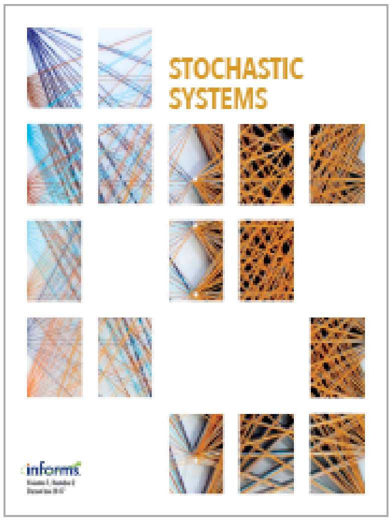

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Polynomial Jump-Diffusion Models

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Abstract. We develop a comprehensive mathematical framework for polynomial jump diffusions in a semimartingale context, which nest affine jump diffusions and have broad applications in finance. We show that the polynomial property is preserved under polynomial transformations and Lévy time change. We present a generic method for option pricing based on moment expansions. As an application, we introduce a large class of novel financial asset pricing models with excess log returns that are conditional Lévy based on polynomial jump diffusions.

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Keywords: polynomial jump diffusions • affine jump diffusions • polynomial transformations • conditional Lévy processes • Lévy time change • asset pricing models • stochastic volatility

1. Introduction

Polynomial jump diffusions have broad applications in finance. A jump diffusion is polynomial if its extended generator maps any polynomial to a polynomial of equal or lower degree. As a consequence, its conditional moments can be computed in closed form. This property renders polynomial jump diffusions computationally tractable and perfectly suited for financial asset pricing models. Many commonly occurring jump diffusions are polynomial, for example, Ornstein–Uhlenbeck processes, square-root diffusions, Jacobi or Wright–Fisher diffusions, Lévy processes, and geometric Lévy processes as well as multidimensional analogues and combinations of these processes.

In this paper, we develop a comprehensive mathematical framework for polynomial jump diffusions. We characterize the polynomial property in terms of the coefficients of the extended generator, and we establish the moment formula. We show that the polynomial property of a jump diffusion is preserved under polynomial transformations and Lévy time change. These transformations allow us to easily construct polynomial jump diffusions from simple building blocks. They also allow us to efficiently specify nonlinearities in financial models, which renders them flexible in capturing many of the empirical features of financial time series.

We also revisit affine jump diffusions, which have been widely used in financial asset pricing models for the last two decades. We provide a relaxed definition of an affine jump diffusion in terms of the pointwise action of its extended generator on exponential affine functions, and we establish the affine transform formula. We show that, modulo integrability conditions on the jumps, affine jump diffusions are polynomial. The converse does not hold, so polynomial jump diffusions truly extend the class of affine jump diffusions. We find that the affine property of a jump diffusion is not invariant under polynomial transformations or Lévy time change in general, which may explain why these transformations have not been widely applied in affine financial models.

In contrast to earlier work on polynomial and affine processes, we do not require the Markov property. Instead, we work in a special semimartingale framework, which is more flexible and amenable to practical applications. Non-Markovian polynomial jump diffusions exist and can be constructed using the counterexamples of Kallsen and Krühner (2016, section 3), and most of our results apply to them. Indeed, Markovianity is only assumed for the invariance of the polynomial property under Lévy time change; all other arguments rely on Itô calculus rather than functional analysis.

We present a generic method for option pricing in polynomial jump-diffusion models. This method builds on the expansion of the likelihood ratio function with respect to an orthonormal basis of polynomials in some conveniently weighted L^2 space. As an application, we introduce a large class of novel financial asset pricing models that are based on polynomial jump diffusions and that are beyond the affine class. In these models, the excess log return processes are conditional Lévy in the sense of Çinlar (2003). This extends several well-known univariate diffusion volatility models, such as the Jacobi model (Ackerer et al. 2018), the extended Stein–Stein model (Stein and Stein 1991, Schöbel and Zhu 1999), and the extended Hull–White model (Hull and White 1987, Lions and Musiela 2007).

Because of their inherent tractability, polynomial jump diffusions have played a prominent and growing role in a wide range of applications in finance. Examples include interest rates (Delbaen and Shirakawa 2002, Zhou 2003, Filipović et al. 2017), stochastic volatility (Gourieroux and Jasiak 2006, Ackerer et al. 2018), exchange rates (Larsen and Sørensen 2007), life insurance liabilities (Biagini and Zhang 2016), variance swaps (Filipović et al. 2016), credit risk (Ackerer and Filipović 2020a), dividend futures (Filipović and Willems 2019), commodities and electricity (Filipović et al. 2018), stochastic portfolio theory (Cuchiero 2019), and economic equilibrium (Guasoni and Wong 2018). Properties of polynomial jump diffusions can also be brought to bear on computational and statistical methods, such as generalized method of moments and martingale estimating functions (Forman and Sørensen 2008), variance reduction (Cuchiero et al. 2012), cubature (Filipović et al. 2019), and quantization (Callegaro et al. 2017). This recent body of research primarily relies on polynomial jump diffusions that are not necessarily affine. Focusing on the affine case, one finds an even richer history in the finance literature, in which affine jump diffusions have long been used to address a large number of problems in asset pricing, optimal investment, equilibrium analysis, and so on.

In addition to their usefulness in applications, polynomial jump diffusions are of theoretical interest because of their rich mathematical structure. Work in this direction has primarily focused on the diffusion case, going back to Wong (1964). Mazet (1997) and Forman and Sørensen (2008) characterize one-dimensional polynomial diffusions. Bakry et al. (2014) describe compact state spaces with nonempty interiors that can support two-dimensional reversible polynomial diffusions. Larsson and Pulido (2017) study polynomial diffusions on compact quadric sets and relate their probabilistic properties to sum-of-squares representations of certain positive biquadratic forms. Existence and uniqueness for a large class of polynomial diffusions on semialgebraic state spaces are developed by Filipović and Larsson (2016). Probability measure-valued polynomial diffusions are studied by Cuchiero et al. (2019). The theoretical literature in the jump-diffusion case is less abundant. The first systematic accounts are those of Cuchiero (2011) and Cuchiero et al. (2012) in a Markovian framework. Gallardo and Yor (2006) study Dunkl processes, which are polynomial jump diffusions; see also Dunkl (1992). Affine jump diffusions with compact state spaces are analyzed by Krühner and Larsson (2018). A large class of specifications of simplex-valued polynomial jump diffusions is developed by Cuchiero et al. (2018). Our paper adds to this applied and theoretical literature by providing a unifying framework for polynomial and affine jump diffusions, enabling new approaches to financial modeling.

We end this introduction with some conventions that are used throughout this paper. We fix a stochastic basis $(\Omega, \mathscr{F}, \mathscr{F}_t, \mathbb{P})$. Equalities between random variables are understood to hold almost surely. Following Jacod and Shiryaev (2003, equation (I.1.1)), we use the notion of generalized conditional expectation, which is defined for any σ-field $\mathscr{F}' \subset \mathscr{F}$ and all random variables X by

$$\mathbb{E}\left[X \mid \mathscr{F}'\right] = \begin{cases} \mathbb{E}\left[X^+ \mid \mathscr{F}'\right] - \mathbb{E}\left[X^- \mid \mathscr{F}'\right], & \text{on } \{\mathbb{E}[|X| | \mathscr{F}'] < \infty\}, \\ +\infty, & \text{elsewhere.} \end{cases}$$

For a multi-index $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d$, we write $|\boldsymbol{\alpha}| = \alpha_1 + \cdots + \alpha_d$ and $x^{\boldsymbol{\alpha}} = x_1^{\alpha_1} \cdots x_d^{\alpha_d}$ for $x \in \mathbb{R}^d$. A *polynomial* p on $\mathbb{R}^d$ is a function $p : \mathbb{R}^d \to \mathbb{R}$ of the form $p(x) = \sum_{\boldsymbol{\alpha}} c_{\boldsymbol{\alpha}} x^{\boldsymbol{\alpha}}$, where the sum runs over all $\boldsymbol{\alpha} \in \mathbb{N}_0^d$, and only finitely, many of the coefficients $c_{\boldsymbol{\alpha}}$ are nonzero. Such a representation is unique. The *degree* of p is the number $\deg p = \max\{|\boldsymbol{\alpha}| : c_{\boldsymbol{\alpha}} \neq 0\}$. We let $\mathrm{Pol}(\mathbb{R}^d)$ denote the ring of all polynomials on $\mathbb{R}^d$ and $\mathrm{Pol}_n(\mathbb{R}^d)$ the subspace consisting of polynomials of degree at most n. Let E be a subset of $\mathbb{R}^d$. A *polynomial on E* is the restriction $p = q|_E$ to E of a polynomial $q \in \mathrm{Pol}(\mathbb{R}^d)$. Its degree is $\deg p = \min\{\deg q : p = q|_E,\, q \in \mathrm{Pol}(\mathbb{R}^d)\}$. We let $\mathrm{Pol}(E)$ denote the algebra of polynomials on E and $\mathrm{Pol}_n(E)$ the subspace of polynomials on E of degree at most n. Both $\mathrm{Pol}_n(\mathbb{R}^d)$ and $\mathrm{Pol}_n(E)$ are finite-dimensional real vector spaces, but if there are nontrivial polynomials that vanish on E, their dimensions are different. If E has a nonempty interior, then $\mathrm{Pol}_n(\mathbb{R}^d)$ and $\mathrm{Pol}_n(E)$ can be identified. For simplicity of notation, we write $f \in \mathrm{Pol}_n(E)$ for any function $f = (f_1, \ldots, f_k) : \mathbb{R}^d \to \mathbb{R}^k$ with $k \in \mathbb{N}$ such that the restrictions $f_i|_E \in \mathrm{Pol}_n(E)$ for all $i = 1, \ldots, k$. The set of real symmetric $d \times d$ matrices is denoted $\mathbb{S}^d$, and the subset of positive semidefinite matrices is denoted $\mathbb{S}^d_+$. For any map $\varphi : E \to F$, we denote the pullback operator by φ^*, which maps a function f on F to a function $\varphi^* f$ on E by

$$\varphi^* f(x) = f(\varphi(x)). \tag{1}$$

The remainder of this paper is as follows. In Section 2, we define polynomial jump diffusions, give a characterization in terms of the coefficients of the extended generator, and establish the moment formula. In Section 3, we revisit affine jump diffusions. In Section 4, we study the invariance of the polynomial property under polynomial transformations. In Section 5, we introduce polynomial conditional Lévy processes. In Section 6, we show that the polynomial property is invariant under Lévy time change. In Section 7, we present a generic method for option pricing in polynomial jump-diffusion models. In Section 8, we introduce a large class of polynomial asset pricing models building on polynomial conditional Lévy processes. In Section 9, we focus on linear volatility models. Section 10 concludes. Appendix A, B, C, and D contain additional results and all proofs.

2. Polynomial Jump Diffusions

We consider a jump-diffusion operator on $\mathbb{R}^d$ of the form

$$\mathscr{G}f(x) = \frac{1}{2}\mathrm{Tr}\Big(a(x)\nabla^2 f(x)\Big) + b(x)^\top \nabla f(x) + \int_{\mathbb{R}^d}\big(f(x+\xi) - f(x) - \xi^\top \nabla f(x)\big)\nu(x, d\xi)\,, \tag{2}$$

for some measurable maps $a : \mathbb{R}^d \to \mathbb{S}^d_+$ and $b : \mathbb{R}^d \to \mathbb{R}^d$, and a transition kernel $\nu(x, d\xi)$ from $\mathbb{R}^d$ into $\mathbb{R}^d$ satisfying $\nu(x, \{0\}) = 0$ and $\int_{\mathbb{R}^d} \|\xi\| \wedge \|\xi\|^2 \nu(x, d\xi) < \infty$, for all $x \in \mathbb{R}^d$.

We let X_t be an E-valued jump diffusion with extended generator $\mathscr{G}$ for some state space $E \subseteq \mathbb{R}^d$. That is, X_t is an E-valued special semimartingale, and

$$f(X_t) - f(X_0) - \int_0^t \mathscr{G}f(X_s)\, ds \tag{3}$$

is a local martingale for any bounded C^2 function $f(x)$ on $\mathbb{R}^d$.[1]

We say that $\mathscr{G}$ is *well defined on* $\mathrm{Pol}(E)$ if

$$\int_{\mathbb{R}^d} \|\xi\|^n \,\nu(x, d\xi) < \infty \text{ for all } x \in E \text{ and all } n \geq 2, \text{ and} \tag{4}$$

$$\mathscr{G}f(x) = 0 \text{ on } E \text{ for any } f \in \mathrm{Pol}(\mathbb{R}^d) \text{ with } f(x) = 0 \text{ on } E. \tag{5}$$

This property ensures that $\mathscr{G}$ is well defined as a linear operator from $\mathrm{Pol}(E)$ into the space of measurable functions on E.[2]

Definition 1. The operator $\mathscr{G}$ is called *polynomial on E* if it is well defined on $\mathrm{Pol}(E)$ and maps $\mathrm{Pol}_n(E)$ to itself for each $n \in \mathbb{N}$. In this case, we call X_t a *polynomial jump diffusion on E*.

Thus, a polynomial jump diffusion on E is an E-valued jump diffusion whose jump measure admits moments of all orders with an extended generator that maps polynomials on E to polynomials on E of lower or equal degree. Polynomial jump diffusions admit closed-form conditional moments and have broad applications in finance, as we see.

The polynomial property of $\mathscr{G}$ on E has a simple characterization in terms of its coefficients $a(x)$, $b(x)$, and $\nu(x, d\xi)$.

Lemma 1. *Assume that $\mathscr{G}$ is well defined on* $\mathrm{Pol}(E)$. *Then the following are equivalent:*

(i) *$\mathscr{G}$ is polynomial on E.*

(ii) *$a(x)$, $b(x)$, and $\nu(x, d\xi)$ in (2) satisfy*

$$b \in \mathrm{Pol}_1(E),$$

$$a + \int_{\mathbb{R}^d} \xi\xi^\top \nu(\cdot, d\xi) \in \mathrm{Pol}_2(E),$$

$$\int_{\mathbb{R}^d} \xi^\alpha \nu(\cdot, d\xi) \in \mathrm{Pol}_{|\alpha|}(E),$$

for all $|\alpha| \geq 3$. *In this case, the polynomials on E listed in property* (ii) *are uniquely determined by the action of* $\mathscr{G}$ *on* $\mathrm{Pol}(E)$. *Moreover,* $a(x)$, $b(x)$, *and* $\int_{\mathbb{R}^d} \xi^\alpha \nu(x, d\xi)$ *are locally bounded in* x *on* E *for all* $|\alpha| \geq 2$.

Proof. The implication (ii) $\Rightarrow$ (i) is immediate, and the implication (i) $\Rightarrow$ (ii) follows by applying $\mathscr{G}$ to all monomials. In particular, the polynomials on E listed in property (ii) are uniquely determined by the action of $\mathscr{G}$ on $\mathrm{Pol}(E)$. It remains to show that $a(x)$ and $\int_{\mathbb{R}^d} \|\xi\|^2 \nu(x, d\xi)$ are locally bounded in x on E. But this follows because $a(x) \in \mathbb{S}^d_+$ and $a_{ii} + \int_{\mathbb{R}^d} \xi_i^2 \nu(\cdot, d\xi) \in \mathrm{Pol}_2(E)$ and, hence, $a_{ii}(x) \geq 0$ and $\int_{\mathbb{R}^d} \xi_i^2 \nu(x, d\xi) \geq 0$ are locally bounded in x on E. □

Although the moments of $\nu(x, d\xi)$ of order three and beyond are determined by the action of $\mathscr{G}$ on $\mathrm{Pol}(E)$, the measure $\nu(x, d\xi)$ itself need not be uniquely determined. The following example illustrates this.

Example 1. Consider the compensated compound Poisson process X_t with unit intensity and log-normal jump distribution, whose extended generator is given by $\mathscr{G}f(x) = \int_{\mathbb{R}} (f(x+\xi) - f(x) - \xi f'(x)) g(\xi) d\xi$, where $g(\xi)$ is the standard log-normal density. It is well known that the log-normal distribution is indeterminate in the sense of the moment problem so that there exists a density $h(\xi)$ different from $g(\xi)$ but with the same moments. The extended generator $\widetilde{\mathscr{G}}f(x) = \int_{\mathbb{R}} (f(x+\xi) - f(x) - \xi f'(x)) h(\xi) d\xi$ then coincides with $\mathscr{G}$ on $\mathrm{Pol}(E)$.

Remark 1. Non-Markovian polynomial jump diffusions exist and can be constructed using the counterexamples of Kallsen and Krühner (2016, section 3). This is why we take a semimartingale approach to polynomial jump diffusions rather than a Markovian approach as in Cuchiero et al. (2012). The semimartingale approach has several advantages. First, we only have to assume existence but do not require uniqueness of the local martingale problem (3). Indeed, uniqueness is tantamount to the Markovianity of X_t. Furthermore, it is straightforward to develop a time-inhomogeneous version of our framework, in which $a(t, x)$, $b(t, x)$, and $\nu(t, x, d\xi)$ depend explicitly and not necessarily polynomially on t. Finally, our framework immediately accommodates polynomial jump diffusions on finite time intervals. Indeed, we did not specify the time set for the stochastic basis and preceding semimartingales, so rather than $[0, \infty)$, one can simply understand the time set to be $[0, T]$ for some finite time horizon T. This flexibility is useful in asset pricing applications, and we make use of it in Section 8.3.

To make the polynomial property of $\mathscr{G}$ on E operational, we now introduce a coordinate system on $\mathrm{Pol}_n(E)$ for a generic $n \in \mathbb{N}$ that we use throughout this paper. We define $N = \dim \mathrm{Pol}_n(E) - 1$ and note that $1 + N \leq \binom{n+d}{n}$ with equality if E has a nonempty interior. We fix polynomials $h_1(x), \ldots, h_N(x)$ on $\mathbb{R}^d$ such that $\{1, h_1, \ldots, h_N\}$ forms a basis of $\mathrm{Pol}_n(E)$ and define the vector-valued function

$$H : \mathbb{R}^d \to \mathbb{R}^N, \quad H(x) = \big(h_1(x), \ldots, h_N(x)\big)^\top. \tag{6}$$

For each $p \in \mathrm{Pol}_n(E)$, we denote its coordinate vector by $\vec{p} \in \mathbb{R}^{1+N}$ so that

$$p(x) = \big(1, H(x)^\top\big)\vec{p} \tag{7}$$

on E. The $(1+N) \times (1+N)$ matrix representation G of $\mathscr{G}$ restricted to $\mathrm{Pol}_n(E)$ is determined by $\mathscr{G}(1, H^\top)(x) = (1, H(x)^\top)G$ on E so that

$$\mathscr{G}p(x) = \big(1, H(x)^\top\big)G\vec{p} \tag{8}$$

on E. We now show that $\mathbb{E}[p(X_T) \mid \mathscr{F}_t]$ is a polynomial function of X_t.

Theorem 1. *Assume that* $\mathscr{G}$ *is polynomial on E. Then, for any* $p \in \mathrm{Pol}_n(E)$, *the moment formula holds,*

$$\mathbb{E}[p(X_T) \mid \mathscr{F}_t] = \big(1, H(X_t)^\top\big)\mathrm{e}^{(T-t)G}\,\vec{p}, \quad \text{for } t \leq T.$$

Theorem 1 implies that X_T has finite $\mathscr{F}_t$-conditional moments of all orders. Note that we do not assume that X_T has any finite unconditional moments. The following example illustrates this.

Example 2. The generalized autoregressive conditional heteroskedasticity (GARCH) diffusion $dX_t = \kappa(\theta - X_t)\,dt + \sqrt{2\kappa}X_t\,dW_t$ for some parameters $\kappa, \theta > 0$ has a unique ergodic solution on $(0, \infty)$, which is a polynomial diffusion. The invariant distribution is an inverse Gamma distribution with shape parameter two and scale parameter $1/\theta$. Hence, in the stationary case, when X_0 has the invariant distribution, we have $\mathbb{E}[X_t] = \theta$ and $\mathbb{E}[X_t^2] = +\infty$. See Forman and Sørensen (2008, case 4).

Here is a large class of polynomial jump diffusions extending the GARCH diffusion. Let W_t be a standard m-dimensional Brownian motion and $N(du, dt)$ a Poisson random measure with compensator $F(du)dt$ on $U \times \mathbb{R}_+$ for some mark space U; see Jacod and Shiryaev (2003, definition II.1.20). We consider the linear stochastic differential equation (SDE)

$$dX_t = b(X_t)\,dt + \sigma(X_t)\,dW_t + \int_U \delta(X_{t-}, u)(N(du, dt) - F(du)dt), \tag{9}$$

with drift, volatility, and jump size functions

$$b(x) = \beta_0 + \sum_{i=1}^d x_i\beta_i, \quad \sigma(x) = \Gamma_0 + \sum_{i=1}^d x_i\Gamma_i, \quad \delta(x, u) = \delta_0(u) + \sum_{i=1}^d x_i\delta_i(u), \tag{10}$$

for parameters $\beta_i \in \mathbb{R}^d$, $\Gamma_i \in \mathbb{R}^{d\times m}$, and functions $\delta_i : U \to \mathbb{R}^d$ with $\int_U \|\delta_i(u)\|^n F(du) < \infty$ for all $n \geq 2$ for $i = 0, \dots, d$. Because of the global Lipschitz continuity of the coefficients, the linear SDE (9) has a unique strong $\mathbb{R}^d$-valued solution X_t for every $\mathscr{F}_0$-measurable initial random variable X_0; see Jacod and Shiryaev (2003, theorem III.2.32). It follows by inspection that X_t is a polynomial jump diffusion on $\mathbb{R}^d$ with linear drift $b \in \mathrm{Pol}_1(\mathbb{R}^d)$, diffusion function $a = \sigma\sigma^\top \in \mathrm{Pol}_2(\mathbb{R}^d)$, and jump measure $\nu(x, d\xi)$ given by $\int_{\mathbb{R}^d} f(\xi)\nu(x, d\xi) = \int_U f(\delta(x, u))F(du)$ so that

$$\int_{\mathbb{R}^d} \xi^{\boldsymbol{\alpha}} \nu(\cdot, d\xi) \in \mathrm{Pol}_{|\boldsymbol{\alpha}|}\left(\mathbb{R}^d\right) \quad \text{for all } |\boldsymbol{\alpha}| \geq 2.$$

We now derive the matrix representation (8) of $\mathscr{G}$ for the linear SDEs (9) and (10) in the univariate case, $d = 1$ when $E \subseteq \mathbb{R}$ has nonempty interior. Straightforward calculations show that for any $j \in \{0, \dots, n\}$,

$$\mathscr{G}x^j = \sum_{i=0}^n G_{ij}x^i,$$

where

$$G_{ij} = \begin{cases} \binom{j}{i}\int_U \delta_0(u)^{j-i}\left(1 + \delta_1(u)\right)^i F(du), & i \leq j-3, \\ \frac{j(j-1)}{2}\Gamma_0^2 + \frac{j(j-1)}{2}\int_U \delta_0(u)^2\left(1 + \delta_1(u)\right)^{j-2}F(du), & i = j-2, \\ j\beta_0 + \frac{j(j-1)}{2}\Gamma_0\Gamma_1 + j\int_U \delta_0(u)\left(\left(1 + \delta_1(u)\right)^{j-1} - 1\right)F(du), & i = j-1, \\ j\beta_1 + \frac{j(j-1)}{2}\Gamma_1^2 + \int_U\left(\left(1 + \delta_1(u)\right)^j - 1 - j\delta_1(u)\right)F(du), & i = j, \\ 0, & i > j. \end{cases}$$

Hence, the $(1+n)\times(1+n)$ upper triangular matrix $G = (G_{ij})_{0\leq i,j\leq n}$ represents $\mathscr{G}$ restricted to $\mathrm{Pol}_n(E)$ with respect to the basis $\{1, x, \dots, x^n\}$.

Example 3. A special case of the linear SDEe (9) and (10) is the Lévy-driven SDE

$$dX_t = b(X_t)\,dt + \sigma(X_{t-})\,dL_t,$$

where $b(x)$ and $\sigma(x)$ are as in (10), and L_t is an m-dimensional Lévy process with $\mathbb{E}[\|L_t\|^n] < \infty$ for all $n \geq 2$; see Sato (1999, theorem 25.3).

3. Affine Jump Diffusions

Affine jump diffusions have been widely studied and applied in finance; see, for example, Duffie et al. (2003). We provide a novel approach to affine jump diffusions and show that they constitute examples of polynomial jump diffusions.

Let $\mathscr{G}$ be a jump-diffusion operator on $\mathbb{R}^d$ of the form (2), and let X_t be an E-valued jump-diffusion with extended generator $\mathscr{G}$ for some state space $E \subseteq \mathbb{R}^d$.

Definition 2. The operator $\mathscr{G}$ is called *affine on E* if there exist complex-valued functions $F(u)$ and $R(u) = (R_1(u), \dots, R_d(u))^\top$ on $\mathrm{i}\mathbb{R}^d$ such that

$$\mathscr{G}e^{u^\top x} = \left(F(u) + R(u)^\top x\right)e^{u^\top x} \tag{11}$$

holds for all $x \in E$ and $u \in \mathrm{i}\mathbb{R}^d$. In this case, we call X_t an *affine jump diffusion on E*.

Note that this is a relaxed definition compared with the definition of an affine process in Duffie et al. (2003) because it is directly given in terms of the pointwise action of $\mathscr{G}$ on exponential affine functions. Indeed, the affine jump diffusion in Example 4 is not an affine process in the sense of Duffie et al. (2003).

In analogy to Lemma 1, the affine property of $\mathscr{G}$ on E has a simple characterization in terms of its coefficients $a(x)$, $b(x)$, and $\nu(x,d\xi)$.

Lemma 2. *The operator $\mathscr{G}$ is affine on E if and only if $a(x)$, $b(x)$, and $\nu(x,d\xi)$ are affine of the form*

$$a(x) = a_0 + \sum_{i=1}^{d} x_i a_i, \quad b(x) = b_0 + \sum_{i=1}^{d} x_i b_i, \quad \nu(x,d\xi) = \nu_0(d\xi) + \sum_{i=1}^{d} x_i \nu_i(d\xi) \tag{12}$$

on E for some matrices $a_i \in \mathbb{S}^d$, vectors $b_i \in \mathbb{R}^d$, and signed measures $\nu_i(d\xi)$ on $\mathbb{R}^d$ such that $\nu_i(\{0\}) = 0$ and $\int_{\mathbb{R}^d} \|\xi\| \wedge \|\xi\|^2\, |\nu_i|(d\xi) < \infty$, $i = 0,\dots,d$. In this case, the functions $F(u)$ and $R(u)$ in (11) can be chosen to be of the form

$$\begin{aligned} F(u) &= \frac{1}{2} u^\top a_0 u + b_0^\top u + \int_{\mathbb{R}^d} \left(e^{u^\top \xi} - 1 - u^\top \xi\right)\nu_0(d\xi), \\ R_i(u) &= \frac{1}{2} u^\top a_i u + b_i^\top u + \int_{\mathbb{R}^d} \left(e^{u^\top \xi} - 1 - u^\top \xi\right)\nu_i(d\xi). \end{aligned} \tag{13}$$

It follows from (12) that an affine jump diffusion X_t cannot be realized as solutions of linear SDEs (9) and (10) in general unless diffusion and jump coefficients $a(x) = a_0$ and $\nu(x,d\xi) = \nu_0(d\xi)$ do not depend on x.

From (12) and Lemma 1, we immediately obtain that affine jump diffusions are polynomial on E subject to being well defined on $\mathrm{Pol}(E)$.

Corollary 1. *If X_t is an affine jump diffusion on E such that $\mathscr{G}$ is well defined on $\mathrm{Pol}(E)$, then X_t is a polynomial jump diffusion on E.*

Affine jump diffusions on E not only satisfy the moment formula in Theorem 1 subject to being well defined on $\mathrm{Pol}(E)$, but their characteristic functions are also analytically tractable.

Theorem 2. *Assume that X_t is an affine jump diffusion on E. For $u \in \mathrm{i}\mathbb{R}^d$ and $T > 0$, let $\phi(\tau)$ and $\psi(\tau) = (\psi_1(\tau),\dots,\psi_d(\tau))^\top$ be functions that solve the generalized Riccati equations*

$$\begin{aligned} \phi'(\tau) &= F\big(\psi(\tau)\big), \quad \phi(0) = 0, \\ \psi'(\tau) &= R\big(\psi(\tau)\big), \quad \psi(0) = u, \end{aligned} \tag{14}$$

for $0 \le \tau \le T$, where $F(u)$ and $R(u)$ are the functions in (13). If

$$\mathrm{Re}\,\phi(T-t) + \mathrm{Re}\,\psi(T-t)^\top X_t \le 0, \qquad \text{for } t \le T, \tag{15}$$

then the affine transform formula holds,

$$\mathbb{E}\left[\mathrm{e}^{u^\top X_T} \mid \mathscr{F}_t\right] = \mathrm{e}^{\phi(T-t)+\psi(T-t)^\top X_t}, \qquad \text{for } t \le T.$$

Remark 2. Inequality (15) is a necessary condition for the affine transform formula to hold. Indeed, the affine transform formula and Jensen's inequality yield, for $u \in \mathrm{i}\mathbb{R}^d$,

$$\exp\big(\mathrm{Re}\,\phi(T-t) + \mathrm{Re}\,\psi(T-t)^\top X_t\big) = \big|\mathbb{E}\big[\exp(u^\top X_T)\big|\mathscr{F}_t\big]\big| \le \mathbb{E}\big[\big|\exp(u^\top X_T)\big|\big|\mathscr{F}_t\big] = 1.$$

There exist affine jump diffusions for which the generalized Riccati equation (14) does not admit global solutions for all $u \in \mathrm{i}\mathbb{R}^d$. The following example illustrates this.

Example 4. Consider the two-point state space $E = \{0,1\} \subseteq \mathbb{R}$ and the process X_t that jumps from one to zero with intensity λ and is absorbed once it reaches zero. This is a jump diffusion with an extended generator

$$\mathscr{G}f(x) = \lambda x\big(f(x-1) - f(x)\big),$$

which is of the form (2) with $a(x) = 0$, $b(x) = \lambda x$, $\nu(x,d\xi) = \lambda x \delta_{-1}(d\xi)$ and. Thus, X_t is an affine jump diffusion, and $F(u) = 0$ and $R(u) = \lambda(e^{-u} - 1)$. The associated generalized Riccati equation (14) is

$$\phi'(\tau) = 0, \quad \psi'(\tau) = \lambda\left(e^{-\psi(\tau)} - 1\right). \tag{16}$$

We claim that this equation does not have a global solution for the initial condition $u = \mathrm{i}\pi$. We argue by contradiction and assume that $\psi(\tau)$ is a global solution of (16). Then $\Psi(\tau) = e^{\psi(\tau)}$ satisfies the linear equation

$$\Psi'(\tau) = -\lambda\Psi(\tau) + \lambda, \qquad \Psi(0) = -1,$$

whose unique solution is $\Psi(\tau) = 1 - 2e^{-\lambda\tau}$, which becomes zero for $\tau = \lambda^{-1}\log 2$, which is absurd.

The deeper reason behind this fact is that the characteristic function of X_T given $X_0 = 1$, $\mathbb{E}[\mathrm{e}^{uX_T}] = 1 - \mathrm{e}^{-\lambda T} + \mathrm{e}^{u-\lambda T}$, for $u = \mathrm{i}\pi$ and $T = \lambda^{-1}\log 2$, becomes zero and, hence, cannot be written as exponential as in the affine transform formula.

4. Polynomial Transformations

The class of polynomial jump diffusions is shown to be invariant under polynomial transformations after an extension of the dimension. This allows us to build a large class of polynomial jump diffusions from basic building blocks, including Brownian motion, Lévy processes, or more general affine processes. This turns out to be a useful and flexible method for introducing nonlinearities and jumps in all kinds of financial models.

Let X_t be a polynomial jump diffusion on $E \subseteq \mathbb{R}^d$ with extended generator $\mathscr{G}$. Fix $n \in \mathbb{N}$ and a basis $\{1, h_1, \ldots, h_N\}$ of $\mathrm{Pol}_n(E)$ as in (6)–(8). Notice that $H : E \to H(E) \subseteq \mathbb{R}^N$ is injective. Indeed, the restriction to E of any linear monomial x_i lies in $\mathrm{Pol}_n(E)$ and is, therefore, a linear combination of $1, h_1(x), \ldots, h_N(x)$. Thus, there exist linear polynomials $\ell_i \in \mathrm{Pol}_1(\mathbb{R}^N)$ such that $\ell_i(H(x)) = x_i$ for all $x \in E$ and all i. Define the vector-valued function

$$L : \mathbb{R}^N \to \mathbb{R}^d, \quad L(\overline{x}) = \big(\ell_1(\overline{x}), \ldots, \ell_d(\overline{x})\big)^\top. \tag{17}$$

Then $L(H(x)) = x$ for all $x \in E$, and $H(L(\overline{x})) = \overline{x}$ for all $\overline{x} \in H(E)$. We define the pullbacks H^* and L^* as in (1).

Lemma 3. *For every $m \in \mathbb{N}$, the pullback $H^* : \mathrm{Pol}_m(H(E)) \to \mathrm{Pol}_{mn}(E)$ is a linear isomorphism with inverse L^*.*

Here is the main result of this section.[3]

Theorem 3. *The process $\overline{X}_t = H(X_t)$ is a polynomial jump diffusion on $H(E)$ with extended generator $\overline{\mathscr{G}} = L^*\mathscr{G}H^*$, and for every $m \in \mathbb{N}$, the following diagram commutes:*

$$\begin{tikzcd}
\mathrm{Pol}_m(H(E)\times\mathbb{R}^e) \arrow[r, "\overline{\mathscr{G}}"] \arrow[d, "\varphi^*"'] & \mathrm{Pol}_m(H(E)\times\mathbb{R}^e) \\
V_m \arrow[r, "\mathscr{G}"] & V_m \arrow[u, "\psi^*"']
\end{tikzcd} \tag{18}$$

As an immediate application of Theorem 3, we can easily infer the action of $\overline{\mathscr{G}}$ on $\mathrm{Pol}_m(H(E))$. Let $1 + \overline{N} = \dim \mathrm{Pol}_m(H(E)) = \dim \mathrm{Pol}_{mn}(E)$, and extend the basis of $\mathrm{Pol}_n(E)$ to a basis

$$\{h_0 = 1, h_1, \ldots, h_N, h_{N+1}, \ldots, h_{\overline{N}}\} \tag{19}$$

of $\mathrm{Pol}_{mn}(E)$ for some polynomials $h_{N+1}(x), \ldots, h_{\overline{N}}(x)$ on $\mathbb{R}^d$. In view of the commuting diagram (18), this induces a basis $\overline{h}_i = L^*h_i$ on $\mathrm{Pol}_m(H(E))$ for $i = 0, \ldots, \overline{N}$. Let $\overline{G}$ be the $(1+\overline{N}) \times (1+\overline{N})$ matrix representing $\mathscr{G}$ on $\mathrm{Pol}_{mn}(E)$ according to (6)–(8), which can be determined using symbolic calculus applied to $\mathscr{G}h_i(x)$. Then $\overline{G}$ is the matrix representing $\overline{\mathscr{G}}$ on $\mathrm{Pol}_m(H(E))$, and Theorem 1 can readily be applied to compute all $\mathscr{F}_t$-conditional moments of $\overline{X}_T$ up to order m.

The following example shows that the affine property is not invariant under polynomial transformations.

Example 5. Consider the square-root process $dX_t = (b + \beta X_t)\,dt + \sigma\sqrt{X_t}\,dW_t$, which is an affine diffusion. The augmented process $\overline{X}_t = (X_t, X_t^2)$ satisfies

$$d\overline{X}_{1t} = \left(b + \beta\overline{X}_{1t}\right)dt + \sigma\sqrt{\overline{X}_{1t}}\,dW_t,$$
$$d\overline{X}_{2t} = \left(\left(2b + \sigma^2\right)\overline{X}_{1t} + 2\beta\overline{X}_{2t}\right)dt + 2\sigma\sqrt{\overline{X}_{1t}\overline{X}_{2t}}\,dW_t.$$

Although the drift function of $\overline{X}_t$ is affine of the form (12), the diffusion function is not. In view of Lemma 2, this shows that $\overline{X}_t$ is not affine, although it is polynomial on $H(\mathbb{R})$ for $H(x) = (x, x^2)^\top$ in line with Theorem 3.

5. Polynomial Conditional Lévy Processes

In financial applications, we often encounter the following situation. Let X_t be a polynomial jump diffusion X_t on $E \subseteq \mathbb{R}^d$, and let Y_t be an $\mathbb{R}^e$-valued semimartingale for some $e \in \mathbb{N}$ whose characteristics are functions of X_t. The process Y_t could model the excess log returns of assets whose stochastic volatilities and jump characteristics are given in terms of the latent factor process X_t. Drawing on Section 4, we develop a polynomial framework that accommodates a large class of such models. The following example illustrates the kind of situation we have in mind. We elaborate on this example further, including jumps, in Section 9.

Example 6. For $\gamma > 0$, $\kappa\theta > 0$, $X_0 > 0$, and $Y_0 = 0$, we consider the following model specified under the risk-neutral measure:

$$dX_t = \kappa(\theta - X_t)\,dt + \gamma X_t\,dW_{1t},$$
$$dY_t = -\frac{1}{2}X_t^2\,dt + X_t\,dW_{2t},$$

where Y_t models the excess log return of an asset and X_t its volatility. In view of Lemma 1 and Example 2, X_t is a polynomial diffusion on $E = (0, \infty)$. Moreover, the drift and diffusion functions of Y_t are both quadratic in X_t. In particular, Lemma 1 shows that the joint process $Z_t = (X_t, Y_t)$ is not polynomial on $E \times \mathbb{R}$. However, the augmented process $\overline{Z}_t = (H(X_t), Y_t)$ with $H(x) = (x, x^2)$ is a polynomial diffusion on $H(E) \times \mathbb{R}$. Thus, the moment formula in Theorem 1 can still be used to compute conditional moments of Y_T.

Returning to the general discussion, we assume that the joint semimartingale $Z_t = (X_t, Y_t)$ is an $E \times \mathbb{R}^e$-valued jump diffusion with extended generator of the form

$$\mathcal{G}f(z) = \frac{1}{2}\mathrm{Tr}\Big(a(x)\nabla^2 f(z)\Big) + b(x)^\top \nabla f(z) + \int_{\mathbb{R}^{d+e}} \big(f(z+\zeta) - f(z) - \zeta^\top \nabla f(z)\big)\nu(x, d\zeta), \tag{20}$$

where we write $z = (x, y)$ for some measurable maps $a : \mathbb{R}^d \to \mathbb{S}_+^{d+e}$ and $b : \mathbb{R}^d \to \mathbb{R}^{d+e}$ and a transition kernel $\nu(x, d\zeta)$ from $\mathbb{R}^d$ into $\mathbb{R}^{d+e}$ satisfying $\nu(x, \{0\}) = 0$ and $\int_{\mathbb{R}^{d+e}} \|\zeta\| \wedge \|\zeta\|^2 \nu(x, d\zeta) < \infty$ for all $x \in \mathbb{R}^d$.

According to the decomposition $Z_t = (X_t, Y_t)$, and, accordingly, $\zeta = (\xi, \eta)$, we write

$$a(x) = \begin{pmatrix} a^X(x) & a^{XY}(x) \\ a^{YX}(x) & a^Y(x)) \end{pmatrix}, \quad b(x) = \begin{pmatrix} b^X(x) \\ b^Y(x) \end{pmatrix}, \quad \nu(x, d\zeta) = \nu\big(x, d\xi \times d\eta\big) \tag{21}$$

and denote by $\nu^X(x, d\xi)$ and $\nu^Y(x, d\eta)$ the marginal measures of $\nu(x, d\xi \times d\eta)$ given by

$$\nu^X(x, A) = \nu(x, A \times \mathbb{R}^e), \quad \nu^Y(x, A) = \nu\big(x, \mathbb{R}^d \times A\big). \tag{22}$$

Then $a^X(x)$, $b^X(x)$, and $\nu^X(x, d\xi)$ are the coefficients of the extended generator $\mathcal{G}^X$ of X_t, which is polynomial on E by assumption. Note that Y_t is a conditional Lévy process in the sense of Çinlar (2003). That is, conditional on the process X_t, the semimartingale Y_t has independent increments.

Fix $n \in \mathbb{N}$ and a basis $\{1, h_1, \ldots, h_N\}$ of $\mathrm{Pol}_n(E)$ as in (6)–(8) for $\mathcal{G}^X$ in lieu of $\mathcal{G}$. Extending (17), we define the maps $\varphi : \mathbb{R}^{d+e} \to \mathbb{R}^{N+e}$ and $\psi : \mathbb{R}^{N+e} \to \mathbb{R}^{d+e}$ by

$$\varphi(x, y) = \big(H(x), y\big), \quad \psi(\overline{x}, y) = \big(L(\overline{x}), y\big),$$

so that $\psi \circ \varphi = \mathrm{id}$ on $E \times \mathbb{R}^e$. The pullbacks φ^* and ψ^* are defined in (1). Here is our first main result of this section, which extends Example 6.

Theorem 4. *Assume that*

$$\int_{\mathbb{R}^e} \|\eta\|^k \,\nu^Y\big(x, d\eta\big) < \infty \text{ for all } x \in E \text{ and all } k \geq 2. \tag{23}$$

Then the augmented process $\overline{Z}_t = (H(X_t), Y_t)$ is a jump diffusion on $H(E) \times \mathbb{R}^e$ with extended generator $\overline{\mathcal{G}} = \psi^* \mathcal{G} \varphi^*$, and the operators $\mathcal{G}$ and $\overline{\mathcal{G}}$ are well defined on $\mathrm{Pol}(E \times \mathbb{R}^e)$ and $\mathrm{Pol}(H(E) \times \mathbb{R}^e)$, respectively. Furthermore, the properties

$$b^Y \in \mathrm{Pol}_n(E), \tag{24}$$

$$a^Y + \int_{\mathbb{R}^e} \eta\eta^\top \nu^Y(\cdot, d\eta) \in \mathrm{Pol}_{2n}(E), \tag{25}$$

$$a^{XY} + \int_{\mathbb{R}^{d+e}} \xi\eta^\top \nu(\cdot, d\xi \times d\eta) \in \mathrm{Pol}_{1+n}(E), \tag{26}$$

$$\int_{\mathbb{R}^{d+e}} \xi^{\boldsymbol{\alpha}}\eta^{\boldsymbol{\beta}} \nu(\cdot, d\xi \times d\eta) \in \mathrm{Pol}_{|\boldsymbol{\alpha}|+n|\boldsymbol{\beta}|}(E), \quad \text{for all } |\boldsymbol{\alpha}| + |\boldsymbol{\beta}| \geq 3, \tag{27}$$

together imply that

$$\overline{Z}_t \text{ is polynomial on } H(E) \times \mathbb{R}^e. \tag{28}$$

Conversely, (28) *implies* (24), (25), *and* (27) *for* $\boldsymbol{\alpha} = 0$.[4]

Remark 3. Because $\mathcal{G}^X$ is polynomial on E and thus well defined on $\mathrm{Pol}(E)$, condition (23) is equivalent to

$$\int_{\mathbb{R}^{d+e}} \|\zeta\|^k \nu(x, d\zeta) < \infty \text{ for all } x \in E \text{ and all } k \geq 2.$$

As an application of Theorem 4, we show how to construct large classes of polynomial jump diffusions by specifying Y_t in terms of the polynomial jump diffusion X_t on E.

Corollary 2. *Let $e = e' + e''$ for some $e', e'' \geq 0$, and consider the maps $P : E \to \mathbb{R}^{e'}$ and $Q : E \to \mathbb{R}^{e'' \times d}$ with polynomial components $P \in \mathrm{Pol}_n(E)$ and $Q \in \mathrm{Pol}_{n-1}(E)$. Then, for*

$$dY_t = \begin{pmatrix} P(X_t)\, dt \\ Q(X_{t-})\, dX_t \end{pmatrix},$$

conditions (23)–(27) *in Theorem* 4 *are satisfied so that $\overline{Z}_t = (H(X_t), Y_t)$ is a polynomial jump diffusion on $H(E) \times \mathbb{R}^e$. For $n \geq 2$, this covers the quadratic covariations $d[X_i, X_j]_t = d(X_{i,t}X_{j,t}) - X_{i,t-}dX_{j,t} - X_{j,t-}dX_{i,t}$ and their predictable compensator, $\Gamma^X(x_i, x_j)(X_t)\, dt$, where Γ^X denotes the carré-du-champ operator related to $\mathcal{G}^X$ (see Appendix A).*

To compute conditional moments of Y_T using the moment formula in Theorem 1, we must understand the structure of $\mathrm{Pol}_m(H(E) \times \mathbb{R}^e)$ and how $\overline{\mathcal{G}}$ acts on it. To this end, we introduce the subspace $V_m \subseteq \mathrm{Pol}_{nm}(E \times \mathbb{R}^e)$ defined by

$$V_m = \mathrm{span}\Big\{p(x)y^{\boldsymbol{\beta}} : p \in \mathrm{Pol}(E),\ \deg p \leq n\big(m - |\boldsymbol{\beta}|\big),\ |\boldsymbol{\beta}| \leq m\Big\}. \tag{29}$$

Extending Lemma 3, we have the following result.

Lemma 4. *For every $m \in \mathbb{N}$, the pullback $\varphi^* : \mathrm{Pol}_m(H(E) \times \mathbb{R}^e) \to V_m$ is a linear isomorphism with inverse ψ^*.*

Here is our second main result of this section.

Theorem 5. *Assume* (23). *Then either of the following statements is equivalent to* (28)*:*

(i) *The operator $\mathcal{G}$ maps the space V_m to itself for each $m \in \mathbb{N}$.*

(ii) *$\mathcal{G}(y^{\boldsymbol{\beta}})$ and $\Gamma(x^{\boldsymbol{\alpha}}, y^{\boldsymbol{\beta}})$ lie in V_m whenever $|\boldsymbol{\alpha}| \leq n(m - |\boldsymbol{\beta}|)$ and $|\boldsymbol{\beta}| \leq m$, where Γ denotes the carré-du-champ operator related to $\mathcal{G}$ (see Appendix A).*

In either case, for every $m \in \mathbb{N}$, the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Pol}_m(H(E) \times \mathbb{R}^e) & \xrightarrow{\overline{\mathcal{G}}} & \mathrm{Pol}_m(H(E) \times \mathbb{R}^e) \\ {\scriptstyle \varphi^*}\big\downarrow & & \big\uparrow{\scriptstyle \psi^*} \\ V_m & \xrightarrow{\quad \mathcal{G} \quad} & V_m \end{array} \tag{30}$$

Remark 4. Note that for $n = 1$ and $H(x) = x$, we have $\overline{Z}_t = Z_t$ and $V_m = \mathrm{Pol}_m(E \times \mathbb{R}^e)$, in which case Theorem 5 simply recovers the definition of Z_t being polynomial on $E \times \mathbb{R}^e$.

As an application of Theorem 5, we can infer the action of $\overline{\mathscr{G}}$ on $\mathrm{Pol}_m(H(E)\times\mathbb{R}^e)$. This allows us to compute conditional moments of Y_T using the moment formula in Theorem 1. Assume (23) and (28). Let $1+\overline{N}=\dim \mathrm{Pol}_{mn}(E)$, and extend the basis of $\mathrm{Pol}_n(E)$ to a basis of $\mathrm{Pol}_{mn}(E)$ as in (19). This induces a basis of V_m of the form

$$v_i(x,y)=h_j(x)y^{\beta},\quad \deg h_j\le n(m-|\beta|),\quad |\beta|\le m,\quad i=0,\dots,M,$$

where $1+M=\dim V_m$. In view of the commuting diagram (30), this induces a basis $\overline{v}_i=\psi^* v_i$, $i=0,\dots,M$, of $\mathrm{Pol}_m(H(E)\times\mathbb{R}^e)$. Let G be the $(1+M)\times(1+M)$ matrix representing $\mathscr{G}$ on V_m according to (6)–(8) with v_i in lieu of h_i, which can be determined using symbolic calculus applied to $\mathscr{G}v_i(x,y)$ for $v_i(x,y)=h_j(x)y^{\beta}$. Then G is the matrix representing $\overline{\mathscr{G}}$ on $\mathrm{Pol}_m(H(E)\times\mathbb{R}^e)$, and Theorem 1 can be readily applied to compute all conditional moments of $\overline{Z}_T$ up to order m.

The following corollary is useful for applications because it helps to reduce the dimension for moment computations. For example, if we only need the conditional moments of Y_{1T}, this does not involve the remaining components Y_{it} for $i\neq 1$.

Corollary 3. *Assume (23) and (28). Let $P:\mathbb{R}^e\to\mathbb{R}^{e'}$ be a linear map for some $e'\in\mathbb{N}$. Then $Z'_t=(X_t,PY_t)$ satisfies (23) and (28) in lieu of $Z_t=(X_t,Y_t)$ with dimension e replaced by e'.*

6. Lévy Time Change

The class of polynomial jump diffusions is shown to be invariant under time change by a Lévy subordinator. Let X_t be a polynomial jump diffusion on $E\subseteq\mathbb{R}^d$ with extended generator $\mathscr{G}$. Let Z_t be an independent nondecreasing Lévy process (subordinator) with Lévy measure $\nu^Z(d\zeta)$ and drift $b^Z\ge 0$ so that its generator is

$$\mathscr{G}^Z f(z)=b^Z f'(z)+\int_0^\infty (f(z+\zeta)-f(z))\nu^Z(d\zeta).$$

The law of Z_t is denoted by $\mu^t(dz)$. A heuristic argument suggests that the time-changed process $\widetilde{X}_t=X_{Z_t}$ is again a polynomial jump diffusion on E. Indeed, the moment formula Theorem 1, the independence of X_t and Z_t, and the Lévy property of Z_t give, for any polynomial $f(x)=(1,H(x)^\top)\vec{f}$ in $\mathrm{Pol}_n(E)$,

$$\begin{aligned}\mathbb{E}\left[f\left(\widetilde{X}_T\right)\mid \widetilde{X}_t\right]&=\mathbb{E}[\mathbb{E}[f(X_{Z_T})|X_{Z_t},Z_t,Z_T]|X_{Z_t}]=\mathbb{E}\left[\left(1,H(X_{Z_t})^\top\right)\mathrm{e}^{(Z_T-Z_t)G}\vec{f}\,\middle|X_{Z_t}\right]\\&=\left(1,H(X_{Z_t})^\top\right)\int_0^\infty \mathrm{e}^{zG}\mu^{T-t}(dz)\vec{f}=\left(1,H\left(\widetilde{X}_t\right)^\top\right)\mathrm{e}^{(T-t)\widetilde{G}}\vec{f},\end{aligned}$$

where the matrix $\widetilde{G}$ is given in (37), subject to $\mu^t(dz)$-integrability conditions. Hence, $\widetilde{X}_t$ satisfies the moment formula. However, it turns out to be surprisingly difficult, if not impossible, to prove without any further assumptions that $\widetilde{X}_t$ is a jump diffusion.[5] Assuming Markovianity, we can prove the following result.

Theorem 6. *Assume that X_t is a Feller process with transition kernel $p_t(x,dy)$, the domain of its generator contains $C_c^\infty(E)$, and the generator coincides with $\mathscr{G}$ on $C_c^\infty(E)$. Assume also that the Lévy measure $\nu^Z(d\zeta)$ admits exponential moments,*

$$\int_1^\infty e^{\zeta\lambda}\nu^Z(d\zeta)<\infty, \tag{31}$$

for any λ from the set of real parts of eigenvalues of $\mathscr{G}$ restricted to $\mathrm{Pol}(E)$. Then the time-changed process $\widetilde{X}_t=X_{Z_t}$ is a polynomial jump diffusion on E and a Feller process with transition kernel

$$\widetilde{p}_t(x,dy)=\int_0^\infty p_z(x,dy)\mu^t(dz) \tag{32}$$

with respect to the usual augmentation $\widetilde{\mathscr{F}}_t$ of its natural filtration. Its extended generator

$$\begin{aligned}\widetilde{\mathscr{G}}f(x)&=\frac{1}{2}\mathrm{Tr}\left(\widetilde{a}(x)\nabla^2 f(x)\right)+\widetilde{b}(x)^\top\nabla f(x)\\&\quad+\int_{\mathbb{R}^d}\left(f(x+\xi)-f(x)-\xi^\top\nabla f(x)\right)\widetilde{\nu}(x,d\xi)\end{aligned} \tag{33}$$

is given by

$$\widetilde{a}(x) = b^Z a(x), \tag{34}$$

$$\widetilde{b}(x) = b^Z b(x) + \int_0^\infty \int_E (y - x) p_\zeta(x, dy) \nu^Z(d\zeta), \tag{35}$$

$$\widetilde{\nu}(x, d\xi) = b^Z \nu(x, d\xi) + \int_0^\infty 1_{\{\xi \neq 0\}} p_\zeta(x, x + d\xi) \nu^Z(d\zeta). \tag{36}$$

The matrix representations G and $\widetilde{G}$ of $\mathscr{G}$ and $\widetilde{\mathscr{G}}$ on $\mathrm{Pol}_n(E)$ *are related by*

$$\widetilde{G} = b^Z G + \int_0^\infty \left(\mathrm{e}^{\zeta G} - \mathrm{id}\right) \nu^Z(d\zeta) \quad \textit{and} \quad \mathrm{e}^{t\widetilde{G}} = \int_0^\infty \mathrm{e}^{zG} \mu^t(dz). \tag{37}$$

Remark 5. It is shown in the proof of Theorem 6 that $p_t(x, A)$ and $p_t(x, x + A)$ are jointly measurable in $(t, x) \in [0, \infty) \times E$ for all measurable $A \subseteq \mathbb{R}^d$ with the obvious extension $p_t(x, A) = p_t(x, A \cap E)$. Hence, (32) and (36) specify well-defined transition kernels from $\mathbb{R}^d$ into $\mathbb{R}^d$, defined to be zero for $x \notin E$.

Remark 6. Theorem 1 yields that

$$\int_E (y - x)^\top p_\zeta(x, dy) = \left(1, H(x)^\top\right) \left(\mathrm{e}^{\zeta G} - \mathrm{id}\right) M,$$

where G is the matrix representation of $\mathscr{G}$ on $\mathrm{Pol}_1(E)$, and M is the matrix whose ith column is the corresponding vector representation of x_i in $\mathrm{Pol}_1(E)$. In view of (31), it thus follows that (35) specifies a well-defined first-order polynomial drift function.

Remark 7. Sato (1999, theorem 25.3) states that condition (31) is equivalent to $\mathbb{E}[\mathrm{e}^{\lambda Z_t}] = \int_0^\infty \mathrm{e}^{z\lambda} \mu^t(dz) < \infty$ for all $t \geq 0$. Hence, the integrals in (37) are well defined. If E is compact, then all eigenvalues of $\mathscr{G}$ restricted to $\mathrm{Pol}(E)$ have a nonpositive real part such that (31) trivially holds.

The following example shows that the affine property is not invariant with respect to Lévy time change.

Example 7. Consider the Ornstein–Uhlenbeck process $dX_t = -\kappa X_t\, dt + \sigma\, dW_t$, which is an affine Feller process with normal transition kernel $p_t(x, dy)$ with mean $\mathrm{e}^{-\kappa t} x$ and variance $\frac{\sigma^2}{2\kappa}(1 - \mathrm{e}^{-2\kappa t})$. Now consider an independent Poisson subordinator Z_t with $\beta^Z = 0$ and $\nu^Z(d\zeta) = \delta_{\{1\}}(d\zeta)$. According to Theorem 6, the Lévy time changed jump diffusion $\widetilde{X}_t = X_{Z_t}$ is polynomial. But $\widetilde{X}_t$ is not affine if $\kappa \neq 0$. Indeed, straightforward integration shows that

$$\widetilde{\mathscr{G}} \mathrm{e}^{ux} = \int_E (\mathrm{e}^{uy} - \mathrm{e}^{ux}) p_1(x, dy) = \left(\mathrm{e}^{(\mathrm{e}^{-\kappa t} - 1)ux + C(t)} - 1\right) \mathrm{e}^{ux}$$

for $C(t) = \frac{\sigma^2 u^2}{4\kappa}(1 - \mathrm{e}^{-2\kappa t})$, which is not of the form (11).

Applications of Lévy time-changed Ornstein–Uhlenbeck processes as in Example 7 to commodity derivatives pricing are given in Li and Linetsky (2014).

7. Polynomial Expansions

We study the generic pricing problem in finance, which can be cast as follows. Let X_t be a polynomial jump diffusion on state space $E \subseteq \mathbb{R}^d$. Pricing a possibly path-dependent option boils down to computing the conditional expectation

$$I_{t_0} = \mathbb{E}\left[F(\boldsymbol{X}) \middle| \mathscr{F}_{t_0}\right],$$

where $\boldsymbol{X} = P(X_{t_1}, \ldots, X_{t_n})$ for some linear map $P : \mathbb{R}^{d \times n} \to \mathbb{R}^m$ with $m \leq d \times n$ for some time partition $0 \leq t_0 < t_1 < \cdots < t_n$ and some discounted payoff function $F(\boldsymbol{x})$ on $\mathbb{R}^m$. For example, $\boldsymbol{X} = (X_{1,t_1}, \ldots, X_{1,t_n})$ may only depend on the first component of X_t so that $m = n$. In the following, we present a method that extends the approach in Filipović et al. (2013).

We denote by $g(\boldsymbol{dx})$ the regular conditional distribution of $\boldsymbol{X}$ on $\mathbb{R}^m$ given $\mathscr{F}_{t_0}$. We let $w(\boldsymbol{dx})$ be an auxiliary probability kernel from $(\Omega, \mathscr{F}_{t_0})$ to $\mathbb{R}^m$ that dominates $g(\boldsymbol{dx})$ with a likelihood ratio function denoted by $\ell(\boldsymbol{x})$ such that

$$g(\boldsymbol{dx}) = \ell(\boldsymbol{x}) w(\boldsymbol{dx}). \tag{38}$$

We define the Hilbert space L^2_w as the set of (equivalence classes of) measurable real functions $f(\boldsymbol{x})$ on $\mathbb{R}^m$ with finite L^2_w-norm given by

$$\| f \|_w^2 = \int_{\mathbb{R}^m} f(\boldsymbol{x})^2 w(\boldsymbol{dx}).$$

The corresponding scalar product is $\langle f, h \rangle_w = \int_{\mathbb{R}^m} f(\boldsymbol{x}) h(\boldsymbol{x}) w(\boldsymbol{dx})$. We assume that L^2_w contains all polynomials on $\mathbb{R}^m$,

$$\mathrm{Pol}(\mathbb{R}^m) \subset L^2_w, \tag{39}$$

and let $q_0(\boldsymbol{x}) = 1, q_1(\boldsymbol{x}), q_2(\boldsymbol{x}), \ldots$ form an orthonormal basis of polynomials spanning the closure $\overline{\mathrm{Pol}(\mathbb{R}^m)}$ in L^2_w. We also assume that the likelihood ratio function lies in L^2_w,

$$\ell \in L^2_w. \tag{40}$$

As a consequence, its Fourier coefficients

$$\ell_k = \langle \ell, q_k \rangle_w = \int_{\mathbb{R}^m} q_k(\boldsymbol{x}) \ell(\boldsymbol{x}) w(\boldsymbol{dx}) = \mathbb{E}\left[q_k(\boldsymbol{X}) \middle| \mathscr{F}_{t_0} \right] \tag{41}$$

are given in closed form by iterating the moment formula in Theorem 1.

We finally assume that the discounted payoff function lies in L^2_w,

$$F \in L^2_w. \tag{42}$$

We denote by $\bar{F}$ the orthogonal projection of F onto $\overline{\mathrm{Pol}(\mathbb{R}^m)}$ in L^2_w. Elementary functional analysis then gives that the price approximation $\bar{I}_{t_0} = \mathbb{E}[\bar{F}(\boldsymbol{X}) \mid \mathscr{F}_{\boldsymbol{t_0}}]$ equals

$$\bar{I}_{t_0} = \int_{\mathbb{R}^m} \bar{F}(\boldsymbol{x}) g(\boldsymbol{dx}) = \langle \bar{F}, \ell \rangle_w = \sum_{k \geq 0} F_k \ell_k, \tag{43}$$

with Fourier coefficients given by

$$F_k = \langle \bar{F}, q_k \rangle_w = \langle F, q_k \rangle_w = \int_{\mathbb{R}^m} F(\boldsymbol{x}) q_k(\boldsymbol{x}) w(\boldsymbol{dx}). \tag{44}$$

The approximation equals the true price $\bar{I}_{t_0} = I_{t_0}$ if the projection $\bar{F}$ equals F in L^2_w. This statement is of more theoretical than practical interest for two reasons. First, depending on the choice of the auxiliary kernel $w(\boldsymbol{dx})$, we have that $\mathrm{Pol}(\mathbb{R}^m)$ is dense in L^2_w such that $\bar{F} = F$ holds anyway. Second, in practice, we approximate the price by truncating the series in (43) at some finite order K,

$$I_{t_0}^{(K)} = \sum_{k=0}^{K} F_k \ell_k, \tag{45}$$

such that the pricing error is $\epsilon^{(K)} = I_{t_0} - I_{t_0}^{(K)}$. Although it is good to know that $\epsilon^{(K)} \to 0$ asymptotically as $K \to \infty$ if $\bar{F} = F$ in L^2_w, the hard work consists in controlling the error $\epsilon^{(K)}$ for finite K.

Computation of the approximation $I_{t_0}^{(K)}$ can be cast as numerical integration over $\mathbb{R}^m$,

$$I_{t_0}^{(K)} = \sum_{k=0}^{K} \langle F, \ell_k q_k \rangle_w = \int_{\mathbb{R}^m} F(\boldsymbol{x}) \ell^{(K)}(\boldsymbol{x}) w(\boldsymbol{dx}), \tag{46}$$

for the likelihood ratio approximation

$$\ell^{(K)}(\boldsymbol{x}) = \sum_{k=0}^{K} \ell_k q_k(\boldsymbol{x}).$$

Note that the approximation $g^{(K)}(d\boldsymbol{x}) = \ell^{(K)}(\boldsymbol{x})w(d\boldsymbol{x})$ of the measure $g(d\boldsymbol{x})$ integrates to one, $g^{(K)}(\mathbb{R}^m) = 1$, because q_k is orthogonal to $q_0 = 1$ in L^2_w for $k \geq 1$. But $g^{(K)}(d\boldsymbol{x})$ is only a signed measure in general.

How do we choose the auxiliary probability kernel $w(d\boldsymbol{x})$? Necessarily, $w(d\boldsymbol{x})$ has to satisfy conditions (38)–(40) and (42), whereof (40) is arguably the most difficult to verify in practice.[6] The following criteria indicate desirable further properties of $w(d\boldsymbol{x})$ from a computational point of view:

1. $w(d\boldsymbol{x})$ admits closed-form $\mathscr{F}_{t_0}$-conditional moments. Then we obtain the orthonormal polynomials $q_0(\boldsymbol{x}) = 1, q_1(\boldsymbol{x}), q_2(\boldsymbol{x}), \dots$ in L^2_w in closed form and without numerical integration. Indeed, we let $\tilde{q}_0(\boldsymbol{x}) = 1, \tilde{q}_1(\boldsymbol{x}), \tilde{q}_2(\boldsymbol{x}), \dots$ be any basis of $\mathrm{Pol}(\mathbb{R}^m)$. We obtain all scalar products $\langle \tilde{q}_k, \tilde{q}_l \rangle_w$ in terms of the $\mathscr{F}_{t_0}$-conditional moments of $w(d\boldsymbol{x})$. This allows us to perform an exact Gram–Schmidt orthonormalization, and we obtain an orthonormal basis of $\overline{\mathrm{Pol}(\mathbb{R}^m)}$ in L^2_w in closed form.
2. There exist closed-form formulas for the Fourier coefficients F_k, and no numerical integration is needed for the computation of $I^{(K)}_{t_0}$. See, for example, the option pricing in Ackerer et al. (2018) and Ackerer and Filipović (2020b). Otherwise, one has to numerically integrate (44) or, equivalently, (46) with respect to $w(d\boldsymbol{x})$. This should then at least be amendable by cubature or Monte Carlo methods.
3. $w(d\boldsymbol{x})$ matches the moments of $g(d\boldsymbol{x})$ of order n and less, $\int_{\mathbb{R}^m} \boldsymbol{x}^{\boldsymbol{\alpha}} w(d\boldsymbol{x}) = \int_{\mathbb{R}^m} \boldsymbol{x}^{\boldsymbol{\alpha}} g(d\boldsymbol{x})$, for all $|\boldsymbol{\alpha}| \leq n$. We already know this always holds for $\boldsymbol{\alpha} = \boldsymbol{0}$ so that $\ell_0 = 1$. Then $\ell_k = \int_{\mathbb{R}^m} q_k(\boldsymbol{x}) g(d\boldsymbol{x}) = \int_{\mathbb{R}^m} q_k(\boldsymbol{x}) w(d\boldsymbol{x}) = \langle 1, q_k \rangle_w = 0$, for all $k \geq 1$ with $\deg q_k \leq n$. This can improve the convergence of the approximation (45). A numerically efficient method for constructing a probability density matching the first n moments in the univariate case, $m = 1$, is presented by Filipović and Willems (2019, section 3.2).

8. Polynomial Asset Pricing Models

Building on Section 5, we develop a polynomial framework that accommodates a large class of asset pricing models. It nests all affine asset pricing models, subject to integrability of jumps. First, we introduce the financial market model with excess log returns that are conditional Lévy based on a polynomial jump-diffusion factor process. Then we discuss option pricing and equivalent measure change and provide closed-form expressions for the return volatility, vol of vol, and leverage.

8.1. Conditional Lévy Excess Log Returns

We consider a financial market with e primary assets with price processes given as

$$S_{i,t} = S_{i,0}\, \mathrm{e}^{\int_0^t r_s ds + Y_{i,t}},$$

where r_t is the risk-free rate, and $Y_t = (Y_{1,t}, \dots, Y_{e,t})$ are the excess log return processes with $Y_0 = 0$. We let X_t be a polynomial jump diffusion on some state space $E \subseteq \mathbb{R}^d$ such that $Z_t = (X_t, Y_t)$ is a an $E \times \mathbb{R}^e$-valued jump diffusion with extended generator $\mathscr{G}$ of the form (20). That is, Y_t is a conditional Lévy process. We follow the conventions (21) and (22) and assume that

$$\int_{\mathbb{R}^e} (\mathrm{e}^{\eta_i} - 1 - \eta_i)\nu^Y(x, d\eta) < \infty, \quad \text{for all } x \in E,\ i = 1, \dots, e. \tag{47}$$

It then follows that the price processes are special semimartingales with the decomposition

$$\frac{dS_{i,t}}{S_{i,t-}} = (r_t + \epsilon_i(X_t))\,dt + dY^c_{i,t} + \int_{\mathbb{R}^e} (\mathrm{e}^{\eta_i} - 1)\big(\mu^Y(d\eta, dt) - \nu^Y(X_{t-}, d\eta)\big),$$

where Y^c_t denotes the continuous martingale part of Y_t, $\mu^Y(d\eta, dt)$ is the integer-valued random measure associated to the jumps of Y_t, and the excess rates of return are given by

$$\epsilon_i(x) = b^Y_i(x) + \frac{1}{2} a^Y_{ii}(x) + \int_{\mathbb{R}^e} (\mathrm{e}^{\eta_i} - 1 - \eta_i)\nu^Y(x, d\eta).$$

We have not specified the measure $\mathbb{P}$ yet. For derivatives pricing, we assume that $\mathbb{P}$ is a risk-neutral measure so that the discounted price processes $\mathrm{e}^{-\int_0^t r_s ds} S_{i,t}$ are local martingales. This is achieved by setting $\epsilon_i(x) = 0$ for all $x \in E$ and $i = 1, \dots, e$. In view of Lemma 1, $\epsilon_i(x)$ cannot be zero if $Z_t = (X_t, Y_t)$ is a polynomial jump diffusion, other than affine, in general. The next result shows how to embed Z_t into a higher-dimensional polynomial jump diffusion such that $\epsilon_i(x) = 0$. It follows immediately from Theorem 4, so we omit its proof.

Lemma 5. *Let $n \in \mathbb{N}$. Assume that (47) holds and*

$$\begin{aligned} a^Y &\in \mathrm{Pol}_n(E), \\ a^{XY} &\in \mathrm{Pol}_{1+n}(E), \\ \int_{\mathbb{R}^{d+e}} \xi^\alpha f(\eta)\,\nu(\cdot, d\xi \times d\eta) &\in \mathrm{Pol}_{|\alpha|+n}(E), \quad \text{for all } |\alpha| \geq 0, \end{aligned} \tag{48}$$

for all functions $f: \mathbb{R}^e \to \mathbb{R}$ for which the integral is finite. Then one can choose

$$b_i^Y = -\frac{1}{2}a_{ii}^Y - \int_{\mathbb{R}^e} (\mathrm{e}^{\eta_i} - 1 - \eta_i)\nu^Y(\cdot, d\eta) \in \mathrm{Pol}_n(E),$$

so that $\epsilon_i(x) = 0$ on E and $\mathbb{P}$ is a risk-neutral measure, and $\overline{Z}_t = (H(X_t), Y_t)$ is a polynomial jump diffusion on $H(E) \times \mathbb{R}^e$.

8.2. Option Pricing

To illustrate how to price options on the primary assets $S_{i,t}$, we now assume that $\mathbb{P}$ is a risk-neutral measure. Consider first a European call option written on asset $S_{i,t}$ with strike K and maturity T. Its price at time $t = 0$ is given by

$$\mathbb{E}\left[\mathrm{e}^{-\int_0^T r_s ds}(S_{i,T} - K)^+ \middle| \mathscr{F}_0\right] = \mathbb{E}\left[\left(S_{i,0}\mathrm{e}^{Y_{i,T}} - K\mathrm{e}^{-\int_0^T r_s ds}\right)^+ \middle| \mathscr{F}_0\right].$$

If the risk-free rates r_t are deterministic, the pricing operation reduces to computing expectations of the form $\mathbb{E}[(\mathrm{e}^{Y_{i,T}} - c)^+ \mid \mathscr{F}_0]$, where c is a constant. Pricing path-dependent derivatives boils down to computing conditional expectations of the form $\mathbb{E}[F(Y_{i,t_1}, \ldots, Y_{i,t_n}) \mid \mathscr{F}_0]$, as discussed in Section 5 for $\overline{Z}_t$ in lieu of X_t.

8.3. Equivalent Measure Change

In general, $\mathbb{P}$ may be any measure, such as the real-world measure, as long as there exists a locally equivalent risk-neutral measure $\mathbb{Q}$ such that the discounted price processes are $\mathbb{Q}$-local martingales. This is a standard condition on asset pricing models to be arbitrage-free; see Harrison and Pliska (1981). In case $\mathbb{P}$ is not a risk-neutral measure, we specify the market price of risk such that X_t is a polynomial jump diffusion under the corresponding risk-neutral measure. Thereto we fix a finite time horizon T and consider equivalent probability measures $\mathbb{Q} \sim \mathbb{P}$ under which Z_t, $t \in [0, T]$, is a jump diffusion with diffusion, drift, and jump coefficients $a^{\mathbb{Q}}(x)$, $b^{\mathbb{Q}}(x)$, and $\nu^{\mathbb{Q}}(x, d\zeta)$ given in terms of the $\mathbb{P}$-coefficients $a(x)$, $b(x)$, and $\nu(x, d\zeta)$ as

$$\begin{aligned} a(x) &= a^{\mathbb{Q}}(x), \\ b(x) &= b^{\mathbb{Q}}(x) + a(x)\phi(x) + \int_{\mathbb{R}^{d+e}} \big(1 - 1/\psi(x,\zeta)\big)\zeta\,\nu(x, d\zeta), \\ \nu(x, d\zeta) &= \psi(x,\zeta)\,\nu^{\mathbb{Q}}(x, d\zeta), \end{aligned} \tag{49}$$

for some $\mathbb{R}^{d+e}$-valued function $\phi(x)$ and real function $\psi(x,\zeta) > 0$. Here $\phi(x)$ is the market price of diffusion risk, and $\psi(x,\zeta)$ is the market price of risk of the jump event of size ζ associated with Z_t.[7] And $\mathbb{Q}$ is a risk-neutral measure if $\epsilon_i^{\mathbb{Q}}(x) = 0$ for all $x \in E$ and $i = 1, \ldots, e$.

In order that the change of measure is well defined, we assume that

$$\mathscr{E}_t(L), t \in [0, T], \text{is a positive martingale} \tag{50}$$

for

$$dL_t = -\phi(X_t)^\top dZ_t^c - \int_{\mathbb{R}^{d+e}} \big(1 - 1/\psi(X_{t-},\zeta)\big)\Big(\mu^Z(d\zeta, dt) - \nu(X_{t-}, d\zeta)dt\Big)$$

with $L_0 = 0$, where Z_t^c is the continuous martingale part of Z_t, and $\mu^Z(d\zeta, dt)$ denotes the integer-valued random measure associated with the jumps of Z_t.[8] Then $d\mathbb{Q}/d\mathbb{P} = \mathscr{E}_T(L)$ defines an equivalent probability measure $\mathbb{Q} \sim \mathbb{P}$. We also assume that

$$\int_0^T \int_{\mathbb{R}^{d+e}} \big(\|\zeta\|^2 \wedge \|\zeta\|\big)/\psi(X_t,\zeta)\,\nu(X_t, d\zeta)dt < \infty. \tag{51}$$

Girsanov's theorem then implies that Z_t, $t \in [0,T]$, is a jump diffusion with diffusion, drift, and jump coefficients $a^{\mathbb{Q}}(x)$, $b^{\mathbb{Q}}(x)$, and $\nu^{\mathbb{Q}}(x,d\zeta)$ given in (49); see Jacod and Shiryaev (2003, theorem III.3.24).

In addition to (50) and (51), assume that $\int_{\mathbb{R}^{d+e}} \|\zeta\|^k/\psi(x,\zeta)\,\nu(x,d\zeta) < \infty$ for all $x \in E$ and all $k \geq 2$. Let $\mathscr{G}^{\mathbb{Q}}$ denote the generator of Z_t under $\mathbb{Q}$ so that

$$\mathscr{G}^{\mathbb{Q}} f(z) = \mathscr{G} f(z) - \phi(x)^\top a(x)\nabla f(z) - \int_{\mathbb{R}^{d+e}} \big(f(z+\zeta) - f(z)\big(1 - 1/\psi(x,\zeta)\big)\nu(x,d\zeta).$$

Because $\mathscr{G}$ is well defined on $\mathrm{Pol}(E\times\mathbb{R}^e)$, Lemma A.2 implies that $\mathscr{G}^{\mathbb{Q}}$ is well defined on $\mathrm{Pol}(E\times\mathbb{R}^e)$. On a case-by-case basis, it is now straightforward to derive conditions from (49) and Lemma 1 such that X_t, $t \in [0,T]$, is a polynomial jump diffusion under $\mathbb{Q}$ and such that Lemma 5 applies so that $\mathbb{Q}$ is a risk-neutral measure and $\overline{Z}_t = (H(X_t), Y_t)$, $t \in [0,T]$, is a polynomial jump diffusion on $H(E)\times\mathbb{R}^e$ under $\mathbb{Q}$.

8.4. Volatility, Vol of Vol, and Leverage

The spot variance $v_i(X_{t-})$ of the ith excess log return $dY_{i,t}$ is defined as the time derivative of the predictable compensator of its quadratic variation $[Y_i, Y_i]_t$ (modified second characteristic) given by

$$v_i(x) = \Gamma\big(y_i, y_i\big)(x) = a^Y_{ii}(x) + \int_{\mathbb{R}^e} \eta_i^2 \nu^Y\big(x, d\eta\big),$$

where Γ denotes the carré-du-champ operator related to $\mathscr{G}$ (see Appendix A).

The volatility of $dY_{i,t}$ is defined as the square root of its spot variance $\mathrm{vol}_i(X_{t-}) = \sqrt{v_i(X_{t-})}$. The vol of vol is defined as the square root of the spot variance of the volatility process $\mathrm{vol}_i(X_t)$,

$$\mathrm{volvol}_i(X_{t-}) = \sqrt{\Gamma(\mathrm{vol}_i, \mathrm{vol}_i)(X_{t-})}.$$

The leverage effect refers to the generally negative correlation between $dY_{i,t}$ and changes of its spot variance $dv_i(X_t)$. It is captured by the time derivative of the predictable compensator of the quadratic covariation between $Y_{i,t}$ and $v_i(X_t)$,

$$\mathrm{lev}_i(X_{t-}) = \frac{\Gamma\big(y_i, v_i\big)(X_{t-})}{\sqrt{v_i(X_{t-})}\sqrt{\Gamma(v_i,v_i)(X_{t-})}}.$$

Note that in the presence of jumps, the jump measure $\nu^Y(x,d\eta)$ and thus the spot variance, volatility, vol of vol, and the leverage depend on the measure $\mathbb{P}$ so that we distinguish risk-neutral and real-world volatility, vol of vol, and leverage.[9]

9. Linear Volatility

We introduce a large class of polynomial asset pricing models based on the linear SDEs (9) and (10), extending Example 6. Throughout this section, we assume that $\mathbb{P}$ is a risk-neutral measure. Let W_t be a standard m-dimensional Brownian motion. Let $N(du,dt)$ be a Poisson random measure with compensator $F(du)dt$ on $U\times\mathbb{R}_+$ for some mark space U.[10] We assume that X_t is the E-valued solution of a linear SDE

$$dX_t = b^X(X_t)\,dt + \sigma^X(X_t)\,dW_t + \int_U \delta^X(X_{t-},u)(N(du,dt) - F(du)dt), \tag{52}$$

where drift, volatility, and jump size functions $b^X(x)$, $\sigma^X(x)$, and $\delta^X(x,u)$ are linear in x of the form (10) for some state space $E \subseteq \mathbb{R}^d$. We then specify the excess log returns by

$$dY_t = b^Y(X_t)\,dt + \sigma^Y(X_t)\,dW_t + \int_U \delta^Y(u)(N(du,dt) - F(du)dt), \tag{53}$$

with drift $b^Y(x)$ to be determined such that $\mathbb{P}$ is a risk-neutral measure. The volatility function is linear,

$$\sigma^Y(x) = \Gamma_0^Y + \sum_{i=1}^d x_i \Gamma_i^Y,$$

for parameters $\Gamma_i^Y \in \mathbb{R}^{e\times m}$, $i=0,\dots,d$. Jumps of Y_t are captured by the state-independent jump size function $\delta^Y(u)$ and can be isolated or simultaneous with jumps of X_t.[11] We assume that the push-forward $\delta_*^Y F(d\eta)$ of $F(du)$ under $\delta^Y(u)$ satisfies (47),

$$\int_U \left(e^{\delta_i^Y(u)} - 1 - \delta_i^Y(u)\right) F(du) < \infty, \quad \text{for all } i = 1,\dots, e. \tag{54}$$

The resulting coefficients $a(x)$, $b(x)$, and $\nu(x, d\xi \times d\eta)$ of the generator of the jump diffusion $Z_t = (X_t, Y_t)$ are functions of x given by

$$a = \begin{pmatrix} \sigma^X \sigma^{X\top} & \sigma^X \sigma^{Y\top} \\ \sigma^Y \sigma^{X\top} & \sigma^Y \sigma^{Y\top} \end{pmatrix} \in \mathrm{Pol}_2(E), \quad b = \begin{pmatrix} b^X \\ b^Y \end{pmatrix},$$

with $b^X \in \mathrm{Pol}_1(E)$ and

$$\int_{\mathbb{R}^{d+e}} f(\xi,\eta)\, \nu(x, d\xi \times d\eta) = \int_U f\left(\delta^X(x,u), \delta^Y(u)\right) F(du).$$

It follows by inspection that the assumptions of Lemma 5 are met for $n = 2$. Hence, we can set

$$b_i^Y = -\frac{1}{2} a_{ii}^Y - \int_U \left(e^{\delta_i^Y(u)} - 1 - \delta_i^Y(u)\right) F(du) \in \mathrm{Pol}_2(E),$$

so that $\mathbb{P}$ is a risk-neutral measure as desired, and $Z_t = (X_t, Y_t)$ satisfies properties (23)–(27) in Theorem 4 for $n = 2$.

Here is an example for the specification of the $N(du, dt)$-driven jumps of (X_t, Y_t).

Example 8. Let $U = U_0 \cup U_1 \cup \cdots \cup U_k$ for pairwise disjoint sets U_j such that $\lambda_j = F(U_j) < \infty$. Then $N_{j,t} = N(U_j \times [0,t))$ are independent Poisson processes with intensities λ_j for $j = 0,\dots,k$. Define the piecewise constant $\delta^X(x,u) = 0$ for $u \in U_0$ and $\delta^X(x,u) = \delta_0^X + \sum_{i=1}^d x_i \delta_{ij}^X$ for $u \in U_j$, $j = 1,\dots,k$, for some parameters $\delta_0^X, \delta_{ij}^X \in \mathbb{R}^d$. Then the $N(du, dt)$-driven jump term in (52) reads

$$\int_U \delta^X(X_{t-}, u)(N(du,dt) - F(du)dt) = \sum_{j=1}^k \left(\delta_0 + \sum_{i=1}^d X_{i,t-} \delta_{ij}^X\right)(dN_{j,t} - \lambda_j\, dt).$$

The $N(du, dt)$-driven jump term in (53) accordingly is of the form

$$\int_0^t \int_U \delta^Y(u)(N(du,ds) - F(du)ds) = \sum_{j=0}^k \left(\sum_{i\in\mathbb{N}} H_j^{(i)} 1_{\{i \le N_{j,t}\}} - \mathbb{E}\left[H_j^{(1)}\right]\lambda_j t\right),$$

where $\delta^Y(u)$ is any jump size function on U satisfying (54), $H_j^{(i)}$ are mutually independent $\mathbb{R}^e$-valued random variables, independent of the Poisson processes $N_{0,t},\dots,N_{k,t}$ and the Brownian motion W_t and such that $H_j^{(i)}$ has distribution given by the push forward of $F(u)/\lambda_j$ under the restriction $\delta^Y|_{U_j}$, that is, $\mathbb{E}[f(H_j^{(i)})] = \int_{U_j} f(\delta^Y(u)) F(du)/\lambda_j$, for $i \in \mathbb{N}$ and $j = 0,\dots,k$. Note that $N_{0,t}$ drives isolated jumps of Y_t. By contrast, $N_{j,t}$ drives isolated jumps of X_t if $H_j^{(i)} = 0$ for some $j = 1,\dots,k$.

Examples of linear diffusion volatility models include the extended Stein–Stein model (Stein and Stein 1991, Schöbel and Zhu 1999) and the extended Hull–White model (Hull and White 1987, Lions and Musiela 2007), which are discussed in detail in Ackerer and Filipović (2020b).

10. Conclusion

We have developed a mathematical framework for polynomial jump diffusions in a relaxed semimartingale context as opposed to a Markovian setup based on the pointwise action of the extended generator on polynomials only. We have established various features, including the moment formula and the invariance with respect to polynomial transformations and Lévy time change. We have also revisited affine jump diffusions, which are nested in the polynomial class, in that relaxed context. We have then constructed a large class of novel asset pricing models based on polynomial jump diffusions and presented a generic method for option pricing. Our results provide the basis for new asset pricing models. Several extensions are possible and left for future research. This includes discrete time and time-inhomogeneous polynomial jump diffusions.

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Appendix A. Carré-du-Champ Operator
Let X_t be a jump diffusion with extended generator $\mathscr{G}$ on $\mathbb{R}^d$ of the form (2). An object closely related to $\mathscr{G}$ is the carré-du-champ operator, which is a bilinear operator Γ given by

$$\Gamma(f,g)(x) = \mathscr{G}(fg)(x) - f(x)\mathscr{G}g(x) - g(x)\mathscr{G}f(x) \tag{A.1}$$

for any C^2 functions $f(x)$ and $g(x)$ on $\mathbb{R}^d$ such that

$$\int_{\mathbb{R}^d} (h(x+\xi) - h(x))^2 \nu(x, d\xi) < \infty, \text{ for all } x \in \mathbb{R}^d \text{ for } h = f, g.$$

Using the product rule, one can express $\Gamma(f,g)$ in terms of $a(x)$ and $\nu(x, d\xi)$ as

$$\Gamma(f,g)(x) = \nabla f(x)^\top a(x) \nabla g(x) + \int_{\mathbb{R}^d} \big(f(x+\xi) - f(x)\big)\big(g(x+\xi) - g(x)\big)\nu(x, d\xi). \tag{A.2}$$

In particular, $\Gamma(f,f) \geq 0$. Just as the extended generator captures the drift of a process $f(X_t)$, the carré-du-champ operator gives information about the quadratic variation of $f(X_t)$.

Lemma A.1. *The predictable compensator of the quadratic covariation $[f(X), g(X)]_t$ is given by*

$$\int_0^t \Gamma(f,g)(X_s)\, ds$$

for any C^2 functions $f(x)$ and $g(x)$ on $\mathbb{R}^d$ such that

$$\int_0^t \int_{\mathbb{R}^d} |f(X_s+\xi) - f(X_s)|\,|g(X_s+\xi) - g(X_s)| \nu(X_s, d\xi)\, ds < \infty. \tag{A.3}$$

Proof. In view of Jacod and Shiryaev (2003, theorem I.4.52) and the fact that $d\langle X^c, X^c\rangle_t = a(X_t)dt$, we have

$$[f(X), g(X)]_t = \int_0^t \nabla f(X_s)^\top a(X_s) \nabla g(X_s)\, ds + \sum_{s\leq t} \big(f(X_s) - f(X_{s-})\big)\big(g(X_s) - g(X_{s-})\big).$$

The result now follows from (A.2) and Jacod and Shiryaev (2003, theorem II.1.8). □

With the help of the carré-du-champ operator, we can qualify the property of $\mathscr{G}$ being well defined on $\mathrm{Pol}(E)$ for some state space $E \subseteq \mathbb{R}^d$.

Lemma A.2. *Assume that $\mathscr{G}$ is well defined on $\mathrm{Pol}(E)$. Let $f \in \mathrm{Pol}(\mathbb{R}^d)$ with $f(x) = 0$ on E. Then $a(x)\nabla f(x) = 0$ and $\int_{\mathbb{R}^d}(f(x+\xi) - f(x))^2 \nu(x, d\xi) = 0$ on E.*

Proof. We have $f(x)^2 = 0$ on E, so that (A.1) implies $\Gamma(f,f)(x) = 0$ on E. Because $a(x) \in \mathbb{S}^d_+$, the lemma follows from identity (A.2). □

Appendix B. Polynomial Transformations of Jump Diffusions
We let X_t be an E-valued jump diffusion with extended generator $\mathscr{G}$ of the form (2) for some state space $E \subseteq \mathbb{R}^d$. We show that an invertible polynomial transformation of X_t is again a jump diffusion, subject to technical conditions, and we identify its extended generator.

Lemma B.1. *Let $\varphi : \mathbb{R}^d \to \mathbb{R}^k$ be a polynomial map that admits a measurable inverse on E in the sense that there exists a measurable map $\psi : \mathbb{R}^k \to \mathbb{R}^d$ such that $\psi \circ \varphi = \mathrm{id}$ on E. Assume that $\mathscr{G}$ is well defined on $\mathrm{Pol}(E)$; that is, (4) and (5) hold. Assume also that the process $\overline{X}_t = \varphi(X_t)$ is a special semimartingale. Then $\overline{X}_t$ is a jump diffusion with extended generator $\overline{\mathscr{G}} = \psi^*\mathscr{G}\varphi^*$, which is well defined on $\mathrm{Pol}(\varphi(E))$.*[12] *It is of the form*

$$\overline{\mathscr{G}}f(\overline{x}) = \frac{1}{2}\mathrm{Tr}\big(\overline{a}(\overline{x})\nabla^2 f(\overline{x})\big) + \overline{b}(\overline{x})^\top \nabla f(\overline{x}) + \int_{\mathbb{R}^k} \Big(f(\overline{x}+\overline{\xi}) - f(\overline{x}) - \overline{\xi}^\top \nabla f(\overline{x})\Big)\overline{\nu}\Big(\overline{x}, d\overline{\xi}\Big),$$

where writing $\overline{x} = \varphi(x)$ and $\varphi_i(x)$ for the ith component of $\varphi(x)$,

$$\begin{aligned}\overline{a}_{ij}(\overline{x}) &= \nabla\varphi_i(x)a(x)\nabla\varphi_j(x)^\top,\\ \overline{b}_i(\overline{x}) &= \mathscr{G}\varphi_i(x),\\ \overline{\nu}(\overline{x}, A) &= \int_{\mathbb{R}^k} \mathbf{1}_A\big(\varphi(x+\xi) - \varphi(x)\big)\nu(x, d\xi).\end{aligned}$$

Proof. Because $\overline{X}_t$ is a special semimartingale, Kallsen (2006, proposition 3) in conjunction with a direct calculation shows that it is a jump diffusion with extended generator $\overline{\mathscr{G}} = \psi^*\mathscr{G}\varphi^*$ of the stated form. In particular, the jump measure satisfies

$$\int_{\mathbb{R}^k} \|\overline{\xi}\|^n\, \overline{\nu}(\overline{x}, d\overline{\xi}) = \int_{\mathbb{R}^d} \|\varphi(x+\xi) - \varphi(x)\|^n\, \nu(x, d\xi), \quad \text{where } \overline{x} = \varphi(x),$$

which is finite for every $\overline{x} \in \varphi(E)$ because of (4) and because $\varphi(x)$ is polynomial. Finally, if $f \in \mathrm{Pol}(\mathbb{R}^k)$ vanishes on $\varphi(E)$, then $\varphi^* f \in \mathrm{Pol}(\mathbb{R}^d)$ vanishes on E; therefore, $\mathscr{G}\varphi^* f$ also vanishes on E in view of (5), and hence, $\overline{\mathscr{G}}f$ vanishes on $\varphi(E)$. Thus, $\overline{\mathscr{G}}$ is well defined on $\mathrm{Pol}(\varphi(E))$. □

Appendix C. Locally Absolutely Continuous Measure Change

We sketch a situation that may occur in applications for the choice of an auxiliary probability kernel $w(\boldsymbol{dx})$ satisfying assumption (40).

Let $\mathbb{Q}$ be a probability measure that is equivalent to $\mathbb{P}$ on each $\mathscr{F}_t$ with Radon–Nikodym density D_t. We define $w(\boldsymbol{dx})$ as the $\mathbb{Q}$-regular conditional distribution of X given $\mathscr{F}_{t_0}$. Then (38) holds with likelihood ratio function given by the $\mathbb{Q}$-regular conditional distribution of D_{t_m}/D_{t_0} given $\mathscr{F}_{t_0} \vee \sigma(\boldsymbol{X})$,

$$\ell(\boldsymbol{x}) = \mathbb{E}_{\mathbb{Q}}\left[\frac{D_{t_m}}{D_{t_0}} \mid \mathscr{F}_{t_0}, \boldsymbol{X} = \boldsymbol{x}\right], \tag{C.1}$$

where we set $D_{t_m}/D_{t_0} = 0$ if $D_{t_0} = 0$. Indeed, let $f(\boldsymbol{x})$ be a bounded measurable function on $\mathbb{R}^m$. Taking conditional expectation gives

$$\begin{aligned}\int_{\mathbb{R}^m} f(\boldsymbol{x})g(\boldsymbol{dx}) &= \mathbb{E}_{\mathbb{P}}\left[f(\boldsymbol{X})\middle|\mathscr{F}_{t_0}\right] = \mathbb{E}_{\mathbb{Q}}\left[f(\boldsymbol{X})\frac{D_{t_m}}{D_{t_0}}\middle|\mathscr{F}_{t_0}\right],\\ &= \mathbb{E}_{\mathbb{Q}}\left[\mathbb{E}_{\mathbb{Q}}\left[f(\boldsymbol{X})\frac{D_{t_m}}{D_{t_0}}\middle|\mathscr{F}_{t_0} \vee \sigma(\boldsymbol{X})\right]\middle|\mathscr{F}_{t_0}\right],\\ &= \mathbb{E}_{\mathbb{Q}}\left[f(\boldsymbol{X})\ell(\boldsymbol{X})\middle|\mathscr{F}_{t_0}\right] = \int_{\mathbb{R}^m} f(\boldsymbol{x})\ell(\boldsymbol{x})w(\boldsymbol{dx}),\end{aligned}$$

which proves the claim (C.1).

The likelihood ratio function satisfies the estimate

$$\int_{\mathbb{R}^m} \ell(\boldsymbol{x})^2 w(\boldsymbol{dx}) = \mathbb{E}_{\mathbb{Q}}\left[\mathbb{E}_{\mathbb{Q}}\left[\frac{D_{t_m}}{D_{t_0}}\middle|\mathscr{F}_{t_0} \vee \sigma(\boldsymbol{X})\right]^2\middle|\mathscr{F}_{t_0}\right] \le \mathbb{E}_{\mathbb{Q}}\left[\left(\frac{D_{t_m}}{D_{t_0}}\right)^2 \mid \mathscr{F}_{t_0}\right]. \tag{C.2}$$

The bound in (C.2) is sharp to the extent that D_{t_m}/D_{t_0} could be $\mathscr{F}_{t_0} \vee \sigma(\boldsymbol{X})$-measurable such that we have equality in (C.2). The estimate (C.2) can be useful for verifying assumption (40).

In practice, we could approximate $w(\boldsymbol{dx})$, the $\mathbb{Q}$-regular conditional distribution of $\boldsymbol{X}$ given $\mathscr{F}_{t_0}$ by simulating X_t under $\mathbb{Q}$. Specifically, we would estimate the Fourier coefficients

$$F_k = \mathbb{E}_{\mathbb{Q}}\left[q_k(\boldsymbol{X})F(\boldsymbol{X})\middle|\mathscr{F}_{t_0}\right] \tag{C.3}$$

in (44) by (nested) Monte Carlo methods. This addresses property (ii). If we further assume that X_t is a polynomial jump diffusion with respect to $\mathbb{Q}$, then $w(\boldsymbol{dx})$ admits closed-form $\mathscr{F}_{t_0}$-conditional moments as indicated in property (i).

Appendix D. Proofs

This appendix contains the proofs of the lemmas and theorems in the main text.

D.1. Proof of Theorem 1

The proof of Theorem 1 builds on the following four lemmas.

Lemma D.1. *The local martingale property (3) holds for any C^2 function $f(x)$ on $\mathbb{R}^d$ satisfying*

$$V_t = \int_0^t \int_{\mathbb{R}^d} \big|f(X_s+\xi) - f(X_s) - \xi^\top \nabla f(X_s)\big|\nu(X_s, d\xi)\, ds < \infty. \tag{D.1}$$

Proof. Property (D.1) states that V_t is in $\mathscr{A}^+_{loc}$. The lemma now follows from Jacod and Shiryaev (2003, theorem ii.1.8 and proof of theorem II.2.42). □

For the rest of this section, we assume that $\mathscr{G}$ is polynomial on E, and we let $f \in \mathrm{Pol}_n(E)$. Then the process

$$M_t^f = f(X_t) - f(X_0) - \int_0^t \mathscr{G}f(X_s)\,ds$$

is well defined.

Lemma D.2. *M_t^f is a local martingale.*

Proof. In view of Lemma D.1, it is enough to show that (D.1) holds. But $W(x,\xi) = f(x+\xi) - f(x) - \xi^\top \nabla f(x)$ is a linear combination of monomials $x^\beta \xi^\gamma$ with $2 \le |\gamma| \le n$. Hence, $|W(x,\xi)| \le C(x)\big(\|\xi\|^2 + \|\xi\|^{2n}\big)$ for some polynomial $C(x)$. Now (D.1) follows from Lemma 1. □

Lemma D.3. *For any $k \in \mathbb{N}$, there is a finite constant C such that*

$$\mathbb{E}\left[1 + \|X_t\|^{2k} \middle| \mathscr{F}_0\right] \le \left(1 + \|X_0\|^{2k}\right) e^{Ct}, \quad t \ge 0.$$

Proof. We recall the argument in Cuchiero et al. (2012, theorem 2.10) or Filipović and Larsson (2016, lemma B.1). Let $f(x) = 1 + \|x\|^{2k}$, and let C be a finite constant such that $|\mathscr{G}f(x)| \le Cf(x)$ on E. Such a constant exists by the polynomial property of $\mathscr{G}$. Let $0 \le T_1 \le T_2 \le \cdots$ be a localizing sequence for the local martingale M_t^f (see Lemma D.2) such that $\|X_t\| \le m$ for $t < T_m$. Then

$$\mathbb{E}\left[f\left(X_{t\wedge T_m}\right)\middle|\mathscr{F}_0\right] = f(X_0) + \mathbb{E}\left[\int_0^{t\wedge T_m} \mathscr{G}f(X_s)\,ds \middle| \mathscr{F}_0\right] \le f(X_0) + C\int_0^t \mathbb{E}\left[f\left(X_{s\wedge T_m}\right)\middle|\mathscr{F}_0\right] ds.$$

Gronwall's inequality and Fatou's lemma now yield the result. □

Lemma D.4. *For any finite c, the process $M_t^f 1_{\{\|X_0\| \le c\}}$ is a martingale.*

Proof. Let c be a finite number. Then $N_t^f = M_t^f 1_{\{\|X_0\|\le c\}}$ is a local martingale by Lemma D.2 with quadratic variation $[N^f, N^f]_t = [f(X), f(X)]_t 1_{\{\|X_0\|\le c\}}$. We claim that its predictable compensator is given by

$$\left\langle N^f, N^f\right\rangle_t = \int_0^t \Gamma\big(f,f\big)(X_s)\,ds 1_{\{\|X_0\|\le c\}}.$$

Indeed, in view of Lemma A.1, the claim follows as soon as (A.3) holds for $g = f$. But $W(x,\xi) = (f(x+\xi) - f(x))^2$ is a linear combination of monomials $x^\beta\xi^\gamma$ with $2 \le |\gamma| \le 2n$. Hence, $|W(x,\xi)| \le C(x)\big(\|\xi\|^2 + \|\xi\|^{2n}\big)$ for some polynomial $C(x)$, and (A.3) follows from Lemma 1.

By (A.1), $\Gamma(f,f)(x)$ is a polynomial on E. Combining this with Lemma D.3, we infer that $\mathbb{E}[\langle N^f, N^f\rangle_t] < \infty$ for all $t \ge 0$, and hence, N_t^f is a square-integrable martingale. □

We now prove Theorem 1. Fix a finite c and $t \ge 0$. By Lemma D.4, the row vector-valued function $F(T) = \mathbb{E}[(1, H(X_T)^\top)1_{\{\|X_0\|\le c\}} \mid \mathscr{F}_t]$ satisfies for $T \ge t$

$$F(T) = \big(1, H(X_t)^\top\big)1_{\{\|X_0\|\le c\}} + \int_t^T \mathbb{E}\left[\mathscr{G}\big(1, H^\top\big)(X_s)1_{\{\|X_0\|\le c\}}\middle|\mathscr{F}_t\right] ds = F(t) + \int_t^T F(s)G\,ds.$$

Hence, $\mathbb{E}[(1, H(X_T)^\top) \mid \mathscr{F}_t]1_{\{\|X_0\|\le c\}} = F(T) = (1, H(X_t)^\top)\mathrm{e}^{(T-t)G}1_{\{\|X_0\|\le c\}}$. Theorem 1 now follows by letting $c \uparrow \infty$.

D.2. Proof of Lemma 2

We first assume that $0 \in E$ and that the affine span of E is all of $\mathbb{R}^d$.

Assume that $\mathscr{G}$ is affine. Straightforward calculations show that

$$\mathscr{G}\mathrm{e}^{u^\top x} = \left(\frac{1}{2}u^\top a(x)u + b(x)^\top u + \int_{\mathbb{R}^d}\left(e^{u^\top\xi} - 1 - u^\top\xi\right)\nu(x,d\xi)\right)\mathrm{e}^{u^\top x}$$

so that, by virtue of the assumed relation (11), we obtain

$$F(u) + R(u)^\top x = \frac{1}{2}u^\top a(x)u + b(x)^\top u + \int_{\mathbb{R}^d}\left(e^{u^\top\xi} - 1 - u^\top\xi\right)\nu(x,d\xi), \quad \text{for all } x \in E, u \in \mathrm{i}\mathbb{R}^d. \tag{D.2}$$

We claim that $F(u)$ and $R(u)$ are of the form (13). Because $0 \in E$, this is clear for $F(u)$, setting $a_0 = a(0)$, $b_0 = b(0)$, and $\nu_0(d\xi) = \nu(0, d\xi)$. Next, because the affine span of E is all of $\mathbb{R}^d$, there exist numbers $\lambda_1, \ldots, \lambda_d$ with $\sum_{k=1}^d \lambda_k = 1$ and points

$x^1, \dots, x^d \in E$ such that $\lambda_1 x^1 + \cdots + \lambda_d x^d = e_1$, the first canonical unit vector. Evaluating both sides of (D.2) at $x = x^k$, multiplying by λ_k, summing over k, and using the form of $F(u)$, it follows that $R_1(u)$ is of the form (13) with

$$a_1 = \sum_{k=1}^d \lambda_k a\left(x^k\right) - a_0, \qquad b_1 = \sum_{k=1}^d \lambda_k b\left(x^k\right) - b_0, \qquad \nu_1(d\xi) = \sum_{k=1}^d \lambda_k \nu\left(x^k, d\xi\right) - \nu_0(d\xi).$$

The same argument shows that $R_2(u), \dots, R_d(u)$ are also of the form (13).

It remains to prove (12). Given $F(u)$ and $R(u)$ just obtained, it is clear that taking $a(x)$, $b(x)$, $\nu(x, d\xi)$ and as in (12) is consistent with (D.2). Furthermore, for each fixed $x \in E$, knowing the right-hand side of (D.2) for all $u \in \mathrm{i}\mathbb{R}^d$ uniquely determines $a(x)$, $b(x)$, and $\nu(x, d\xi)$; see Jacod and Shiryaev (2003, lemma II.2.44). Thus (12) is, in fact, the only possibility, completing the proof of the forward direction.

For the converse, assume that $a(x)$, $b(x)$, and $\nu(x, d\xi)$ are of the form (12). A calculation then shows that $\mathscr{G}$ satisfies (11) with $F(u)$ and $R(u)$ given by (13) and thus is affine.

In the general case, in which either $0 \notin E$ or the affine span of E is not $\mathbb{R}^d$, we apply an invertible affine transformation $T : \mathbb{R}^d \to \mathbb{R}^d$ such that $0 \in T(E)$ and the affine span of $T(E)$ is $\mathbb{R}^{d'} \times \{0\}$ for some $d' \le d$. In these new coordinates, we set the corresponding a_i, b_i, and $\nu_i(d\xi)$ to zero for $i > d'$ and then transform back by T^{-1}. □

D.3. Proof of Theorem 2

Define the function $f(t, x) = \exp(\phi(T - t) + \psi(T - t)^\top x)$ and the complex-valued process $M_t = f(t, X_t)$. Then (15) yields $|M_t| \le 1$. Moreover, a calculation using (14) yields

$$\partial_t f(t, x) + \mathscr{G} f(t, x) = 0, \qquad 0 \le t \le T, \quad x \in E,$$

where $\mathscr{G}$ acts on the real and imaginary parts of $f(t, \cdot)$ separately. Thus, M_t is a martingale on $[0, T]$ with $M_T = \exp(u^\top X_T)$. The affine transform formula is now just the equality $M_t = \mathbb{E}[M_T \mid \mathscr{F}_t]$. □

D.4. Proof of Lemma 3

The proof of Lemma 3 builds on the following lemma.

Lemma D.5. *Any polynomial $p \in \mathrm{Pol}_{mn}(E)$ is of the form $p(x) = f(H(x))$ for some $f \in \mathrm{Pol}_m(H(E))$.*

Proof. It suffices to consider monomials $p(x) = x^{\boldsymbol{\alpha}}$ with $|\boldsymbol{\alpha}| \le mn$. It follows by inspection that $\boldsymbol{\alpha} = \boldsymbol{\alpha}_1 + \cdots + \boldsymbol{\alpha}_k$ for some multi-indices $\boldsymbol{\alpha}_i \in \mathbb{N}_0^d$ with $|\boldsymbol{\alpha}_i| \le n$. Thus, $x^{\boldsymbol{\alpha}_i} = f_i(H(x))$ on E for some linear polynomial $f_i \in \mathrm{Pol}_1(\mathbb{R}^N)$ for each i. We deduce that

$$p(x) = \prod_{i=1}^m x^{\boldsymbol{\alpha}_i} = f(H(x)) \quad \text{on } E,$$

where $f(\bar{x}) = \prod_{i=1}^m f_i(\bar{x})$ is of degree at most m. □

We now prove Lemma 3. If a function $f : \mathbb{R}^N \to \mathbb{R}$ vanishes on $H(E)$, then $H^* f(x) = f(H(x))$ vanishes on E. Thus, H^* is well defined as a map from $\mathrm{Pol}(H(E))$ to $\mathrm{Pol}(E)$, and it is linear with inverse L^*. It is clear that H^* maps $\mathrm{Pol}_m(H(E))$ to $\mathrm{Pol}_{mn}(E)$ for each $m \in \mathbb{N}$. To see that L^* maps $p \in \mathrm{Pol}_{mn}(E)$ to an element of $\mathrm{Pol}_m(H(E))$, observe that $p(x) = f(H(x))$ for some $f \in \mathrm{Pol}_m(H(E))$ by Lemma D.5 so that $L^* p(\bar{x}) = f(H(L(\bar{x}))) = f(\bar{x})$. This proves Lemma 3. □

D.5. Proof of Theorem 3

Because $\overline{X}_t$ is a special semimartingale by Lemma D.2, Lemma B.1 implies that $\overline{X}_t$ is an $H(E)$-valued jump diffusion with extended generator $\overline{\mathscr{G}} = L^* \mathscr{G} H^*$, which is well defined on $\mathrm{Pol}(H(E))$. Lemma 3 implies that $\overline{\mathscr{G}}$ is polynomial on $H(E)$, and the diagram (18) commutes. This completes the proof of Theorem 3.

D.6. Proof of Theorem 4

The proof of Theorem 4 builds on the following lemma.

Lemma D.6. *Assume (23). Then the augmented process $\overline{Z}_t = (H(X_t), Y_t)$ is a jump diffusion on $H(E) \times \mathbb{R}^e$ with extended generator $\overline{\mathscr{G}} = \psi^* \mathscr{G} \varphi^*$, and the operators $\mathscr{G}$ and $\overline{\mathscr{G}}$ are well defined on $\mathrm{Pol}(E \times \mathbb{R}^e)$ and $\mathrm{Pol}(H(E) \times \mathbb{R}^e)$, respectively.*

Proof. We first prove that $\mathscr{G}$ is well defined on $\mathrm{Pol}(E \times \mathbb{R}^e)$. Because of (23), we only need to verify (5). Let $f(x, y)$ be a polynomial that vanishes on $E \times \mathbb{R}^e$. Collecting the monomials in y yields the representation

$$f(x, y) = \sum_{\beta} p_{\beta}(x) y^{\beta}$$

for finitely many polynomials $p_\beta(x)$ on $\mathbb{R}^d$. For each fixed $x \in E$, $f(x,y)$ is the zero polynomial on $\mathbb{R}^e$, and hence, $p_\beta(x) = 0$ for all $\boldsymbol{\beta}$. Thus, we may suppose $f(x,y) = p(x)q(y)$, where $p(x)$ vanishes on E and $q(y) = y^\beta$. One has

$$\mathcal{G}(pq)(x,y) = p(x)\mathcal{G}q(x,y) + q(y)\mathcal{G}p(x,y) + \Gamma(p,q)(x,y).$$

The first term is zero for any $x \in E$. So is the second term because $\mathcal{G}p(x,y) = \mathcal{G}^X p(x)$ and $\mathcal{G}^X$ is well defined on Pol(E). For the third term, note that the carré-du-champ operator is bilinear and positive semidefinite and, hence, satisfies the Cauchy–Schwarz inequality. That is,

$$|\Gamma(p,q)(x,y)|^2 \le \Gamma(p,p)(x,y)\,\Gamma(q,q)(x,y).$$

But $\Gamma(p,p)(x,y) = \Gamma^X(p,p)(x) = 0$ because $\mathcal{G}$ is well defined on Pol(E). Thus, $\mathcal{G}(pq)(x,y) = 0$ on $E \times \mathbb{R}^e$, and we deduce that $\mathcal{G}$ is well defined on Pol$(E \times \mathbb{R}^e)$.

Next, because $H(X_t)$ is a special semimartingale by Lemma D.2 and because Y_t is a special semimartingale by assumption, $\overline{Z}_t$ is also a special semimartingale. Therefore, because $\mathcal{G}$ is well defined on Pol$(E \times \mathbb{R}^e)$, it follows from Lemma B.1 that $\overline{Z}_t$ is a jump diffusion with extended generator $\overline{\mathcal{G}} = \psi^*\mathcal{G}\varphi^*$, which is well defined on Pol$(H(E) \times \mathbb{R}^e)$. □

We now prove Theorem 4. Because of Lemma D.6, it remains to prove that (24)–(27) together imply (28) and, conversely, that (28) implies (24), (25), and (27) for $\boldsymbol{\alpha} = 0$. To do this, we make use of Theorem 5.

We first assume that (24)–(27) hold and prove property (i) in Theorem 5. Fix $m \in \mathbb{N}$, and consider any monomial $f(z) = f(x,y) = x^\alpha y^\beta$ with $|\boldsymbol{\alpha}| \le n(m - |\boldsymbol{\beta}|)$ and $|\boldsymbol{\beta}| \le m$, which then lies in V_m. It suffices to show that $\mathcal{G}f$ again lies in V_m. Let $\widetilde{a}^X(x)$, $\widetilde{a}^{XY}(x)$, and $\widetilde{a}^Y(x)$ denote the modified second characteristics of $\mathcal{G}$, that is,

$$\widetilde{a}^X(x) = a^X(x) + \int_{\mathbb{R}^d} \xi\xi^\top \nu^X(x,d\xi), \qquad \widetilde{a}^Y(x) = a^Y(x) + \int_{\mathbb{R}^{d+e}} \eta\eta^\top \nu(x,d\xi \times d\eta),$$
$$\widetilde{a}^{XY}(x) = a^{XY}(x) + \int_{\mathbb{R}^{d+e}} \xi\eta^\top \nu(x,d\xi \times d\eta).$$

Furthermore, write $\nabla_x f(x,y)$ for the first d components of $\nabla f(x,y)$ and similarly for $\nabla_y f(x,y)$, $\nabla^2_{xx} f(x,y)$, $\nabla^2_{xy} f(x,y)$, and $\nabla^2_{yy} f(x,y)$. We then get

$$\mathcal{G}f(x,y) = \frac{1}{2}\mathrm{Tr}\left(\widetilde{a}^X(x)\nabla^2_{xx} f(x,y)\right) + \mathrm{Tr}\left(\widetilde{a}^{XY}(x)\nabla^2_{xy} f(x,y)\right) + \frac{1}{2}\mathrm{Tr}\left(\widetilde{a}^Y(x)\nabla^2_{yy} f(x,y)\right) \tag{D.3}$$
$$+ b^X(x)^\top \nabla_x f(x,y) + b^Y(x)^\top \nabla_y f(x,y) \tag{D.4}$$
$$+ \int_{\mathbb{R}^{d+e}} \left(f(z+\zeta) - f(z) - \zeta^\top \nabla f(z) - \frac{1}{2}\zeta^\top \nabla^2 f(z)\zeta\right)\nu(x,d\zeta). \tag{D.5}$$

Consider first (D.3). Because $\nabla^2_{xx} f(x,y) = y^\beta \nabla^2(x^\alpha)$, and because $\mathcal{G}^X$ is polynomial, the first term in (D.3) is of degree at most $|\boldsymbol{\alpha}|$ in x and $|\boldsymbol{\beta}|$ in y and, thus, lies in V_m. Next, $\nabla^2_{xy} f(x,y)$ is of degree at most $|\boldsymbol{\alpha}| - 1$ in x and $|\boldsymbol{\beta}| - 1$ in y, which, together with (26), implies that the second term in (D.3) is of degree at most $n + 1 + |\boldsymbol{\alpha}| - 1 = n + |\boldsymbol{\alpha}|$ in x and $|\boldsymbol{\beta}| - 1$ in y. This term therefore also lies in V_m. Finally, $\nabla^2_{yy} f(x,y)$ is of degree at most $|\boldsymbol{\alpha}|$ in x and $|\boldsymbol{\beta}| - 2$ in y, which, together with (25), implies that the third term in (D.3) is of degree at most $2n + |\boldsymbol{\alpha}|$ in x and $|\boldsymbol{\beta}| - 2$ in y. Again, this yields membership in V_m.

Consider now (D.4). Because $\nabla_x f(x,y) = y^\beta \nabla(x^\alpha)$, and because $\mathcal{G}^X$ is polynomial, the first term in (D.4) is of degree at most $|\boldsymbol{\alpha}|$ in x and $|\boldsymbol{\beta}|$ in y and, thus, lies in V_m. Similarly, as before, the second term also lies in V_m because of (24).

Consider finally (D.5). It follows from the multibinomial theorem that the expression in parentheses is a linear combination of monomials $x^\gamma y^\delta \xi^\epsilon \eta^\upsilon$ with $|\boldsymbol{\gamma}| + |\boldsymbol{\epsilon}| \le |\boldsymbol{\alpha}|$, $|\boldsymbol{\delta}| + |\boldsymbol{\upsilon}| \le |\boldsymbol{\beta}|$, and $|\boldsymbol{\epsilon}| + |\boldsymbol{\upsilon}| \ge 3$. Thus, (D.5) is a linear combination of expressions of the form

$$x^\gamma y^\delta \int_{\mathbb{R}^{d+e}} \xi^\epsilon \eta^\upsilon \nu(x,d\xi \times d\eta).$$

Because of (27), these expressions are polynomial of degree at most $|\boldsymbol{\gamma}| + |\epsilon| + n|\upsilon|$ in x and $|\boldsymbol{\delta}|$ in y. Because $|\boldsymbol{\gamma}| + |\boldsymbol{\epsilon}| + n|\upsilon| + n|\boldsymbol{\delta}| \le |\boldsymbol{\alpha}| + n|\boldsymbol{\beta}| \le nm$, it follows that (D.5) lies in V_m. This completes the proof of property (i) in Theorem 5, showing that (28) holds.

Conversely, assume that that (28) holds, so property (ii) in Theorem 5 holds as well. Because $\mathcal{G}(y^\beta) \in V_{|\beta|}$, the identity

$$\mathcal{G}(y^\beta) = \frac{1}{2}\mathrm{Tr}\left(\widetilde{a}^Y(x)\nabla^2_{yy}\left(y^\beta\right)\right) + b^Y(x)^\top \nabla_y\left(y^\beta\right)$$
$$+ \int_{\mathbb{R}^{d+e}} \left((y+\eta)^\beta - y^\beta - \eta^\top \nabla\left(y^\beta\right) - \frac{1}{2}\eta^\top \nabla^2\left(y^\beta\right)\right)\nu(x,d\xi \times d\eta)$$

applied with $\boldsymbol{\beta} = e_i$ yields $b_i^Y \in V_1$, which gives (24). Taking $\beta = e_i + e_j$, we similarly obtain $\widetilde{a}^Y_{ij} \in V_2$, which gives (25). By considering $|\boldsymbol{\beta}| \ge 3$, we obtain (27) for $\boldsymbol{\alpha} = 0$. □

D.7. Proof of Corollary 2

The process $Z_t = (X_t, Y_t)$ is given by

$$dZ_t = \begin{pmatrix} dX_t \\ P(X_t)\,dt \\ Q(X_{t-})\,dX_t \end{pmatrix} = K(X_{t-})\,d\begin{pmatrix} t \\ X_t \end{pmatrix}, \quad \text{where} \quad K(x) = \begin{pmatrix} 0 & \mathrm{id} \\ P(x) & 0 \\ 0 & Q(x) \end{pmatrix}.$$

Its differential characteristics can then be computed using Kallsen (2006, proposition 2). One finds that they are deterministic functions $a(x)$, $b(x)$, and $\nu(x, d\zeta)$ of X_t, where

$$\begin{aligned} a(x) &= K(x)\begin{pmatrix} 0 & 0 \\ 0 & a^X(x) \end{pmatrix}K(x)^\top = \begin{pmatrix} a^X(x) & 0 & a^X(x)Q(x)^\top \\ 0 & 0 & 0 \\ Q(x)a^X(x) & 0 & Q(x)a^X(x)Q(x)^\top \end{pmatrix}, \\ b(x) &= K(x)\begin{pmatrix} 1 \\ b^X(x) \end{pmatrix}, \\ \nu(x, A) &= \int_{\mathbb{R}^d} \mathbf{1}_A\left(K(x)\begin{pmatrix} 0 \\ \xi \end{pmatrix}\right)\nu^X(x, d\xi). \end{aligned} \tag{D.6}$$

In particular, we have

$$\int_{\mathbb{R}^{d+e}} \|\zeta\|^n\,\nu(x, d\zeta) = \int_{\mathbb{R}^d} \left(\|\xi\|^2 + \|Q(x)\xi\|^2\right)^{n/2} \nu^X(x, d\xi) \le \left(1 + \|Q(x)\|^2\right)^{n/2} \int_{\mathbb{R}^d} \|\xi\|^n\,\nu^X(x, d\xi) < \infty$$

for all $x \in E$ and all $n \ge 2$, where $\|Q(x)\|$ denotes the operator norm of $Q(x)$. Thus, (23) holds.

Next, (24) holds because the components of $b(x)$ are polynomials of degree at most n. To verify (25)–(27), we first observe the identity

$$\int_{\mathbb{R}^{d+e}} f(\xi)g(\eta', \eta'')\nu(x, d\xi \times d\eta' \times d\eta'') = \int_{\mathbb{R}^d} f(\xi)g(0, Q(x)\xi)\nu^X(x, d\xi), \tag{D.7}$$

where we write $\eta = (\eta', \eta'')$ for a generic vector in $\mathbb{R}^e = \mathbb{R}^{e'+e''}$. Let $f(\xi) = x^{\boldsymbol{\alpha}}$ and $g(\eta) = \eta^{\boldsymbol{\beta}} = \eta'^{\boldsymbol{\beta}'}\eta''^{\boldsymbol{\beta}''}$, where we decompose $\boldsymbol{\beta} = (\boldsymbol{\beta}', \boldsymbol{\beta}'')$ according to the decomposition $\eta = (\eta', \eta'')$. Because $g(0, Q(x)s\xi) = s^{|\boldsymbol{\beta}''|}g(0, Q(x)\xi)$ for any $s \in \mathbb{R}$, it follows that

$$g(0, Q(x)\xi) = \sum_{\gamma:\,|\gamma|=|\boldsymbol{\beta}''|} r_\gamma(x)\xi^\gamma$$

for some polynomials $r_\gamma(x)$. Because the left-hand side vanishes if $\boldsymbol{\beta}' \ne 0$, we take $r_\gamma(x) = 0$ in this case. Moreover, because the components of $Q(x)$ are of degree at most $n-1$, the degree of $g(0, Q(x)\xi)$, regarded as a polynomial in x, is at most $(n-1)\beta''_1 + \cdots + (n-1)\beta''_{e''} = (n-1)|\boldsymbol{\beta}''|$. This is, therefore, also an upper bound on the degrees of the polynomials $r_\gamma(x)$.

These observations readily yield (27). To see this, we write

$$\int_{\mathbb{R}^{d+e}} \xi^{\boldsymbol{\alpha}}\eta^{\boldsymbol{\beta}}\nu(x, d\xi \times d\eta) = \sum_{\gamma:\,|\gamma|=|\boldsymbol{\beta}''|} r_\gamma(x) \int_{\mathbb{R}^d} \xi^{\boldsymbol{\alpha}+\gamma}\nu^X(x, dx). \tag{D.8}$$

Either $\boldsymbol{\beta}' \ne 0$, in which case the right-hand side of (D.8) vanishes, or $\boldsymbol{\beta}' = 0$, hence, $|\boldsymbol{\beta}''| = |\boldsymbol{\beta}|$, and thus, the right-hand side of (D.8) is a polynomial of degree at most $(n-1)|\boldsymbol{\beta}| + |\boldsymbol{\alpha} + \boldsymbol{\gamma}| = |\boldsymbol{\alpha}| + n|\boldsymbol{\beta}|$, provided that $|\boldsymbol{\alpha} + \boldsymbol{\gamma}| = |\boldsymbol{\alpha}| + |\boldsymbol{\beta}| \ge 3$. Consequently, (27) holds.

Finally, (25) and (26) follow in a similar manner. For $i \in \{1, \ldots, d\}$ and $j \in \{e'+1, \ldots, e'+e''\}$, we apply (D.6) and (D.7) with $f(\xi) = \xi_i$ and $g(\eta) = \eta_j$ to get

$$a^{XY}_{ij}(x) + \int_{\mathbb{R}^{d+e}} \xi_i\eta_j\nu(x, d\xi \times d\eta) = \sum_{k=1}^d q_{j-e',k}(x)\left(a_{ik}(x) + \int_{\mathbb{R}^d} \xi_i\xi_k\,\nu^X(x, d\xi)\right),$$

which is a polynomial of degree at most $n - 1 + 2 = 1 + n$. If instead $j \in \{1, \ldots, e'\}$, then the left-hand side vanishes. It follows that (26) holds. Property (25) is proved similarly. This completes the proof of the corollary. □

D.8. Proof of Lemma 4

Similarly, as in the proof of Lemma 3, the pullback maps φ^* and ψ^* are well defined as operators

$$\mathrm{Pol}(H(E) \times \mathbb{R}^e) \to \mathrm{Pol}(E \times \mathbb{R}^e) \qquad \text{and} \qquad \mathrm{Pol}(E \times \mathbb{R}^e) \to \mathrm{Pol}(H(E) \times \mathbb{R}^e).$$

Viewed with these domain and range spaces, φ^* and ψ^* are each other's inverses. It is clear that φ^* is a linear map. We show that it maps $\mathrm{Pol}_m(H(E)\times\mathbb{R}^e)$ to V_m, and for this, it suffices to consider monomials $f(\bar{x},y)=\bar{x}^{\boldsymbol{\alpha}}y^{\boldsymbol{\beta}}$, where $|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|\le m$. One then has

$$\varphi^* f(x,y)=p(x)y^{\boldsymbol{\beta}},\quad \text{where}\quad p(x)=\prod_{i=1}^N h_i(x)^{\alpha_i}.$$

Because $\deg h_i\le n$ for all i, and because $|\boldsymbol{\alpha}|\le m-|\boldsymbol{\beta}|$, it follows that $\deg p\le n(m-|\boldsymbol{\beta}|)$, showing that $\varphi^* f\in V_m$, as claimed.

Injectivity of φ^* follows from the identity $\psi^*\circ\varphi^*=\mathrm{id}$ on $\mathrm{Pol}_m(H(E)\times\mathbb{R}^e)$. To see that φ^* maps $\mathrm{Pol}_m(H(E)\times\mathbb{R}^e)$ surjectively to V_m, let $p(x)y^{\boldsymbol{\beta}}$ with $\deg p\le n(m-|\boldsymbol{\beta}|)$ and $|\boldsymbol{\beta}|\le m$ be an element of V_m. Lemma D.5 implies that $p(x)=f(H(x))$ for some polynomial $f\in\mathrm{Pol}_{m-|\boldsymbol{\beta}|}(H(E))$. Thus, $p(x)y^{\boldsymbol{\beta}}=\varphi^* g(x,y)$, where $g(\bar{x},y)=f(\bar{x})y^{\boldsymbol{\beta}}$ is of degree at most $m-|\boldsymbol{\beta}|+|\boldsymbol{\beta}|=m$ and, thus, lies in $\mathrm{Pol}_m(H(E)\times\mathbb{R}^e)$. Because any element of V_m is a linear combination of polynomials $p(x)y^{\boldsymbol{\beta}}$ as before, this proves surjectivity. □

D.9. Proof of Theorem 5

We now prove Theorem 5. Because of Lemma D.6, $\bar{Z}_t$ is a jump diffusion on $H(E)\times\mathbb{R}^e$ with extended generator $\bar{\mathscr{G}}=\psi^*\mathscr{G}\varphi^*$, which is well defined on $\mathrm{Pol}(H(E)\times\mathbb{R}^e)$. Thus, property (28) is, by definition, equivalent to

$$\bar{\mathscr{G}} \text{ maps } \mathrm{Pol}_m(H(E)\times\mathbb{R}^e) \text{ to itself for each } m\in\mathbb{N}. \tag{D.9}$$

The equivalence (D.9) ⇔ (i) follows from Lemma 4 and the expression $\bar{\mathscr{G}}=\psi^*\mathscr{G}\varphi^*$, which show that in the diagram (30) each horizontal arrow holds if and only the other one does. In particular, the diagram commutes if either condition holds. It remains to prove (ii) ⇔ (i). By definition of the carré-du-champ operator, we have the identity

$$\mathscr{G}\left(x^{\boldsymbol{\alpha}}y^{\boldsymbol{\beta}}\right)=y^{\boldsymbol{\beta}}\mathscr{G}(x^{\boldsymbol{\alpha}})+x^{\boldsymbol{\alpha}}\mathscr{G}\left(y^{\boldsymbol{\beta}}\right)+\Gamma\left(x^{\boldsymbol{\alpha}},y^{\boldsymbol{\beta}}\right).$$

Moreover, if $|\boldsymbol{\alpha}|\le n(m-|\boldsymbol{\beta}|)$ and $|\boldsymbol{\beta}|\le m$, then $y^{\boldsymbol{\beta}}\mathscr{G}(x^{\boldsymbol{\alpha}})=y^{\boldsymbol{\beta}}\mathscr{G}^X(x^{\boldsymbol{\alpha}})\in V_m$ because $\mathscr{G}^X$ is polynomial. Hence, (i) is equivalent to

$$x^{\boldsymbol{\alpha}}\mathscr{G}(y^{\boldsymbol{\beta}})+\Gamma(x^{\boldsymbol{\alpha}},y^{\boldsymbol{\beta}})\in V_m \text{ whenever } |\boldsymbol{\alpha}|\le n(m-|\boldsymbol{\beta}|) \text{ and } |\boldsymbol{\beta}|\le m. \tag{D.10}$$

It suffices to argue that (D.10) is equivalent to (ii). To this end, first observe that

$$x^{\boldsymbol{\alpha}}\mathscr{G}(y^{\boldsymbol{\beta}})\in V_m \text{ whenever } |\boldsymbol{\alpha}|\le n(m-|\boldsymbol{\beta}|), |\boldsymbol{\beta}|\le m, \text{ and } \mathscr{G}(y^{\boldsymbol{\beta}})\in V_{|\boldsymbol{\beta}|}. \tag{D.11}$$

Indeed, $\mathscr{G}(y^{\boldsymbol{\beta}})\in V_{|\boldsymbol{\beta}|}$ is a linear combination of monomials of the form $x^{\boldsymbol{\gamma}}y^{\boldsymbol{\delta}}$ with $|\boldsymbol{\gamma}|\le n(|\boldsymbol{\beta}|-|\boldsymbol{\delta}|)$ and $|\boldsymbol{\delta}|\le|\boldsymbol{\beta}|$. Thus, $x^{\boldsymbol{\alpha}}\mathscr{G}(y^{\boldsymbol{\beta}})$ is a linear combination of monomials of the form $x^{\boldsymbol{\alpha}+\boldsymbol{\gamma}}y^{\boldsymbol{\delta}}$ with $|\boldsymbol{\alpha}+\boldsymbol{\gamma}|=|\boldsymbol{\alpha}|+|\boldsymbol{\gamma}|\le n(m-|\boldsymbol{\beta}|)+n(|\boldsymbol{\beta}|-|\boldsymbol{\delta}|)=n(m-|\boldsymbol{\delta}|)$ and therefore lies in V_m. This proves (D.11).

Assume that (ii) holds. By taking $\boldsymbol{\alpha}=0$, we see that $\mathscr{G}(y^{\boldsymbol{\beta}})\in V_{|\boldsymbol{\beta}|}$ for all $\boldsymbol{\beta}$, which, in view of (D.11) and (ii), implies that (D.10) holds. Conversely, assume that (D.10) holds. Because $\Gamma(1,y^{\boldsymbol{\beta}})=0$, it follows that $\mathscr{G}(y^{\boldsymbol{\beta}})\in V_{|\boldsymbol{\beta}|}$ for all $\boldsymbol{\beta}$, and hence, in view of (D.11), that $x^{\boldsymbol{\alpha}}\mathscr{G}(y^{\boldsymbol{\beta}})\in V_m$ whenever $|\boldsymbol{\alpha}|\le n(m-|\boldsymbol{\beta}|)$ and $|\boldsymbol{\beta}|\le m$. Another application of (D.10) then yields $\Gamma(x^{\boldsymbol{\alpha}},y^{\boldsymbol{\beta}})\in V_m$, and we deduce that (ii) holds. □

D.10. Proof of Corollary 3

Lemma D.1 implies that $Z_t'=(X_t,PY_t)$ is a special semimartingale. We infer that $Z_t'=(X_t,PY_t)$ is an $E\times\mathbb{R}^{e'}$-valued jump diffusion with extended generator $\mathscr{G}'$ of the form (20). Indeed, this follows from a straightforward modification of Lemma B.1 applied to the linear map $\varphi(x,y)=(x,Py)$, observing that P does not need to be invertible to back out the state x from (x,Py). It follows by inspection that $Z_t'=(X_t,PY_t)$ satisfies (23) and, by Lemma B.1 in conjunction with Lemma 1 applied to $(H(X_t),PY_t)$, property (28) with e replaced by e' as claimed. □

D.11. Proof of Theorem 6

The proof of Theorem 6 builds on the following two lemmas.

Lemma D.7. *Let $\mathscr{G}$ be polynomial on E. Then $\mathscr{G}f(x)$ is locally bounded on E for every $f\in C^k(\mathbb{R}^d)$ satisfying the growth condition $|f(x)|\le c(1+\|x\|^k)$ on $\mathbb{R}^d$ for some real c and integer $k\ge 2$.*

Proof. Write

$$\begin{aligned}\mathscr{G}f(x)=&\frac{1}{2}\mathrm{Tr}\big(a(x)\nabla^2 f(x)\big)+b(x)^\top\nabla f(x)+\sum_{2\le|\boldsymbol{\alpha}|\le k-1}\frac{\partial^{\boldsymbol{\alpha}}f(x)}{\boldsymbol{\alpha}!}\int_{\mathbb{R}^d}\xi^{\boldsymbol{\alpha}}\nu(x,d\xi)\\&+\int_{\mathbb{R}^d}g(x,\xi)\nu(x,d\xi),\end{aligned}\tag{D.12}$$

where

$$g(x,\xi) = f(x+\xi) - \sum_{|\alpha|\le k-1} \frac{\partial^\alpha f(x)}{\alpha!}\xi^\alpha. \tag{D.13}$$

By Lemma 1 and the C^k smoothness of $f(x)$, the first three terms on the right-hand side of (D.12) are locally bounded in x on E.

We now bound the remaining term in (D.12). For $\|\xi\|>1$, the assumed polynomial bound on $f(x)$ along with the crude inequality $1+\|x+\xi\|^k \le \|\xi\|^k 2^k(1+\|x\|^k)$ and (D.13) yield

$$|g(x,\xi)| \le \|\xi\|^k \left(c2^k(1+\|x\|^k) + \sum_{|\alpha|\le k-1} \frac{|\partial^\alpha f(x)|}{\alpha!} \right), \qquad \|\xi\|>1.$$

Next, Taylor's theorem with the remainder in integral form yields

$$g(x,\xi) = \sum_{|\alpha|=k} \frac{k}{\alpha!}\xi^\alpha \int_0^1 (1-t)^{k-1}\partial^\alpha f(x+t\xi)dt,$$

and hence

$$|g(x,\xi)| \le \|\xi\|^k \sum_{|\alpha|=k} \frac{1}{\alpha!} \max_{\|\xi\|\le 1} |\partial^\alpha f(x+\xi)|, \qquad \|\xi\|\le 1.$$

Combining these two bounds gives $|g(x,\xi)| \le \|\xi\|^k M(x)$, where $M(x)$ is a continuous function because of the C^k smoothness of $f(x)$. Consequently,

$$\int_{\mathbb{R}^d} |g(x,\xi)|\nu(x,d\xi) \le M(x)\int_{\mathbb{R}^d} \|\xi\|^k \nu(x,d\xi),$$

which is locally bounded in x on E by Lemma 1. □

Lemma D.8. *Let $\mathscr{G}$ be polynomial on E. Then, for every $f\in C_b^\infty(\mathbb{R}^d)$, there exists a sequence of functions $f_n \in C_c^\infty(\mathbb{R}^d)$ such that $f_n \to f$ and $\mathscr{G}f_n \to \mathscr{G}f$ locally uniformly on E.*

Proof. Define $f_n(x)$ to be $f(x)$ multiplied by a smooth cutoff function that is equal to one on $B_n = \{x\in\mathbb{R}^d : \|x\|\le n\}$. Then $f_n \to f$ locally uniformly. For $n>m$ and $x\in B_m$, we have

$$\begin{aligned}\left|\mathscr{G}f_n(x) - \mathscr{G}f(x)\right| &\le \int_{\mathbb{R}^d} \left|f_n(x+\xi) - f(x+\xi)\right|\nu(x,d\xi)\\ &= \int_{\mathbb{R}^d} \left|f_n(x+\xi) - f(x+\xi)\right| \mathbf{1}_{\{\|\xi\|>n-m\}}\nu(x,d\xi)\\ &\le \frac{2\|f\|_\infty}{(n-m)^2}\int_{\mathbb{R}^d}\|\xi\|^2\nu(x,d\xi).\end{aligned}$$

By Lemma 1, the right-hand side is locally bounded in x on E. Hence, $\mathscr{G}f_n \to \mathscr{G}f$ uniformly on $E\cap B_m$ for all m. □

We now prove Theorem 6. We first prove the statement in Remark 5. Because of the Feller property, for any $f\in C_0(\mathbb{R}^d)$, we know that $\int_E f(y)p_t(x,dy)$ is jointly continuous in $(t,x)\in[0,\infty)\times E$. A monotone class argument now yields the claim for $p_t(x,A)$. For $p_t(x,x+A)$, we observe that

$$p_t(x,x+A) = \int_E 1_A(y-x)p_t(x,dy).$$

Because $1_A(y-x)$ is jointly measurable in (x,y), the claim follows also for $p_t(x,x+A)$. Hence, $\widetilde{\mathscr{G}}f(x)$ is well defined by (33)–(36) for any bounded C^2 function $f(x)$ on $\mathbb{R}^d$.

We next claim that

$$\widetilde{\mathscr{G}}f(x) = b^Z\mathscr{G}f(x) + \int_0^\infty \int_E (f(y)-f(x))p_\zeta(x,dy)\nu^Z(d\zeta) \tag{D.14}$$

for all $f\in \mathrm{Pol}(E)$. Indeed, for any polynomial $f(x) = (1,H(x)^\top)\vec{f}$ in $\mathrm{Pol}_n(E)$, Theorem 1 yields

$$\int_E (f(y)-f(x))p_\zeta(x,dy) = (1,H(x)^\top)(\mathrm{e}^{\zeta G} - \mathrm{id})\vec{f}. \tag{D.15}$$

Hence, the right-hand side of (D.14) is well defined because of (31) for all $x \in E$. Moreover, by (33)–(36), we infer that

$$\begin{aligned}\widetilde{\mathscr{G}}f(x) &= b^Z \mathscr{G}f(x) + \int_0^\infty \int_E (y-x)^\top \nabla f(x) p_\zeta(x,dy)\nu^Z(d\zeta) \\ &\quad + \int_0^\infty \int_E \big(f(y) - f(x) - (y-x)^\top \nabla f(x)\big) 1_{\{y\neq x\}} p_\zeta(x,dy)\nu^Z(d\zeta),\end{aligned} \tag{D.16}$$

which proves (D.14).

We next claim that the jump-diffusion operator

$$\widetilde{\mathscr{G}} \text{ is polynomial on } E. \tag{D.17}$$

First, we have

$$\int_{\mathbb{R}^d} \|\xi\|^n \,\widetilde{\nu}(x,d\xi) = \int_{\mathbb{R}^d} \|\xi\|^n \, b^Z \nu(x,d\xi) + \int_0^\infty \int_{\mathbb{R}^d} \|y-x\|^n \, p_\zeta(x,dy)\nu^Z(d\zeta) < \infty$$

for all $x \in E$ and all $n \geq 2$. Indeed, the first term on the right-hand side is finite because of (4). The second term is also finite. This follows from (D.15) for $f(y) = \|y-x\|^n$ and (31) for even $n \geq 2$. Second, $\widetilde{\mathscr{G}}f(x) = 0$ on E for any $f \in \mathrm{Pol}(\mathbb{R}^d)$ with $f(x) = 0$ on E. This follows from (5) and (D.14). Hence, $\widetilde{\mathscr{G}}$ is well defined on $\mathrm{Pol}(E)$. Finally, the polynomial property of $\mathscr{G}$, (D.14), and (D.15) again imply that $\widetilde{\mathscr{G}}$ maps $\mathrm{Pol}_n(E)$ to itself for each $n \in \mathbb{N}$, which proves (D.17).

Now let G and $\widetilde{G}$ be the matrix representations of $\mathscr{G}$ and $\widetilde{\mathscr{G}}$ on $\mathrm{Pol}_n(E)$. The first equation in (37) then follows from (D.14) and (D.15). The right-hand side of the second equation equals $M(t) = \mathbb{E}[\mathrm{e}^{Z_t G}]$, which is well defined by Remark 7. Because of the Lévy property of Z_t, we have that $M(t+s) = M(t)M(s)$. Hence, $M(t) = \exp(t\dot{M}(0))$, where $\dot{M}(0) = \mathscr{G}^Z \mathrm{e}^{zG}|_{z=0} = \widetilde{G}$. This proves (37).

It remains to verify that $\widetilde{X}_t$ is a jump diffusion with respect to $\widetilde{\mathscr{F}}_t$ and that its extended generator is $\widetilde{\mathscr{G}}$. By Jacod and Shiryaev (2003, theorem II.2.42), it suffices to prove that the process

$$M_t^f = f\big(\widetilde{X}_t\big) - f\big(\widetilde{X}_0\big) - \int_0^t \widetilde{\mathscr{G}}f\big(\widetilde{X}_s\big)ds$$

is well defined and a local martingale for every $f \in C_b^\infty(\mathbb{R}^d)$. We do this in three steps.

First, Phillips' theorem (Sato 1999, theorem 32.1) shows that $\widetilde{p}_t(x,dy)$ given in (32) is a Feller transition kernel, and the domain of its generator contains $C_c^\infty(E)$ on which it coincides with the operator

$$\overline{\mathscr{G}}f(x) = b^Z \mathscr{G}f(x) + \int_0^\infty \int_E \big(f(y) - f(x)\big)p_\zeta(x,dy)\nu^Z(d\zeta). \tag{D.18}$$

Here the integral with respect to $\nu^Z(d\zeta)$ is understood as the Bochner integral of the $C_0(E)$-valued map $\zeta \mapsto u_\zeta$, where $u_\zeta(x) = \int_E \big(f(y) - f(x)\big)p_\zeta(x,dy)$; see Sato (1999, comment after theorem 32.1). In particular, when evaluated at a point $x \in E$, this integral coincides with the Lebesgue integral with respect to $\nu^Z(d\zeta)$ of the $\mathbb{R}$-valued function $\zeta \mapsto \int_E \big(f(y) - f(x)\big)p_\zeta(x,dy)$, which is thus well defined and finite. In view of Remark 6, therefore, (D.16) and, thus, (D.14) also hold for all $f \in C_c^\infty(E)$. We conclude that

$$\widetilde{\mathscr{G}}f(x) = \overline{\mathscr{G}}f(x) \text{ on } E \text{ for all } f \in C_c^\infty(E). \tag{D.19}$$

Second, an argument based on Revuz and Yor (1999, proposition III.1.4) shows that $\widetilde{X}_t$ is a Markov process with respect to its natural filtration $\widetilde{\mathscr{F}}_t^0 = \sigma(\widetilde{X}_s, s \leq t)$ with transition kernel $\widetilde{p}_t(x,dy)$. Because of the Feller property, this also holds with respect to the usual right-continuous augmentation $\widetilde{\mathscr{F}}_t$ of $\widetilde{\mathscr{F}}_t^0$; see Revuz and Yor (1999, proposition III.2.10). Therefore, by Revuz and Yor (1999, proposition VII.1.6) and (D.19), it follows that M_t^f is a martingale for every $f \in C_c^\infty(\mathbb{R}^d)$.

Third, let $f \in C_b^\infty(\mathbb{R}^d)$. In view of (D.17) and Lemma D.7, $\widetilde{\mathscr{G}}f(x)$ is locally bounded on E, whence the process M_t^f is well defined. Furthermore, Lemma D.8 yields a sequence of functions $f_n \in C_c^\infty(\mathbb{R}^d)$ such that $f_n \to f$ and $\widetilde{\mathscr{G}}f_n \to \widetilde{\mathscr{G}}f$ locally uniformly on E. Therefore, defining the stopping times $T_m = \inf\{t \geq 0 : \|\widetilde{X}_t\| \geq m\}$, we have

$$\left|M_{t\wedge T_m}^f - M_{t\wedge T_m}^{f_n}\right| \leq \left|f\big(\widetilde{X}_{t\wedge T_m}\big) - f_n\big(\widetilde{X}_{t\wedge T_m}\big) - f\big(\widetilde{X}_0\big) + f_n\big(\widetilde{X}_0\big)\right| + t \max_{x\in E,\, \|x\|\leq m} \left|\widetilde{\mathscr{G}}f(x) - \widetilde{\mathscr{G}}f_n(x)\right|.$$

The right-hand side is bounded and converges to zero as $n \to \infty$, which yields $M_{t\wedge T_m}^{f_n} \to M_{t\wedge T_m}^f$ in L^1. Because each $M_t^{f_n}$ is a martingale, it follows that M_t^f is a local martingale as required. This finishes the proof of Theorem 6. □

Endnotes

[1] The special semimartingale property of X_t implies that

$$\int_0^t \left(\|a(X_s)\| + \|b(X_s)\| + \int_{\mathbb{R}^d} \|\xi\|^2 \wedge \|\xi\| \, \nu(X_s, d\xi) \right) ds < \infty,$$

and its semimartingale characteristics (B, C, ν) associated with the identity truncation function $h(\xi) = \xi$ are given by $B_t = \int_0^t b(X_s)ds$, $C_t = \int_0^t a(X_s)ds$, and $\nu(X_{t-}, d\xi)dt$; see Jacod and Shiryaev (2003, proposition II.2.29). In particular, $X_t - B_t$ is a local martingale.

[2] Property (5) always holds when E has a nonempty interior. An example of a diffusion operator that is not well defined on $\mathrm{Pol}(E)$ can be found in Filipović and Larsson (2016, section 2).

[3] This result was derived in collaboration with Sergio Pulido and is applied in Filipović et al. (2019).

[4] We conjecture that properties (26) and (27) are not necessary for (28) to hold in general. However, we have not found a counterexample.

[5] It is straightforward to prove that $\widetilde{X}_t$ is a semimartingale, but it is not clear that its jump characteristic is a function of the current state only as required for jump diffusions. What is more, the drift and jump characteristics in (35) and (36) are given in terms of the Markov transition kernel of X_t.

[6] In Appendix C, we sketch a situation that one may encounter in applications.

[7] If the continuous martingale part of Z_t is of the form $dZ_t^c = \sigma(X_t)\,dW_t$ for some Brownian motion W_t and $a(x) = \sigma(x)\sigma(x)^\top$, then $\sigma(X_t)^\top \phi(X_t)$ is the market price of risk of W_t.

[8] This assumption entails that $\phi(x)$ and $\psi(x, \zeta) > 0$ are measurable and such that

$$\int_0^T \phi(X_t)^\top a(X_t) \phi(X_t)\, dt + \int_0^T \int_{\mathbb{R}^{d+e}} \left(1 - \sqrt{1/\psi(X_t, \zeta)}\right)^2 \nu(X_t, d\zeta) dt < \infty,$$

so that L_t, $t \in [0, T]$, is a well defined local martingale; see Jacod and Shiryaev (2003, theorem II.1.33d).

[9] Some authors restrict to the diffusive component of $dY_{i,t}$ for the definitions of spot variance, volatility, vol of vol, and leverage, which are the same under both measures.

[10] The Poisson random measure $N(du, dt)$ and the Brownian motion W_t are automatically independent; see Ikeda and Watanabe (1989, theorem II.6.3).

[11] State-dependent jumps of Y_t that are simultaneous with jumps of X_t would violate the structural condition (48).

[12] The pullback is defined in (1).

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