac{k-q^\star+2}{l}}^{s^\star-2} \int_{y=\frac{n-l-q^\star+2}{n-l}}^{\frac{lx-(k-(n-l))}{n-l}} (1-x)^{-1/2} x^{-3/2} \cdot y^{-1/2} (1-y)^{-3/2} dy dx.$$

Because of our choices of q^\star and s^\star , then as $n \rightarrow \infty$, the two integral expressions converge to the same constant. That is,

$$\mathbb{P}(A_n \geq k; A_{l,n} < s^\star; A_{n-l,n} < q^\star) \sim \frac{\beta(1-\beta)\theta^2}{2\pi} \frac{2}{\sqrt{l(n-l)}} \int_{x=\frac{\alpha-(1-\beta)}{\beta}}^1 \frac{x^{-3/2}}{\sqrt{1-x}} \sqrt{\frac{s(x)}{1-s(x)}} dx,$$

as $n \rightarrow \infty$, which asymptotically coincides with (25). ■

4.3. Proof of Theorem 3

Next, we prove Theorem 3 using the approach outlined in Table 2.

Proof of Theorem 3. First consider the case that $\beta \in (0, 1/2)$. Using identity (7), we observe that it suffices to show that the asymptotic behavior provided in Table 2 holds. That is, the result is immediate from Table 3, which in turn only highlights the dominant terms of Table 2.

Let $s_\star = o(\log(l/(|r| + 1)))$ be growing if $-r > 0$ is growing or $|r|$ fixed, and $s_\star = r + o(\min\{r, \log(l/r)\})$ such that $s_\star - r$ is growing if $r > 0$ is growing. Let $s^\star = o(l)$ be growing such that $s^\star = \omega(l/\log(l/(|r| + 1)))$. Note that because of this choice, $s^\star > s_\star$ for all n large enough. Then, Lemmas 8 and 9 yield

$$\sum_{j=1}^{s_\star} \mathbb{P}(A_{l,n} \geq j) \mathbb{P}(A_{n-l,n} = k - j) = o\left(\frac{\log\left(\frac{l}{|r|+1}\right)}{k}\right),$$

$$\sum_{j=s^\star}^l \mathbb{P}(A_{l,n} \geq j) \mathbb{P}(A_{n-l,n} = k - j) = o\left(\frac{\log\left(\frac{l}{|r|+1}\right)}{k}\right).$$

Moreover, uniformly as $n \rightarrow \infty$,

$$\sum_{j=s_\star}^{s^\star} \mathbb{P}(A_{l,n} \geq j) \mathbb{P}(A_{n-l,n} = k - j) \sim \frac{\beta(1 - \beta)\theta^2}{\pi} \sum_{j=s_\star}^{s^\star} j^{-1/2} \frac{(j - r)^{-1/2}}{n - l}.$$

By assumption, $k \sim n - l$ and $\log l \sim \log k$. Invoking Lemma B.1 hence yields

$$\sum_{j=s_\star}^{s^\star} \mathbb{P}(A_{l,n} \geq j) \mathbb{P}(A_{n-l,n} = k - j) \sim \frac{\beta(1 - \beta)\theta^2 \log(k/(|r| + 1))}{\pi k},$$

as $n \rightarrow \infty$. Using (17), we obtain as $n \rightarrow \infty$,

$$\mathbb{P}(A_n \geq k) \sim \mathbb{P}(A_{n-l,n} \geq k) + \frac{\beta(1 - \beta)\theta^2 \log\left(\frac{k}{|r|+1}\right)}{\pi k},$$

where the asymptotic behavior of $\mathbb{P}(A_{n-l,n} \geq k)$ is given by Equation (17). The result follows by observing that phase transitions occur when $-r \propto \log^2 k$. In words, the threshold is most likely exceeded in the larger component alone or both components have a significant number of line failures. The latter turns dominant as soon as the difference between the threshold and larger component size becomes small enough.

Next, we prove the second case of the theorem with $\beta = 1/2$. That is, the two component sizes are approximately the same, making the analysis more delicate. Effectively, we follow the same strategy as before, but make some modifications, as the smaller component is approximately of the same size as the bigger component. Equation (17) provides the asymptotic behavior of $\mathbb{P}(A_{n-l,n} \geq k)$. Again, let s_\star and s^\star be as above. Using the analysis above shows that

$$\sum_{j=\max\{1,r\}}^{s^\star} \mathbb{P}(A_{l,n} \geq j) \mathbb{P}(A_{n-l,n} = k - j) \sim \frac{\beta(1 - \beta)\theta^2 \log(k/(|r| + 1))}{\pi k} = \frac{\theta^2 \log(k/(|r| + 1))}{4\pi k},$$

remains valid in this case, covering the asymptotic behavior of terms II and III.

Let $q_\star = l - o(l)$ satisfy $l - q_\star = \omega(l/\log(l/(|t| + 1)))$ and is growing. Let $q^\star = k - o(l - k) = l - (|t| + o(|t|))$ be growing such that $k - q^\star = \omega(|t|/\log(l/|t|))$ is growing if $-t = l - k > 0$ is growing, and $q^\star = l - o(l)$ such that $l - q^\star = o(\log(l/(|t| + 1)))$ is growing otherwise. We observe that for this choice of q_\star , term IV yields

$$\begin{aligned} \sum_{j=s^\star}^{q_\star} \mathbb{P}(A_{l,n} \geq j) \mathbb{P}(A_{n-l,n} = k - j) &\leq \mathbb{P}(A_{l,n} \geq s^\star) \mathbb{P}(A_{n-l,n} \geq k - q_\star) \\ &= O\left(\frac{1}{\sqrt{s^\star(k - q_\star)}}\right) = o\left(\frac{\sqrt{\log(l/(|r| + 1)) \log(l/(|t| + 1))}}{k}\right) \\ &= o\left(\frac{\log(k/(|r| + 1)) + \log(k/(|t| + 1))}{k}\right). \end{aligned}$$

It follows from Theorem 1 that uniformly as $n \rightarrow \infty$,

$$\sum_{j=q_\star}^{q_\star} \mathbb{P}(A_{l,n} \geq j) \mathbb{P}(A_{n-l,n} = k-j) \sim \frac{\theta^2}{4\pi} \sum_{j=q_\star}^{q_\star} \frac{(l-j)^{1/2}}{l} (k-j)^{-3/2}.$$

Applying Lemma B.2 results into $q_\star < q^\star$ for all n large enough, and

$$\sum_{j=q_\star}^{q^\star} \mathbb{P}(A_{l,n} \geq j) \mathbb{P}(A_{n-l,n} = k-j) \sim \frac{\theta^2 \log(k/(|l|+1))}{4\pi k}.$$

That is, it describes the asymptotic behavior of all events where almost all lines have failed in the smaller component, whereas the number of failures in the larger component is substantial, yet relatively small. Finally, Lemma 9 implies as $n \rightarrow \infty$,

$$\sum_{j=q_\star}^l \mathbb{P}(A_{l,n} \geq j) \mathbb{P}(A_{n-l,n} = k-j) \sim \frac{\theta}{\sqrt{2\pi}} \frac{\sqrt{-t}}{k},$$

if $-t > 0$ is growing, and

$$\sum_{j=q_\star}^l \mathbb{P}(A_{l,n} \geq j) \mathbb{P}(A_{n-l,n} = k-j) = o\left(\frac{\log\left(\frac{l}{|l|+1}\right)}{k}\right),$$

otherwise. In other words, the event that the threshold is exceeded in the smaller component alone contributes to the dominant behavior only if k is significantly smaller than l . Combining the above results then concludes the result for $\beta = 1/2$.

Finally, we have to show the result for $\alpha = 1$. The threshold is close to n itself, and hence both components can only have a few surviving lines after the cascading failure process. Recall (9) and the results of Lemmas 8 and 9. Observe that $r > 0$ is of order n , and hence for any $s_\star = r + o(n-k)$ and $s^\star = l - o(n-k)$ satisfying the conditions in Lemmas 8 and 9 yield

$$\begin{aligned} \sum_{j=r}^{s_\star} \mathbb{P}(A_{l,n} \geq j) \mathbb{P}(A_{n-l,n} = k-j) &= o(k^{-1}), \\ \sum_{j=s^\star}^l \mathbb{P}(A_{l,n} \geq j) \mathbb{P}(A_{n-l,n} = k-j) &= o\left(\frac{n-k}{k^2}\right). \end{aligned}$$

To finalize the proof, we hence have to show that for suitable s_\star and s^\star satisfying the conditions above,

$$\sum_{j=s_\star}^{s^\star} \mathbb{P}(A_{l,n} \geq j) \mathbb{P}(A_{n-l,n} = k-j) \sim \frac{\theta^2}{2} (n-k)k^{-2}, \tag{26}$$

as $n \rightarrow \infty$. Fix $\epsilon > 0$, then for large enough n ,

$$\begin{aligned} \sum_{j=s_\star}^{s^\star} \mathbb{P}(A_{l,n} \geq j) \mathbb{P}(A_{n-l,n} = k-j) &= \sum_{j=s_\star-r}^{s^\star-r} \mathbb{P}(A_{l,n} \geq j+r) \mathbb{P}(A_{n-l,n} = n-l-j) \\ &\leq (1+\epsilon)^2 \int_{x=0}^{n-k} \frac{2\beta\theta}{\sqrt{2\pi}} \frac{\sqrt{l-r-x}(1-\beta)\theta}{l} \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{x(n-l)}} dx \\ &= (1+\epsilon)^2 \frac{\beta(1-\beta)\theta^2}{\pi l(n-l)} \int_{x=0}^{n-k} (n-k-x)^{1/2} x^{-1/2} dx \\ &= (1+\epsilon)^2 \frac{\beta(1-\beta)\theta^2}{2} \frac{n-k}{l(n-l)}. \end{aligned}$$

For the lower bound, note that we can set $s_\star - r = l - s^\star$, which is done to simplify the integration term in the lower bound. That is,

$$\begin{aligned} \sum_{j=s_\star}^{s^\star} \mathbb{P}(A_{l,n} \geq j) \mathbb{P}(A_{n-l,n} = k - j) &\geq (1 - \epsilon)^2 \frac{\beta(1 - \beta)\theta^2}{\pi l(n - l)} \int_{x=s_\star - r + 1}^{s^\star - r - 1} (n - k - x)^{1/2} x^{-1/2} dx \\ &\geq (1 - \epsilon)^2 \frac{\beta(1 - \beta)\theta^2(n - k)}{\pi l(n - l)} \arctan\left(\frac{n - k - 2(s_\star - r)}{2\sqrt{(n - k - (s_\star - r))(s_\star - r)}}\right). \end{aligned}$$

We note that

$$\lim_{n \rightarrow \infty} \arctan\left(\frac{n - k - 2(s_\star - r)}{2\sqrt{(n - k - (s_\star - r))(s_\star - r)}}\right) = \frac{\pi}{2}.$$

Letting $\epsilon \downarrow 0$ shows that the bounds coincide, and hence (26) holds. ■

4.4. Proof of Theorem 4

Lastly, we prove Theorem 4.

Proof of Theorem 4. In order to see that (10) holds for $\alpha < 1$, choose $s_\star = o(\min\{k, l\})$ and $s^\star = \min\{k, l\} - o(\min\{k, l\})$. In addition, let $s^\star = \min\{k, l\} - \omega(\min\{k, l\}/s_\star)$ if $k = O(l)$. Lemma 6 and (17) yield

$$\mathbb{P}(A_n \geq k; A_{l,n} \leq s_\star) \sim \mathbb{P}(A_{n-l,n} \geq k) \sim \frac{2\theta}{\sqrt{2\pi}} \sqrt{1 - \frac{k}{n-l}} k^{-1/2},$$

and

$$\mathbb{P}(A_n \geq k; A_{l,n} \in (s_\star, s^\star)) \leq \underbrace{\mathbb{P}(A_{l,n} \geq s_\star)}_{=O(l/ns_\star^{-1/2})} \underbrace{\mathbb{P}(A_{n-l,n} \geq k - s^\star)}_{=O((k-s^\star)^{-1/2})} = o(k^{-1/2}).$$

Moreover, if $k < l$,

$$\mathbb{P}(A_n \geq k; A_{l,n} \geq s_\star) \leq \mathbb{P}(A_{l,n} \geq s_\star) = O\left(\frac{l}{n} \sqrt{\frac{l - s^\star}{s^\star l}}\right) = O\left(\frac{\sqrt{l(l - s^\star)}}{n} k^{-1/2}\right) = o(k^{-1/2}),$$

and if $k \geq l$,

$$\mathbb{P}(A_n \geq k; A_{l,n} \geq s_\star) \leq \mathbb{P}(A_{l,n} \geq s_\star) = O\left(\frac{l}{n} \sqrt{\frac{l - s^\star}{s^\star l}}\right) = O\left(\frac{\sqrt{l - s^\star}}{n}\right) = o(k^{-1/2}).$$

Because of (7), we can therefore conclude that (10) holds when $\alpha < 1$.

Next, suppose $\alpha = 1$. Equation (17) then translates to

$$\mathbb{P}(A_{n-l,n} \geq k) \begin{cases} \sim \frac{2\theta}{\sqrt{2\pi}} \frac{\sqrt{-r}}{k} & \text{if } -r > 0 \text{ growing,} \\ \sim \chi(r) k^{-1} & \text{if } r \text{ fixed,} \\ = 0 & \text{if } r > 0, \end{cases}$$

where

$$\chi(r) = \sum_{m=0}^{\max\{-r, \lceil \theta \rceil - 1\}} \frac{\theta(m - \theta)^m}{m!} e^{-(m-\theta)}.$$

If $-r = \Omega(l)$, we have the bound

$$\mathbb{P}(A_n \geq k; A_{n-l,n} < k) \leq \mathbb{P}(A_{l,n} \geq 1) \mathbb{P}(A_{n-l,n} \geq k - l).$$

Because $A_{l,n}$ obeys a quasibinomial distribution (Dobson et al. 2005),

$$\mathbb{P}(A_{l,n} \geq 1) = 1 - \left(1 - \frac{\theta}{n}\right)^l = o(1),$$

and the second term is bounded by

$$\mathbb{P}(A_{n-l,n} \geq k - l) = O\left(\frac{\sqrt{n-k}}{n-l}\right) = O\left(\frac{\sqrt{-r}}{k}\right).$$

Again, because of identity (7), we observe that (10) holds in this case as well.

Next, suppose $\alpha = 1$ with $|r| = o(l)$. Choose s_\star small enough and s^\star large enough such that the conditions in Lemmas B.1 and 10 are satisfied. Then, uniformly,

$$\sum_{j=s_\star+1}^{s^\star-1} \mathbb{P}(A_{l,n} \geq j) \mathbb{P}(A_{n-l,n} = k - j) \sim \sum_{j=s_\star+1}^{s^\star-1} \frac{\theta^2}{\pi} \frac{l}{k^2} j^{-1/2} (j-r)^{-1/2} \sim \frac{\theta^2}{\pi} \frac{l}{k^2} \log\left(\frac{l}{|r|+1}\right).$$

Recalling (7) and (17) and applying Lemmas 10 and 11 then yields that as $n \rightarrow \infty$,

$$\mathbb{P}(A_n \geq k) \sim \mathbb{P}(A_{n-l,n} \geq k) + \frac{\theta^2 l \log(l/(|r|+1))}{\pi k^2}. \quad (27)$$

It follows immediately that (13) holds if $r > 0$. Moreover, if $r \leq 0$ and fixed, the exceedance of the threshold in the larger component alone already yields a term of order k^{-1} . Because $\log(l/(|r|+1)) < \log(n)$ and $k \sim n$, it is necessary that $l = \Omega(n/\log n)$ for the second term in (27) to be nonnegligible. It is also sufficient because $l = n/\log n$ yields

$$\frac{l \log(l/(|r|+1))}{k^2} \sim \frac{n \log(n/\log n)}{k^2 \log n} \sim \frac{n \log n}{k n \log n} = \frac{1}{k}.$$

That is, (12) holds as well. When $-r > 0$ is growing, the same reasoning as before shows that it is necessary that $l = \Omega(n\sqrt{-r}/\log n)$ for the second term in (27) to be nonnegligible. Note that this implies $\sqrt{-r} = O(l/n \log n) = o(\log n)$. This condition is also sufficient: When $l = n\sqrt{-r}/\log n$,

$$\frac{l \log(l/(|r|+1))}{k^2} = \frac{n\sqrt{-r} \log(n/(\sqrt{-r} \log n))}{k^2 \log n} \sim \frac{\sqrt{-r} \log n}{k \log n} = \frac{\sqrt{-r}}{k}.$$

In conclusion, also the case (11) holds.

Next, suppose $\alpha = 1$ with $\gamma = \lim_{n \rightarrow \infty} r/l \in (0, 1)$. Choose $s_\star = r + o(l)$ such that $s_\star - r$ is growing and $s^\star = l - o(l)$ such that $l - s^\star$ is growing. Then,

$$\mathbb{P}(A_n \geq k; A_{n-l,n} \geq k - s_\star) \leq \mathbb{P}(A_{l,n} \geq r) \mathbb{P}(A_{n-l,n} \geq k - s_\star) = O\left(\frac{l}{n} r^{-1/2} \frac{(k - s_\star)^{1/2}}{n - l}\right) = o\left(\frac{l}{k^2}\right),$$

and

$$\mathbb{P}(A_n \geq k; A_{n-l,n} \leq k - s^\star) \leq \mathbb{P}(A_{l,n} \geq s^\star) \mathbb{P}(A_{n-l,n} \geq k - l) = O\left(\frac{l(l - s^\star)^{1/2} \sqrt{n - k}}{n} \frac{1}{n - l}\right) = o\left(\frac{l}{k^2}\right).$$

Using (17), we obtain that as $n \rightarrow \infty$,

$$\begin{aligned} \mathbb{P}(A_n \geq k; A_{n-l,n} \in (k - s^\star, k - s_\star)) &= \sum_{j=s_\star+1}^{s^\star-1} \mathbb{P}(A_{n-l,n} = k - j) \mathbb{P}(A_{l,n} \geq j) \\ &\sim \sum_{j=s_\star+1}^{s^\star-1} \frac{\theta}{\sqrt{2\pi}} \frac{1}{(n-l)\sqrt{n-l-k+j}} \cdot \frac{2\theta}{\sqrt{2\pi}} \frac{l}{n} \sqrt{\frac{l-j}{l \cdot j}} \sim \frac{\theta^2 \sqrt{l}}{\pi k^2} \sum_{j=s_\star+1}^{s^\star-1} \sqrt{\frac{l-j}{(j-r)j}}. \end{aligned}$$

Note that the function within the summation is (strictly) decreasing on $(r, l]$. Hence, an upper bound for the summation term is given by

$$\sum_{j=s_\star+1}^{s_\star-1} \sqrt{\frac{l-j}{(j-r)j}} \leq \int_{x=s_\star}^l \sqrt{\frac{l-j}{(j-r)j}} dx = \sqrt{l} \int_{y=s_\star/l}^1 \sqrt{\frac{1-y}{(y-r/l)y}} dy \sim \sqrt{l} \int_{y=\gamma}^1 \sqrt{\frac{1-y}{(y-\gamma)y}} dy,$$

and a lower bound is given by

$$\sum_{j=s_\star+1}^{s_\star-1} \sqrt{\frac{l-j}{(j-r)j}} \geq \int_{x=s_\star+1}^{s_\star} \sqrt{\frac{l-j}{(j-r)j}} dx = \int_{y=(s_\star+1)/l}^{s_\star/l} \sqrt{\frac{l(1-y)}{(y-r/l)y}} dy \sim \sqrt{l} \int_{y=\gamma}^1 \sqrt{\frac{1-y}{(y-\gamma)y}} dy.$$

As the asymptotic behavior of the upper and lower bound coincides, we obtain

$$\mathbb{P}(A_n \geq k; A_{n-l,n} \in (k - s_\star, k - s_\star)) \sim \frac{\theta^2}{\pi} \int_{y=\gamma}^1 \sqrt{\frac{1-y}{(y-\gamma)y}} dy \frac{\sqrt{l}}{k^2}.$$

We observe that $\int_{y=\gamma}^1 \sqrt{\frac{1-y}{(y-\gamma)y}} dy$ is a constant, because

$$\int_{y=\gamma}^1 \sqrt{\frac{1-y}{(y-\gamma)y}} dy \leq \int_{y=\gamma}^1 \sqrt{\frac{1}{(y-\gamma)\gamma}} dy < \infty.$$

Recalling (9) yields the result in this case.

Finally, we consider $\alpha = 1$ with $r = l - o(l)$, and hence both components can only have a few surviving lines after the cascading failure process. The proof is analogous to the case where $\beta \neq 0$ and is merely adapted below to account for the disparity between the component sizes. Choose $s_\star = r + o(n - k)$ and $s^\star = l - o(n - k)$. Then,

$$\mathbb{P}(A_n \geq k; A_{n-l,n} \geq k - s_\star) \leq \mathbb{P}(A_{l,n} \geq r) \mathbb{P}(A_{n-l,n} \geq k - s_\star) = O\left(\frac{l \sqrt{l-r} \sqrt{k-s_\star}}{n l n-l}\right) = o\left(\frac{n-k}{k^2}\right),$$

and

$$\mathbb{P}(A_n \geq k; A_{n-l,n} \leq k - s^\star) \leq \mathbb{P}(A_{l,n} \geq s^\star) \mathbb{P}(A_{n-l,n} \geq k - l) = O\left(\frac{l \sqrt{l-s^\star} \sqrt{n-k}}{n l n-l}\right) = o\left(\frac{n-k}{k^2}\right).$$

Fix $\epsilon > 0$. Using (17) yields

$$\begin{aligned} \sum_{j=s_\star+1}^{s_\star-1} \mathbb{P}(A_{l,n} \geq j) \mathbb{P}(A_{n-l,n} = k - j) &= \sum_{j=s_\star+1-r}^{s_\star-1-r} \mathbb{P}(A_{l,n} \geq j+r) \mathbb{P}(A_{n-l,n} = n-l-j) \\ &\leq (1+\epsilon)^2 \int_{x=0}^{n-k} \frac{l}{n} \frac{2\theta}{\sqrt{2\pi}} \frac{\sqrt{l-r-x}}{l} \frac{\theta}{\sqrt{2\pi}} \frac{1}{\sqrt{x}(n-l)} dx \\ &= (1+\epsilon)^2 \frac{\theta^2}{\pi n(n-l)} \int_{x=0}^{n-k} (n-k-x)^{1/2} x^{-1/2} dx \\ &= (1+\epsilon)^2 \frac{\theta^2}{2} \frac{n-k}{n(n-l)}. \end{aligned}$$

For the lower bound, note that we can set $s_\star - r = l - s^\star$ without violating the assumptions on s_\star and s^\star . This is done to simplify the integration term in the lower bound—that is,

$$\begin{aligned} \sum_{j=s_\star+1}^{s_\star-1} \mathbb{P}(A_{l,n} \geq j) \mathbb{P}(A_{n-l,n} = k - j) &\geq (1-\epsilon)^2 \frac{\theta^2}{\pi n(n-l)} \int_{x=s_\star-r+2}^{s^\star-r-2} (n-k-x)^{1/2} x^{-1/2} dx \\ &\geq (1-\epsilon)^2 \frac{\theta^2}{\pi n(n-l)} (n-k) \arctan\left(\frac{n-k-2(s_\star-r)}{2\sqrt{(n-k-(s_\star-r))(s_\star-r)}}\right). \end{aligned}$$

We note that

$$\lim_{n \rightarrow \infty} \arctan \left(\frac{n - k - 2(s_\star - r)}{2\sqrt{(n - k - (s_\star - r))(s_\star - r)}} \right) = \frac{\pi}{2}.$$

Letting $\epsilon \downarrow 0$ shows that the bounds coincide and hence

$$\sum_{j=s_\star+1}^{s_\star-1} \mathbb{P}(A_{l,n} \geq j) \mathbb{P}(A_{n-l,n} = k - j) \sim \frac{\theta^2}{2} (n - k) k^{-2}.$$

Combining the results shows that the theorem also holds in this final case. ■

5. Discussion

The results of Theorems 2–4 identify how the power-law exponent and its prefactor are affected when a single immediate split occurs. A highly relevant and interesting problem is to move to other network structures that also account for more general properties. In this section, we discuss a few possible generalizations/extensions and offer some ideas how to deal with the analysis in these settings.

5.1. Load Surge Function

In general, we are interested in the question of why and how the number of failures exhibits power-law behavior in cascading failure models. In this paper, we consider the impact of a single immediate disconnection that leads to no further *edge disconnections* (i.e., every possible consecutive edge failure only causes an isolated node to be disconnected). The reason that power-law behavior appears in this setting is due to the way the load surge behaves. More specifically, the surplus capacities are uniformly distributed, and the corresponding order statistics $U_{(1)}^1, \dots, U_{(l)}^1$ and $U_{(1)}^2, \dots, U_{(n-l)}^2$ describe their values from smallest to largest in the first and second component, respectively. We note that in expectation, $\mathbb{E}(U_{(i)}^1) = i/l$ and $\mathbb{E}(U_{(i)}^2) = i/(n-l)$. That is, the expected spacings after an edge failure between two order statistics (i.e., $1/l$ and $1/(n-l)$) are exactly equal to the additional load surges in both components. This causes the evolution of the failure to exhibit some form of *criticality*, leading to the heavy-tailed behavior of the number of failed edges.

In Sloothaak et al. (2018a), the robustness of the power-law behavior is studied for the single star topology. This setting allows small perturbations on the total load surge function. More specifically, recall (16). The approximation operator can be specified as

$$l^n(i) = \frac{\theta + i - 1 + \Delta(i)}{n}.$$

The results in Sloothaak et al. (2018a) and Sloothaak et al. (2018b) suggest that if $\Delta(i) = o(\sqrt{i})$ as $i \rightarrow \infty$, then for all $k := k_n$ such that $k/n \in [0, 1)$

$$\mathbb{P}(A_n \geq k) \sim c(\theta, \Delta) k^{-1/2},$$

as $n \rightarrow \infty$, where $c(\theta, \Delta) \in (0, \infty)$ is a constant that depends on the values θ and the perturbations $\Delta(\cdot)$. In a similar way, we can extend the results in this paper to allow for perturbations of this size.

We point out that if the load surges move beyond this critical window, the behavior of the number of failed edges becomes significantly different. If the load surges are much smaller, the distribution of the failure size becomes light-tailed: The probability distribution of the number of failed edges decays exponentially in the tail. On the other hand, if the load surge is much larger, then for both components, there is a strictly positive (nonvanishing) probability for the entire component to fail (for both components). In fact, given that the number of failed edges is $\omega(1)$ in one component, then with high probability, all edges in that component have failed. For the power-law behavior to prevail, it is therefore essential that the load surges are appropriately close to the expected spacings of two order statistics.

5.2. General Network Topologies

Our focus in this paper is on the impact of a single immediate disconnection that leads to no further *edge disconnections*. For the power-law behavior to appear in more general topologies, we need to understand how the load surges need to behave such that this criticality property prevails. In other words, we require that the

load surges are equal (or close) to the expected spacings between the ordered statistics corresponding to the surplus capacity values, even when edge disconnections are taking place. This leads to the following cascading failure procedure. Suppose we consider a network with n nodes and m edges, where each edge has a surplus capacity that is uniformly distributed on $[0, 1]$. There is an initial disturbance—for example, due to a failure of one or multiple lines—that causes an additional loading of $l_j(1) = \theta/m$ at every edge $j \in [m] = \{1, \dots, m\}$. Edge failures occur consecutively whenever an edge's load exceeds its capacity, and the load surge is defined recursively. If line j fails when it inherits a load surge for the i th time, then all edges for which there exists a path to edge j experience an additional load surge of $(1 - l_j(i-1))/|E_j(i-1)|$, where $|E_j(i-1)|$ is the number of edges in the component containing edge j after experiencing an additional load surge $i-1$ times, and $l_j(i-1)$ the total load surge at edge j . The cascading failure process ends whenever the surplus capacity exceeds the total load surge at all edges. We point out that this procedure ensures that the load surges at two edges are the same as long as they continue to be in the same component. Therefore, we can define the total load surge at any line $j \in [m]$ through the recursive relation

$$\begin{cases} l_j(1) = \theta/m, \\ l_j(i+1) = l_j(i) + \frac{1 - l_j(i)}{|E_j(i)|}, \end{cases} \quad (28)$$

which is well defined as long as line j has not failed.

In case of the topology illustrated in Figure 1, the disturbance is initiated by the failure of the connecting line between two star-shaped components with $l_j(i) = \theta/n$. For every (surviving) edge j in the first component, it holds that $|E_j(i)| = l - i + 1$. Solving recursion (28) yields

$$l_j(i) = \frac{\theta}{n} + \frac{1 - \theta/n}{l} (i-1) = \frac{\theta}{n} + \frac{i-1}{l} - \frac{\theta i-1}{n l} \approx \frac{\theta}{n} + \frac{i-1}{l}.$$

We note that the term $\frac{\theta i-1}{n l}$ is relatively very small and does not impact the asymptotic result for the exceedance probability as $n \rightarrow \infty$. In other words, the described failure mechanism indeed includes the main setting we consider in this paper.

An immediate question that arises for the network in Figure 1 is what happens when the initial disturbance does not necessarily cause the bridging edge to fail immediately, but is due to the failure of one edge (or several) chosen uniformly at random. The failure of the bridging edge is a random variable itself in this setting and (only) likely to occur at the σ th step with $\lim_{n \rightarrow \infty} \sigma/n \in (0, 1)$. Using the results for the star topology and the insights obtained in this paper, one can show that for every k such that $k = \omega(1)$ and $\alpha = \lim_{n \rightarrow \infty} k/n \in [0, 1)$,

$$\mathbb{P}(A_n \geq k) \sim c(\theta, \alpha, \beta) k^{-1/2}, \quad (29)$$

as $n \rightarrow \infty$, where $c(\theta, \alpha, \beta) \in (0, \infty)$ is some constant depending on the initial disturbance θ , $\alpha \in [0, 1)$ and $\beta = \lim_{n \rightarrow \infty} l/n \in [0, 1/2]$. In particular, if $\beta = 0$ and/or $\alpha = 0$, then $c(\theta, \alpha, \beta) = 2\theta/\sqrt{2\pi}$. Indeed, as is illustrated in Figure 4, simulation experiments seem to confirm the asymptotic behavior as in (29).

An interesting question is how the tail probability of A_n behaves for the setting with load surges as in (28) for other topologies. Naturally, the tail behavior depends heavily on the network topology itself, as well as the way the initial disturbance is caused—for example, random or specific/targeted first failure(s). Still, the power law with exponent $-1/2$ for A_n (i.e., the number of edge failures) is a property that holds for a wide range of network topologies up to a certain threshold α . A necessary condition for this property to hold is that the network does not disintegrate in many components too quickly, where the different components have sizes that are of similar order. In particular, if the network is sufficiently dense, the first disconnection occurs only after a significant number of edge failures and causes the tail of A_n to exhibit power-law behavior up till that moment.

As an example, we consider the standard 2D-lattice graph with $\sqrt{n} \times \sqrt{n}$ nodes and $m = 2(n-1)n$ edges. It is well known from percolation theory that for large n , the first edge disconnections are likely to take place after a significant number of other edge failures, and the network then disintegrates into a unique giant (largest) component and relatively tiny other components (Grimmett 1999). At a critical threshold $k = m/2$ the uniqueness of the giant is no longer guaranteed—that is, the network disintegrates further into many small components (Kesten 1980). This suggests that, as $n \rightarrow \infty$,

$$\mathbb{P}(A_n \geq k) \sim c(\theta, \alpha) k^{-1/2}, \quad (30)$$

holds for all k such that $k = \omega(1)$ and $\alpha = \lim_{m \rightarrow \infty} k/m \in [0, 1/2)$. Simulation experiments seem to confirm this type of behavior (30), as is illustrated in Figure 5.

We conducted more simulation experiments on other sufficiently dense graphs, for which similar behavior seems to appear. In these cases, the first edge disconnections are likely to occur after a significant time such that the tail behavior is the same as for the star topology for any threshold before that time. In addition, as long as relatively small components are disconnected from the largest one, the power-law behavior with exponent $-1/2$ prevails. This property is satisfied by many (dense) network topologies, such as the lattice network, but also the erased configuration model with positive proportion of nodes of degree 2 and 3 or larger, or (the largest component in) a Erdős–Rényi graph with average degree strictly larger than one.

In the main setting of this paper, we were able to track the asymptotic tail behavior for (almost) the complete interval of k after the initial disconnection, even at its phase transitions. This was because of the fact that no edge disconnections occur after the initial failure. For simple and specific network topologies, similar rigorous asymptotic results may be derived. However, for more general network settings, one would need a significantly different approach that goes beyond the proof strategy and insights obtained in this paper. In particular, this is the case for network topologies for which at some point the network disintegrates rapidly into many different components whose sizes are not easily tractable.

5.3. Heterogeneous Edge Capacities

In many applications, the capacities are not the same at every edge. One possible way to include heterogeneity between edges is introducing edge classes. That is, suppose there are I classes of edges. An edge in class i has a surplus capacity that is uniformly distributed between $[0, a_i]$ with $a_1 \leq \dots \leq a_I$. For power-law behavior to appear in the tail distribution, we require that some form of criticality is induced by the load surge function. To understand what kinds of load surge functions result in power-law behavior for the tail, one needs to define the correlation between the edge classes—that is, the effect of edge failure to edges of other classes. We consider the two extreme scenarios in this section: one with no correlation and one with full correlation. For simplicity, we consider these scenarios only for the star topology where class i has l_i edges and $l_1 + \dots + l_I = n$.

First, suppose that an edge failure of class i does not yield an additional load surge to other edges of different classes. As we have observed in the previous section with general network topologies, the load surges need to be approximately equal to the expected spacings between two uniformly distributed order statistics. More specifically, suppose there is an initial disturbance of θ_i/n at every edge of class $i = 1, \dots, I$. Every edge failure of class i causes an additional load surge of a_i/l_i at every surviving edge of the same class and none at edges of other classes. In this case, the tail of the number of edge failures obeys a power-law distribution with an exponent (and prefactor) that depends on a_1, \dots, a_I and threshold k . In fact, we note that the special case where $I = 2$ and $\theta_1 = \theta_2 = \theta$ reduces to the main setting of this paper. A similar proof strategy can be used to rigorously derive the asymptotic behavior of this slightly more general setting.

Second, suppose that an edge failure of class i yields an additional load surge to all other edges of different classes. We suppose that this occurs in a fair way—that is, the load surge is proportional to the surplus capacity distributions. More specifically, suppose there is an initial disturbance of $\theta a_i/n$ at every edge in class i and every class $i = 1, \dots, I$. Every edge failure (regardless of the class) causes an additional load surge of a_i/n to

Figure 4. Scaled Tail Probability for Network as in Figure 1 with Random First Failure over 100,000 Runs

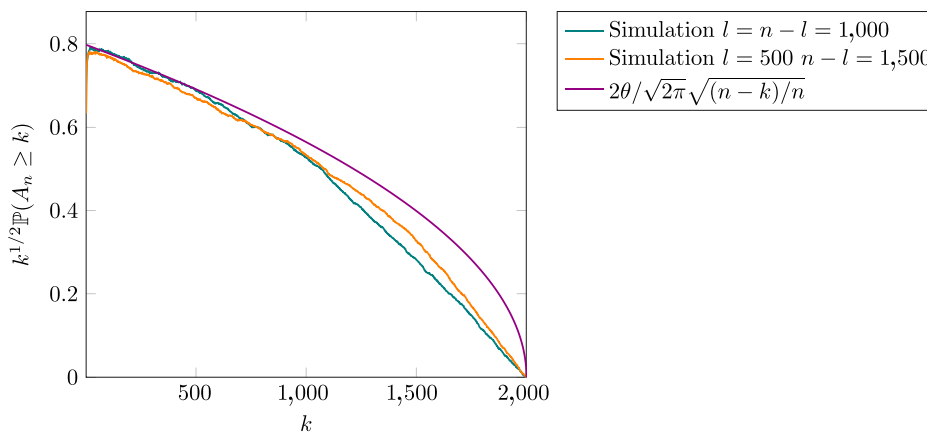
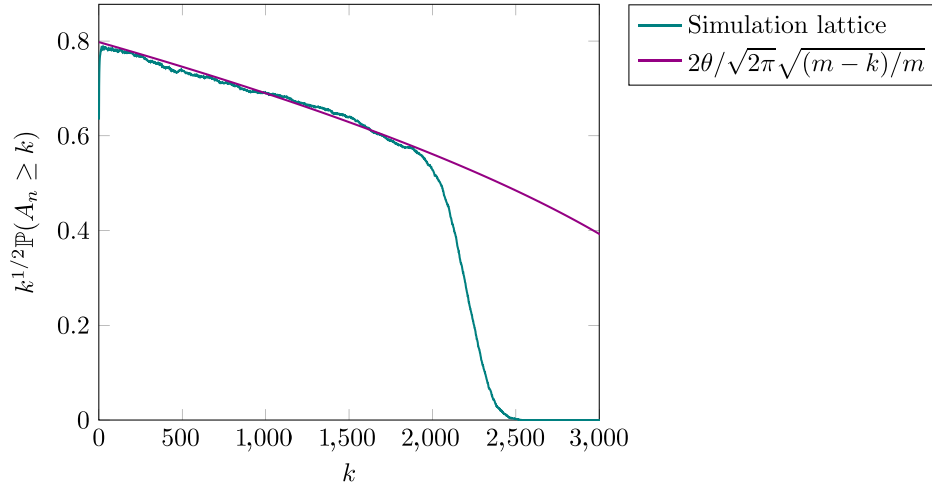


Figure 5. Scaled Tail Probability for Lattice Network with $45 \times 45 = 2,025$ and $m = 3,960$ and Random First-Edge Failure over 100,000 Runs

every surviving edge of class i . Write $U_{(i)}^l$ for the i th uniformly distributed order statistic on $[0, 1]$ with $1 \leq i \leq l$. We observe that

$$\mathbb{P}(A_n \geq k) = \mathbb{P}\left(\bigcup_{j_1+\dots+j_l=j} \bigcap_{i=1}^l U_{(i)}^{j_i} \leq \frac{\theta + (j_i - 1)}{n}, j = 1, \dots, k\right) = \mathbb{P}\left(U_{(j)}^n \leq \frac{\theta + (j - 1)}{n}, j = 1, \dots, k\right).$$

In other words, this case reduces to the setting with a single class of edges, and hence

$$\mathbb{P}(A_n \geq k) \sim \frac{2\theta}{\sqrt{2\pi}} \sqrt{\frac{n-k}{kn}}.$$

To bridge the two extreme scenarios discussed in this section in a general framework is an extremely interesting problem. A key element is how to define the dependence between edge classes. The challenging part of the problem is then determining the way the load surge function needs to behave such that power-law behavior prevails.

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Appendix A: A Proof of Theorem 1

Proof of Theorem 1. Equation (1) is theorem 1 of Sloothaak et al. (2018a). The second statement follows the lines of the proof of theorem 2 of Sloothaak et al. (2018a), but it is adapted here to show it holds uniformly in $k \in [k_*, k^*]$.

Choose $\tilde{k} = n - \log(n - k^*)$ and fix $\epsilon > 0$. Following the proof of theorem 2 in Sloothaak et al. (2018a), we observe that for every $k \in [k_*, k^*]$ that

$$\sqrt{\frac{kn}{n-k}} \mathbb{P}(A_n \geq k) \leq e \frac{\theta}{\theta/k + 1} \frac{n}{k(n-k)} + (1 + \epsilon) \frac{2\theta}{\sqrt{2\pi}} + c \cdot \frac{\sqrt{k/n}}{1 - \log(n - \tilde{k})/n} \frac{\log(n - \tilde{k})}{\sqrt{n-k}},$$

for some positive constant c , and

$$\sqrt{\frac{kn}{n-k}} \mathbb{P}(A_n \geq k) \geq (1 - \epsilon) \frac{2\theta}{\sqrt{2\pi}} \left(1 - \sqrt{\frac{k}{\tilde{k}}} \sqrt{\frac{n - \tilde{k}}{n - k}}\right),$$

for large enough n . Therefore,

$$\sup_{k \in [k_*, k^*]} \left| \sqrt{\frac{kn}{n-k}} \mathbb{P}(A_n \geq k) - \frac{2\theta}{\sqrt{2\pi}} \right| \leq \sup_{k \in [k_*, k^*]} \max \left\{ e \frac{2\theta}{\sqrt{2\pi}} \left(1 - \sqrt{\frac{k}{\tilde{k}}} \sqrt{\frac{n - \tilde{k}}{n - k}}\right) + \frac{2\theta}{\sqrt{2\pi}} \sqrt{\frac{k}{\tilde{k}}} \sqrt{\frac{n - \tilde{k}}{n - k}}, \right. \\ \left. e \frac{2\theta}{\sqrt{2\pi}} + e \frac{\theta}{\theta/k + 1} \frac{n}{k(n-k)} + \frac{c \cdot \sqrt{k/n}}{1 - \log(n - \tilde{k})/n} \frac{\log(n - \tilde{k})}{\sqrt{n-k}} \right\}.$$

We see that as $n \rightarrow \infty$, this gives

$$\lim_{n \rightarrow \infty} \sup_{k \in [k_*, k^*]} \left| \sqrt{\frac{kn}{n-k}} \mathbb{P}(A_n \geq k) - \frac{2\theta}{\sqrt{2\pi}} \right| \leq \epsilon \frac{2\theta}{\sqrt{2\pi}}.$$

Letting $\epsilon \downarrow 0$ concludes the proof. ■

Appendix B: Asymptotic Behavior of Some Summation Terms

In our analysis, determining the asymptotic behavior often boils down to deriving the asymptotics of some summation terms. In this section, we provide two of such results that are used. Henceforth, write $a_n \lesssim b_n$ if $\lim_{n \rightarrow \infty} a_n/b_n \leq 1$, and, similarly, write $a_n \gtrsim b_n$ if $\lim_{n \rightarrow \infty} a_n/b_n \geq 1$.

Lemma B.1. *Suppose $|r| = o(l)$. Let s_* be such that $s_* = o(\log(l/(|r| + 1)))$ is growing if $-r > 0$ is growing or $|r|$ fixed, and $s_* = r + o(r)$ be such that $s_* - r$ is growing if $r > 0$ is growing. Let $s^* = o(l)$ be growing such that $s^* = \omega(l/\log(l/(|r| + 1)))$. Then, as $l \rightarrow \infty$, $s_* \lesssim s^*$ and*

$$\sum_{j=s_*}^{s^*} j^{-1/2} (j-r)^{-1/2} \sim \log\left(\frac{l}{|r|+1}\right). \quad (\text{B.1})$$

Proof. First, we have that $s_* \leq s^*$ as $l \rightarrow \infty$. That is, if $-r > 0$ is growing or $|r|$ fixed,

$$s_* \lesssim \log\left(\frac{l}{|r|+1}\right) \leq \log(l) \lesssim \frac{l}{\log l} \leq \frac{l}{\log\left(\frac{l}{|r|+1}\right)} \lesssim s^*,$$

and if $r > 0$ is growing,

$$s_* \sim r = l \frac{r}{l} \lesssim \frac{l}{\log(l/r)} \lesssim s^*.$$

Next, observe that the expression in the summation is a decreasing function, and therefore

$$\sum_{j=s_*}^{s^*} j^{-1/2} (j-r)^{-1/2} \leq \int_{j=s_*-1}^{s^*} j^{-1/2} (j-r)^{-1/2} = 2 \log\left(\frac{\sqrt{s^*} + \sqrt{s^*-r}}{\sqrt{s_*-1} + \sqrt{s_*-r-1}}\right),$$

and

$$\sum_{j=s_*}^{s^*} j^{-1/2} (j-r)^{-1/2} \geq \int_{j=s_*}^{s^*} j^{-1/2} (j-r)^{-1/2} = 2 \log\left(\frac{\sqrt{s^*} + \sqrt{s^*-r}}{\sqrt{s_*} + \sqrt{s_*-r}}\right).$$

It is apparent that the asymptotic behavior of the upper and lower bounds is the same. It remains to derive this behavior in terms of l and r .

For an asymptotic upper bound, we observe that $\sqrt{s^*} + \sqrt{s^*-r} \lesssim \sqrt{l}$ and $\sqrt{s_*} + \sqrt{s_*-r} \gtrsim \sqrt{|r|+1}$ due to our choice of s^* . Therefore,

$$\sum_{j=s_*}^{s^*} j^{-1/2} (j-r)^{-1/2} \lesssim 2 \log\left(\sqrt{\frac{l}{|r|+1}}\right) \sim \log\left(\frac{l}{|r|}\right).$$

For a lower bound, recall that $|r| \lesssim l/\log(l/(|r|+1))$ and thus $\sqrt{s^*} + \sqrt{s^*-r} \gtrsim 2\sqrt{l/\log(l/(|r|+1))}$. Because $\log \log x = o(\log x)$, we derive that as $l \rightarrow \infty$,

$$\sum_{j=s_*}^{s^*} j^{-1/2} (j-r)^{-1/2} \gtrsim \log\left(\frac{4 \cdot l/\log(l/(|r|+1))}{\max\{|r|, \log(l/(|r|+1))\}}\right) \sim \log\left(\frac{l}{|r|+1}\right). \quad \blacksquare$$

Lemma B.2. *Suppose $|r| = o(l)$. Let s_* be such that $s_* = o(\log(l/(|r| + 1)))$ is growing if $-r > 0$ is growing or $|r|$ fixed, and if $r > 0$ is growing, let $s_* = r + o(r)$ be such that $s_* - r = \omega(r/\log(l/r))$ is growing. Let $s^* = o(l)$ be growing such that $s^* = \omega(l/\log(l/(|r| + 1)))$. Then, there exists a s_* satisfying the assumptions, as $l \rightarrow \infty$, $s_* \lesssim s^*$ and*

$$\sum_{j=s_*}^{s^*} j^{-1/2} (j-r)^{-3/2} \sim \log\left(\frac{l}{|r|+1}\right). \quad (\text{B.2})$$

Proof. It is not immediate that if $r > 0$ is growing, there exists a s_* that satisfies both $s_* = r + o(r)$ and $s_* - r = \omega(r/\log(l/r))$. Yet, we observe that $\log(l/r) \rightarrow \infty$ as $l \rightarrow \infty$ and hence $r/\log(l/r) = o(r)$. Therefore, there exists a s_* that satisfies the stated conditions.

The claim that $s_\star \leq s^\star$ as $l \rightarrow \infty$ is already proven in Lemma B.1.

Finally, we have to show (B.2). Note that the expression in the summation is a decreasing function, and therefore

$$\sum_{j=s_\star}^{s^\star} j^{1/2}(j-r)^{-3/2} \leq \int_{j=s_\star-1}^{s^\star} j^{1/2}(j-r)^{-3/2} = 2\sqrt{\frac{s_\star-1}{s_\star-1-r}} - 2\sqrt{\frac{s^\star}{s_\star-r}} + 2\log\left(\frac{\sqrt{s^\star} + \sqrt{s^\star-r}}{\sqrt{s_\star-1} + \sqrt{s_\star-1-r}}\right).$$

Similarly,

$$\begin{aligned} \sum_{j=s_\star}^{s^\star} j^{-1/2}(j-r)^{-1/2} &\geq \int_{j=s_\star}^{s^\star} j^{-1/2}(j-r)^{-1/2} \\ &= 2\sqrt{\frac{s_\star}{s_\star-r}} - 2\sqrt{\frac{s^\star}{s_\star-r}} + 2\log\left(\frac{\sqrt{s^\star} + \sqrt{s^\star-r}}{\sqrt{s_\star} + \sqrt{s_\star-r}}\right). \end{aligned}$$

It is apparent that the bounds asymptotically coincide, and it remains to express the asymptotics in terms of l and r . First, as we have seen in the proof of Lemma B.1, $r = O(s^\star)$, and hence

$$2\sqrt{\frac{s^\star}{s^\star-r}} = O(1) = o\left(\log\left(\frac{l}{|r|+1}\right)\right).$$

Next, if $r \leq 0$ or $|r|$ fixed, then clearly,

$$2\sqrt{\frac{s_\star}{s_\star-r}} = O(1) = o\left(\log\left(\frac{l}{|r|+1}\right)\right).$$

If $r > 0$, then

$$2\sqrt{\frac{s_\star}{s_\star-r}} \sim 2\frac{r}{s_\star-r} = o\left(\log\left(\frac{l}{r}\right)\right).$$

Finally, it follows from the proof of Lemma B.1 that

$$2\log\left(\frac{\sqrt{s^\star} + \sqrt{s^\star-r}}{\sqrt{s_\star} + \sqrt{s_\star-r}}\right) \sim \log\left(\frac{l}{|r|+1}\right),$$

as $l \rightarrow \infty$. Adding the above expressions yields the result. ■

Appendix C: Proofs of Lemmas 5–10

Proof of Lemma 6. Note that if $1 - \beta < \alpha < 1$, we must be in the balanced case. Therefore, for large enough n , $s_\star \leq k - (n - l)$, which proves the second assertion.

Next, suppose $\alpha < 1 - \beta$ (this can be either the disparate or the balanced case). We then have to prove that the joint event that the threshold is exceeded and the smaller component has few line failures is dominated by the event that k is exceeded in the larger component. Note that $\{A_n \geq k; A_{l,n} \leq s_\star\}$ implies that at least $\{A_{n-l,n} \geq k - s_\star\}$. Moreover, $\{A_{n-l,n} \geq k\}$ implies $\{A_n \geq k\}$. Then,

$$\underbrace{(1 - \mathbb{P}(A_{l,n} > s_\star))}_{=1-o(1)} \underbrace{\mathbb{P}(A_{n-l,n} \geq k)}_{\sim \frac{2(1-\beta)\theta}{\sqrt{2\pi}}} \leq \mathbb{P}(A_n \geq k; A_{l,n} \leq s_\star) \leq \underbrace{\mathbb{P}(A_{n-l,n} \geq k - s_\star)}_{\sim \frac{2(1-\beta)\theta}{\sqrt{2\pi}}}.$$

Therefore,

$$\mathbb{P}(A_n \geq k; A_{l,n} \leq s_\star) \sim \frac{2(1-\beta)\theta}{\sqrt{2\pi}} \sqrt{\frac{n-l-k}{k(n-l)}} \sim \mathbb{P}(A_{n-l,n} \geq k). \quad \blacksquare$$

Proof of Lemma 7. First, suppose $\alpha < \beta$. Then, we must be in the balanced case, and $k < l$ and $s^\star = k - o(k)$ for n large enough. Basically, we want to show in this case that it is most likely that $A_{l,n}$ already exceeds k , given that it exceeds s^\star . Note

$$\mathbb{P}(A_{l,n} \geq k) \leq \mathbb{P}(A_n \geq k; A_{l,n} \geq s^\star) \leq \mathbb{P}(A_{l,n} \geq s^\star).$$

Equation (17), where l is balanced and $\alpha < \beta$, yields that $\mathbb{P}(A_{l,n} \geq s^*) \sim \mathbb{P}(A_{l,n} \geq k)$ as $n \rightarrow \infty$. This coincides with the lower bound, and hence

$$\mathbb{P}(A_n \geq k; A_{l,n} \geq s^*) \sim \mathbb{P}(A_{l,n} \geq k).$$

If $\beta \leq \alpha < 1 - \beta$, we can have both the disparate and the balanced case. When the component sizes are disparate,

$$\mathbb{P}(A_n \geq k; A_{l,n} \geq s^*) \leq \mathbb{P}(A_{l,n} \geq s^*) = O\left(\frac{l}{n} \sqrt{\frac{l-s^*}{l}} s^{*-1/2}\right) = o(k^{-1/2}).$$

When the component sizes are balanced, note that the condition $\beta \leq \alpha < 1 - \beta$ implies that $(l - s^*)/l = o(1)$, and hence

$$\mathbb{P}(A_n \geq k; A_{l,n} \geq s^*) \leq \mathbb{P}(A_{l,n} \geq s^*) = O\left(\sqrt{\frac{l-s^*}{l}} s^{*-1/2}\right) = o(k^{-1/2}).$$

Finally, if $1 - \beta < \alpha < 1$, we have a balanced case and $k > l$ for n large enough. Then,

$$\mathbb{P}(A_n \geq k; A_{l,n} \geq s_\star) \leq \underbrace{\mathbb{P}(A_{l,n} \geq s_\star)}_{=o(k^{-1/2})} \underbrace{\mathbb{P}(A_{n-l,n} \geq k-l)}_{=O(k^{-1/2})} = o(k^{-1}). \quad \blacksquare$$

Proof of Lemma 8. For the first claim, note that $s_\star - r > 0$ is growing, and

$$\mathbb{P}(A_n \geq k, A_{n-l,n} \in [k - s_\star, \min\{k - 1, n - l\}]) \leq s_\star \sup_{i \in [0, s_\star - r]} \mathbb{P}(A_{n-l,n} = n - l - i) = O\left(\frac{s_\star}{n-l}\right).$$

Next, in the second case,

$$\mathbb{P}(A_n \geq k, A_{n-l,n} \geq k - s_\star) \leq \underbrace{\mathbb{P}(A_{l,n} \geq r)}_{=O(r^{-1/2})} \underbrace{\mathbb{P}(A_{n-l,n} \geq k - s_\star)}_{=O(\sqrt{s_\star - r}/(n-l))} = o\left(\frac{\sqrt{s_\star - r}}{n-l}\right).$$

For the final case, observe that $r = k - (n - l) = l - (n - k) = l - o(l)$, and hence

$$\mathbb{P}(A_n \geq k, A_{n-l,n} \geq k - s_\star) \leq \underbrace{\mathbb{P}(A_{l,n} \geq r)}_{=O(\sqrt{n-k}/l)} \underbrace{\mathbb{P}(A_{n-l,n} \geq k - s_\star)}_{=o(\sqrt{n-k}/(n-l))} = o\left(\frac{n-k}{k^2}\right). \quad \blacksquare$$

Proof of Lemma 9. For (21), note that $k - l$ is of order n and hence,

$$\mathbb{P}(A_n \geq k, A_{n-l,n} \in [k - l, k - s^*]) \leq \mathbb{P}(A_{l,n} \geq s^*) \mathbb{P}(A_{n-l,n} \geq k - l) = O(s^{*-1/2} k^{-1/2}).$$

For (22), observe $l \sim k$ as $n \rightarrow \infty$ and hence,

$$\mathbb{P}(A_n \geq k, A_{n-l,n} \in [k - l, k - s^*]) \leq \mathbb{P}(A_{l,n} \geq s^*) = O\left(\frac{\sqrt{l - s^*}}{k}\right).$$

For (23), we thus want to show that it is most likely that if $A_{n-l,n}$ is at most $k - s^* = o(l - k)$, the threshold is exceeded in the smaller component itself. As $n \rightarrow \infty$,

$$\mathbb{P}(A_n \geq k, A_{n-l,n} \in [k - l, k - s^*]) \leq \mathbb{P}(A_{l,n} \geq s^*) \sim \frac{2\theta \cdot 1/2}{\sqrt{2\pi}} \sqrt{\frac{l - s^*}{ls^*}} \sim \frac{\theta}{\sqrt{2\pi}} \frac{\sqrt{l - k}}{k}.$$

For the lower bound,

$$\mathbb{P}(A_n \geq k, A_{n-l,n} \in [k - l, k - s^*]) \geq \mathbb{P}(A_{l,n} \geq k) \mathbb{P}(A_{n-l,n} \leq k - s^*) \sim \frac{\theta}{\sqrt{2\pi}} \frac{\sqrt{l - k}}{k}.$$

Because the lower and upper bounds coincide, we observe that (23) holds. Finally, if $\alpha = 1$ and $\beta \in (0, 1/2]$, then

$$\begin{aligned} \mathbb{P}(A_n \geq k, A_{n-l,n} \in [k - l, k - s^*]) &\leq \mathbb{P}(A_{l,n} \geq s^*) \mathbb{P}(A_{n-l,n} \geq n - l - (n - k)) \\ &= O\left(\frac{\sqrt{l - s^*}}{l} \frac{\sqrt{n - k}}{n - l}\right) = o\left(\frac{n - k}{k^2}\right). \quad \blacksquare \end{aligned}$$

Proof of Lemma 10. When $r < 0$ is growing, we have (for n large enough)

$$\mathbb{P}(A_n \geq k; A_{n-l,n} \in [k - s_\star, k - 1]) = \sum_{j=1}^{s_\star} \mathbb{P}(A_{l,n} \geq j) \mathbb{P}(A_{n-l,n} = k - j) \leq \mathbb{P}(A_{l,n} \geq 1) s_\star \sup_{j \in [1, s_\star]} \mathbb{P}(A_{n-l,n} = k - j).$$

Because of our choice of s_\star , we obtain the inequality,

$$\mathbb{P}(A_n \geq k; A_{n-l,n} \in [k - s_\star, k - 1]) = O\left(\frac{l}{k} s_\star \frac{1}{\sqrt{-rk}}\right) = o\left(\frac{l \log l}{k^2}\right).$$

When r is fixed,

$$\mathbb{P}(A_n \geq k; A_{n-l,n} \in [k - s_\star, k - 1]) \leq \mathbb{P}(A_{l,n} \geq 1) s_\star \sup_{j \in [1, s_\star]} \mathbb{P}(A_{n-l,n} = k - j) = O\left(\frac{l}{k} s_\star \frac{1}{k}\right) = o\left(\frac{l \log l}{k^2}\right).$$

When $r > 0$ is growing,

$$\mathbb{P}(A_n \geq k; A_{n-l,n} \in [k - s_\star, k - 1]) \leq \mathbb{P}(A_{l,n} \geq r) \mathbb{P}(A_{n-l,n} \geq n - l - v) = O\left(\frac{l}{k} r^{-1/2} \frac{\sqrt{v}}{k}\right) = o\left(\frac{l \log(l/r)}{k^2}\right). \quad \blacksquare$$

Proof of Lemma 11. Observe that

$$\mathbb{P}(A_n \geq k; A_{n-l,n} \in [k - l, k - s^\star]) \leq \mathbb{P}(A_{l,n} \geq s^\star) \cdot \mathbb{P}(A_{n-l,n} \geq k - l) = O\left(\frac{l}{k} s^{\star-1/2} \frac{\sqrt{l}}{k}\right) \cdot o\left(\mathbb{1}_{\{r \leq 0\}} \left(\frac{l \log l}{k^2}\right) + \mathbb{1}_{\{r > 0\}} \frac{l \log(l/r)}{k^2}\right). \quad \blacksquare$$

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