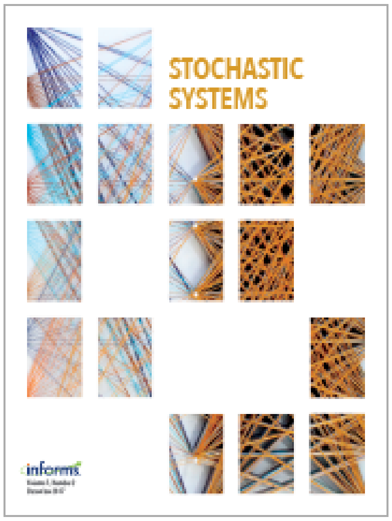

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Erratum: "Transform Methods for Heavy-Traffic Analysis"

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Abstract. In Hurtado-Lange and Maguluri (2020) [Transform methods for heavy-traffic analysis. *Stochastic Systems* 10(4):275–309], the statement of Claim 4 has a typo; it should say that the expression therein is bounded by a constant. The statement of Lemma 11, and Lemma 14 is incorrect. The lemmas should show the existence of the moment generating function in an interval around the origin with the length being independent of the heavy-traffic parameter. The correct statements and proofs are provided.

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Keywords: drift method • heavy-traffic analysis • state space collapse • complete resource pooling

1. Correction of Lemma 11

Lemma 11. *Consider a load balancing system operating under JSQ, parametrized by $\epsilon \in (0, \mu_\Sigma)$ as described in* Corollary 1 *by Hurtado-Lange and Maguluri (2020). Then, there exists $\Theta > 0$ (which is independent of ϵ) such that $E[e^{\theta\epsilon\sum_{i=1}^n \bar{q}_i^{(\epsilon)}}] < \infty$ for all $\theta \in [-\Theta, \Theta]$.*

Proof of Lemma 11. We omit the dependence on ϵ of the variables for ease of exposition. First observe that if $\theta \le 0$, then $E[e^{\theta\epsilon\sum_{i=1}^n \bar{q}_i}] < \infty$ trivially because $\bar{\boldsymbol{q}} \ge \boldsymbol{0}$ by definition of queue length.

In the rest of this proof, we assume $\theta > 0$. Observe that the function $f(x) = e^{\theta\epsilon x}$ is convex. Then, by Jensen's inequality, we have that, for all $\boldsymbol{q} \ge \boldsymbol{0}$,

$$e^{\frac{\theta\epsilon}{n}\sum_{i=1}^n q_i} \le \frac{1}{n}\sum_{i=1}^n e^{\theta\epsilon q_i}.$$

Hence, it suffices to show that $\sum_{i=1}^n E[e^{\theta\epsilon\bar{q}_i}] < \infty$ for $\theta \le \Theta$. We show that the conditions of the Foster-Lyapunov theorem (Hajek 2015, proposition 6.13) are satisfied with Lyapunov function $V(\boldsymbol{q}) = \sum_{i=1}^n e^{\theta\epsilon q_i}$. Then, one can easily obtain a finite bound on $E[V(\bar{\boldsymbol{q}})]$ (Hajek 2015, proposition 6.14).

Using Lemma 1 by Hurtado-Lange and Maguluri (2020), for each of the n queues (i.e., that $(e^{\theta\epsilon q_i(k+1)} - 1)(e^{-\theta\epsilon u_i(k)} - 1) = 0$ with probability one) and rearranging terms we obtain that, for each $i \in [n]$ and $k \ge 1$,

$$e^{\theta\epsilon q_i(k+1)} = 1 - e^{-\theta\epsilon u_i(k)} + e^{\theta\epsilon(q_i(k) + a_i(k) - s_i(k))}.$$

Then, using the notation $E_{\boldsymbol{q}}[\cdot] \triangleq E[\cdot \mid \boldsymbol{q}(k) = \boldsymbol{q}]$, we obtain

$$\begin{aligned} E_{\boldsymbol{q}}[V(\boldsymbol{q}(k+1)) - V(\boldsymbol{q}(k))] &= \sum_{i=1}^n E_{\boldsymbol{q}}[e^{\theta\epsilon q_i(k+1)} - e^{\theta\epsilon q_i(k)}] \\ &= \sum_{i=1}^n \Big(1 - E_{\boldsymbol{q}}[e^{-\theta\epsilon u_i(k)}]\Big) + \sum_{i=1}^n e^{\theta\epsilon q_i}\Big(E_{\boldsymbol{q}}[e^{\theta\epsilon(a_i(k) - s_i(k))}] - 1\Big). \end{aligned}$$

Because $E_{\boldsymbol{q}}[e^{-\theta\epsilon u_i(k)}] \ge 0$, we have

$$\sum_{i=1}^n \Big(1 - E_{\boldsymbol{q}}[e^{\theta\epsilon u_i(k)}]\Big) \le n$$

Then, it suffices to show that for some Θ and some $\eta > 0$, we have

$$\sum_{i=1}^{n} e^{\theta\epsilon q_i}\Big(E_{\boldsymbol{q}}[e^{\theta\epsilon(a_i(k)-s_i(k))}]-1\Big) \le -\eta\sum_{i=1}^{n} e^{\theta\epsilon q_i} \qquad \forall\,\theta\in(0,\Theta].$$

Given $\boldsymbol{q}(k)=\boldsymbol{q}$, let $i^*\in\arg\min_{i\in\{1,\dots,n\}}\{q_i(k)\}$ be the queue where arrivals in time slot k are routed. Then,

$$\begin{aligned}\sum_{i=1}^{n} e^{\theta\epsilon q_i}\Big(E_{\boldsymbol{q}}[e^{\theta\epsilon(a_i(k)-s_i(k))}]-1\Big) &= e^{\theta\epsilon q_{i^*}}\Big(E[e^{\theta\epsilon(a(k)-s_{i^*}(k))}]-1\Big)+\sum_{\substack{i=1\\ i\neq i^*}}^{n} e^{\theta\epsilon q_i}\Big(E[e^{-\theta\epsilon s_i(k)}]-1\Big)\\ &= e^{\theta\epsilon q_{i^*}}M_{a-s_{i^*}}(\theta)+\sum_{\substack{i=1\\ i\neq i^*}}^{n} e^{\theta\epsilon q_i}M_{-s_i}(\theta),\end{aligned}$$

where we used the notation $M_X(\theta)=E[e^{\theta\epsilon X}]-1$. Next we take the truncated Taylor series around zero. Then, for some $\xi_i\in(0,\theta)$ for each $i\in 1,\dots,n$, we have

$$\begin{aligned}\sum_{i=1}^{n} e^{\theta\epsilon q_i}\Big(E_{\boldsymbol{q}}[e^{\theta\epsilon(a_i(k)-s_i(k))}]-1\Big) &\overset{(a)}{=} e^{\theta\epsilon q_{i^*}}\Big(\theta\epsilon(\lambda-\mu_{i^*})+\frac{\theta^2}{2}M''_{a-s_{i^*}}(\xi_{i^*})\Big)+\sum_{\substack{i=1\\ i\neq i^*}}^{n} e^{\theta\epsilon q_i}\Big(-\theta\epsilon\mu_i+\frac{\theta^2}{2}M''_{s_i}(\xi_i)\Big)\\ &\overset{(b)}{\le} e^{\theta\epsilon q_{i^*}}\theta\epsilon(\lambda-\mu_{i^*})-\sum_{\substack{i=1\\ i\neq i^*}}^{n} e^{\theta\epsilon q_i}\theta\epsilon\mu_i+\frac{\theta^2\epsilon^2}{2}(A_{\max^2}+S_{\max^2})e^{\theta\mu_\Sigma A_{\max}}\sum_{i=1}^{n} e^{\theta\epsilon q_i},\end{aligned} \tag{1}$$

where (a) holds by definition of i^* and the routing algorithm and (b) holds after bounding the second derivatives as follows. Because $\epsilon<\mu_\Sigma$, by definition, we have

$$\begin{aligned}M''_{a-s_{i^*}}(\theta) &= E[\epsilon^2(a-s_{i^*})^2e^{\theta\epsilon(a-s_{i^*})}]\le\epsilon^2(A^2_{\max}+S^2_{\max})e^{\theta\mu_\Sigma A_{\max}},\\ M''_{-s_i}(\theta) &= E[\epsilon^2 s_i^2 e^{-\theta\epsilon s_i}]\le\epsilon^2S^2_{\max}\le\epsilon^2(A^2_{\max}+S^2_{\max})e^{\theta\mu_\Sigma A_{\max}}.\end{aligned}$$

For each $i\in 1,\dots,n$, define $\lambda_i\overset{\triangle}{=}\mu_i-\varepsilon/n$ and observe $\lambda=\sum_{i=1}^n\lambda_i$. Then, we obtain

$$\begin{aligned}(1) &= \theta\epsilon\sum_{i=1}^{n} e^{\theta\epsilon q_i}(\lambda_i-\mu_i)+\frac{\theta^2\epsilon^2}{2}(A^2_{\max}+S^2_{\max})e^{\theta\mu_\Sigma A_{\max}}\sum_{i=1}^{n} e^{\theta\epsilon q_i}+\theta\epsilon\sum_{i=1}^{n}\lambda_i(e^{\theta\epsilon q_{i^*}}-e^{\theta\epsilon q_i})\\ &\overset{(a)}{\le} \theta\epsilon^2\Big(-\frac{1}{n}+\frac{\theta}{2}(A^2_{\max}+S^2_{\max})e^{\theta\mu_\Sigma A_{\max}}\Big)\sum_{i=1}^{n} e^{\theta\epsilon q_i},\end{aligned}$$

where (a) holds because $q_{i^*}\le q_i$ for all $i\in 1,\dots,n$ by definition of i^* because $\lambda_i-\mu_i=-\varepsilon/n$ and by reorganizing terms. Therefore, it suffices to show the existence of $\Theta>0$ such that

$$-\frac{1}{n}+\frac{\theta}{2}(A^2_{\max}+S^2_{\max})e^{\theta\mu_\Sigma A_{\max}}\le-\frac{1}{2n}\quad\forall\,\theta\le\Theta.$$

Solving the inequality yields

$$\Theta=\frac{1}{\mu_\Sigma A_{\max}}W_0\Big(\frac{A_{\max}\mu_\Sigma}{n(A^2_{\max}+S^2_{\max})}\Big),$$

where $W_0(\cdot)$ is the principal branch of the Lambert W function, which has been studied by Corless et al. (1996) among others. □

2. Correction of Lemma 14

Lemma 14. *Consider a generalized switch parametrized by ϵ as described in Theorem 3 by Hurtado-Lange and Maguluri (2020). Then, there exists $\Theta>0$ (which is independent of ϵ) such that $E[e^{\theta\epsilon\langle c^{(\ell)},\bar{q}\rangle}]<\infty$ for all $\theta\in[-\Theta,\Theta]$.*

Lemma 14 can be proved using the Foster-Lyapunov theorem, following similar steps to the proof of Lemma 11. However, here we propose a different approach. We use Lemma 15, which presents explicit bounds for the setting of Lemma 10, stated by Hurtado-Lange and Maguluri (2020). We present the proof of the lemma at the end of this section.

Lemma 15. *For an irreducible and aperiodic Markov chain $\{X(k): k \geq 1\}$ over a countable state space $\mathcal{X}$, suppose $Z: \mathcal{X} \to \mathbb{R}_+$ is a nonnegative valued Lyapunov function and consider its drift $\Delta Z(x)$ as defined in Lemma* 10, *stated by Hurtado-Lange and Maguluri* (2020). *Suppose the following conditions are satisfied:*

(C1) *There exists* $\eta > 0$ and $\kappa < \infty$ *such that*

$$E[\Delta Z(x) \,|\, X(k) = x] \leq -\eta \quad \textit{for all } x \in \mathcal{X} \textit{ with } Z(x) \geq \kappa.$$

(C2) *There exists* $D < \infty$ *such that*

$$|\Delta Z(x)| \leq D \quad \textit{with probability } 1 \textit{ for all } x \in \mathcal{X},$$

and $\eta^2 \leq 4(e^D - 1 - D)$. *Define*

$$\Theta^* \triangleq \min\left\{1, \frac{\eta}{2(e^D - 1 - D)}\right\}.$$

Then, for any $\theta^* \in (0, \Theta^*)$, *we have*

$$\lim_{k\to\infty} E[e^{\theta^* V(X_k)}] \leq 2\left(1 + \frac{D}{\eta}\right) e^{\theta^* \kappa}.$$

Now we prove the existence of MGF for the generalized switch.

Proof of Lemma 14. First observe that if $\theta \leq 0$, the lemma holds trivially. Therefore, in this proof we assume $\theta > 0$. We use Lemma 15 with $Z(\boldsymbol{q}) = \|\boldsymbol{q}\|$.

To show that condition *(C1)* is satisfied, we first observe that $f(x) = \sqrt{x}$ is a concave function and $Z(\boldsymbol{q}) = \sqrt{\|\boldsymbol{q}\|^2}$. Then,

$$E_{\boldsymbol{q}}[\Delta Z(\boldsymbol{q})] \leq \frac{1}{2\|\boldsymbol{q}\|} E_{\boldsymbol{q}}[\|\boldsymbol{q}(k+1)\|^2 - \|\boldsymbol{q}\|^2]. \tag{2}$$

We now compute an upper bound on $E_{\boldsymbol{q}}[\|\boldsymbol{q}(k+1)\|^2 - \|\boldsymbol{q}\|^2]$. For every $k \geq 1$, we have

$$\begin{aligned}
\|\boldsymbol{q}(k+1)\|^2 - \|\boldsymbol{q}(k)\|^2 &= \|\boldsymbol{q}(k+1) - \boldsymbol{u}(k) + \boldsymbol{u}(k)\|^2 - \|\boldsymbol{q}(k)\|^2 \\
&\overset{(a)}{=} \|\boldsymbol{q}(k) + \boldsymbol{a}(k) - \boldsymbol{s}(k)\|^2 + \|\boldsymbol{u}(k)\|^2 + 2\langle \boldsymbol{q}(k+1) - \boldsymbol{u}(k), \boldsymbol{u}(k)\rangle - \|\boldsymbol{q}(k)\|^2 \\
&\overset{(b)}{\leq} \|\boldsymbol{a}(k) - \boldsymbol{s}(k)\|^2 + 2\langle \boldsymbol{q}(k), \boldsymbol{a}(k) - \boldsymbol{s}(k)\rangle,
\end{aligned} \tag{3}$$

where (a) holds by the dynamics of the queues described in Equation (2) by Hurtado-Lange and Maguluri (2020) and (b) holds because $q_i(k+1)u_i(k) = 0$ with probability one for all i, because $\|\boldsymbol{u}(k)\|^2 \geq 0$ and expanding the first term. Next we bound the conditional expectation of each of the terms in (3). For the first term, we have

$$E_{\boldsymbol{q}}[\|\boldsymbol{a}(k) - \boldsymbol{s}(k)\|^2] \overset{(a)}{\leq} E[\|\boldsymbol{a}(k)\|^2] + E_{\boldsymbol{q}}[\|\boldsymbol{s}(k)\|^2] \overset{(b)}{\leq} \sum_{i=1}^n \left((\lambda_i^{(\epsilon)})^2 + \sigma_{a_i}^2\right) + nS_{\max^2}, \tag{4}$$

where (a) holds by triangle inequality and because the arrival process is independent of the queue length process and (b) holds by definition of variance and because the potential service to each of the queues is upper bounded by $S_{\max}$. Define $K_1 \triangleq \sum_{i=1}^n ((\lambda_i^{(\epsilon)})^2 + \sigma_{a_i}^2) + nS_{\max}^2$.

For the second term in (3), we obtain

$$E_{\boldsymbol{q}}[\langle \boldsymbol{q}, \boldsymbol{a}(k) - \boldsymbol{s}(k)\rangle] \overset{(a)}{=} \langle \boldsymbol{q}, \boldsymbol{r}^{(\ell)} - \epsilon \boldsymbol{c}^{(\ell)}\rangle - \langle \boldsymbol{q}, E_{\boldsymbol{q}}[\boldsymbol{s}(k)]\rangle \overset{(b)}{\leq} \langle \boldsymbol{q}, \boldsymbol{r}^{(\ell)} - \epsilon \boldsymbol{c}^{(\ell)}\rangle - \langle \boldsymbol{q}, \boldsymbol{s}^*\rangle,$$

where $\boldsymbol{s}^* \in \mathcal{C}$. Here, (a) holds because the arrival processes are independent of the queue lengths and by definition of $\boldsymbol{\lambda}^{(\epsilon)}$; and (b) holds for any $\boldsymbol{s}^* \in \mathcal{C}$ because scheduling occurs according to the MaxWeight algorithm, and $E_{\boldsymbol{q}}[\boldsymbol{s}(k)]$ is the maximizer of MaxWeight over the capacity region $\mathcal{C}$, i.e., $\langle \boldsymbol{q}, E_{\boldsymbol{q}}[\boldsymbol{s}(k)]\rangle = \max_{\boldsymbol{x}\in\mathcal{C}} \langle \boldsymbol{q}, \boldsymbol{x}\rangle$. This last fact is proved by Hurtado-Lange and Maguluri (2019, lemma 1).

We now compute a vector $\boldsymbol{s}^* \in \mathcal{C}$. Define a vector $\boldsymbol{d}^{(\ell)}$ with elements $d_i^{(\ell)} \triangleq 1_{\{c_i^{(\ell)} = 0\}}$ for each $i \in \{1, \ldots, n\}$ and observe that $\boldsymbol{d}^{(\ell)}$ is orthogonal to $\boldsymbol{c}^{(\ell)}$. Because $\boldsymbol{r}^{(\ell)}$ is in the relative interior of the facet $\mathcal{F}^{(\ell)}$, there exists a positive number $\delta \in (0, 1)$ such that

$$\boldsymbol{r}^{(\ell)} + \epsilon\delta \boldsymbol{d}^{(\ell)} \in \textit{RelativeInterior}(\mathcal{F}^{(\ell)}).$$

Set $\boldsymbol{s}^* = \boldsymbol{r}^{(\ell)} + \epsilon\delta\boldsymbol{d}^{(\ell)}$; and note that if all the components of $\boldsymbol{c}^{(\ell)}$ are strictly positive, then $\boldsymbol{d}^{(\ell)} = \boldsymbol{0}$ and $\boldsymbol{s}^* = \boldsymbol{r}^{(\ell)}$. Then, we obtain

$$\begin{aligned}
E_{\boldsymbol{q}}[\langle \boldsymbol{q}, \boldsymbol{a}(k) - \boldsymbol{s}(k)\rangle] &\leq -\epsilon \langle \boldsymbol{q}, \boldsymbol{c}^{(\ell)} + \delta \boldsymbol{d}^{(\ell)}\rangle \\
&\overset{(a)}{\leq} -\epsilon \rho^{(\ell)} \sqrt{n} \|\boldsymbol{q}\|,
\end{aligned} \tag{5}$$

where $\rho^{(\ell)} \overset{\Delta}{=} \min_{i\in\{1,\dots,n\}} c_i^{(\ell)} + \delta d_i^{(\ell)}$. Here, (a) holds by the Cauchy-Schwarz inequality. Observe that $\rho^{(\ell)}$ is a constant independent of ϵ and that $\rho^{(\ell)} > 0$ by definition of δ and $\boldsymbol{d}^{(\ell)}$.

Using (3), (4), and (5) in (2), and rearranging terms, we obtain

$$E_{\boldsymbol{q}}[\Delta Z(\boldsymbol{q})] \leq \frac{K_1}{2\|\boldsymbol{q}\|} - \epsilon\rho^{(\ell)}\sqrt{n}.$$

Therefore, condition *(C1)* is satisfied with

$$\eta \overset{\Delta}{=} \frac{\epsilon\rho^{(\ell)}\sqrt{n}}{2}, \text{ and } \quad \kappa \overset{\Delta}{=} \frac{K_1}{\epsilon\rho^{(\ell)}\sqrt{n}}.$$

Now we show condition *(C2)*. We have

$$\begin{aligned}
|\Delta Z(\boldsymbol{q})| &= \big|\|\boldsymbol{q}(k+1)\| - \|\boldsymbol{q}(k)\|\big|\, \mathbb{1}_{\{\boldsymbol{q}(k)=\boldsymbol{q}\}} \\
&\overset{(a)}{\leq} \|\boldsymbol{q}(k+1) - \boldsymbol{q}(k)\| \mathbb{1}_{\{\boldsymbol{q}(k)=\boldsymbol{q}\}} \\
&\overset{(b)}{\leq} \|\boldsymbol{a}(k) - \boldsymbol{s}(k) + \boldsymbol{u}(k)\| \mathbb{1}_{\{\boldsymbol{q}(k)=\boldsymbol{q}\}} \\
&\overset{(c)}{\leq} (\|\boldsymbol{a}(k)\| + \|\boldsymbol{s}(k) - \boldsymbol{u}(k)\|) \mathbb{1}_{\{\boldsymbol{q}(k)=\boldsymbol{q}\}} \\
&\overset{(d)}{\leq} \sqrt{n}(A_{\max} + S_{\max}),
\end{aligned}$$

where (a) holds by triangle inequality; (b) holds by the dynamics of the queues; (c) holds by triangle inequality; and (d) because for each $i \in \{1,\dots,n\}$, we have $a_i(k) \leq A_{\max}$ and $s_i(k) - u_i(k) \leq s_i(k) \leq S_{\max}$ with probability one. Hence, condition *(C2)* is satisfied with $D \overset{\Delta}{=} \sqrt{n}(A_{\max} + S_{\max})$. Observe that $\eta \leq 1/2$ and $D \geq 2$ because we must have $A_{\max} \geq 1$ and $S_{\max} \geq 1$. Then, $\eta^2 \leq 4(e^D - 1 - D)$ holds trivially.

Therefore, setting $\theta^* \overset{\Delta}{=} \theta\epsilon$ in Lemma 15, we obtain

$$E[e^{\theta\epsilon\|\bar{\boldsymbol{q}}\|}] \leq 2\left(1 + \frac{2\sqrt{n}(A_{\max} + S_{\max})}{\epsilon\rho^{(\ell)}\sqrt{n}}\right) e^{\frac{\theta K_1}{\rho^{(\ell)}\sqrt{n}}}$$

for any $\theta \leq \Theta$, where, using that $\epsilon < 1$, we obtain

$$\Theta = \min\left\{1,\ \frac{\rho^{(\ell)}\sqrt{n}}{2\left(e^{\sqrt{n}(A_{\max}+S_{\max})} - 1 - \sqrt{n}(A_{\max} + S_{\max})\right)}\right\}.$$

Because $\langle \boldsymbol{c}^{(\ell)}, \bar{\boldsymbol{q}}\rangle = \|\bar{\boldsymbol{q}}_{\|}\|$ and the projection is nonexpansive, we have $\langle \boldsymbol{c}^{(\ell)}, \bar{\boldsymbol{q}}\rangle \leq \|\bar{\boldsymbol{q}}\|$. Hence, for every $\theta \in (0, \Theta]$, we have

$$E[e^{\theta\epsilon\langle \boldsymbol{c}^{(\ell)}, \bar{\boldsymbol{q}}\rangle}] \leq 2\left(1 + \frac{2\sqrt{n}(A_{\max} + S_{\max})}{\epsilon\rho^{(\ell)}\sqrt{n}}\right) e^{\frac{\theta K_1}{\rho^{(\ell)}\sqrt{n}}}. \quad \square$$

We finish this section with the proof of Lemma 15, which is based on Hajek 1982 (lemma 2.2). We state this lemma for completeness.

Lemma 16. (Hajek 1982, lemma 2.2) *Suppose Y_1 and Y_2 are random variables such that $|Y_1|$ is stochastically dominated by Y_2 and $E[e^{\theta_{\max} Y_2}] < \infty$ for some $\theta_{\max} > 0$. Then, for $\theta^* \in [0, \theta_{\max}]$,*

$$E[e^{\theta^* Y_1}] \leq 1 + \theta^* E[Y_1] + (\theta^*)^2 \sum_{i=2}^{\infty} \frac{(\theta_{\max})^{i-2}}{i!} E[Y_2^i].$$

Now we prove Lemma 15.

Proof of Lemma 15. We use Lemma 16 with $Y_1 = \Delta Z(x)$ and $Y_2 = D$. Because D is a constant, we have $E[e^{\theta_{\max} D}] < \infty$ for any finite $\theta_{\max}$. For simplicity, we pick $\theta_{\max} = 1$. Then, for any $\theta^* \in [0, 1]$, we have

$$\begin{aligned}
E[e^{\theta^* \Delta Z(X_k)} | X_k = x] &\leq 1 + \theta^* E[\Delta Z(X_k) \,|\, X_k = x] + (\theta^*)^2 \sum_{i=2}^{\infty} \frac{D^i}{i!} \\
&\overset{(a)}{\leq} 1 + \theta^* E[-\eta \mathbb{1}_{\{Z(X_k)\geq\kappa\}} + D\mathbb{1}_{\{Z(X_k)<\kappa\}} \,|\, X_k = x] + (\theta^*)^2(e^D - 1 - D) \\
&\overset{(b)}{=} 1 - \theta^*\eta + \theta^*(\eta + D)1_{\{Z(x)<\kappa\}} + (\theta^*)^2(e^D - \mathbb{1} - D),
\end{aligned} \tag{6}$$

where (a) holds by conditions *(C1)* and *(C2)*, solving the series in the last term; and (b) holds reorganizing terms.

Now we compute a bound for $E[e^{\theta^* Z(X_{k+1})}]$ as follows. Observe

$$\begin{aligned}
&E[e^{\theta^* Z(X_{k+1})}]\\
&\quad = E[E[e^{\theta^* Z(X_k)}e^{\theta^* \Delta Z(X_k)}|X_k]]\\
&\quad \overset{(a)}{\le} (1-\eta\theta^* + (\theta^*)^2(e^D - 1 - D))E[e^{\theta^* Z(X_k)}] + \theta^*(\eta + D)e^{\theta^*\kappa}\\
&\quad \overset{(b)}{\le} (1-\eta\theta^* + (\theta^*)^2(e^D - 1 - D))^{k+1}E[e^{\theta^* Z(X_0)}] + \theta^*(\eta + D)e^{\theta^*\kappa}\sum_{i=0}^{k}(1-\eta\theta^* + (\theta^*)^2(e^D - 1 - D))^i,
\end{aligned}$$

where (a) holds by (8) and reorganizing terms; and (b) holds by using the recursion from the previous line $k+1$ times because the condition $\eta^2 \le 4(e^D - 1 - D)$ ensures that $(1-\eta\theta^* + (\theta^*)^2(e^D - 1 - D)) \ge 0$.

The upper bound converges only if $1-\eta\theta^* + (\theta^*)^2(e^D - 1 - D) < 1$, so we solve for θ^* such that

$$1-\eta\theta^* + (\theta^*)^2(e^D - 1 - D) \le 1 - \frac{\eta\theta^*}{2}.$$

We obtain that

$$\theta^* \le \frac{\eta}{2(e^D - 1 - D)}.$$

For such θ^*, we have

$$E[e^{\theta^* Z(X_{k+1})}] \le \left(1 - \frac{\eta\theta^*}{2}\right)^{k+1} E[e^{\theta^* Z(X_0)}] + \theta^*(\eta + D)e^{\theta^*\kappa}\sum_{i=0}^{k+1}\left(1 - \frac{\eta\theta^*}{2}\right)^i.$$

Then, taking the limit of the recursion as $k \to \infty$, we obtain the result. □

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