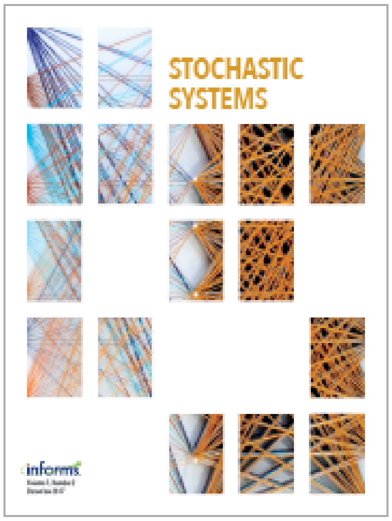

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Special Section: Open Problems in Applied Probability

Open Problem—Weakly Interacting Particle Systems on Dense Random Graphs

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Background on Weakly Interacting Particle Systems

We first briefly introduce the following classic mean-field particle system given as a collection of $\mathbb{R}^d$-valued stochastic processes $\{X_i^n(t) : t \in [0,T]\}_{i=1}^n$.

$$X_i^n(t) = x_0 + \frac{1}{n}\sum_{j=1}^n \int_0^t b\left(X_i^n(s), X_j^n(s)\right) ds + W_i(t), i = 1,\ldots,n. \tag{1}$$

Here, $\{W_i\}$ are independent and identically distributed standard d-dimensional Brownian motions, and b: $\mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ is assumed to be bounded and Lipschitz, but one could also allow for more general initial states, drift coefficient b, and a state-dependent diffusion coefficient. Denote the empirical measure by

$$\mu^n \doteq \frac{1}{n}\sum_{i=1}^n \delta_{X_i^n(\cdot)} \in \mathcal{P}(\mathcal{C}_d),$$

where $\mathcal{C}_d \doteq \mathbb{C}([0,T] : \mathbb{R}^d)$ is the path space of continuous functions and $\mathcal{P}(\cdot)$ is the space of probability measures endowed with the weak convergence topology. The system (1) could be rewritten as

$$X_i^n(t) = x_0 + \int_0^t \langle b\left(X_i^n(s), \cdot\right), \mu^n(s)\rangle\, ds + W_i(t),\ \ i = 1,\ldots,n. \tag{2}$$

Because the contribution of each particle state X_i^n to the empirical measure μ^n is of the order $1/n$, which goes to zero as $n \to \infty$, such systems are also referred to as weakly interacting particle systems. The well posedness of this system and various scaling limits of μ^n have been well established, including the law of large numbers (LLN), central limit theorems (CLTs), and large and moderate deviation principles (LDP and MDP). In particular, the following LLN and CLT hold (cf. Sznitman 1991 and Bhamidi et al. 2018 and references therein).

Theorem 1. *As $n \to \infty$, $\mu^n \to \mu$ in $\mathcal{P}(\mathcal{C}_d)$ in probability, where $\mu = \mathcal{L}(X^i)$ is the unique solution to the following McKean–Vlasov equation:*

$$X_i(t) = x_0 + \int_0^t \langle b(X_i(s), \cdot)\, ,\, \mu_s\rangle\, ds + W_i(t), i \in \mathbb{N}.$$

Let $\nu \doteq \mathcal{L}(X_i, W_i)$ and A: $L_2(\mathcal{C}_d \times \mathcal{C}_d, \nu) \to L_2(\mathcal{C}_d \times \mathcal{C}_d, \nu)$ be some integral operator depending on the drift b. Let

$$\xi^n(\phi) \doteq \frac{1}{\sqrt{n}}\sum_{i=1}^n \left(\phi\left(X_i^n\right) - \langle \phi, \mu\rangle\right), \quad \phi \in L_2(\mathcal{C}_d, \mu),$$

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and $\{\xi(\phi) : \phi \in L_2(\mathscr{C}_d, \mu)\}$ *be a mean-zero Gaussian random field with*

$$\mathbb{E}[\xi(\phi)\xi(\psi)] = \int (I - A)^{-1}\phi(x, w)(I - A)^{-1}\psi(x, w)\, \nu(dx, dw),$$

where $\phi(x, w) \doteq \phi(x) - \langle \phi, \mu \rangle$ *and I is the identity operator. Then, the collection*

$$\left\{\xi^n(\phi) : \phi \in L_2(\mathscr{C}_d, \mu)\right\} \to \left\{\xi(\phi) : \phi \in L_2(\mathscr{C}_d, \mu)\right\} \tag{3}$$

as $n \to \infty$ *in the sense of the convergence of finite dimensional distributions.*

One could also show the convergence of the whole process $n^{-1/2}(\mu^n - \mu)$ to an infinite-dimensional Gaussian process in a nuclear space under suitable conditions (cf. Hitsuda and Mitoma 1986).

Systems on Dense Random Graphs

Consider an Erdős–Rényi random graph with n vertices $\{1, \ldots, n\}$ and edges $\xi^n_{ij} \equiv \xi^n_{ji}$, which are independent $\{0, 1\}$-valued random variables such that

$$p_n \doteq P\big(\xi^n_{ij} = 1\big) = 1 - P\big(\xi^n_{ij} = 0\big), \quad \xi^n_{ii} \equiv 1.$$

Let $N^n_i \doteq \sum_{j=1}^n \xi^n_{ij} \geq 1$ be the number of neighbors of the vertex i (including itself). Consider the following weakly interacting particle system on this random graph with global empirical measure μ^n in the drift of (2) replaced by the local one:

$$X^n_i(t) = x_0 + \frac{1}{N^n_i}\sum_{j=1}^n \xi^n_{ij} \int_0^t b\big(X^n_i(s), X^n_j(s)\big)\, ds + W_i(t) \tag{4}$$

$$= x_0 + \int_0^t \langle b\big(X^n_i(s), \cdot\big), \mu^n_i(s)\rangle\, ds + W_i(t), \tag{5}$$

$$\mu^n_i = \frac{1}{N^n_i}\sum_{j=1}^n \xi^n_{ij}\delta_{X^n_j(\cdot)}.$$

It could be shown that Theorem 1 is still valid when the graph is suitably dense.

Theorem 2 (Bhamidi et al. 2018)**.** *Recall* μ, X_i, $\xi^n(\phi)$, *and* $\xi(\phi)$ *introduced in Theorem 1. Suppose* $np_n \to \infty$. *Then* $\mu^n, \mu^n_i \to \mu \doteq \mathscr{L}(X^i)$ *as* $n \to \infty$ *in probability. Further, suppose* $p_n \to p > 0$. *Then* (3) *holds as* $n \to \infty$ *in the same sense as in Theorem 1.*

Theorem 2 implies that the edge probability p_n does not affect the LLN (when $np_n \to \infty$) or CLT (when $p_n \to p > 0$) for μ^n. Indeed, with the interaction in (4) scaled by np_n instead of N_i, Delattre et al. (2016) gets annealed and quenched LLN under the condition $np_n \to \infty$, Coppini et al. (2019) shows quenched LDP assuming $np_n/\log n \to \infty$, and Reis and Oliveira (2019) proves LDP when $np_n \to \infty$. All of these scaling limits are the same as those in the classic weakly interacting particle systems. This means the assumption $np_n \to \infty$, as the weakest dense graph assumption here, is sufficient to get the same scaling limits of μ^n.

Problem 1

Prove the CLT result (3) under the condition $np_n \to \infty$.

The original proof in Bhamidi et al. (2018) relies on an application of Girsanov's theorem and a careful analysis of the asymptotics of the Radon–Nikodym derivative. This is a classic approach, but the stronger assumption $p_n \to p > 0$ is imposed for technical reasons. An alternative approach in Hitsuda and Mitoma (1986) starts by getting a Markovian stochastic differential equation (SDE) for $n^{-1/2}(\mu^n - \mu)$ and then applies tightness and weak convergence arguments. Therefore, the following subproblem would be sufficient to answer the preceding one (although one may have to assume stronger regularity conditions on coefficients).

Problem 2

Get a Markovian SDE for $n^{-1/2}(\mu^n - \mu)$ (probably the same as in Hitsuda and Mitoma 1986 but with different lower order terms), although the interaction in (5) cannot be written as a closed expression of the empirical measure μ^n.

The last problem, which is related to the stochastic block model and community detection, is partially motivated by Detering et al. (2019), in which a system with both mean-field and sparse local interactions on a circle is studied and a detection problem is discussed.

Problem 3

Suppose, for simplicity, $p_n = p$ and all trajectories $\{X_i^n(t) : t \in [0, T]\}_{i=1}^n$ are observed in (4). Find a consistent estimator $\hat{p}_n$ of p and analyze its asymptotic distribution.

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