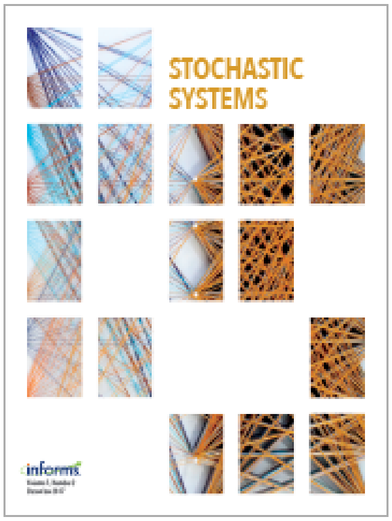


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# Martin Boundary of a Degenerate Reflected Brownian Motion in a Wedge

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**Abstract.** We consider an outward degenerate drifted Brownian motion in the quarter plane with oblique reflections on the boundaries. In this article, we explicitly compute the Laplace transforms of the Green's functions associated with the process. These Laplace transforms are expressed as an infinite sum of products by iterating a functional equation, which is deeply linked to the compensation method. We also derive the asymptotics of the Green's functions along all possible paths and determine the (minimal) Martin boundary. Finally, we provide explicit formulae for all the corresponding positive harmonic functions.



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**Keywords:** asymptotics of Green's functions • compensation approach • Martin boundary • reflected Brownian motion

## 1. Introduction and Main Results

### 1.1. Context

The semimartingale reflecting Brownian motion (SRBM) in two-dimensional convex cones is a classical topic in probability theory. Problems such as existence and uniqueness (Harrison and Reiman 1981, Taylor and Williams 1993), recurrence and transience conditions (Williams 1985, Hobson and Rogers 1993), the study of stationary distribution properties (Harrison and Williams 1987b, Dieker and Moriarty 2009, Dai and Miyazawa 2011, Franceschi and Kourkova 2017), and many others have been studied extensively in the literature, mostly under the assumption of a nondegenerate covariance matrix.

An important problem in transient SRBM is the analysis of Green's functions, which can be divided into two parts:

$(P_1)$ Obtaining the Laplace transforms of the Green's functions;

$(P_2)$ Computing the asymptotics of the Green's functions along all trajectories of the SRBM.

Solutions to $(P_1)$ in the half-plane can be expressed directly in terms of a rational function of two variables, $(x, Y(x))$, where $Y(x)$ is a branch of a certain two-valued algebraic function, as detailed in Ernst and Franceschi (2021). However, solving $(P_1)$ in a general cone presents a significantly greater challenge. Specifically, for nondegenerate SRBM in the quarter plane with three domains, the Laplace transforms are obtained as singular integral representations via boundary value problems, as shown in Franceschi and Raschel (2019) and Franceschi (2020). Although these expressions are explicit, they are not particularly amenable to in-depth analysis. Fortunately, they are not required to resolve the second issue $(P_2)$. In fact, only the locations of the dominant singularities of unknown Laplace transforms are necessary to compute the asymptotics of the Green's functions. Problem $(P_2)$ for nondegenerate SRBM has been solved in the half-plane in Ernst and Franceschi (2021) and in an arbitrary wedge in Franceschi et al. (2024b). The approach followed in these articles has been developed in Malyshev (1973), Kurkova and Malyshev (1998), Dai and Miyazawa (2011), Franceschi and Kourkova (2017), Kurkova and Raschel (2011), and Fayolle et al. (2017) and can be considered as a version of the so-called kernel method. For more information, see the survey by Zhao (2022). The kernel $\gamma(x,y)$ of the SRBM is given by one half of the quadratic form of the covariance matrix plus the linear form of the drift inside the cone. The interplay between the branches of algebraic functions $X(y)$ and $Y(x)$, defined by the kernel equation $\gamma(x,y)=0$, allows us to analytically continue unknown Laplace transforms and to determine their singularities. The inverse Laplace transforms, combined with the saddle point method, then yield asymptotic expansions for the Green's functions. This procedure provides asymptotic developments of Green's functions with arbitrarily many terms, but with unknown multiplicative constants. These constants may be derived—albeit somewhat indirectly—from the solutions to $(P_1)$.

The degenerate SRBM in two-dimensional cones, that is, with a covariance matrix of rank one, has been studied far less extensively. In Ichiba and Karatzas (2022) and Franceschi et al. (2024a), it arises as the gap process between three particles moving and colliding in $\mathbb{R}^1$. The construction of this three-particle process relies on the Skorokhod reflection approach, as developed in Harrison and Reiman (1981), to define pathwise reflected Brownian motion.

In the present article, we consider a class of degenerate transient SRBMs in the quadrant, defined by conditions (1.1)–(1.3), and solve both problems $(P_1)$ and $(P_2)$. The Laplace transforms of the Green's functions are expressed in terms of infinite series in product form. This result follows from the compensation method, initially introduced in Adan et al. (1993) to obtain the stationary measure for certain degenerate random walks in a quadrant. This approach has since been successfully applied to queueing systems (Adan et al. 1991, Adan 1994). It has also been used to derive generating functions for random walks with small steps Adan et al. (2013) and, more recently, to determine the harmonic functions of singular walks in the quadrant (Hoang et al. 2023). In Franceschi et al. (2024a), for instance, it was used to derive the explicit form of the stationary distribution and in Franceschi (2024) to determine the Martin boundary of killed degenerate Brownian motion in a two-dimensional cone.

In this article, we compute the asymptotics of the Green's functions along all trajectories. To achieve this, we adapt the approach described earlier and developed in Kurkova and Malyshev (1998), Dai and Miyazawa (2011), Fayolle et al. (2017), Franceschi and Kourkova (2017), Ernst and Franceschi (2021), and Franceschi et al. (2024b) to this class of degenerate SRBMs. A key difference is that, unlike the nondegenerate case—in which the kernel equation for the process defines an ellipse—the kernel equation for the degenerate case defines a parabola in $\mathbb{R}^2$. The multiplicative constants in the asymptotic expressions of the Green's functions, derived from the solution to $(P_1)$, are made explicit in terms of infinite series in product form. The significance of these constants—viewed as functions of the starting point of the process—extends beyond asymptotic precision; they also yield all positive harmonic functions for the DRBM via the Martin boundary theory.

Initiated by Martin (1941) and further developed by Hunt (1957), Doob (1959), and Kunita and Watanabe (1965), this theory is summarized in Doob (1984) and Chung and Walsh (2005). Its aim is to describe the asymptotic behavior of the process and to characterize all nonnegative superharmonic and harmonic functions. The limits of the Martin kernel along the trajectories of the process, when they exist, compactify the state space and form the so-called Martin boundary. This procedure allows every nonnegative harmonic function to be expressed as an integral representation over the Martin boundary. In Ney and Spitzer (1966), Ignatiouk-Robert (2009, 2010), Ignatiouk-Robert and Loree (2010), and Duraj et al. (2022), the Martin boundary is identified via large deviation principles. It has also been obtained from the asymptotics of Green's functions in Kurkova and Malyshev (1998), Kurkova and Raschel (2011), Ernst and Franceschi (2021), and Franceschi et al. (2024b). In this article, using the solutions to problems $(P_1)$ and $(P_2)$ described above, we determine the Martin boundary, the minimal one, and provide explicit expressions for all positive harmonic functions.

## 1.2. Main Results

**The Degenerate Reflected Brownian Motion, Assumptions.** We consider a degenerate Brownian motion $(Z_t)_{t\geq 0}$ in a quadrant, with oblique reflection at the boundaries. By *degenerate*, we mean that the covariance matrix is of rank 1. This obliquely reflected process was studied in Ichiba and Karatzas (2022), and its rigorous definition is provided in Section 2. The parameters of the degenerate reflected Brownian motion are given by

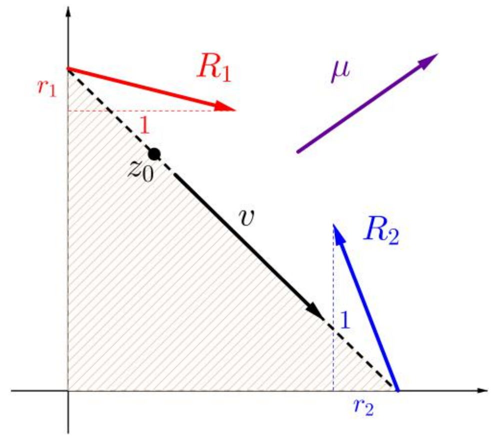
$$\Sigma = \begin{pmatrix} \sigma_1^2 & -\sigma_1\sigma_2 \\ -\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}, \mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, R = \begin{pmatrix} 1 & r_2 \\ r_1 & 1 \end{pmatrix} = \begin{pmatrix} R_1 & R_2 \end{pmatrix}, \tag{1.1}$$

where $\Sigma$ is the degenerate covariance matrix ($\det(\Sigma) = 0$), $\mu$ is the drift, and columns of $R$ represent the reflection directions from the axes. The direction $v = (v_1, v_2)^T = (\sigma_1, -\sigma_2)$ is antidiagonal, that is, $v_1 v_2 < 0$ (see Figure 1). When the process does not hit the boundaries, it behaves like a one-dimensional Brownian motion along the direction $v$ (plus the drift). Our main assumptions in this article are as follows:

$$\mu_1 > 0, \quad \mu_2 > 0, \tag{1.2}$$

$$r_1 > -\frac{\sigma_2}{\sigma_1}, \quad r_2 > -\frac{\sigma_1}{\sigma_2}. \tag{1.3}$$

**Figure 1.** Reflections $R_1, R_2$ on the Edges, the Drift $\mu$, and the Direction $v$ of the Degenerate Brownian Motion




*Note.* The process starting from $z_0$ never reaches the hatched region.

Assumption (1.2) ensures that the process is transient, whereas (1.3) specifies that the reflection vectors $R_1 = (1, r_1)^T$ (on $\{x = 0\}$) and $R_2 = (1, r_2)^T$ (on $\{y = 0\}$) point outward from the direction $v$ of the Brownian motion (see Figure 1).

In Sections 1–8, we state and prove our results under the additional assumption

$$\sigma_1 = \sigma_2 = 1, \quad \mu_1 + \mu_2 = 1. \tag{1.4}$$

Results for the general case, that is, without Assumption (1.4), are stated and proved in Section 9. In fact, they are easily deduced from the results under (1.4) by means of a simple space-time transformation.

**Green's Functions.** We show that for any starting point $z_0 \in \mathbb{R}_+^2$, there exists a density $g^{z_0}(\cdot)$ of the Green's measure $G(z_0, \cdot)$ on the quadrant defined as

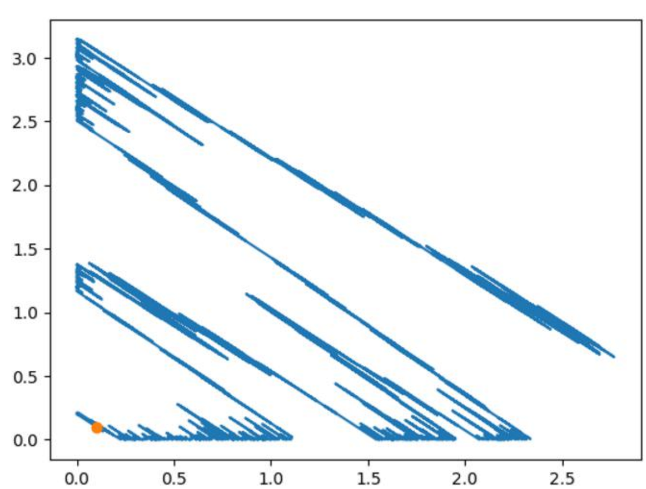
$$G(z_0, A) := \int_0^{+\infty} \mathbb{P}_{z_0}(Z_t \in A) dt = \int_A g^{z_0}(z) dz. \tag{1.5}$$

Functions $g^{z_0}(\cdot)$ are called the Green's functions. We also define the Green's measures on the sides of the wedge

$$H_i(z_0, A) := \mathbb{E}_{z_0}\left[\int_0^\infty \mathbb{1}_A(Z_t) dL_t^i\right], \quad i = 1, 2, \tag{1.6}$$

where $(L_t^1)_{t\geq 0}$ (resp. $(L_t^2)_{t\geq 0}$) is the local time of the process on the axis $\{x = 0\}$ (resp. $\{y = 0\}$). The measure $H_1$ has its support on the vertical axis, and $H_2$ has its support on the horizontal axis. Laplace transforms $\varphi(x, y)$ of $G(z_0, \cdot)$

**Figure 2.** Example of a Typical Path (Over a Finite Time Horizon) of the Drifted Degenerate Brownian Motion




*Note.* The initial point is marked in orange.

and $\varphi_1(y), \varphi_2(x)$ of $H_1(z_0,\cdot), H_2(z_0,\cdot)$ are related by the functional equation,

$$-\gamma(x,y)\varphi(x,y) = \gamma_1(x,y)\varphi_1(y) + \gamma_2(x,y)\varphi_2(x) + e^{(x,y)\cdot z_0}, \ \ Re(x) < 0, Re(y) < 0, \tag{1.7}$$

where

$$\gamma(x,y) = \frac{1}{2}(x-y)^2 + \mu_1 x + \mu_2 y, \tag{1.8}$$

and

$$\gamma_1(x,y) = R^1 \cdot (x,y) = x + r_1 y, \qquad \gamma_2(x,y) = R^2 \cdot (x,y) = r_2 x + y. \tag{1.9}$$

It can be viewed as a balance equation for Green's measures between the interior and the edges of the quadrant. Let us define

$$\mathcal{P} = \{(x,y) \in \mathbb{R}^2, \ \gamma(x,y) = 0\}. \tag{1.10}$$

The functional Equation (3.1) is similar to that in Franceschi et al. (2024b); however, an important difference is that $\mathcal{P}$ is now a parabola rather than an ellipse. This distinction is what allows the compensation method to be effective, leading to explicit expressions for the Laplace transforms and positive harmonic functions.

**Explicit Expressions for Laplace Transforms.** The first results of the article provide explicit expressions for Laplace transforms $\varphi_1$ and $\varphi_2$ in terms of infinite series of product forms, given by formulae (4.24) and (4.23), which we do not specify here. Function $\varphi$ is derived from $\varphi_1$ and $\varphi_2$ via the functional Equation (1.7).

**Asymptotics of Green's Functions.** We now focus on the asymptotics of $g^{z_0}(r\cos(\alpha), r\sin(\alpha))$ as $r \to +\infty$ and $\alpha \to \alpha_0 \in [0, \pi/2]$. For any direction $\alpha$, we denote by $(x(\alpha), y(\alpha))$ a corresponding point on the parabola given by

$$(x(\alpha), y(\alpha)) = \arg\max_{(x,y)\in\mathcal{P}} (\cos(\alpha)x + \sin(\alpha)y), \tag{1.11}$$

see Figure 3(a). It can be computed explicitly as

$$(x(\alpha), y(\alpha)) = \left( \frac{(\mu_2 - \tan(\alpha)\mu_1)(\mu_2 + \tan(\alpha))(1+\mu_2)}{2(1+\tan(\alpha))^2}, \frac{(\mu_2 - \tan(\alpha)\mu_1)(1+\mu_1(1+\tan(\alpha)))}{2(1+\tan(\alpha))^2} \right). \tag{1.12}$$

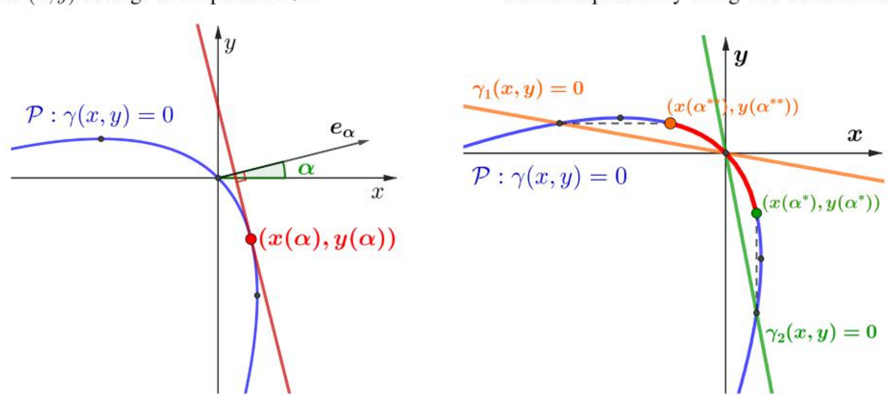
Let us define two particular directions

$$\alpha^* := \begin{cases} 0 & \text{if} \quad (r_1+1)\mu_2 \le 2 \\ \arctan\left(\dfrac{(1+r_1)\mu_2 - 2}{2+(1+r_1)\mu_1}\right) & \text{if} \quad (r_1+1)\mu_2 > 2. \end{cases} \tag{1.13}$$

**Figure 3.** Geometric Interpretation of $(x(\alpha), y(\alpha))$, $\alpha^*$, and $\alpha^{**}$

(a) The point $(x(\alpha), y(\alpha))$ maximises the scalar product $\langle (x,y), e_\alpha \rangle$ where $e_\alpha = (\cos(\alpha), \sin(\alpha))$ and $(x,y)$ belongs to the parabola $\mathcal{P}$.

(b) In the case $0 < \alpha^* < \alpha^{**} < \pi/2$, angles $\alpha^*$ and $\alpha^{**}$ introduced in (1.13), (1.14) can be defined equivalently using this construction.



$$\alpha^{**} := \begin{cases} \arctan\left(\dfrac{(1+r_2)\mu_2+2}{(1+r_2)\mu_1-2}\right) & \text{if} \quad (r_2+1)\mu_1 > 2 \\ \pi/2 & \text{if} \quad (r_2+1)\mu_1 \leq 2, \end{cases} \tag{1.14}$$

see Figure 3(b) for their geometric interpretation. We always have $\alpha^* < \alpha^{**}$, as will be proved in Section 5.

In the following theorem, we summarize the asymptotics of Green's functions for directions $\alpha_0 \in (0, \pi/2)\backslash\{\alpha^*, \alpha^{**}\}$. The ones for $\alpha_0 \in \{0, \alpha^*, \alpha^{**}, \pi/2\}$ are given later in Theorems 6, 7, and 8.

**Theorem 1** (Asymptotics in the Quadrant, General Case). *Assume* (1.2) *to* (1.4). *Then, the Green's density function $g^{z_0}$ of this process has the following asymptotics as $\alpha \to \alpha_0$ and $r \to \infty$.*

- *If $\alpha^* < \alpha_0 < \alpha^{**}$, then*

$$g^{z_0}(r\cos(\alpha), r\sin(\alpha)) \underset{\substack{r\to\infty \\ \alpha\to\alpha_0}}{\sim} c_{\alpha_0} h_{\alpha_0}(z_0) \frac{e^{-r(\cos(\alpha)x(\alpha)+\sin(\alpha)y(\alpha))}}{\sqrt{r}}. \tag{1.15}$$

- *If $\alpha_0 < \alpha^*$, then*

$$g^{z_0}(r\cos(\alpha), r\sin(\alpha)) \underset{\substack{r\to\infty \\ \alpha\to\alpha_0}}{\sim} c^* h_{\alpha^*}(z_0) e^{-r(\cos(\alpha)x(\alpha^*)+\sin(\alpha)y(\alpha^*))}. \tag{1.16}$$

- *If $\alpha_0 > \alpha^{**}$, then*

$$g^{z_0}(r\cos(\alpha), r\sin(\alpha)) \underset{\substack{r\to\infty \\ \alpha\to\alpha_0}}{\sim} c^{**} h_{\alpha^{**}}(z_0) e^{-r(\cos(\alpha)x(\alpha^{**})+\sin(\alpha)y(\alpha^{**}))}. \tag{1.17}$$

*where $c_{\alpha_0} = \frac{1}{\sqrt{2\pi(\cos(\alpha_0)+\sin(\alpha_0))}}$, $c^*$ and $c^{**}$ are positive explicit constants depending only on the parameters of the degenerate reflected Brownian motion (see* (6.1)*), and $h_\alpha(z_0), h_{\alpha^*}(z_0), h_{\alpha^{**}}(z_0)$ are harmonic functions given in Theorem 2. Furthermore, $h_\alpha(z_0), h_{\alpha^*}(z_0), h_{\alpha^{**}}(z_0)$ are nonzero.*

**Explicit Expressions for Positive Harmonic Functions with the Compensation Method.** Let us recall the following definition: a function $h : \mathbb{R}_+^2 \to \mathbb{R}$ is harmonic if and only if for all $t \geq 0$ and $z_0 \in \mathbb{R}_+^2$,

$$\mathbb{E}_{z_0}[h(Z_t)] = h(z_0). \tag{1.18}$$

All functions $h_\alpha, \alpha \in [\alpha^*, \alpha^{**}]$ are harmonic. These functions are explicitly stated in Theorem 2 below and will be derived in this article using the compensation method. The essence of this method is to construct functions that satisfy the partial differential equation along with boundary conditions:

$$\begin{cases} (H_0) & \mathcal{G}h = 0 & \text{on} \quad (0, +\infty)^2, \\ (H_1) & \partial_{R_1} h(0, y) = 0, & y \geq 0 \\ (H_2) & \partial_{R_2} h(x, 0) = 0, & x \geq 0 \end{cases} \tag{1.19}$$

where $\mathcal{G} = \frac{1}{2}\nabla \cdot \Sigma\nabla + \mu \cdot \nabla$. Those function are harmonic, as will be noticed in Section 4.1.

For $(a_0, b_0) \in \mathcal{P}$ and $k \in \mathbb{Z}\backslash\{0\}$, we set

$$a_{2k} = -2k^2 + 2(a_0 - b_0 - \mu_2)k + a_0, \ a_{2k+1} = a_{2k} \tag{1.20}$$

$$b_{2k} = -2k^2 + 2(a_0 - b_0 + \mu_1)k + b_0, \ b_{2k+1} = b_{2k+2} \tag{1.21}$$

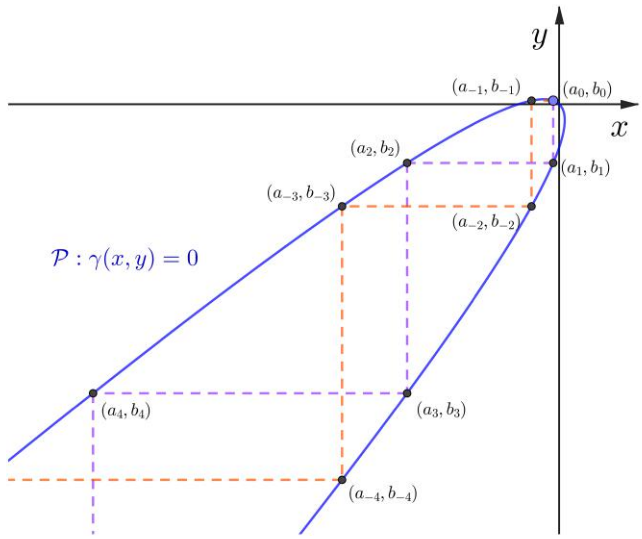
As illustrated in Figure 4, points $(a_p, b_p) \in \mathcal{P}$ are constructed by following the "downstairs" path on the parabola, applying successively automorphisms that leave invariant the first or the second coordinate, respectively.

**Theorem 2** (Explicit Expressions for Harmonic Functions $(h_\alpha)_{\alpha\in[\alpha^*,\alpha^{**}]}$). *Assume* (1.2) *to* (1.4). *Then, the functions $(h_\alpha)_{\alpha\in[\alpha^*,\alpha^{**}]}$ are harmonic and are given by the following formulae:*

- *For $\alpha \in (\alpha^*, \alpha^{**})$, taking $(a_0, b_0) = (x(\alpha), y(\alpha))$, we have*

$$h_\alpha : z_0 \longmapsto \sum_{m=-\infty}^{+\infty} \kappa_m(\alpha) e^{z_0\cdot(a_m, b_m)} \tag{1.22}$$

**Figure 4.** Parabola $\mathcal{P}$ and Points $(a_n, b_n)$ on the Parabola




*where* $\kappa_0(\alpha) = 1$ *and*

$$\kappa_m(\alpha) = \begin{cases} (-1)^m \left[\prod_{k=0}^{\lfloor \frac{m}{2} \rfloor -1} \frac{\frac{\gamma_1}{\gamma_2}(a_{2k+1}, b_{2k+1})}{\frac{\gamma_1}{\gamma_2}(a_{2k+2}, b_{2k+2})}\right] \frac{\gamma_2(a_0, b_0)}{\gamma_2(a_m, b_m)} & \text{if } m > 0 \\ (-1)^m \left[\prod_{k=0}^{\lfloor \frac{-m}{2} \rfloor -1} \frac{\frac{\gamma_2}{\gamma_1}(a_{-2k-1}, b_{-2k-1})}{\frac{\gamma_2}{\gamma_1}(a_{-2k-2}, b_{-2k-2})}\right] \frac{\gamma_1(a_0, b_0)}{\gamma_1(a_m, b_m)} & \text{if } m < 0 \end{cases} \tag{1.23}$$

*(with the convention* $\prod_{k=0}^{-1} = 1$*).*

- *For* $\alpha = \alpha^*$,
  - *If* $\frac{2}{r_2+1} < \mu_2$, *then* $\alpha^* = 0$ *and* $h_0 : z_0 \longmapsto \partial_\alpha [h_\alpha(z_0)]_{\alpha=0^+}$.
  - *If* $\frac{2}{r_2+1} > \mu_2$, *then* $\alpha^* > 0$ *and taking* $(a_0, b_0) = (x(\alpha^*), y(\alpha^*))$,

$$h_{\alpha^*} : z_0 \longmapsto e^{z_0 \cdot (a_1, b_1)} + \sum_{m=2}^{+\infty} \widehat{\kappa}_m(\alpha^*) e^{z_0 \cdot (a_m, b_m)}, \tag{1.24}$$

*where*

$$\widehat{\kappa}_m(\alpha^*) = (-1)^{m+1} \frac{\gamma_1(a_1, b_1)}{\frac{\gamma_1}{\gamma_2}(a_2, b_2)} \left[\prod_{k=1}^{\lfloor \frac{m}{2} \rfloor -1} \frac{\frac{\gamma_1}{\gamma_2}(a_{2k+1}, b_{2k+1})}{\frac{\gamma_1}{\gamma_2}(a_{2k+2}, b_{2k+2}}\right] \frac{1}{\gamma_2(a_m, b_m)}.$$

  - *If* $\frac{2}{r_2+1} = \mu_2$, *then* $\alpha^* = 0$ *and taking* $(a_0, b_0) = (x(0), y(0))$,

$$h_0 : z_0 \longmapsto 2e^{z_0 \cdot (a_0, b_0)} + \sum_{m=-\infty}^{-1} \kappa_m(\alpha^*) e^{z_0 \cdot (a_m, b_m)} + \sum_{m=2}^{+\infty} \tilde{\kappa}_m(\alpha^*) e^{z_0 \cdot (a_m, b_m)} \tag{1.25}$$

*where*

$$\tilde{\kappa}_m(\alpha^*) = (-1)^{m+1} \frac{\gamma_1(a_1, b_1)}{\frac{\gamma_1}{\gamma_2}(a_2, b_2)} \left[\prod_{k=1}^{\lfloor \frac{m}{2} \rfloor -1} \frac{\frac{\gamma_1}{\gamma_2}(a_{2k+1}, b_{2k+1})}{\frac{\gamma_1}{\gamma_2}(a_{2k+2}, b_{2k+2})}\right] \frac{1}{\gamma_2(a_m, b_m)}.$$

- *For* $\alpha = \alpha^{**}$, *symmetrical formulae hold replacing* $r_1$ *by* $r_2$, $\mu_1$ *by* $\mu_2$, *and* $0$ *by* $\frac{\pi}{2}$.

Note that if $\alpha < \alpha^*$ or $\alpha > \alpha^{**}$, expression (1.22) may define a harmonic function that is not necessarily nonnegative everywhere.

The Martin boundary and its minimality are derived from Theorems 1 and 2, together with the further technical results in Theorems 6, 7, and 8 concerning the asymptotics of Green functions along the directions $0, \alpha^*, \alpha^{**}$ and $\pi/2$.

**Theorem 3** (Martin Boundary). *Under (1.2) to (1.4), the Martin boundary $\Gamma$ of the degenerate reflected Brownian motion is homeomorphic to $[\alpha^*, \alpha^{**}]$ via the mapping*

$$\alpha \in [\alpha^*, \alpha^{**}] \longmapsto h_\alpha(\cdot)/h_\alpha(0) \in \Gamma. \tag{1.26}$$

*Furthermore, the Martin boundary is minimal.*

**Remark 1** (On Assumptions (1.2) and (1.3) and Possible Extensions). Some results may be generalised with extended parameters

- Regarding assumption (1.2), similar results could be established under the more general condition $\mu_1\sigma_2 + \mu_2\sigma_1 > 0$. This condition is equivalent to the orientation of the parabola toward $x \to -\infty$ and $y \to -\infty$. It is also necessary for the convergence of the expressions defining $h_\alpha$—specifically, equation (1.22). Namely, if $\mu_2 < 0$, then the Laplace transform $\varphi_2$ would have a pole at zero. Because of the technical nature of this paper, we have chosen to restrict our analysis to Assumption (1.2). Investigating how the Martin boundary is affected by the presence of such a pole could be an interesting direction for future work.

- If (1.3) is not satisfied, then the arguments that yield the explicit expressions of the harmonic functions fail. In particular, attempts to construct the functions $h_\alpha$ without this assumption often lead to signed functions that, although possibly harmonic, are not necessarily nonnegative. For interested readers, the only step in our argument that fails for general reflection vectors is equation (4.18), which may offer a direction for future investigation.

### 1.3. Plan of the Article

In Section 2, we define the degenerate reflected Brownian motion. We then derive the functional Equation (1.7) in Section 3 and meromorphically extend Laplace transforms on the edges up to their singularities. In Section 4, we obtain the explicit form of the Laplace transforms iterating the functional Equation (1.7). Next, in Section 5, we carry out preparatory work to derive the asymptotics of Green's functions. These asymptotics are computed in all directions in Sections 6 and 7 by the saddle point method. This enables us to prove Theorems 1 and 2 by employing the explicit expressions from Section 4. In Section 8, we establish the asymptotics of the Martin kernel and identify all the harmonic functions. We also prove the minimality of the Martin boundary and conclude the proof of Theorem 3. Finally, in Section 9, we treat the general case of the model without Assumption (1.4) via a linear transformation of space and time.

## 2. Definition of the Process

Throughout the following, the filtered space we consider is always the space of continuous functions $\mathcal{C}(\mathbb{R}_+, \mathbb{R}_+^2)$ with the standard $\sigma-$field and the usual filtration. The following background definition is taken from Taylor and Williams (1993), where the nondegenerate reflected Brownian motion is studied.

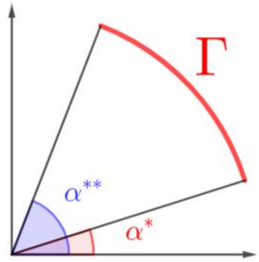
**Definition 1** (Degenerate Reflecting Brownian Motion). Let $\Sigma, R$, and $\mu$ be defined as in (1.1). A degenerate reflecting Brownian motion (DRBM) associated with the data $(\Sigma, \mu, R)$ is a process $(Z_t)_{t\geq 0}$ and a family of measures $(\mathbb{P}_{z_0})_{z_0 \in \mathbb{R}_+^2}$ such that $(Z_t)_{t\geq 0}$ can be written as

$$Z_t = X_t + RL_t \in \mathbb{R}_+^2, \ t \geq 0, \tag{2.1}$$

where

- $(X_t - \mu t)_{t\geq 0}$ is an adapted degenerate Brownian motion (with zero drift) of covariance $\Sigma$ starting from $z_0$ under $\mathbb{P}_{z_0}$.

**Figure 5.** Martin Boundary $\Gamma$ When $0 < \alpha^*$ and $\alpha^{**} < \pi/2$



- L is an adapted two-dimensional process starting from 0 such that $\mathbb{P}_{z_0}-$ almost surely, and its components $L^1, L^2$ are continuous and nondecreasing with $\mathrm{supp}(dL^i) \subset \{t \geq 0, Z_t^i = 0\}$; that is, $L^i$ increases only when $Z_t^i = 0$.

Note that under $\mathbb{P}_{z_0}$, $Z$ can be written as $(Z_t)_{t\geq 0} = (z_0 + vB_t + \mu t + RL_t)_{t\geq 0}$, where $(B_t)_{t\geq 0}$ is a one-dimensional Brownian motion and $v = (\sigma_1, -\sigma_2)$ $(= (1, -1)$ under (1.4)) is the unique eigenvector (up to a scalar multiplication) associated with the positive eigenvalue of the covariance matrix.

**Theorem 4** (Existence, Uniqueness, and Strong Markov Property). *Suppose that $|r_1 r_2| < 1$. Then, for any starting point $z_0$, there exists a DRBM associated with $(\Sigma, \mu, R)$. The processes $Z$ and $(Z, L)$ are pathwise unique (according to the associated degenerate Brownian motion). Furthermore, $Z$ is a semimartingale, a Feller process (i.e., for any $t \geq 0$, $x \longmapsto \mathbb{E}_x[f(Z_t)]$ is continuous whenever $f$ is bounded and continuous), and a Strong Markov process.*

**Proof.** Define the matrix $Q = I - R$, whose spectral radius is $\rho(Q) = \sqrt{|r_1 r_2|} < 1$. By theorem 1 in Harrison and Reiman (1981), for any continuous path $x = (x_t)_{t\geq 0} \subset \mathbb{R}^2$, there exists a unique solution $(z_t)_{t\geq 0} = \psi(x)$ of the Skorokod problem

$$z_t = x_t + R(l_t^1, l_t^2)^T, \quad t \geq 0$$

where $(z_t)_{t\geq 0} \subset \mathbb{R}_+^2$, and for $i \in \{1, 2\}$, $(l^i)_{t\geq 0}$ is a continuous, increasing function with $\mathrm{supp}(dl^i) \subset \{t \geq 0, z_t^i = 0\}$. Moreover, $\psi$ is continuous in the topology of uniform convergence on compact sets. This yields the stated result with $Z = \psi(X)$. □

As in the nondegenerate case, there may be existence and uniqueness in law if $R$ is a general $\mathcal{S}$-matrix Taylor and Williams (1993) (without assuming $|r_1 r_2| < 1$) but not path-wise uniqueness (Bass and Burdzy 2024). To avoid excessive technicality, we work under assumption $|r_1 r_2| < 1$.

**Proposition 1** (Transience). *Under conditions (1.2) and (1.3), the DRBM is a transient Markov process.*

**Proof.** Consider $w = (\sigma_2, \sigma_1)$, which is orthogonal to the direction of the Brownian motion. It suffices to note that $(Z_t \cdot w)_{t\geq 0}$ is almost surely strictly increasing and tends to $+\infty$ because $Z_t \cdot w \geq \mu \cdot wt$ by (1.3). □

We recall the definition of Green's measure $G(z_0, \cdot)$ and $H_i(z_0, \cdot)$ from (1.5) and (1.6). Assumption (1.2) on the drift is crucial for the following proposition.

**Proposition 2** (Densities and Laplace Transforms). *Suppose that assumptions (1.2) and (1.3) hold. Then, Green's measure $G(z_0, \cdot)$ has a density $g^{z_0}(\cdot)$ with respect to the Lebesgue measure. Functions $g^{z_0}(\cdot)$ are called Green's functions. Furthermore, measures $H_i(z_0, \cdot)$ $(i = 1, 2)$ have densities $f_i^{z_0}(\cdot)$ with respect to the one-dimensional Lebesgue measure.*

**Proof.** Let $A$ be a compact set of $\mathbb{R}_+^2$ at a positive distance of the edges. Define the stopping times:

$$\sigma = \inf\{t \geq 0, Z_t \in A\}, \quad \tau = \inf\{t \geq \sigma, Z_t \in \partial\mathbb{R}_+^2\}.$$

Considering the back-and-forth trajectories between $A$ and $\partial\mathbb{R}_+^2$ (see Harrison and Williams 1987a and Lemma 9 in Section 7), we can reduce the proof to showing that

$$\mathbb{E}_{z_0}\left[\int_\sigma^\tau 1_A(Z_s) ds\right] = 0.$$

Then, by the Strong Markov property, it suffices to prove the result for a nonreflected degenerate Brownian motion. By Assumption (1.2), rotating the plane so that the $x$-axis aligns with the drift direction reduces the problem to one-dimensional Brownian motion. The proposition then follows from elementary properties of the latter. □

**Definition 2** (Laplace Transforms of Green's Measures). We denote the Laplace transforms of $G(z_0, \cdot)$ by

$$\varphi(x, y) := \mathbb{E}_{z_0}\left[\int_0^\infty e^{(x,y)\cdot Z_t} dt\right] = \int_{\mathbb{R}_+^2} e^{(x,y)\cdot z} g^{z_0}(z) dz$$

and the Laplace transforms of $H_1(z_0, \cdot), H_2(z_0, \cdot)$ by

$$\varphi_1(y) := \mathbb{E}_{z_0}\left[\int_0^\infty e^{(0,y)\cdot Z_t} dL_t^1\right], \quad \varphi_2(x) := \mathbb{E}_{z_0}\left[\int_0^\infty e^{(x,0)\cdot Z_t} dL_t^2\right].$$

For brevity, we omit the dependence on the starting point in the notation for the Laplace transforms. However, when relevant, we will denote this dependence explicitly as $\varphi^{z_0}(x, y), \varphi_1^{z_0}(y)$ and $\varphi_2^{z_0}(x)$.

## 3. Functional Equation, Kernel, and Analytic Continuation

From now on, we assume (1.2) to (1.4). As mentioned in the introduction, Laplace transforms $\varphi, \varphi_1, \varphi_2$ are linked by a functional equation.

**Proposition 3** (Functional Equation). *If $Re(x) < 0$ and $Re(y) < 0$, then $\varphi_1(y)$, $\varphi_2(x)$, and $\varphi(x,y)$ converge, and the following equation holds:*

$$-\gamma(x,y)\varphi(x,y) = \gamma_1(x,y)\varphi_1(y) + \gamma_2(x,y)\varphi_2(x) + e^{(x,y)\cdot z_0}, \tag{3.1}$$

*where $\gamma, \gamma_1$, and $\gamma_2$ are defined in* (1.8) *and* (1.9).

**Proof.** We apply Itô's formula to the semimartingale $(Z_t)_{t\geq 0}$ and the function $(u,v) \to e^{xu+yv}$. Then,

$$e^{(x,y)\cdot Z_t} - e^{(x,y)\cdot z_0} = \int_0^t e^{(x,y)\cdot Z_t}(x,y)^T \cdot dB_s + \gamma(x,y)\int_0^t e^{(x,y)\cdot Z_s} ds + \sum_{i=1}^{2}\gamma_i(x,y)\int_0^t e^{(x,y)\cdot Z_s} dL_s^i\cdot, \tag{3.2}$$

where $(B_t)_{t\geq 0} = (X_t - \mu t)_{t\geq 0}$ is the non-reflected degenerate Brownian motion associated with the process (see Definition 1). Next, taking the expectation and letting $t$ to $+\infty$, we derive (3.1). See (Franceschi et al. 2024b, proposition 2.7) for a detailed version of the proof in the nondegenerate case. □

Considering $\gamma(x,y)$ as a polynomial in $x$ (resp. $y$) with coefficients depending on $y$ (resp. $x$), we obtain two complex branches $Y^+(x)$, $Y^-(x)$ (resp. $X^+(y)$, $X^-(y)$) satisfying $\gamma(x, Y^\pm(x)) = \gamma(X^\pm(y), y) = 0$:

$$Y^\pm(x) = x - \mu_2 \pm \sqrt{-2x + \mu_2^2},\ X^\pm(y) = y - \mu_1 \pm \sqrt{-2y + \mu_1^2}. \tag{3.3}$$

We have one branching point $x_{max} = \frac{\mu_2^2}{2} > 0$ (resp. $y_{max} = \frac{\mu_1^2}{2} > 0$) for $Y^\pm$ (resp. $X^\pm$). The square roots are chosen to be defined as holomorphic functions on $\mathbb{C}\backslash(-\infty, 0)$ and take nonnegative values on the nonnegative reals.

**Lemma 1.** *Let $u, v \in \mathbb{R}$ such that $u + iv \notin [x_{max}, +\infty[$. Then, we have*

$$\mathrm{Re}(Y^\pm(u+iv)) = u - \mu_2 \pm \frac{1}{\sqrt{2}}\sqrt{\mu_2^2 - 2u + \sqrt{(\mu_2^2 - 2u)^2 + 4v^2}} \tag{3.4}$$

*If $u, v \in \mathbb{R}$ satisfy $u + iv \notin [y_{max}, +\infty[$, then*

$$\mathrm{Re}(X^\pm(u+iv)) = u - \mu_1 \pm \frac{1}{\sqrt{2}}\sqrt{\mu_1^2 - 2u + \sqrt{(\mu_1^2 - 2u)^2 + 4v^2}}. \tag{3.5}$$

*Let $\delta = \min(\mu_1, \mu_2) > 0$. Then, $\mathrm{Re}(Y^-(x)) < 0$ for all $x$ such that $\mathrm{Re}(x) < x_{max} + \delta, x \notin [x_{max}, +\infty[$. Similarly, $\mathrm{Re}(X^-(y)) < 0$ for all $y$ such that $\mathrm{Re}(y) < y_{max} + \delta, y \notin [y_{max}, +\infty[$.*

**Proof.** Equations (3.4) and (3.5) follow directly from the expression (1.8) of $\gamma$. The last statements come from the inequalities $x_{max} = \frac{\mu_2^2}{2} < \mu_2$ and $y_{max} = \frac{\mu_1^2}{2} < \mu_1$. □

**Corollary 1** (Continuation of Laplace Transforms). *The Laplace transforms $\varphi_1$ and $\varphi_2$ can be extended as meromorphic functions on $\{y \in \mathbb{C}, \mathrm{Re}(y) < y_{max} + \delta\}\backslash[y_{max}, y_{max} + \delta]$ and $\{x \in \mathbb{C}, \mathrm{Re}(x) < x_{max} + \delta\}\backslash[x_{max}, x_{max} + \delta]$, respectively via the formulae*

$$\varphi_1(y) = \frac{-\gamma_2(X^-(y), y)\varphi_2(X^-(y)) - \exp(a_0 X^-(y) + b_0 y)}{\gamma_1(X^-(y), y)} \tag{3.6}$$

$$\varphi_2(x) = \frac{-\gamma_1(x, Y^-(x))\varphi_1(Y^-(x)) - \exp(a_0 x + b_0 Y^-(x))}{\gamma_2(x, Y^-(x))}. \tag{3.7}$$

**Proof.** This follows directly from Lemma 1 and the functional Equation (3.1). □

From now on, $\varphi_1$ and $\varphi_2$ will be considered over their extended domains. Let us define

$$x^* = 2\frac{\mu_2 r_2 - \mu_1}{(1 + r_2)^2}, \qquad y^{**} = 2\frac{\mu_1 r_1 - \mu_2}{(r_1 + 1)^2}. \tag{3.8}$$

If equation $\gamma_2(x, Y^-(x)) = 0$ (resp. $\gamma_1(X^-(y), y) = 0$) has a solution in the complex plane, then it is unique and is given by $x = x^*$ (resp. $y = y^{**}$). We also define

$$y^* = Y^+(x^*), \;\; x^{**} = X^+(y^{**}), \tag{3.9}$$

see Figure 6.

**Proposition 4** (Poles of Laplace Transform). *The Laplace transforms $\varphi_1$ and $\varphi_2$ satisfy the following properties:*

(i) $x = 0$ (*resp.* $y = 0$) *is not a pole of* $\varphi_2(x)$ (*resp.* $\varphi_1(y)$).

(ii) *If* $x$ (*resp.* $y$) *is a pole of* $\varphi_2(x)$ (*resp.* $\varphi_1(y)$), *then* $x = x^*$ (*resp.* $y = y^{**}$) *and* $\gamma_1(x^*, Y^-(x^*)) = 0$ (*resp.* $\gamma_2(X^-(y^{**}), y^{**}) = 0$). *Furthermore,* $x^*$ *is a pole of* $\varphi_2$ (*resp.* $y^{**}$ *is a pole of* $\varphi_1$) *if and only if* $(r_2+1)\mu_2 > 2$ (*resp.* $(r_1+1)\mu_1 > 2$).

**Proof.** The first point follows from the continuation Equation (3.7) because $\gamma_2(0, Y^-(0)) = -2\mu_2 \neq 0$.

For (ii), if $x$ is a pole of $\varphi_2$, then it follows from (3.7) that $\gamma_2(x, Y^-(x)) = 0$, which implies that $x = x^*$. Moreover, the Laplace transform $\varphi_2$ is holomorphic in $\mathrm{Re}(x) < 0$. Thus $x^*$, being a pole of $\varphi_2$, must be positive. Note that equation $\gamma_2(x, Y^-(x)) = 0$ has a positive solution if and only if $\gamma_2(x_{max}, Y^{\pm}(x_{max})) > 0$. This last condition is equivalent to $(r_2+1)\mu_2 > 2$.

Let us assume that $(r_2+1)\mu_2 > 2$. Then, $x^* > 0$. Because $\gamma_2(x^*, Y^-(x^*)) = 0$, it follows from (3.7) that $x^*$ is a pole of $\varphi_2$ if the numerator of the right-hand side of (3.6) does not vanish at $x^*$. The last fact holds true and is actually equivalent to the nonnullity of the function $h_{\alpha^*}(z_0)$ defined in (3.24); this equivalence and the non-nullity are postponed till the end of Section 6.2. □

The following proposition provides some estimates for the Laplace transforms. These estimates will be useful in Section 5.

**Proposition 5** (Decay of Laplace Transforms on $\mathrm{Re} = -\epsilon$). *Let* $z_0 = (a_0, b_0) \in \mathbb{R}_+^2$ *be an initial condition with* $a_0 \neq 0, b_0 \neq 0$ *and* $\epsilon > 0$. *Then, there exist constants* $c, C > 0$ *such that for* $l = 1, 2$,

$$\forall v \in \mathbb{R}, \qquad |\varphi_l^{z_0}(-\epsilon + iv)| \leq Ce^{-c\sqrt{|v|}}. \tag{3.10}$$

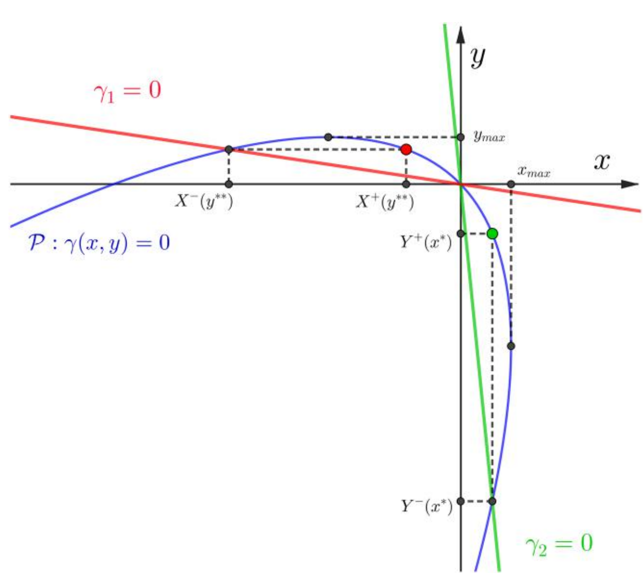
**Proof.** The expressions $\gamma_1(-\epsilon + iv, Y^-(-\epsilon + iv))$ and $\gamma_2(-\epsilon + iv, Y^-(-\epsilon + iv))$ grow linearly with respect to $v$ as $v$ tends to $\pm\infty$. Furthermore, expression (3.4) provides inequality $\exp(b_0 Y^-(-\epsilon + iv)) \leq C_1 e^{-b_0\sqrt{c_1|v|}}$ for some constants $c_1, C_1 > 0$ and

$$\varphi_1^{z_0}(Y^-(-\epsilon + iv)) \leq \varphi_1^{z_0}(0) e^{(a_0+b_0)Re(Y^-(-\epsilon+iv))} \leq C_2 e^{-c_2\sqrt{|v|}}$$

for some constants $c_2, C_2 > 0$. Finally, Equation (3.7) implies the conclusion for $\varphi_2$. The proof for $\varphi_1$ is analogous. □

We also give some further estimates for Laplace transforms that will be useful in Section 4.3.

**Figure 6.** In the Case of the Figure, Both $\varphi_1$ and $\varphi_2$ Have a Pole



**Lemma 2** (Decay of Laplace Transforms). *Assume that* (1.2) *to* (1.4) *hold. For any initial point* $z_0 = (a_0, b_0)$ *and* $p \geq 0$,

$$\varphi_2^{z_0}(-p) \leq e^{-p(a_0+b_0)}\varphi_2^{z_0}(0). \tag{3.11}$$

*The symmetric result holds for* $\varphi_1^{z_0}$.

**Proof.** By (1.3), note that the support of the measure $H_2((a_0, b_0), \cdot)$ is $[a_0 + b_0, +\infty)$. Then,

$$\varphi_2^{z_0}(-p) = \int_{a_0+b_0}^{+\infty} e^{-px} f_2^{z_0}(x)dx \leq e^{-p(a_0+b_0)} \int_{a_0+b_0}^{+\infty} f_2^{z_0}(x)dx = e^{-p(a_0+b_0)}\varphi_2^{z_0}(0). \quad \square$$

# 4. The Compensation Method and the Explicit Expressions of the Laplace Transforms

## 4.1. Heuristic of the Compensation Method

Let $h$ be a smooth function satisfying the following partial differential equation with boundary conditions,

$$\begin{cases} (H_0) & \mathcal{G}h = 0 \qquad\qquad \text{on} \quad (0, +\infty)^2 \\ (H_1) & \partial_{R_1} h(0, y) = 0, \quad y \geq 0 \\ (H_2) & \partial_{R_2} h(x, 0) = 0, \quad x \geq 0 \end{cases} \tag{4.1}$$

with $\mathcal{G} = \frac{1}{2}\nabla \cdot \Sigma\nabla + \mu \cdot \nabla$, and then $h$ is harmonic (see Ernst and Franceschi 2021, section 6). To demonstrate this, one may apply Itô's formula to the process $(Z_t)_{t\geq 0}$ and $h \in C^2(\mathbb{R}_+^2, \mathbb{R})$:

$$h(Z_t) = h(Z_0) + \int_0^t \nabla h(Z_s)dB_s + \int_0^t \mathcal{G}h(s)ds + \sum_{i=1}^{2} \int_0^t R_i \cdot \nabla h(Z_s)dL_s^i,$$

where $(B_t)_{t\geq 0} = (X_t - \mu t)_{t\geq 0}$ is the non-reflected degenerate Brownian motion associated with the process (see Definition 1). If $h$ satisfies (4.1), then $h(Z_t) = h(Z_0) + \int_0^t \nabla h(Z_s)dB_s$, and thus $\mathbb{E}[h(Z_t)] = \mathbb{E}[h(Z_0)]$ (at least formally), which implies that $h$ is harmonic (cf. (1.18)).

The principle of the compensation method is to find functions of the form $h(x, y) = \sum_{n\in\mathbb{Z}} c_n e^{a_n x + b_n y}$ such that each exponential term satisfies condition $(H_0)$: $\mathcal{G}e^{a_n x + b_n y} = 0$ (i.e., $(a_n, b_n) \in \mathcal{P}$, see Figure 4) and to "compensate" the constants $(c_n)_{n\in\mathbb{Z}}$ so as to ensure that conditions $(H_1)$ and $(H_2)$ are satisfied. We require that

$$h(x, y) = \cdots + \overbrace{c_{-2}e^{a_{-2}x+b_{-2}y} + \underbrace{c_{-1}e^{a_{-1}x+b_{-1}y}}_{}}^{\in(H_2)}\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\underbrace{\phantom{c_{-1}e^{a_{-1}x+b_{-1}y}} + c_0 e^{a_0 x+b_0 y}}_{\in(H_1)}\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\overbrace{\phantom{c_0 e^{a_0 x+b_0 y}} + c_1 e^{a_1 x+b_1 y}}^{\in(H_2)}\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\underbrace{\phantom{c_1 e^{a_1 x+b_1 y}} + c_2 e^{a_2 x+b_2 y}}_{\in(H_1)} + \cdots. \tag{4.2}$$

Given that conditions $(H_0), (H_1), (H_2)$ are linear, it follows that $h$ is a harmonic function. By a direct computation, we find that conditions $(H_2)$ on the right-hand side of (4.2) are satisfied if and only if $a_{2k} = a_{2k+1}$ and $c_{2n+1} = -\frac{\gamma_2(a_{2n}, b_{2n})}{\gamma_2(a_{2n+1}, b_{2n+1})} c_{2n}$ for any integer $k$. Similarly, conditions $(H_1)$ in the right-hand side of (4.2) are satisfied if and only if $b_{2n+1} = b_{2n+2}$ and $c_{2n+2} = -\frac{\gamma_1(a_{2n+1}, b_{2n+1})}{\gamma_1(a_{2n+2}, b_{2n+2})} c_{2n+1}$ for any integer $n$.

We will see in Section 6.1 that the harmonic functions we obtain can be written as

$$(x, y) \longmapsto \gamma_1(a_0, b_0)\varphi_1^{(x,y)}(b_0) + \gamma_2(a_0, b_0)\varphi_2^{(x,y)}(a_0) + e^{a_0 x + b_0 y}.$$

The explicit expressions of $\varphi_1$ and $\varphi_2$ in Section 4.3 then provide the exact Equation (4.2) suggested by the compensation method. Moreover, the approach of Section 6.1 justifies why the harmonic functions given by (4.2) are nonnegative when $(a_0, b_0)$ is well chosen.

## 4.2. Parabola and Automorphisms

Let us recall that $\mathcal{P}$ is the parabola defined by $\mathcal{P} = \{(x, y) \in \mathbb{R}^2, \ \gamma(x, y) = 0\}$ (see (1.10)). Before defining the sequence $((a_n, b_n))_{n\in\mathbb{Z}}$ motivated by Section 4.1 (see Figure 4), we first give a parametrization of $\mathcal{P}$.

**Proposition 6** (Parameterization of $\mathcal{P}$). *The parabola $\mathcal{P}$ (see (1.10)) admits the following parameterization:*

$$\begin{cases} x(s) = -\dfrac{1}{2}s(s-2\mu_2) \\ y(s) = -\dfrac{1}{2}s(s+2\mu_1), \end{cases} \quad s \in \mathbb{R}. \tag{4.3}$$

*This means that $\{(x,y) \in \mathbb{R}^2, \gamma(x,y) = 0\} = \{(x(s), y(s)), s \in \mathbb{R}\}$.*

**Proof.** The relation $\gamma(x(s), y(s)) = 0$ is easily verified by substituting $x(s), y(s)$ into the expression (1.8) of $\gamma(x,y)$. Furthermore, the parameterization is injective. To show this, assume that

$$\begin{cases} s(s-2\mu_2) = s'(s'-2\mu_2) \\ s(s+2\mu_1) = s'(s'+2\mu_1). \end{cases}$$

Subtracting the second equation from the first gives $2s(\mu_1+\mu_2) = 2s'(\mu_1+\mu_2)$, which implies $s = s'$. Similarly, surjectivity can be verified by elementary considerations. □

To define the "downstairs" as in Figure 4, we introduce two transformations on the parabola that leave the first (resp. second) coordinate invariant. This is the aim of the following proposition (which also serves as a definition). This proposition is illustrated by Figure 7.

**Proposition 7** (Automorphisms $\eta, \zeta$). *For $s \in \mathbb{R}$, we define*

$$\begin{cases} \zeta s = -s + 2\mu_2 \\ \eta s = -s - 2\mu_1. \end{cases} \tag{4.4}$$

*Then, $x(\zeta s) = x(s)$ and $y(\eta s) = y(s)$ for all $s \in \mathbb{R}$. Therefore, $\varphi_2(x(\zeta s)) = \varphi_2(x(s))$ and $\varphi_1(y(\eta s)) = \varphi_1(y(s))$ in their respective domains of definition. Furthermore, for all $n \in \mathbb{Z}$ and $s \in \mathbb{R}$, we have*

$$(\eta\zeta)^n s = s - 2n. \tag{4.5}$$

**Proof.** The formulae $x(-s+2\mu_2) = x(s)$ and $y(-s-2\mu_1) = y(s)$ are easily verified. The expression of $(\eta\zeta)^n$ is a consequence of expressions of $\eta, \zeta$ and of the equation $\mu_1 + \mu_2 = 1$ (see Assumption (1.4)). □

Note that $\zeta^2 = Id$, $\eta^2 = Id$. By the parameterization (4.3), $\zeta$ and $\eta$ can be regarded as reflections (see (4.4)) and their composition as a translation (see (4.5)).

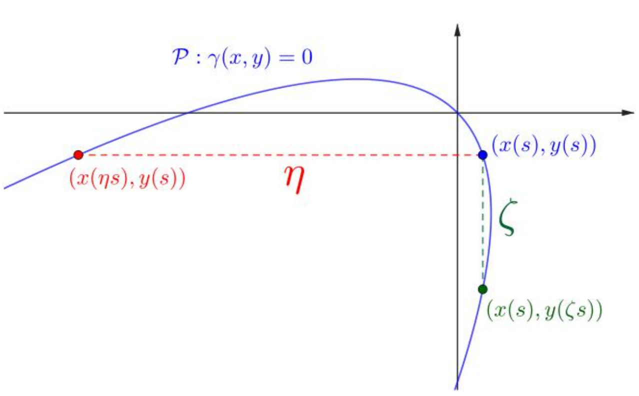
**Lemma 3** (Explicit Form of $(a_n, b_n)$). *Let $s \in \mathbb{R}$ and $(a_0, b_0) = (x(s), y(s))$. For any integer $n \in \mathbb{Z}$, we define*

$$(a_{2n}, b_{2n}) = (x((\eta\zeta)^n s), y((\eta\zeta)^n s)), \quad (a_{2n+1}, b_{2n+1}) = (x(\zeta(\eta\zeta)^n s), y(\zeta(\eta\zeta)^n s))$$

*(see Figure 4). Then, for any $n \in \mathbb{Z}$, the following expressions hold:*

$$a_{2n} = -2n^2 + 2(a_0 - b_0 - \mu_2)n + a_0, \; a_{2n+1} = a_{2n} \tag{4.6}$$
$$b_{2n} = -2n^2 + 2(a_0 - b_0 + \mu_1)n + b_0, \; b_{2n+1} = b_{2n+2}. \tag{4.7}$$

**Figure 7.** Parabola $\mathcal{P}$ and Automorphisms $\eta$ and $\zeta$



**Proof.** The invariance of the first and the second coordinate of $\zeta$ and $\eta$, respectively, implies equalities $a_{2n+1} = a_{2n}$ and $b_{2n+1} = b_{2n+2}$. The explicit expressions of $a_{2n}$ and $b_{2n}$ are obtained from the explicit expression (4.5). □

**Notations 1.** For $s \in \mathbb{R}$, we define

$$z(s) = (x(s), y(s)) \tag{4.8}$$

as the point of the parabola corresponding to the parameter $s \in \mathbb{R}$. We also define

$$s_{max} = \mu_2, \; s_{min} = -\mu_1 \tag{4.9}$$

and write $\gamma_i(s)$ instead of $\gamma_i(x(s), y(s))$ for $i = 1, 2$. Finally, let $s^*$ and $s^{**}$ be defined as

$$s^* = \frac{2}{r_2 + 1}, \; s^{**} = \frac{-2}{r_1 + 1}. \tag{4.10}$$

With these notations,

$$z(s_{max}) = (x_{max}, Y^{\pm}(x_{max})) \quad \text{and} \quad z(s_{min}) = (X^{\pm}(y_{max}), y_{max}).$$

Note that the curve $(z(s))_{s \in [s_{min}, s_{max}]}$ is the portion of the parabola from $(x_{max}, Y^+(x_{max}))$ to $(X^+(y_{max}), y_{max})$ going counterclockwise (see Figure 6). Furthermore, $x(s^*) = x^*$ and $y(s^{**}) = y^{**}$ with definition (3.8). We can now provide explicit expressions for the Laplace transforms $\varphi_1$ and $\varphi_2$.

### 4.3. Explicit Expression of Laplace Transforms via the Compensation Approach

**Theorem 5** (Explicit Expressions for Laplace Transforms). *Let $z_0 \in \mathbb{R}_+^2 \setminus \{(0,0)\}$ be the initial condition. Then, for any $s \in (\max(s_{min}, s^{**}), \min(s_{max}, s^*))$,*

$$\varphi_2^{z_0}(x(s)) = \frac{-1}{\gamma_2(\zeta s)} e^{z_0 \cdot z(\zeta s)} + \sum_{n=1}^{+\infty} \left[ \prod_{k=0}^{n-1} G(s-2k) \right] \left[ \frac{e^{z_0 \cdot z(s-2n)}}{\gamma_2(s-2n)} - \frac{e^{z_0 \cdot z(\zeta(s-2n))}}{\gamma_2(\zeta(s-2n))} \right] \tag{4.11}$$

*where*

$$G(s) = \frac{\frac{\gamma_1}{\gamma_2}(\zeta s)}{\frac{\gamma_1}{\gamma_2}(s-2)}. \tag{4.12}$$

*Similarly, for all $s \in (\max(s_{min}, s^{**}), \min(s_{max}, s^*))$*

$$\varphi_1^{z_0}(y(s)) = \frac{-1}{\gamma_1(\eta s)} e^{z_0 \cdot z(\eta s)} + \sum_{n=1}^{+\infty} \left[ \prod_{k=0}^{n-1} \tilde{G}(s+2k) \right] \left[ \frac{e^{z_0 \cdot z(s+2n)}}{\gamma_1(s+2n)} - \frac{e^{z_0 \cdot z(\eta(s+2n))}}{\gamma_1(\eta(s+2n))} \right] \tag{4.13}$$

*where*

$$\tilde{G}(s) = \frac{\frac{\gamma_2}{\gamma_1}(\eta s)}{\frac{\gamma_2}{\gamma_1}(s+2)}.$$

Before proving Theorem 5, we establish a technical lemma.

**Lemma 4.** *For all $n \geq 1$ and $s \in (\max(s_{min}, s^{**}), \min(s_{max}, s^*))$, we have $\gamma_1(s-2n) \neq 0$ and $\gamma_2(\zeta(s-2n)) \neq 0$. Furthermore, $\gamma_2(s), \gamma_2(\zeta s), \gamma_1(s), \gamma_1(\eta s)$ are also nonzero.*

**Proof.** We define two portions of the parabola $E^+$ and $E^-$ given by

$$E^+ = \{(x, Y^+(x)), x \leq X^+(0)\} \quad \text{and} \quad E^- = \{(X^-(y)), y), \; y \leq Y^-(0)\}.$$

By Assumption (1.3), the line $\{\gamma_2 = 0\}$ (resp. $\{\gamma_1 = 0\}$) cannot pass through $E^-$ (resp. $E^+$). Additionally, note that $\eta(E^-) \subset E^+$ and $\zeta(E^+) \subset E^-$. Because$s \in (s_{min}, s_{max})$, $z((\eta\zeta)^n s) = z(s-2n)$ belongs to $E^+$ for all $n \geq 1$. Thus, $\gamma_2(\zeta(s-2n)) \neq 0$ for any $n \geq 0$. By similar reasoning, $\gamma_1(s-2n) \neq 0$ for any $n \geq 0$. The last statement comes from the fact that $s \in (s^{**}, s^*)$. □

**Proof of Theorem 5.** The main idea of the proof is to get a recursive formula for Laplace transforms. To do this, we rewrite the functional Equation (3.1) in $z(\zeta s)$ and $z(\eta\zeta s) = z(s-2)$, which holds because $x(\zeta s), y(\zeta s), x(s-2)$

and $y(s-2)$ are negative:

$$\begin{cases} 0 = \gamma_1(\zeta s)\varphi_1(y(\zeta s)) + \gamma_2(\zeta s)\varphi_2(x(\zeta s)) + e^{z_0\cdot z(\zeta s)} \\ 0 = \gamma_1(s-2)\varphi_1(y(s-2)) + \gamma_2(s-2)\varphi_2(x(s-2)) + e^{z_0\cdot z(s-2)}. \end{cases}$$

By the invariance of $\varphi_2$ (resp. $\varphi_1$) under $\zeta$ (resp. $\eta$), we have $\varphi_2(x(\zeta s)) = \varphi_2(x(s))$ and $\varphi_1(y(s-2)) = \varphi_1(y(\zeta s))$. Then, by eliminating $\varphi_1(y(\zeta s))$ from the equations (which is possible by Lemma 4), we obtain

$$\varphi_2(x(s)) = \frac{\frac{\gamma_1}{\gamma_2}(\zeta s)}{\frac{\gamma_1}{\gamma_2}(s-2)}\varphi_2(x(s-2)) + \left[\frac{\frac{\gamma_1}{\gamma_2}(\zeta s)e^{z(s-2)\cdot z_0}}{\gamma_1(s-2)} - \frac{e^{z(\zeta s)\cdot z_0}}{\gamma_2(\zeta s)}\right] \tag{4.14}$$

$$= G(s)\varphi_2(x(s-2)) + \left[\frac{G(s)}{\gamma_2(s-2)}e^{z_0\cdot z(s-2)} - \frac{e^{z_0\cdot z(\zeta s)}}{\gamma_2(\zeta s)}\right]. \tag{4.15}$$

Similarly, we get

$$\varphi_2(x(s-2)) = G(s-2)\varphi_2(x(s-4)) + \left[\frac{G(s-2)}{\gamma_2(s-4)}e^{z_0\cdot z(s-4)} - \frac{e^{z_0\cdot z(\zeta(s-2))}}{\gamma_2(\zeta(s-2))}\right]. \tag{4.16}$$

Substituting this into (4.15), we get

$$\varphi_2(x(s)) = G(s)G(s-2)\varphi_2(x(s-4)) + G(s)G(s-2)\frac{e^{z_0\cdot z(s-4)}}{\gamma_2(s-4)} - G(s)\frac{e^{z_0\cdot z(\zeta(s-2))}}{\gamma_2(\zeta(s-2))} + \frac{G(s)}{\gamma_2(s-2)}e^{z_0\cdot z(s-2)} - \frac{e^{z_0\cdot z(\zeta s)}}{\gamma_2(\zeta s)}.$$

Then, by induction on $N$, we obtain the following equality for all $N \geq 1$:

$$\varphi_2(x(s)) = \left[\prod_{k=0}^{N} G(s-2k)\right]\varphi_2(x(s-2(N+1))) - \frac{e^{z_0\cdot z(\zeta s)}}{\gamma_2(\zeta s)} + \left[\prod_{k=0}^{N} G(s-2k)\right]\frac{e^{z_0\cdot z(s-2(N+1))}}{\gamma_2(s-2(N+1))} + \sum_{n=1}^{N}\left[\prod_{k=0}^{n-1} G(s-2k)\right]\left[\frac{e^{z_0\cdot z(s-2n)}}{\gamma_2(s-2n)} - \frac{e^{z_0\cdot z(\zeta(s-2n))}}{\gamma_2(\zeta(s-2n))}\right] \tag{4.17}$$

The proof is then reduced to proving the following limit:

$$\left[\prod_{k=0}^{n} G(s-2k)\right]\varphi_2(x(s-2(n+1))) \underset{n\to+\infty}{\longrightarrow} 0. \tag{4.18}$$

To justify this, note using Equation (4.12) and Lemma 3 that

$$G(s-2k) = \frac{(k+a)(k+b)}{(k+c)(k+d)}. \tag{4.19}$$

for some constants $a$, $b$, $c$, and $d$ defined by

$$a = \frac{-s}{2} + \frac{r_1}{1+r_1}, \quad b = 1 - \frac{s}{2} + \frac{\mu_2 r_2 - \mu_1}{1+r_2}$$
$$c = \frac{-s}{2} + \frac{1}{1+r_2}, \quad d = 1 - \frac{s}{2} + \frac{\mu_2 - \mu_1 r_1}{1+r_1}.$$

By elementary considerations, the following asymptotic behavior holds:

$$\left[\prod_{k=0}^{n} G(s+2k)\right] \underset{n\to\infty}{\sim} Cn^{a-c+b-d}, \tag{4.20}$$

where $C$ is a real constant. Moreover,

$$a+b-c-d=2-2\left(\frac{1}{1+r_1}+\frac{1}{1+r_2}\right) \tag{4.21}$$

because $\mu_1+\mu_2=1$. Then, the exponential decay in (3.11) for $\varphi_2$, together with the polynomial rate of expression (4.20), yields (4.11). Note that inequality (3.11) is the only (and crucial) reason why we work under Assumption (1.3). Equation (4.13) is obtained with symmetric arguments. □

**Remark 2.** The exponent given by (4.20) is exactly the parameter $-2\gamma$ introduced in Dreyfus et al. (2025), which determines the algebraic nature of the Laplace transforms for the same degenerate particle model in the recurrent case. Furthermore, the constants $\kappa_m=\kappa_m(\alpha)$ in (1.22) satisfy

$$\kappa_m \underset{m\to\pm\infty}{\sim} C_\pm m^{-2\gamma-2}. \tag{4.22}$$

for some constant $C_\pm>0$, where $-2\gamma-2<0$ by (1.3).

In (4.11) (resp. (4.13)), $\varphi_2$ (resp. $\varphi_1$) is not given as a function of $x$ (resp. $y$) but of $s$. Therefore, we establish the following corollary.

**Corollary 2.** *The following expressions hold in the domains $Re(x)<x_{max}$ and $Re(y)<y_{max}$, respectively:*

$$\varphi_2(x)=\frac{-1}{\gamma_2(x,Y^-(x))}e^{z_0\cdot(x,Y^-(x))}+\sum_{n=1}^{+\infty}\left[\prod_{k=1}^{n}\frac{\frac{\gamma_1}{\gamma_2}(\psi_{2k-1}(x,Y^+(x))}{\frac{\gamma_1}{\gamma_2}(\psi_{2k}(x,Y^+(x)))}\right]\left[\frac{e^{z_0\cdot\psi_{2n}(x,Y^+(x))}}{\gamma_2(\psi_{2n}(x,Y^+(x))}-\frac{e^{z_0\cdot\psi_{2n+1}(x,Y^+(x))}}{\gamma_2(\psi_{2n+1}(x,Y^+(x))}\right] \tag{4.23}$$

$$\varphi_1(y)=\frac{-1}{\gamma_1(X^+(y),y)}e^{z_0\cdot(X^+(y),y)}+\sum_{n=1}^{+\infty}\left[\prod_{k=1}^{n}\frac{\frac{\gamma_2}{\gamma_1}(\psi_{-2k+1}(X^+(y),y))}{\frac{\gamma_2}{\gamma_1}(\psi_{-2k}(X^+(y),y))}\right]\left[\frac{e^{z_0\cdot\psi_{-2n}(X^+(y),y))}}{\gamma_1(\psi_{-2n}(X^+(y),y))}-\frac{e^{z_0\cdot\psi_{-2n+1}(X^+(y),y)}}{\gamma_1(\psi_{-2n+1}(X^+(y),y))}\right], \tag{4.24}$$

*where*

$$\psi_{2n}(a,b)=(-2n^2+2(a-b-\mu_2)n+a,\ -2n^2+2(a-b+\mu_1)n+b)$$

*and*

$$\psi_{2n+1}(a,b)=(-2n^2+2(a-b-\mu_2)n+a,\ -2(n+1)^2+2(a-b+\mu_1)(n+1)+b).$$

**Proof.** By Lemma 3 and equalities $z(s)=(x(s),Y^+(x(s))=(X^+(y(s)),y(s))$ for $s\in(s_{min},s_{max})$, Equations (4.23) and (4.24) hold on the curve $\{(x,y)=(x(s),y(s)):s\in((\max(s_{min},s^{**}),\min(s_{max},s^*)))\}$. By Corollary 1, Laplace transforms $\varphi_2(x)$ and $\varphi_1(y)$ are meromorphic on $Re(x)<x_{max}$ and $Re(y)<y_{max}$, respectively. Consequently, the explicit expressions (4.23) and (4.24) remain valid in these domains. □

## 5. Laplace Inverse and Saddle Point Method

To avoid certain technical complications, we first derive the asymptotic behavior of the Green functions $g^{z_0}$ for $z_0\neq 0$ and later address the case $z_0=0$ with additional arguments.

### 5.1. Inverse Laplace Theorem and Saddle Point

Let $z_0\neq(0,0)$ be a starting point of the process. The inverse Laplace transform formula (see (Doetsch 1974 (theorems 24.3 and 24.4) and Brychkov 1992) yields the following representation for $g^{z_0}(a,b)$: for $\epsilon>0$ sufficiently small,

$$g^{z_0}(a,b)=\frac{1}{(2\pi i)^2}\int_{-\epsilon-i\infty}^{-\epsilon+i\infty}\int_{-\epsilon-i\infty}^{-\epsilon+i\infty}\varphi^{z_0}(x,y)\exp(-ax-by)dxdy, \tag{5.1}$$

where the convergence is in the sense of principal value. This can be justified by the functional Equation (1) and the decay properties of the Laplace transforms established in Proposition 5.

**Lemma 5** (From Double to Simple Integrals). *Denote by $z_0=(a_0,b_0)$ the starting point of the process. Then, for any $(a,b)\in\mathbb{R}_+^2$ satisfying $a>0$ or $b>0$,*

$$g(a,b)=I_1(a,b)+I_2(a,b)+I_3(a,b),$$

*where*

$$I_1(a,b)=\frac{1}{2\pi i}\int_{-\epsilon-i\infty}^{-\epsilon+i\infty}\varphi_2(x)\gamma_2(x,Y^+(x))\exp(-ax-bY^+(x))\frac{dx}{\partial_y\gamma(x,Y^+(x))},$$

$$I_2(a,b)=\frac{1}{2\pi i}\int_{-\epsilon-i\infty}^{-\epsilon+i\infty}\varphi_1(y)\gamma_1(X^+(y),y)\exp(-aX^+(y)-by)\frac{dy}{\partial_x\gamma(X^+(y),y)},$$

$$I_3(a,b)=\frac{1}{2\pi i}\int_{-\epsilon-i\infty}^{-\epsilon+i\infty}\exp(a_0x+b_0Y^+(x))\exp(-ax-bY^+(x))\frac{dx}{\partial_y\gamma(x,Y^+(x))}\quad \text{if } b>b_0,$$

$$I_3(a,b)=\frac{1}{2\pi i}\int_{-\epsilon-i\infty}^{-\epsilon+i\infty}\exp(a_0X^+(y)+b_0y)\exp(-aX^+(y)-by)\frac{dy}{\partial_x\gamma(X^+(y),y)}\quad \text{if } a>a_0.$$

**Proof.** By the functional Equation (3.1), $\varphi(x,y)$ can be decomposed as

$$\varphi(x,y)=-\frac{\gamma_1(x,y)\varphi_1(y)}{\gamma(x,y)}-\frac{\gamma_2(x,y)\varphi_2(x)}{\gamma(x,y)}-\frac{e^{(x,y)\cdot z_0}}{\gamma(x,y)}. \tag{5.2}$$

Substituting this expression into the double integral (5.1), $g^{z_0}(a,b)$ is written as the sum of three double integrals. Let us consider the first term, given by

$$\frac{-1}{(2\pi i)^2}\int_{-\epsilon-i\infty}^{-\epsilon+i\infty}\varphi_2(x)\int_{-\epsilon-i\infty}^{-\epsilon+i\infty}\frac{\gamma_2(x,y)}{\gamma(x,y)}e^{-ax-by}\,dy\,dx. \tag{5.3}$$

Let $C_R$ be the closed oriented contour defined by

$$C_R=\{-\epsilon+it\,|\,t\in[-R,R]\}\cup\{-\epsilon+Re^{-i\theta}\,|\,\theta\in[-\pi/2,\pi/2]\}.$$

By applying the residue theorem along the contour $C_R$ and considering the asymptotics as $R\to+\infty$ (see Franceschi et al. 2024b (lemma 4.1) for more details), we obtain the identity

$$\int_{-\epsilon-i\infty}^{-\epsilon+i\infty}\frac{\gamma_2(x,y)}{\gamma(x,y)}e^{-ax-by}\,dy=\frac{\gamma_2(x,Y^+(x))}{\partial_y\gamma(x,Y^+(x))}e^{-ax-bY^+(x)},$$

so that expression (5.3) equals $I_1(a,b)$. The remaining terms are handled analogously. □

To find the asymptotics of these integrals as $a,b\to+\infty$, we use the saddle point method. For any $\alpha\in[0,\pi/2]$, let $(x(\alpha),y(\alpha))$ be defined as

$$(x(\alpha),y(\alpha))=\underset{(x,y)\in\mathcal{P}}{\arg\max}\,(\cos(\alpha)x+\sin(\alpha)y), \tag{5.4}$$

see Figure 3(a). For $\alpha\in[0,\pi/2]$, we define the real number $\mathfrak{s}(\alpha)\in\mathbb{R}$ by

$$\mathfrak{s}(\alpha)=\underset{s\in\mathbb{R}}{\arg\max}\,(\cos(\alpha)x(s)+\sin(\alpha)y(s)).$$

Note that $(x(\alpha),y(\alpha))=(x(\mathfrak{s}(\alpha)),y(\mathfrak{s}(\alpha)))$, using notation (4.3). By studying the variations of the function $s\longmapsto x(s)\cos(\alpha)+y(s)\sin(\alpha)$, we prove that

$$\mathfrak{s}:\begin{cases}[0,\pi/2]\longrightarrow[s_{min},s_{max}]\\ \alpha\longmapsto\dfrac{\mu_2-\tan(\alpha)\mu_1}{1+\tan(\alpha)}\end{cases}\qquad(\text{with}\quad \mathfrak{s}(\pi/2)=-\mu_1=s_{min}) \tag{5.5}$$

is a $C^\infty$ diffeomorphism, and

$$\mathfrak{s}^{-1}:\begin{cases}[s_{min},s_{max}]\longrightarrow[0,\pi/2]\\ s\longmapsto\arctan\left(\dfrac{\mu_2-s}{s+\mu_1}\right)\end{cases}\qquad(\text{with}\quad \mathfrak{s}^{-1}(-\mu_1)=\pi/2). \tag{5.6}$$

Using the definitions of $\alpha^*,\alpha^{**},x^*$, and $y^{**}$ given by (1.13), (1.14), and (3.8), if $x^*$ (resp. $y^*$) is a pole of $\varphi_2$ (resp. $\varphi_1$), then $x(\alpha^*)=x^*$ (resp. $y(\alpha^{**})=y^{**}$). Because $s^{**}<0<s^*$ (see Notation 1), then the monotonicity of (5.6) implies that

$0 \le \alpha^* < \alpha_\mu < \alpha^{**} \le \pi/2$, where $\alpha_\mu = \arctan(\mu_2/\mu_1) \in (0, \pi/2)$ is the angle of the drift. We follow the notation of Franceschi et al. (2024b) and define

$$F(x,\alpha) = -\cos(\alpha)x - \sin(\alpha)Y^+(x) + \cos(\alpha)x(\alpha) + \sin(\alpha)y(\alpha) \tag{5.7}$$

$$G(y,\alpha) = -\cos(\alpha)X^+(y) - \sin(\alpha)y + \cos(\alpha)x(\alpha) + \sin(\alpha)y(\alpha). \tag{5.8}$$

By construction, the equations $\partial_x F(x(\alpha),\alpha) = 0$ and $\partial_y G(y(\alpha),\alpha) = 0$ hold. Then, by differentiating Equation (5.7) and $\gamma(x, Y^+(x)) = 0$, we get for any $\alpha \in (0, \pi/2]$:

$$(Y^+(x))'\Big|_{x=x(\alpha)} = -\frac{\partial_x\gamma(x(\alpha),y(\alpha))}{\partial_y\gamma(x(\alpha),y(\alpha))} = -\frac{\cos(\alpha)}{\sin(\alpha)},\ (Y^+(x))''\Big|_{x=x(\alpha)} = -\frac{(1+\mathrm{ctan}\,(\alpha))^2}{\partial_y\gamma(x(\alpha),y(\alpha))}. \tag{5.9}$$

Therefore,

$$\partial^2_{xx}F(x(\alpha),\alpha) = \frac{(\sin(\alpha)+\cos(\alpha))^2}{\partial_y\gamma(x(\alpha),y(\alpha))\sin(\alpha)} > 0, \quad \alpha \in (0,\pi/2]. \tag{5.10}$$

Similarly,

$$\partial^2_{yy}G(y(\alpha),\alpha) = \frac{(\sin(\alpha)+\cos(\alpha))^2}{\partial_x\gamma(x(\alpha),y(\alpha))\cos(\alpha)} > 0, \quad \alpha \in [0,\pi/2).$$

## 5.2. Contour of Steepest Descent

Let $\alpha_0 \in (0, \pi/2]$. The key idea of the saddle point method is to use the parameterized Morse lemma. Because $\partial^2_{xx}F(x(\alpha),\alpha) > 0$, lemma A.1 from Franceschi et al. (2024b) yields some $\epsilon > 0, \eta > 0$ and a family of smooth paths $\Gamma_{x,\alpha} = \{x(it,\alpha) | t \in [-\epsilon,\epsilon]\}$, $|\alpha - \alpha_0| < \eta$ such that

$$\forall t \in [-\epsilon,\epsilon], \quad F(x(it,\alpha),\alpha) = -t^2. \tag{5.11}$$

For further details on the construction, please refer to appendix A in Franceschi et al. 2024b. Define

$$x^+_\alpha = x(i\epsilon,\alpha), \quad x^-_\alpha = x(-i\epsilon,\alpha). \tag{5.12}$$

In particular,

$$F(x^+_\alpha,\alpha) = -\epsilon^2,\ F(x^-_\alpha,\alpha) = -\epsilon^2. \tag{5.13}$$

Furthermore, $Im(x^+_\alpha) > 0$ and $Im(x^-_\alpha) < 0$ (see Figure 8 and construction in Franceschi et al. 2024b). The same construction holds for $\Gamma_{y,\alpha} = \{y(it,\alpha) | t \in [-\epsilon,\epsilon]\}$ for $G$ and $\alpha_0 \in [0, \pi/2)$. These paths satisfy

$$\Gamma_{x,\alpha} = \underrightarrow{\overleftarrow{X^+(\Gamma_{y,\alpha})}} \quad \text{and} \quad \Gamma_{y,\alpha} = \underrightarrow{\overleftarrow{Y^+(\Gamma_{x,\alpha})}}, \quad 0 < \alpha < \pi/2.$$

The arrows above and below the paths indicate reversed orientations; this notation is taken from (Fayolle et al. 2017, chapter 5.3).

## 5.3. Shift of the Integration Contours and Contribution of the Poles

We now apply the saddle point method. To do this, we shift the integration contours of $I_1$, $I_2$, and $I_3$ to contours passing through the saddle point and following the steepest descent contours $\Gamma_{x,\alpha}$ and $\Gamma_{y,\alpha}$. We define $T_{x,\alpha} = S^-_{x,\alpha} + \Gamma_{x,\alpha} + S^+_{x,\alpha}$ and $T_{y,\alpha} = S^-_{y,\alpha} + \Gamma_{y,\alpha} + S^+_{y,\alpha}$ for $\alpha \in [0, \pi/2]$, where

$$S^+_{x,\alpha} = \{x^+_\alpha + it | t \ge 0\},\ S^-_{x,\alpha} = \{x^-_\alpha - it | t \ge 0\},$$
$$S^+_{y,\alpha} = \{y^+_\alpha + it | t \ge 0\},\ S^-_{y,\alpha} = \{y^-_\alpha - it | t \ge 0\}.$$

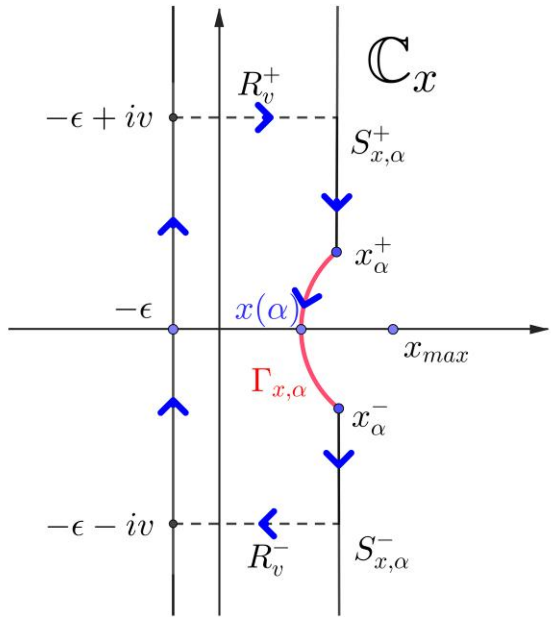
**Lemma 6** (Contour Deformation and Contribution of the Pole).

*Let $\alpha \in [0, \pi/2] \backslash \{\alpha^*, \alpha^{**}\}$ and $z_0 \ne (0,0)$ be the initial condition of the process. Then, for any $a, b > 0$,*

$$I_1(a,b) = \frac{(-\mathrm{res}_{x=x^*}\varphi_2(x))\gamma_2(x^*,y^*)}{\partial_y\gamma(x^*,y^*)}\exp(-ax^* - by^*)\mathbf{1}_{\alpha<\alpha^*} + \frac{1}{2\pi i}\int_{T_{x,\alpha}} \frac{\varphi_2(x)\gamma_2(x,Y^+(x))}{\partial_y\gamma(x,Y^+(x))}\exp(-ax - bY^+(x))dx, \tag{5.14}$$

$$I_2(a,b) = \frac{(-\mathrm{res}_{y=y^{**}}\varphi_1(y))\gamma_1(x^{**},y^{**})}{\partial_x\gamma(x^{**},y^{**})}\exp(-ax^{**} - by^{**})\mathbf{1}_{\alpha>\alpha^{**}} + \frac{1}{2\pi i}\int_{T_{y,\alpha}} \frac{\varphi_1(y)\gamma_1(X^+(y),y)}{\partial_x\gamma(X^+(y),y)}\exp(-aX^+(y) - by)dy, \tag{5.15}$$

**Figure 8.** Changing Path for $I_2$




*Note.* Here, $x(\alpha) < x^*$.

$$I_3(a,b) = \frac{1}{2\pi i}\int_{T_{x,\alpha}} \exp((a_0 - a)x + (b_0 - b)Y^+(x))\frac{dx}{\partial_y\gamma(x, Y^+(x))} \quad \text{if } b > b_0, \tag{5.16}$$

$$I_3(a,b) = \frac{1}{2\pi i}\int_{T_{y,\alpha}} \exp((a_0 - a)X^+(y) + (b_0 - b)y)\frac{dy}{\partial_x\gamma(X^+(y), y)} \quad \text{if } a > a_0. \tag{5.17}$$

**Proof.** The shift of the path is illustrated in Figure 8 and is the same as in (Franceschi et al. 2024b, lemma 6.1). The proof of (5.14) is a direct consequence of the residue theorem, provided that the integrals over the horizontal contours $R_v^+$ and $R_v^-$ tend to 0 as $v$ tends to $+\infty$. Then, it remains to be proven that for any sufficiently small $\eta > 0$,

$$\sup_{u\in[X^+(y_{max})-\eta,\, x^{max}+\eta]}\left|\frac{\varphi_2(u+iv)\gamma_2(u+iv, Y^+(u+iv))}{\gamma_y'(u+iv, Y^+(u+iv))}\exp(-a(u+iv) - bY^+(u+iv))\right| \to 0, \quad \text{as } v \to \infty.$$

By the functional Equation (3.1) and continuation Equation (3.7), the term inside the supremum is equal to

$$\left|\frac{(\gamma_1(u+iv, Y^-(u+iv))\varphi_1(Y^-(u+iv)) + e^{a_0(u+iv)+b_0Y^-(u+iv)})\gamma_2(u+iv, Y^+(u+iv))}{\gamma_2(u+iv, Y^-(u+iv))\gamma_y'(u+iv, Y^+(u+iv))}\right|$$
$$\times\, |\exp(-a(u+iv) - bY^+(u+iv))|.$$

By (3.3), $Re(Y^\pm(u+iv))$ grows like $\pm\sqrt{|v|}$ uniformly in $u \in [X^+(y_{max}) - \eta, x^{max} + \eta]$ as $|v| \to +\infty$. Furthermore, $\gamma_2(u+iv, Y^\pm(u+iv))$ grows linearly in $v$ uniformly in $u \in [X^+(y_{max}) - \eta, x^{max} + \eta]$ as $v \to +\infty$ by Assumption (1.3). The same asymptotics hold for $\gamma_1(u+iv, Y^-(u+iv))$. Moreover, $\partial_y\gamma(u+iv, Y^+(u+iv)) = \sqrt{-2(u+iv) + \mu_2^2}$, so this expression grows with rate $\sqrt{v}$, uniformly in $u \in [X^+(y_{max}) - \eta, x^{max} + \eta]$. Considering the exponential decay of $\varphi_1$ (see Lemma 2) we get the conclusion for $I_1$. Formulae for $I_2$ and $I_3$ are obtained similarly. □

### 5.4. Negligibility of Some Integrals

For any pair $(a,b) \in \mathbb{R}_+^2$ let $\alpha(a,b)$ be the angle in $[0, \pi/2]$ such that $\cos(\alpha) = \frac{a}{\sqrt{a^2+b^2}}$ and $\sin(\alpha) = \frac{b}{\sqrt{a^2+b^2}}$. We now aim to evaluate the asymptotics of the integrals over $T_{x,\alpha}^\pm$ and $T_{y,\alpha}^\pm$ in Lemma 6 as $\sqrt{a^2+b^2} \to +\infty$ and $\alpha(a,b) \to \alpha_0$ for some $\alpha_0 \in [0, \pi/2]$. In the next lemma, we establish exponential bounds for the integrals over the vertical contours $S^{\pm}_{x,\alpha}$, $S^{\pm}_{y,\alpha}$. These bounds imply that the main contribution to the above asymptotics comes from the integrals over the steepest descent contours $\Gamma_{x,\alpha}$, $\Gamma_{y,\alpha}$, whereas those over $S^{\pm}_{x,\alpha}$ and $S^{\pm}_{y,\alpha}$ turn out to be negligible.

**Lemma 7** (Negligibility of the Integrals Over $S^{\pm}_{x,\alpha}$ and $S^{\pm}_{y,\alpha}$). *Suppose $z_0 \neq (0,0)$. Let $K$ be a compact neighborhood of $z_0$ in the quadrant satisfying $d((0,0),K) > 0$. Let $\alpha_0 \in [0,\pi/2]$. Then, for sufficiently small $\eta > 0$, there exist constants $r_0 > 0$ and $D_{\alpha_0} > 0$ such that for any $z \in K$ and any pair $(a,b)$ satisfying $\sqrt{a^2+b^2} > r_0$ and $|\alpha(a,b) - \alpha_0| < \eta$, the following inequalities hold:*

$$\left| \int_{S^{\pm}_{x,\alpha}} \frac{\varphi^z_2(x)\gamma_2(x,Y^+(x))}{\partial_y\gamma(x,Y^+(x))}\exp(-ax - bY^+(x))dx \right| \leq D_{\alpha_0} e^{-ax(\alpha)-by(\alpha)-\epsilon\sqrt{a^2+b^2}}, \tag{5.18}$$

$$\left| \int_{S^{\pm}_{y,\alpha}} \frac{\varphi^z_1(y)\gamma_1(X^+(y),y)}{\partial_x\gamma(X^+(y),y)}\exp(-aX^+(y) - by)dy \right| \leq D_{\alpha_0} e^{-ax(\alpha)-by(\alpha)-\epsilon\sqrt{a^2+b^2}}. \tag{5.19}$$

*If $b > b_0$, then*

$$\left| \int_{S^{\pm}_{x,\alpha}} \exp((a_0 - a)x + (b_0 - b)Y^+(x)) \frac{dx}{\partial_y\gamma(x,Y^+(x))} \right| \leq \frac{D_{\alpha_0}}{b - b_0} e^{-ax(\alpha)-by(\alpha)-\epsilon^2\sqrt{a^2+(b-b_0)^2}}. \tag{5.20}$$

*If $a > a_0$, then*

$$\left| \int_{S^{\pm}_{y,\alpha}} \exp((a_0 - a)X^+(y) + (b_0 - b)y) \frac{dy}{\partial_x\gamma(X^+(y),y)} \right| \leq \frac{D_{\alpha_0}}{a - a_0} e^{-ax(\alpha)-by(\alpha)-\epsilon^2\sqrt{(a-a_0)^2+b^2}}. \tag{5.21}$$

**Proof.** We start by showing (5.18). Using notations (5.7) and (5.12), this inequality can be rewritten as

$$\left| \int_{v>0} \frac{\varphi^z_2(x^+_\alpha + iv)\gamma_2(x^+_\alpha + iv, Y^+(x^+_\alpha + iv))}{\partial_y\gamma(x^+_\alpha + iv, Y^+(x^+_\alpha + iv))} e^{-aiv - b(Y^+(x^+_\alpha+iv) - Y^+(x^+_\alpha))} dx \right| \leq D_{\alpha_0} \tag{5.22}$$

where $\alpha = \alpha(a,b)$.

Suppose first that $\alpha_0 > 0$. Let $\alpha > 0$ and $0 < \eta < \alpha_0/2$. Because $\partial_y\gamma(x^+_\alpha + iv, Y^+(x^+_\alpha + iv)) = \sqrt{-2(x^+_\alpha + iv) + \mu_2^2}$, this expression does not vanish and grows at rate $\sqrt{|v|}$ as $v \longrightarrow +\infty$, uniformly in $\alpha$, with $|\alpha - \alpha_0| < \eta$. Similarly, $\gamma_2(x^+_\alpha + iv, Y^+(x^+_\alpha + iv))$ grows with speed $|v|$ as $v \longrightarrow +\infty$, uniformly in $\alpha$, $|\alpha - \alpha_0| < \eta$. Then, we have, for all $v \geq 0$,

$$\sup_{|\alpha-\alpha_0|<\eta} \frac{\gamma_2(x^+_\alpha + iv, Y^+(x^+_\alpha + iv))}{\partial_y\gamma(x^+_\alpha + iv, Y^+(x^+_\alpha + iv))} \leq C_\eta(1 + \sqrt{v})$$

for some constant $C_\eta > 0$. If $|\alpha - \alpha_0| < \eta$, then by (3.4), there exists a constant $C'_\eta > 0$ such that

$$\begin{aligned} Re(\sqrt{a^2+b^2}(F(x^+_\alpha + iv,\alpha) - F(x^+_\alpha,\alpha))) &= b(Re(Y^+(x^+_\alpha + iv)) - Re(Y^+(x^+_\alpha))) \\ &\geq C'_\eta b\sqrt{v} \end{aligned} \tag{5.23}$$

for any $v \geq 1$. Furthermore, using the continuation Equation (3.7), the estimates (3.11), and the continuity of $\varphi^{z_0}_2(0)$ in $z_0$ (see (4.23)), there exists a constant $D$ such that $|\varphi^z_2(x^+_\alpha + iv)| \leq D$ for all $v \geq 0$, $z \in K$ and $|\alpha - \alpha_0| < \eta$. Then, the left-hand side of (5.22) is bounded by

$$DC_\eta C'_\eta\left(2 + \int_{v>1}(1+\sqrt{v})e^{-b\sqrt{v}}dv\right) = DC_\eta C'_\eta\left(2 + \frac{1}{b^2} + \frac{4}{b^3}\right) \leq D_{\alpha_0}$$

for some constant $D_{\alpha_0} > 0$ because $b \longrightarrow +\infty$ (because $\alpha_0 > 0$). This inequality implies (5.18).

Now suppose that $\alpha_0 = 0$. We no longer use estimate (5.23) because it would produce terms of order $\frac{1}{b}$, and here $b$ may be close to zero. Let $z = (a_1, b_1) \in K$. We write continuation Equation (3.7) for $\varphi^z_2(x^+_\alpha + iv)$, which splits into two terms,

$$\varphi^z_2(x^+_\alpha + iv) = -\frac{\gamma_1(x^+_\alpha + iv, Y^-(x^+_\alpha + iv))\varphi^z_1(Y^-(x^+_\alpha + iv))}{\gamma_2(x^+_\alpha + iv, Y^-(u+iv))} - \frac{e^{a_1(x^+_\alpha+iv)+b_1Y^-(x^+_\alpha+iv)}}{\gamma_2(x^+_\alpha + iv, Y^-(u+iv))}, \tag{5.24}$$

and we substitute into the right-hand side of (5.22). Then, the integral (5.22) can be written as the sum of two terms. For the first term, note that there are some constants, $c, C_0 > 0$ independent on $\alpha \in [0,\eta]$ and $z \in K$, such that

$$\left|\frac{\gamma_2(x_\alpha^+ + iv, Y^+(x_\alpha^+ + iv))\gamma_1(x_\alpha^+ + iv, Y^-(x_\alpha^+ + iv))\varphi_1^z(Y^-(x_\alpha^+ + iv))}{\gamma_2(x_\alpha^+ + iv, Y^-(u+iv))\partial_y\gamma(x_\alpha^+ + iv, Y^+(x_\alpha^+ + iv))}\right| \tag{5.25}$$

$$\times e^{-aiv-b(Y^+(x_\alpha^+ + iv)-Y^+(x_\alpha^+))} \le C_0(\sqrt{v}+1)e^{(a_1+b_1)Re(Y^-(x_\alpha+iv))}\varphi_1^z(0) \le C_0(\sqrt{v}+1)e^{(a_1+b_1)c\sqrt{v}}\varphi_1^z(0).$$

for any $v \ge 0$. We recall that function $z \longmapsto \varphi_1^z(0)$ is continuous and therefore locally bounded. The integral of (5.25) over $v > 0$ can then be bounded by a positive constant that is (locally) independent of $z$ and of $0 \le \alpha \le \eta$. The second term is given by

$$\int_0^{+\infty} \frac{\gamma_2(x_\alpha^+ + iv, Y^+(x_\alpha^+ + iv))e^{a_1(x_\alpha^+ + iv)+b_1Y^-(x_\alpha^+ + iv)}}{\gamma_2(x_\alpha^+ + iv, Y^-(u+iv))\partial_y\gamma(x_\alpha^+ + iv, Y^+(x_\alpha^+ + iv))} e^{-aiv-b(Y^+(x_\alpha^+ + iv)-Y^+(x_\alpha^+))}dv. \tag{5.26}$$

Note that if $b_1 = 0$, then the quotient in the integrand is of order $O(1/\sqrt{v})$ as $v \longrightarrow +\infty$. Moreover, it suffices to bound the integral over $(v_0, +\infty)$ for some $v_0 > 0$ because the integrand is uniformly bounded with respect to $\alpha \in [0,\eta]$ and $z \in K$. By integration by parts, the integral over $(v_0, +\infty)$ equals

$$\begin{aligned}&\frac{\gamma_2(x_\alpha^+ + iv_0, Y^+(x_\alpha^+ + iv_0))e^{a_1(x_\alpha^+ + iv_0)+b_1Y^-(x_\alpha^+ + iv_0)}e^{-aiv_0-b(Y^+(x_\alpha^+ + iv_0)-Y^+(x_\alpha))}}{\gamma_2(x_\alpha^+ + iv, Y^-(x_\alpha^+ + iv_0))\partial_y\gamma(x_\alpha^+ + iv_0, Y^+(x_\alpha^+ + iv_0))(-ai - b\frac{d}{dv}(Y^+(x_\alpha^+ + iv))_{v=v_0})}\\ &-\int_{v_0}^{\infty}\frac{d}{dv}\left(\frac{\gamma_2(x_\alpha^+ + iv, Y^+(x_\alpha^+ + iv))e^{a_1(x_\alpha^+ + iv)+b_1Y^-(x_\alpha^+ + iv)}}{\gamma_2(x_\alpha^+ + iv, Y^-(x_\alpha^+ + iv))\partial_y\gamma(x_\alpha^+ + iv, Y^+(x_\alpha^+ + iv))(-ai - b\frac{d}{dv}(Y^+(x_\alpha^+ + iv))}\right)\\ &\times \exp(-aiv - b(Y^+(x_\alpha^+ + iv) - Y^+(x_\alpha^+)))dv.\end{aligned} \tag{5.27}$$

Furthermore, $\frac{d}{dv}(Y^+(x_\alpha^+ + iv)) = i\left(1 - \frac{1}{\sqrt{\mu_2^2 - 2(x_\alpha + iv)}}\right)$ and $Re\left(1 - \frac{1}{\sqrt{\mu_2^2 - 2(x_\alpha + iv)}}\right) \ge 1/2$ for all $v \ge v_0$ with $v_0$ large enough and $0 < \alpha < \eta$. With some calculations, the integrand of (5.27) is of order $O(1/v^{3/2})$ as $v \longrightarrow +\infty$. Hence, the integral in (5.27) is bounded by a positive constant independent of $\alpha$ and of $z \in K$. This establishes the bound in (5.18). Inequalities (5.19), (5.20), and (5.21) are obtained similarly. □

## 6. Proof of Theorem 1

In Section 6.1, we establish the asymptotics stated in Theorem 1. In Section 6.2, we show that all the constants $h_\alpha(z_0)$ appearing in the asymptotics of Theorem 1 are nonzero, which completes the proof of the theorem.

### 6.1. Asymptotics in Theorem 1

We now have the tools to derive the asymptotics stated in Theorem 1, where $h_{\alpha_0}(z_0)$ is given by (1..22), $h_{\alpha^*}(z_0)$ by (1..24) (with the symmetric formula for $h_{\alpha^{**}}(z_0)$), and

$$c^* = \frac{\gamma_2(x^*, y^*)}{\partial_y\gamma(x^*, y^*)}\frac{x'(s^*)}{\gamma_2(x'(\zeta s^*), y'(\zeta s^*))}, \quad c^{**} = \frac{\gamma_1(x^{**}, y^{**})}{\partial_x\gamma(x^{**}, y^{**})}\frac{y'(s^{**})}{\gamma_1(x(\eta s^{**}), y'(\eta s^{**}))}, \tag{6.1}$$

where $x'(s)$ and $y'(s)$ are the derivatives of $x(s)$ and $y(s)$ (see (4.3) for the definition of $x(s)$ and $y(s)$, (4.10) for $s^*$ and $s^{**}$, (3.8) and (3.9) for $x^*, x^{**}, y^*$, and $y^{**}$, and (4.4) for $\eta s$ and $\zeta s$).

**Proof of the Asymptotics in Theorem 1 When $\boldsymbol{z_0 \ne (0,0)}$.** We use the identity $g(a,b) = I_1(a,b) + I_2(a,b) + I_3(a,b)$, using the expressions provided in Lemma 6. By the classical saddle point method (see details in (Franceschi et al. 2024b, lemma 8.1), the sum of the integrals of Lemma 6 along $\Gamma_{\alpha,x}$ and $\Gamma_{y,\alpha} = \underrightarrow{\overleftarrow{Y^+(\Gamma_{x,\alpha})}}$ has the following asymptotic expansion:

$$\frac{1}{2\pi i}\int_{\Gamma_{x,\alpha}} \frac{\varphi_2(x)\gamma_2(x, Y^+(x))}{\partial_y\gamma(x, Y^+(x))}e^{-ax-bY^+(x)}dx + \frac{1}{2\pi i}\int_{\Gamma_{y,\alpha}} \frac{\varphi_1(y)\gamma_1(X^+(y), y)}{\partial_x\gamma(X^+(y), y)}e^{-aX^+(y)-by}dy + \frac{1}{2\pi i}\int_{\Gamma_{y,\alpha}} e^{(a_0-a)X^+(y)+(b_0-b)y}\frac{dy}{\partial_x\gamma(X^+(y), y)}$$

$$\underset{\substack{r\to\infty\\ \alpha\to\alpha_0}}{=} e^{-r(\cos(\alpha)x(\alpha)+\sin(\alpha)y(\alpha))}\left(\frac{1}{\sqrt{r}}\sum_{k=0}^{n}\frac{c_k^{z_0}(\alpha)}{r^k} + o\left(\frac{1}{r^n\sqrt{r}}\right)\right). \tag{6.2}$$

where $(a,b) = (r\cos(\alpha), r\sin(\alpha))$, and

$$c_0^{z_0}(\alpha) = \frac{1}{\sqrt{2\pi(\cos(\alpha)+\sin(\alpha))^2}}\sqrt{\frac{\sin(\alpha)}{\partial_y\gamma(x(\alpha),y(\alpha))}} \tag{6.3}$$

$$\times(\gamma_1(x(\alpha),y(\alpha))\varphi_1(y(\alpha)) + \gamma_2(x(\alpha),y(\alpha))\varphi_2(x(\alpha)) + e^{(x(\alpha),y(\alpha))\cdot z_0}) \tag{6.4}$$

$$= \frac{1}{\sqrt{2\pi(\cos(\alpha)+\sin(\alpha))}}h_\alpha(z_0) \tag{6.5}$$

by the explicit expressions of $\varphi_1(y(s))$ and $\varphi_2(x(s))$ given in (4.1) and (4.13), evaluated at $s = \mathfrak{s}(\alpha)$ (see (5.5)).

Lemma 7 shows that, when $z_0 \neq (0,0)$, integrals over $S_x^\pm, S_y^\pm$ are negligible compared with those over paths of steepest descent. Finally, Theorem 5 gives the explicit form of residues of Lemma 6 providing $h_{\alpha^*}(z_0)$, $h_{\alpha^{**}}(z_0)$. □

For the case $z_0 = (0,0)$, we establish two preliminary lemmas. The first one is a consequence of the general Martin boundary theory.

**Lemma 8.** *For $\alpha \in [\alpha^*, \alpha^{**}]$, $z \longmapsto h_\alpha(z)$ is harmonic on $\mathbb{R}_+^2 \setminus \{(0,0)\}$.*

**Proof.** For $z_0 = (a_0, b_0) \neq (0,0)$, we may consider the process evolving in $\mathbb{R}_+^2 \cap \{(x,y),\ x+y \geq a_0 + b_0\}$. Because $h_\alpha$ is the limit of the quotient of Green's kernels, Kunita and Watanabe (1965) implies its harmonicity over all these domains and thus over $\mathbb{R}_+^2\setminus\{(0,0)\}$. □

**Lemma 9.** *Let $\Theta$ be the contour defined by $\Theta := \{z \in \mathbb{R}_+^2 : |z| = 1\}$ and $T_\Theta := \inf\{t \geq 0, Z_t \in \Theta\}$ the stopping time at $\Theta$. Then, for all $z_0 \in \mathbb{R}_+^2$ satisfying $|z_0| < 1$,*

$$h_{\alpha_0}(z_0) = \int_\Theta h_{\alpha_0}(z)\mathbb{P}_{z_0}(Z_{T_\Theta} = dz). \tag{6.6}$$

**Proof.** Suppose first that $z_0 \neq (0,0)$. The process $(h_\alpha(Z_t))_{t\geq 0}$ is a martingale; indeed, for $t, s \geq 0$,

$$\mathbb{E}_{z_0}[h_\alpha(Z_{t+s})\,|\,\mathcal{F}_t] = \mathbb{E}_{Z_t}[h_\alpha(Z_s)] = h_\alpha(Z_t)$$

by the strong Markov property and the harmonicity of $h_\alpha$ (see Lemma 8). Furthermore, under $\mathbb{P}_{z_0}$, the process $(h_\alpha(Z_{t\wedge T_\Theta}))_{t\geq 0}$ is bounded above by $\sup_{|z|\leq 1} h_\alpha(z) < \infty$ because $h_\alpha$ is continuous. Then, by the optional stopping theorem for bounded martingales, we obtain $h_{\alpha_0}(z_0) = \mathbb{E}_{z_0}[h(Z_{T_\Theta})]$, which is precisely the desired equality.

Now suppose $z_0 = (0,0)$, and consider a sequence $(z_n)_{n\geq 1}$ in the quarter plane converging to $(0,0)$ such that $0 < |z_n| < 1$. By continuity of $h_\alpha$, $h_\alpha(z_n)$ converges to $h_\alpha(z_0)$ as $n$ goes to $+\infty$. Because Equation (6.6) holds for all nonzero initial conditions, it suffices to show that

$$\int_\Theta h_{\alpha_0}(z)\mathbb{P}_{z_n}(Z_{T_\Theta} = dz) \underset{n\to+\infty}{\longrightarrow} \int_\Theta h_{\alpha_0}(z)\mathbb{P}_{(0,0)}(Z_{T_\Theta} = dz).$$

By continuity and boundedness of $h_\alpha$ on $\{z \in \mathbb{R}_+^2, |z| \leq 1\}$), it is enough to show that $\mathcal{L}_{z_n}Z_{T_\Theta} \underset{n\to+\infty}{\longrightarrow} \mathcal{L}_{z_0}Z_{T_\Theta}$ weakly, where $\mathcal{L}_z Z_{T_\Theta}$ denotes the law of $Z_{T_\Theta}$ with initial condition $Z_0 = z$. This follows from Assumption (1.2), combined with Harrison and Reiman (1981, theorem 1), which ensures the continuity of the mapping from the nonreflected to the reflected path under the topology of uniform convergence on compacts. □

We can now prove Theorem 1 in the case of $z_0 = (0,0)$.

**Proof of the Asymptotics in Theorem 1 for $z_0 = (0,0)$.** By continuity of the process and by the Strong Markov property, if $(a,b)$ lies at a distance $> 1$ from $(0,0)$, then

$$g^{(0,0)}(a,b) = \int_\Theta g^z(a,b)\mathbb{P}(Z_{T_\Theta} = dz). \tag{6.7}$$

Because the constant $C$ from the saddle point method (Franceschi et al. 2024b, lemma 8.1) depends continuously on $z_0$, and because the constants $D_{\alpha_0}$ in Lemma 7 are locally uniform in $z_0$, then for any compact set $K$ in the quadrant $\mathbb{R}_+^2$ with $d((0,0),K) > 0$, we have

$$\sup_{z\in K}\left|g^z(r\cos(\alpha), r\sin(\alpha)) - e^{-r(\cos(\alpha)x(\alpha)+\sin(\alpha)y(\alpha))}\frac{1}{\sqrt{r}}\sum_{k=0}^{n}\frac{c_k^z(\alpha)}{r^k}\right| \underset{\substack{r\to\infty\\ \alpha\to\alpha_0}}{=} o\left(\frac{e^{-r(\cos(\alpha)x(\alpha)+\sin(\alpha)y(\alpha))}}{r^n\sqrt{r}}\right). \tag{6.8}$$

By this expansion,the asymptotics of (6.7) yield

$$g^{(0,0)}(a,b) \underset{\substack{r\to\infty\\ \alpha\to\alpha_0}}{\sim} \frac{e^{-r(\cos(\alpha)x(\alpha)+\sin(\alpha)y(\alpha))}}{\sqrt{r}}\int_{\Theta} h_{\alpha_0}(z)\mathbb{P}(Z_{T_\Theta} = dz). \tag{6.9}$$

Lemma 9 combined with (6.9) gives the result. □

### 6.2. Positivity of $h_\alpha(z_0)$

To make our asymptotics consistent, we prove here the positivity of the constants $h_\alpha(z_0)$.

**Lemma 10.** *Let $\alpha \in (\alpha^*, \alpha^{**})$. Then, for every $R > 0$, there exists $z_0$ such that $|z_0| \geq R$ and $h_\alpha(z_0) > 0$. If $\alpha^* > 0$ (resp. $\alpha^{**} < \pi/2$), then the same result holds for $h_{\alpha^*}$ (resp. $h_{\alpha^{**}}$).*

**Proof.** By the explicit Equation (1.22) for $h_\alpha$, Equation (1.24) for $h_{\alpha^*}$, and its equivalent for $h_{\alpha^{**}}$, the following asymptotics hold as $r \to +\infty$ for $\alpha \in (\alpha^*, \alpha^{**})$, and for $\alpha = \alpha^*$ (resp. $\alpha = \alpha^{**}$), if $\alpha^* > 0$ (resp. $\alpha^{**} < \pi/2$), then

$$h_\alpha(r\cos(\alpha), r\sin(\alpha)) \underset{r\to\infty}{\sim} e^{r(x(\alpha)\cos(\alpha)+y(\alpha)\sin(\alpha))}. \tag{6.10}$$

The conclusion follows with $z_0 = (r\cos(\alpha), r\sin(\alpha))$ for $r$ large enough. □

The following Lemma is inspired by (Franceschi et al. 2024b, lemma 8.3) and establishes the positivity of constants $h_\alpha(z_0)$ in the framework of Theorem 1.

**Lemma 11** (Positivity of $h_\alpha(z)$). *Let $\alpha \in (\alpha^*, \alpha^{**})$ or $\alpha = \alpha^*$ (resp. $\alpha = \alpha^{**}$) if $\alpha^* > 0$ (resp. if $\alpha^{**} < \pi/2$). Let $z \in \mathbb{R}_+^2$. Then, $h_\alpha(z) \neq 0$.*

**Proof.** Let $\alpha \in (\alpha^*, \alpha^{**})$, and let $z_0$ such that both coordinates are larger than those of $z$ and such that $h_\alpha(z_0) > 0$; see Lemma 10. Let $V$ be a compact neighborhood of $z_0$, and denote by $T_V := \inf\{t \geq 0 : Z_t \in V\}$ the hitting time of $V$. By the hypothesis on $z$, $\mathbb{P}_z(T_V < +\infty) > 0$. By the strong Markov property,

$$g^z(r\cos(\alpha), r\sin(\alpha)) \geq \mathbb{P}_z(T_V < +\infty)\inf_{z_0'\in V} g^{z_0'}(r\cos(\alpha), r\sin(\alpha)) \tag{6.11}$$

$$\geq \mathbb{P}_z(T_V < +\infty)\inf_{z_0'\in V}(h_\alpha(z_0') + \varepsilon_{z_0',\alpha}(r))\frac{e^{-r(\cos(\alpha)x(\alpha)+\sin(\alpha)y(\alpha))}}{\sqrt{r}}. \tag{6.12}$$

for $r$ large enough where (6.8) provides the asymptotics

$$\sup_{z_0'\in V}\left|\varepsilon_{z_0',\alpha}(r)\right| \underset{r\to\infty}{\longrightarrow} 0.$$

Furthermore, by continuity of $h_\alpha$, the set $V$ can be chosen to satisfy $\inf_{z_0'\in V} h_\alpha(z_0') > 0$. On the other hand, we also have

$$g^z(r\cos(\alpha), r\sin(\alpha)) = \frac{e^{-r(\cos(\alpha)x(\alpha)+\sin(\alpha)y(\alpha))}}{\sqrt{r}}(h_\alpha(z) + \varepsilon_{z_0',\alpha}(r))$$

where $\varepsilon_{z_0',\alpha}(r) \to 0$ as $r \to +\infty$. Therefore, comparing the two expressions, we conclude that $h_\alpha(z) > 0$. If $\alpha = \alpha^*$ or $\alpha = \alpha^{**}$, then the proof is analogous. □

This is the end of the proof of Proposition 4.

The remaining part of the proof is equivalent to showing that $h_{\alpha^*}(z_0) > 0$; indeed, $h_{\alpha^*}(z_0)$ is equal to $\mathrm{res}_{x=x^*}\varphi_2(x)$ up to a nonnull multiplicative constant (see (1.24) and (4.11)). The positivity is established in the previous lemma. □

## 7. Asymptotics of Green's Kernel in the Particular Directions 0, $\alpha^*$, $\alpha^{**}$, and $\pi/2$

In Section 7.1, we study the asymptotics of Green's functions in the direction $\alpha = 0$ under the assumption that $\gamma_2(x_{max}, Y^{\pm}(x_{max})) \neq 0$. In Section 7.2, we provide these asymptotics in the direction $\alpha^*$ if $\alpha^* > 0$. Then, in Section 7.3, we analyze the limiting case where $\alpha^* = 0$ and $\gamma_2(x_{max}, Y^{\pm}(x_{max})) = 0$. The analysis of the directions $\alpha = \pi/2$, $\alpha = \alpha^{**}$ if $\alpha^* < \pi/2$ and $\alpha^{**} = \pi/2, \gamma_1(X^{\pm}(y_{max}, y_{max})) = 0$ is symmetrical. We then derive the proof of Theorem 2 in Section 7.4.

### 7.1. Case $\alpha \to 0$ If $\gamma_2(x_{max}, Y^{\pm}(x_{max})) \neq 0$

Before deriving the asymptotics, let us relate Green's densities $g^{z_0}$ to $f_1^{z_0}, f_2^{z_0}$.

**Proposition 8** (Link Between Densities). *Let $a, b \geq 0$. Suppose $z_0 \neq (a, 0)$ and $z_0 \neq (0, b)$. Then, we have*

$$f_1^{z_0}(b) = \frac{1}{2} g^{z_0}(0, b) \quad \text{and} \quad f_2^{z_0}(a) = \frac{1}{2} g^{z_0}(a, 0).$$

**Proof.** By the functional Equation (3.1), if $x, y < 0$, then

$$-\frac{\gamma(x,y)}{x}\varphi(x,y) = \frac{\gamma_1(x,y)}{x}\varphi_1(y) + \frac{\gamma_2(x,y)}{x}\varphi_2(x) + \frac{e^{(x,y)\cdot z_0}}{x}. \tag{7.1}$$

Furthermore, by elementary properties of Laplace transforms,

$$x\varphi(x,y) \underset{x\to-\infty}{\longrightarrow} -\int_0^{+\infty} e^{by} g(0,b)db.$$

Then, letting $x \to -\infty$ in (7.1), we get $\int_0^{+\infty} e^{by} g(0,b)db = \varphi_1(y) = \int_0^{+\infty} e^{by} f_1(b)db$. The injectivity of the Laplace transform concludes the proof. The case of $f_2$ is symmetrical. □

**Lemma 12** (Asymptotics at $\alpha = 0$). *Suppose that $\varphi_2$ does not have a pole. Let $\kappa = \left(\frac{1+\mu_2}{2}\Gamma(1/2)\right)^{-1}$, where $\Gamma$ denotes the usual Gamma function. Then,*

$$f_2^{z_0}(x) \underset{x\to+\infty}{\sim} \frac{\kappa \partial_\alpha [h_\alpha(z_0)]_{\alpha=0}}{x^{3/2}} e^{-x_{max}x}.$$

**Proof.** Note that $Y^-(x) = Y^-(x_{max}) - \sqrt{2(x_{max} - x)} + o(\sqrt{x - x_{max}})$ by (3.3). Then, using the continuation Equation (3.7), $\varphi_2$ is continuous at $x_{max}$ and

$$\begin{aligned}
\frac{\varphi_2(x) - \varphi_2(x_{max})}{\sqrt{2(x_{max} - x)}} &= r_1\varphi_1(Y^-(x_{max})) + \gamma_1(x_{max}, Y^{\pm}(x_{max}))\varphi_1'(Y^{\pm}(x_{max})) + b_0 e^{z_0\cdot(x_{max}, Y^{\pm}(x_{max}))} \\
&\quad + \frac{\gamma_1(x_{max}, Y^{\pm}(x_{max}))\varphi_1(Y^-(x_{max})) + e^{z_0\cdot(x_{max}, Y^{\pm}(x_{max}))}}{\gamma_2(x_{max}, Y^{\pm}(x_{max}))} + o_{x\to x_{max}}(1) \\
&= r_1\varphi_1(Y^-(x_{max})) + \gamma_1(x_{max}, Y^{\pm}(x_{max}))\varphi_1'(Y^{\pm}(x_{max})) + b_0 e^{z_0\cdot(x_{max}, Y^{\pm}(x_{max}))} \\
&\quad + \varphi_2(x_{max}) + o_{x\to x_{max}}(1) \\
&=: A + o_{x\to x_{max}}(1),
\end{aligned}$$

where $z_0 = (a_0, b_0)$. Then, by the Tauberian theorem given by Dai and Miyazawa (2011, lemma C.2), we obtain

$$f_2^{z_0}(x) \underset{x\to+\infty}{\sim} \frac{A}{\Gamma(1/2)x^{3/2}} e^{-x_{max}x}.$$

It then remains to be shown that $\frac{A}{\Gamma(1/2)} = \kappa \partial_\alpha [h_\alpha(z_0)]_{\alpha=0}$. From Equation (5.6), we have

$$\begin{aligned}
\partial_\alpha [h_\alpha(z_0)]_{\alpha=0^+} &= (\mathfrak{s}^{-1})'(s_{max})\partial_s \left[\gamma_1(x(s), y(s))\varphi_1(y(s)) + \gamma_2(x(s), y(s))\varphi_2(x(s)) + e^{a_0 x(s) + b_0 y(s)}\right]_{s=s_{max}} \\
&= \left(\frac{-(\mu_1 + \mu_2)}{(\mu_1 + \mu_2)^2}\right)(r_1 y'(s_{max})\varphi_1(y(s_{max})) + \gamma_1(x(s_{max}), y(s_{max}))y'(s_{max})\varphi_1'(y(s)) \\
&\quad + y'(s_{max})\varphi_2(x(s_{max})) + y'(s_{max}) b_0 e^{a_0 x(s_{max}) + b_0 y(s_{max})}) \\
&= \frac{1 + \mu_2}{2} A
\end{aligned}$$

because $y'(s_{max}) = y'(\mu_2) = -\frac{1+\mu_2}{2}$ and $x'(s_{max}) = 0$. The conclusion follows. □

We can now establish the asymptotics of Green's functions as $\alpha \to 0$. We use the notation $c_0(\alpha)$ and $c_1(\alpha)$ for the constants in the first and second terms, respectively, in the asymptotic expansion of Green's functions (cf. (6.8)).

**Theorem 6** (Asymptotics with $\alpha \to 0$). *Suppose $\gamma_2(x_{max}, Y^{\pm}(x_{max})) \neq 0$. Then,*

$$h_\alpha(z_0) \underset{\alpha\to 0}{\sim} \alpha\partial_\alpha[h_\alpha(z_0)]_{\alpha=0} \tag{7.2}$$

*and thus $c_0^{z_0}(\alpha) \underset{\alpha\to 0}{\sim} \frac{1}{\sqrt{2\pi}}\alpha\partial_\alpha[h_\alpha(z_0)]_{\alpha=0}$. Furthermore, if $\alpha^* = 0$, then*

$$c_1^{z_0}(\alpha) \underset{\alpha\to 0}{\longrightarrow} 2\kappa\partial_\alpha[h_\alpha(z_0)]_{\alpha=0}. \tag{7.3}$$

*Moreover,*

- *If $\alpha^* = 0$ (i.e. $\varphi_2$ has no pole), then*

$$g^{z_0}(r\cos(\alpha), r\sin(\alpha)) \underset{\substack{r\to\infty\\ \alpha\to 0}}{\sim} \partial_\alpha[h_\alpha(z_0)]_{\alpha=0}\frac{e^{-r(\cos(\alpha)x(\alpha)+\sin(\alpha)y(\alpha))}}{\sqrt{r}}\left(\frac{\alpha}{\sqrt{2\pi}} + \frac{2\kappa}{r}\right).$$

- *If $\alpha^* > 0$ (i.e. $\varphi_2$ has a pole), then*

$$g^{z_0}(r\cos(\alpha), r\sin(\alpha)) \underset{\substack{r\to\infty\\ \alpha\to 0}}{\sim} c^* h_{\alpha^*}(z_0)e^{-r(\cos(\alpha)x^*+\sin(\alpha)y^*)},$$

*where $h_{\alpha^*}(z_0)$ and $c^*$ are given by (1.24) and (6.1), respectively.*

*Moreover, the constants $\partial_\alpha[h_\alpha(z_0)]_{\alpha=0}$ and $h_{\alpha^*}(z_0)$ are nonzero in the corresponding asymptotics.*

**Proof.** First, (7.2) follows from the regularity of $h_\alpha(z_0)$ in $\alpha$ and from the convergence $h_\alpha(z_0) \to 0$ as $\alpha \to 0$ (see (1.22)). Now we analyze the asymptotics of the sum of the three integrals in (2) along the saddle point curves as $\alpha \to 0$. The integrands in the second and third terms are holomorphic in a neighborhood of the saddle point $Y^+(x_{max})$. The integrand of the first term, namely $\varphi_2$, has a branching point at $x_{max}$. For this reason, we perform the change of variables $\Gamma_{x,\alpha} = \overleftarrow{\underrightarrow{X^+(\Gamma_{y,\alpha})}}$:

$$\int_{\Gamma_{x,\alpha(a,b)}} \frac{\varphi_2(x)\gamma_2(x, Y^+(x))}{\partial_y\gamma(x, Y^+(x))}e^{-ax-bY^+(x)}dx = \int_{\Gamma_{y,\alpha(a,b)}} \frac{\varphi_2(X^+(y))\gamma_2(X^+(y), y)}{\partial_x\gamma(X^+(y), y)}e^{-aX^+(y)-by}dy. \tag{7.4}$$

Additionally, from (3.7),

$$\varphi_2(X^+(y)) = \frac{-\gamma_1(X^+(y), Y^-(X^+(y)))\varphi_1(Y^-(X^+(y))) - e^{a_0X^+(y)+b_0Y^-(X^+(y))}}{\gamma_2(X^+(y), Y^-(X^+(y)))}. \tag{7.5}$$

Note that $X^+(y)$ is holomorphic in a neighborhood of $Y^{\pm}(x_{max})$. The crucial point is that $Y^-(X^+(y))$ is also holomorphic there. Indeed, it can be expressed as

$$Y^-(X^+(y)) = \frac{X^+(y)^2 + 2\mu_1X^+(y)}{y}. \tag{7.6}$$

To see this, note that $Y^-(x)$ and $Y^+(x)$ are the two roots of $y \mapsto \frac{1}{2}(x-y)^2 + \mu_1x + \mu_2y$. Then, by Vieta's equations and because $Y^+(X^+(y)) = y$, (7.6) follows immediately. Because $\gamma_2(x_{max}, Y^{\pm}(x_{max}) \neq 0$, it follows from (7.5) that $\varphi_2(X^+(y))$ is holomorphic at $Y^{\pm}(x_{max})$, so the saddle point method applies to the right-hand side of (7.4). Then, asymptotics of Green's functions become

$$g(r\cos(\alpha), r\sin(\alpha)) \underset{\substack{r\to\infty\\ \alpha\to \alpha_0}}{\sim} c^*h_{\alpha^*}(z_0)e^{-r(\cos(\alpha)x^*+\sin(\alpha)y^*)}\mathbb{1}_{\alpha^*>0} + e^{-r(\cos(\alpha)x(\alpha)+\sin(\alpha)y(\alpha))}\frac{1}{\sqrt{r}}\left(c_0(\alpha) + \frac{c_1(\alpha)}{r}\right) \tag{7.7}$$

for all $\alpha_0 \in [0, \epsilon]$, where $\epsilon > 0$ is sufficiently small. It remains to show that if $\alpha^* = 0$, then (3) holds. With $\alpha = 0$, Equation (7.7) becomes

$$g(r, 0) \underset{r\to\infty}{\sim} \frac{e^{-rx_{max}}}{r^{3/2}}c_1(0). \tag{7.8}$$

By Proposition 8, $g(r, 0) = 2f_2(r)$. Finally, Lemma 12 applies and completes the asymptotic analysis. The nonvanishing of $\partial_\alpha[h_\alpha(z_0)]_{\alpha=0^+}$ is analogous to the case $\alpha \in (\alpha^*, \alpha^{**})$; see Section 6.2. The nonvanishing of $h_{\alpha^*}(z_0)$ if $\alpha^* > 0$ is already proved in Lemma 11. □

### 7.2. Case $\alpha \to \alpha^*$ When $\alpha^* > 0$

**Theorem 7** (Asymptotics with $\alpha \to \alpha^*$). *Suppose $\alpha^* > 0$ (i.e., $\varphi_2$ has a pole). Then,*

- *If $r(\alpha - \alpha^*)^2 \to 0$, then*

$$g^{z_0}(r\cos(\alpha), r\sin(\alpha)) \underset{\substack{r\to\infty \\ \alpha\to\alpha^*}}{\sim} \frac{1}{2} h_{\alpha^*}(z_0) e^{-r(\cos(\alpha)x^* + \sin(\alpha)y^*)}, \tag{7.9}$$

*where $h_{\alpha^*}(z_0)$ is given by (1.24).*

- *If $r(\alpha - \alpha^*)^2 \to K > 0$ for some constant $K$, then for $\alpha < \alpha^*$ (resp. $\alpha > \alpha^*$),*

$$g^{z_0}(r\cos(\alpha), r\sin(\alpha)) \underset{\substack{r\to\infty \\ \alpha\to\alpha^*}}{\sim} c_K h_{\alpha^*}(z_0) e^{-r(\cos(\alpha)x^* + \sin(\alpha)y^*)}$$

$$\left( resp. \quad g^{z_0}(r\cos(\alpha), r\sin(\alpha)) \underset{\substack{r\to\infty \\ \alpha\to\alpha^*}}{\sim} \tilde{c}_K h_{\alpha^*}(z_0) e^{-r(\cos(\alpha)x^* + \sin(\alpha)y^*)} \right), \tag{7.10}$$

*where constants $c_K > 0$, $\tilde{c}_K > 0$ are independent of initial condition $z_0$.*

- *If $r(\alpha - \alpha^*)^2 \to \infty$, then*
  - *If $\alpha < \alpha^*$, then*

$$g^{z_0}(r\cos(\alpha), r\sin(\alpha)) \underset{\substack{r\to\infty \\ \alpha\to\alpha^*}}{\sim} h_{\alpha^*}(z_0) e^{-r(\cos(\alpha)x^* + \sin(\alpha)y^*)}. \tag{7.11}$$

  - *If $\alpha > \alpha^*$, then*

$$g^{z_0}(r\cos(\alpha), r\sin(\alpha)) \underset{\substack{r\to\infty \\ \alpha\to\alpha^*}}{\sim} h_{\alpha^*}(z_0) e^{-r(\cos(\alpha)x(\alpha) + \sin(\alpha)y(\alpha))} \frac{1}{\sqrt{r}} \frac{C}{\alpha - \alpha^*}, \tag{7.12}$$

*where $C$ is a positive constant independent of initial condition $z_0$.*

*Furthermore, $h_{\alpha^*}(z_0) > 0$. Constants $c_K, \tilde{c}_K$, and $C$ are made explicit in Franceschi et al. (2024b, section 10).*

The proof is analogous to Franceschi et al. (2024b, section 10), which compares the asymptotic contribution of the pole term and the saddle point term in the expressions of Lemma 6 for $g^{z_0} = I_1 + I_2 + I_3$. The nonvanishing of $h_{\alpha^*}(z_0)$ was already proved in Lemma 11.

### 7.3. Last Particular Case $\gamma_2(x_{\max}, Y^{\pm}(x_{\max})) = 0$

**Theorem 8.** *Suppose $\gamma_2(x_{max}, Y^{\pm}(x_{max})) = 0$ (so $\alpha^* = 0$). Then,*

$$g^{z_0}(r\cos(\alpha), r\sin(\alpha)) \underset{\substack{r\to\infty \\ \alpha\to 0}}{\sim} e^{-r(\cos(\alpha)x(\alpha) + \sin(\alpha)y(\alpha))} \frac{h_0(z_0)}{\sqrt{r}}, \tag{7.13}$$

*where $h_0(z_0)$ is given by (1.25). Furthermore, $h_0(z_0) \neq 0$.*

**Proof.** The proof follows the same approach as that of Theorem 6, but here $c_0(\alpha) \to h_0(z_0) \neq 0$. Thus, we consider only the first term in (6.2) with the representation (7.4) for $I_1(a,b)$. First, note from (3.6) that $\frac{d}{dy}[X^+(y)]_{y=Y^{\pm}(x_{max})} = 0$. Then, by (7.6),

$$\frac{d}{dy}[\gamma_2(X^+(y), Y^-(X^+(y)))]_{y=Y^{\pm}(x_{max})} = -\left( \frac{(\mu_2/2)^2 + 2\mu_1\mu_2^2/2}{(\mu_2^2/2 - \mu_2)^2} \right) = -1. \tag{7.14}$$

Hence, $\gamma_2(X^+(y), Y^-(X^+(y))) = (y - Y^{\pm}(x_{max}))(-1 + o_{y\to Y^{\pm}(x_{max})}(1))$. Furthermore, $\gamma_2(X^+(y), y) = (y - Y^{\pm}(x_{max}))(1 + o_{y\to Y^{\pm}(x_{max})}(1))$ by similar calculations.

Using the same arguments as in the proof of Theorem 6, the function

$$\frac{\gamma_2(X^+(y), y)\varphi_2(X^+(y))}{\partial_x \gamma(X^+(y), y)} \tag{7.15}$$

is holomorphic in a neighborhood of $Y^{\pm}(x_{max})$, expect possibly in $Y^{\pm}(x_{max})$, where $\varphi_2(X^+(y))$ has a simple pole by (7.5) and (7.14). Because $\gamma_2(X^+(y), y)$ has a zero of the same order at this point, then the quantity (7.15) turns out to be holomorphic at $Y^{\pm}(x_{max})$ as well. Moreover, by (4.23), the following asymptotic expansion holds as $y \to Y^{\pm}(x_{max})$:

$$\frac{\gamma_2(X^+(y),y)\varphi_2(X^+(y))}{\partial_x\gamma(X^+(y),y)} = (-1+o_{y\to Y^{\pm}(x_{max})}(1))\Bigg(-e^{z_0\cdot(X^+(y),Y^-(X^+(y)))} + \frac{\gamma_1(\psi_1(X^+(y),y))}{\frac{\gamma_1}{\gamma_2}(\psi_2(X^+(y),y))}$$
$$\sum_{n=1}^{+\infty}\left[\prod_{k=2}^{n}\frac{\frac{\gamma_1}{\gamma_2}(\psi_{2k-1}(X^+(y),y)}{\frac{\gamma_1}{\gamma_2}(\psi_{2k}(X^+(y),y))}\right]\left[\frac{e^{z_0\cdot\psi_{2n}(X^+(y),y))}}{\gamma_2(\psi_{2n}(X^+(y),y))} - \frac{e^{z_0\cdot\psi_{2n+1}(X^+(y),y)}}{\gamma_2(\psi_{2n+1}(X^+(y),y))}\right]\Bigg). \quad (7.16)$$

This implies (1.25). The proof of the nonvanishing of $h_0(z_0)$ is analogous to Lemma 11. □

### 7.4. Proof of Theorem 2

This is a direct consequence of Theorems 1, 6, 7, and 8.

## 8. Harmonic Functions and Martin Boundary

In this section, we prove Theorem 3. In particular, we show in Section 8.1 that the Martin boundary is homeomorphic to $[\alpha^*, \alpha^{**}]$ and in Section 8.2 that the Martin boundary is minimal.

### 8.1. Context of Martin the Boundary

In this section, we consider the construction of the Martin boundary as presented in (Pinsky 1995, section 7.1) for elliptic processes, and we adapt this approach to reflected degenerate processes. This method allows us to consistently link the harmonic functions $(h_\alpha)_{\alpha\in[\alpha^*,\alpha^{**}]}$ found in Theorem 2 and the Martin boundary. Note that another general construction of the Martin compactification is presented in Kunita and Watanabe (1965).

**Definition 3** (Martin Kernel). For $z_0, z_1 \in \mathbb{R}_+^2$, we define the Martin kernel,

$$k(z_0,z_1) = \begin{cases} \dfrac{g^{z_0}(z_1)}{g^{(0,0)}(z_1)} & \text{if} \quad z_1 \neq (0,0) \\ 0 & \text{if} \quad z_1 = (0,0) \quad \text{or} \quad z_0 = z_1, \end{cases} \quad (8.1)$$

and the Martin metric,

$$\rho(z_1,z_2) = \int_{\mathbb{R}_+^2} \frac{|k(x,z_1)-k(x,z_2)|}{1+|k(x,z_1)-k(x,z_2)|}e^{-|x|^2}dx. \quad (8.2)$$

By usual considerations (see Pinsky 1995), $\rho$ is a metric equivalent to the Euclidean one on $\mathbb{R}_+^2$. A sequence $(y_n)_{n\geq 0}$ of $\mathbb{R}_+^2$ is called a Martin sequence if $(k(\cdot,y_n))_{n\geq 0}$ converges pointwise. Two Martin sequences are said to be equivalent if their limit functions are equal. We then define $M$ as the quotient of the set of all Martin sequences by this equivalence relation. Each $\xi \in M$ is then naturally associated with function denoted by $k(\cdot,\xi)$. The metric $\rho$ extends naturally to $M$ with the same formula so that the map

$$\iota : \begin{cases} \mathbb{R}_+^2 \longrightarrow (M,\rho) \\ z \longmapsto k(\cdot,z) \end{cases} \quad (8.3)$$

is injective and continuous. We define the Martin boundary $\Gamma$ as $\Gamma = M\backslash\iota(\mathbb{R}_+^2)$.

**Lemma 13.** *Let $(h_\alpha)_{\alpha\in[\alpha^*,\alpha^{**}]}$ be defined in Theorem 2. Then, the map*

$$\Phi : \alpha \in [\alpha^*, \alpha^{**}] \longmapsto h_\alpha(\cdot)/h_\alpha(0) \in \Gamma$$

*is a homeomorphism.*

Before proving this lemma, we recall some properties of the family $(h_\alpha)_{\alpha\in[\alpha^*,\alpha^{**}]}$.

**Remark 3.** Note that for $z \in \mathbb{R}_+^2$:

- If $\alpha^* > 0$, then by (1.22),

$$h_\alpha(z) \underset{\substack{\alpha\to\alpha^* \\ \alpha>\alpha^*}}{\longrightarrow} +\infty.$$

- If $\alpha^* = 0$ and $\gamma_2(x_{max}, Y^{\pm}(x_{max}) = 0$, then by (1.25),

$$h_\alpha(z) \underset{\substack{\alpha\to 0 \\ \alpha>0}}{\longrightarrow} h_0(z) > 0.$$

- If $\alpha^* = 0$ and $\gamma_2(x_{max}, Y^{\pm}(x_{max}) \neq 0$, then by (1.24),

$$h_\alpha(z) \underset{\substack{\alpha\to 0 \\ \alpha>0}}{\longrightarrow} 0.$$

**Proof of Lemma 13.** By Theorems 1, 2, 6, 7, and 8, $\Phi$ is surjective. To prove the continuity of $\Phi$, note that a sequence $(\xi_n)_{n\geq 0}$ converges to some $\xi \in M$ if $k(\cdot, \xi_n)$ converges pointwise toward $k(\cdot, \xi)$ almost everywhere. Therefore, the proof of the continuity of $\Phi$ is reduced to showing that, for any $z \in \mathbb{R}_+^2$, the map $\alpha \longmapsto \Phi(\alpha)(z)$ is continuous. Let $z \in \mathbb{R}_+^2$.

- By (1.22), the map $\alpha \longmapsto \Phi(\alpha)(z)$ is continuous on $(\alpha^*, \alpha^{**})$.
- If $\alpha^* = 0$ and $\gamma_2(x_{max}, Y^{\pm}(x_{max})) \neq 0$, then we have

$$\Phi(\alpha)(z) = \frac{h_\alpha(z)}{h_\alpha(0)} = \frac{h_\alpha(z)}{\alpha}\frac{\alpha}{h_\alpha(0)} \underset{\alpha\to 0}{\longrightarrow} \frac{[\partial_\alpha h_\alpha(z)]_{\alpha=0}}{[\partial_\alpha h_\alpha(0)]_{\alpha=0}} = \Phi(0)(z),$$

so $\alpha \longmapsto \Phi(\alpha)(z)$ is continuous at $\alpha^* = 0$.

- If $\alpha^* > 0$, then $\frac{h_\alpha(z)}{h_\alpha(0)}$ can be written as

$$\frac{\sum_{m=-\infty}^{0}\kappa_m(\alpha)e^{z\cdot(a_m(\mathfrak{s}(\alpha)),b_m(\mathfrak{s}(\alpha)))} + \frac{1}{\gamma_2(\zeta\mathfrak{s}(\alpha))}\sum_{m=1}^{+\infty}\kappa_m(\alpha)e^{z\cdot(a_m(\mathfrak{s}(\alpha)),b_m(\mathfrak{s}(\alpha)))}}{\sum_{m=-\infty}^{0}\kappa_m(\alpha) + \frac{1}{\gamma_2(\zeta\mathfrak{s}(\alpha))}\sum_{m=1}^{+\infty}\kappa_m(\alpha)},$$

where $(a_m(\mathfrak{s}(\alpha)), b_m(\mathfrak{s}(\alpha)))$, are defined by (1.20) and (1.21), with $(a_0(\mathfrak{s}(\alpha)), b_0(\mathfrak{s}(\alpha))) = (x(\alpha), y(\alpha))$, and $\kappa_m(\alpha)$ is given by (1.23). Because $\gamma_2(\zeta(\mathfrak{s}(\alpha))) \underset{\alpha\to\alpha^*}{\longrightarrow} 0$ (see Notation 1 and Proposition 4), the expected continuity in $\alpha^*$ follows from standard continuity theorems on series.

- The remaining case $\alpha^* = 0$ and $\gamma_2(x_{max}, Y^{\pm}(x_{max})) = 0$ is analogous.
- The proof of the continuity of $\Phi$ at $\alpha^{**}$ is symmetric.

Next, let us show that $\Phi$ is injective. By the explicit expressions in Theorem 2, the following asymptotics hold as $r \to +\infty$. For $\alpha^* < \alpha < \alpha^{**}$ and $0 \leq \theta \leq \pi/2$, we have

$$h_\alpha(r\cos(\theta), r\sin(\theta)) \underset{r\to\infty}{\sim} e^{r(x(\alpha)\cos(\theta)+y(\alpha)\sin(\theta))}. \tag{8.4}$$

If $\alpha^* = 0$, then for any $0 < \theta \leq \pi/2$,

$$h_0(r\cos(\theta), r\sin(\theta)) \underset{r\to\infty}{\sim} r\sin(\theta)e^{r(x(0)\cos(\theta)+y(0)\sin(\theta))}. \tag{8.5}$$

If $\alpha^* > 0$ and $0 \leq \theta \leq \pi/2$, then

$$h_{\alpha^*}(r\cos(\theta), r\sin(\theta)) \underset{r\to\infty}{\sim} e^{r(x(\alpha^*)\cos(\theta)+y(\alpha^*)\sin(\theta))}. \tag{8.6}$$

The corresponding symmetric asymptotic behavior holds for $h_{\alpha^{**}}$. If $\alpha, \alpha' \in [\alpha^*, \alpha^{**}]$ are distinct, then by (5.4) and the preceding formulae,

$$\frac{h_\alpha(r\cos(\alpha), r\sin(\alpha))}{h_{\alpha'}(r\cos(\alpha), r\sin(\alpha))} \underset{r\to\infty}{\longrightarrow} +\infty.$$

Hence, $h_\alpha \neq Ch_{\alpha'}$ for any constant $C$, and $\Phi$ is injective.

Because $\Phi$ is continuous, and because $[\alpha^*, \alpha^{**}]$ is compact, $\Phi(F)$ is closed in $\Gamma$ for any closed subset $F$ of $[\alpha^*, \alpha^{**}]$. Therefore, $\Phi$ is a homeomorphism. □

**Corollary 2.** *The following properties hold:*

(i) *If $\eta, \xi \in M$ satisfies $k(\cdot, \xi) = k(\cdot, \eta)$, then $\eta = \xi$.*
(ii) *The metric space $(M, \rho)$ is compact.*

(iii) $\iota(\mathbb{R}_+^2)$ *is dense in* $M$ *with respect to* $\rho$.
(iv) *If a sequence* $(y_n)_{n\geq 0} \subset \mathbb{R}_+^2$ *converges to* $\eta \in \Gamma$ *with respect to* $\rho$, *then* $k(\cdot, y_n)$ *converges pointwise to* $k(\cdot, \eta)$.

**Proof.** Properties (i), (iii), and (iv) follow directly from our construction. We now prove (ii). Let $(y_n)_{n\geq 0}$ be a sequence in $M$. Then:

- Either $(y_n)_{n\geq 0}$ has infinitely many points in $\Gamma$, in which case it has a convergent subsequence because $\Gamma$ is compact (see Lemma 13);
- or $(y_n)_{n\geq 0}$ has a bounded subsequence, in which case the conclusion follows because $\rho|_{\mathbb{R}_+^2 \times \mathbb{R}_+^2}$ is equivalent to the Euclidean metric;
- or $(y_n)_{n\geq 0}$ has a subsequence that tends to infinity. Because $[0, \pi/2]$ is compact, $(y_n)_{n\geq 0}$ has a subsequence to infinity in some direction $\alpha \in [0, \pi/2]$. By Theorems 1, 6, 7, and 8, this subsequence converges (with respect to the metric $\rho$) to $\mathbb{1}_{\alpha<\alpha^*} h_{\alpha^*} + \mathbb{1}_{\alpha^* \leq \alpha \leq \alpha^{**}} h_\alpha + \mathbb{1}_{\alpha<\alpha^{**}} h_{\alpha^{**}}$. □

**Remark 4.** By Corollary 2, M is the Martin compactification in the sense of Kunita and Watanabe (1965) and Pinsky (1995), and the Martin boundary $\Gamma$ is homeomorphic to $[\alpha^*, \alpha^{**}]$.

In particular, by Kunita and Watanabe (1965, theorem 4), the following representation theorem holds.

**Theorem 9** (Integral Representation). *If* $h$ *is a nonnegative harmonic function, then there exists a Radon measure* $\mu_h$ *on* $[\alpha^*, \alpha^{**}]$ *satisfying*

$$\forall z \in \mathbb{R}_+^2, \quad h(z) = \int_{[\alpha^*, \alpha^{**}]} h_\alpha(z) d\mu_h(\alpha). \tag{8.7}$$

*Furthermore, every function defined by* (8.7) *is harmonic.*

### 8.2. Minimality of Functions $(h_\alpha)_{\alpha\in[\alpha^*, \alpha^{**}]}$ and Martin Boundary

In this section, we prove that the Martin boundary is minimal.

**Definition 4** (Minimal Harmonic Function). A nonnegative harmonic function h is said to be minimal if, for every pair of nonnegative harmonic functions $f_1$ and $f_2$ satisfying $f_1 + f_2 = h$, both $f_1$ and $f_2$ are proportional to $h$.

**Proposition 9** ($\Gamma$ is Minimal). *The Martin boundary is minimal in the sense that if* $\eta \in \Gamma$, *then* $k(\cdot, \eta)$ *is minimal. In particular, the measure* $\mu_h$ *in representation* (8.7) *is unique.*

To prove this, we state the following lemma.

**Lemma 14.** *Let* $\alpha^* < \alpha_1 < \alpha^{**}$ *and* $\epsilon > 0$. *Then, there exist constants* $\eta > 0$ *and* $r_0 > 0$ *such that*

$$h_\alpha(r\cos(\alpha_1), r\sin(\alpha_1)) \geq \frac{1}{2} e^{r(x(\alpha_1)\cos(\alpha_1)+y(\alpha_1)\sin(\alpha_1)-\epsilon)} \tag{8.8}$$

*for all* $r \geq r_0$ *and* $\alpha \in [\alpha_1 - \eta, \alpha_1 + \eta]$.

**Proof.** This follows from Theorem 2, where explicit formulae of $h_\alpha$ are given. □

**Proof of Proposition 9.** Let $\alpha_0 \in [\alpha^*, \alpha^{**}]$. We aim to prove that if $h_{\alpha_0} = \int_{[\alpha^*, \alpha^{**}]} h_\alpha d\mu(\alpha)$ for some Radon measure $\mu$, then $\mu$ is the Dirac measure at $\alpha_0$. This directly implies the minimality of $\Gamma$ using Definition 4 and Theorem 9. It suffices to show that the support of $\mu$ is exactly $\{\alpha_0\}$. Suppose first that $\alpha^* < \alpha_0 < \alpha^{**}$. Let us prove that $\mu((\alpha^*, \alpha^{**})\backslash\{\alpha_0\}) = 0$. Let $\alpha_1 \in (\alpha^*, \alpha^{**})\backslash\{\alpha_0\}$. First, by (5.4), we can choose $\epsilon > 0$ such that

$$x(\alpha_1)\cos(\alpha_1) + y(\alpha_1)\sin(\alpha_1) - \epsilon > x(\alpha_0)\cos(\alpha_1) + y(\alpha_0)\sin(\alpha_1). \tag{8.9}$$

Second, by Lemma 14, $\eta > 0$ such that

$$h_\alpha(r\cos(\alpha), r\sin(\alpha)) \geq \frac{1}{2} e^{r(x(\alpha_1)\cos(\alpha_1)+y(\alpha_1)\sin(\alpha_1)-\epsilon)}$$

for $\alpha \in [\alpha_1 - \eta, \alpha_1 + \eta]$ and $r \geq r_0$ large enough. Then,

$$h_{\alpha_0}(r\cos(\alpha_1), r\sin(\alpha_1)) \geq \int_{\alpha\in[\alpha_1-\eta, \alpha_1+\eta]} h_\alpha(re_{\alpha_1})\mu(d\alpha) \underset{r\geq r_0}{\geq} \frac{\mu([\alpha_1 - \eta, \alpha_1 + \eta])}{2} e^{r(x(\alpha_1)\cos(\alpha_1)+y(\alpha_1)\sin(\alpha_1)-\epsilon)}.$$

Considering $\theta = \alpha_1$ and $\alpha = \alpha_0$ in (5.4), we obtain, for $r \geq r_1$ large enough,

$$e^{r(x(\alpha_1)\cos(\alpha_0)+y(\alpha_1)\sin(\alpha_0)-\epsilon)} \geq \frac{\mu([\alpha_1 - \eta, \alpha_1 + \eta])}{2} e^{r(x(\alpha_1)\cos(\alpha_1)+y(\alpha_1)\sin(\alpha_1)-\epsilon)}.$$

By (8.9), the asymptotics of the previous inequality as $r \to +\infty$ yield $\mu([\alpha_1 - \eta, \alpha_1 + \eta]) = 0$. Therefore, $\mu$ can be written as $\mu = A\delta_{\alpha_0} + B\delta_{\alpha^*} + C\delta_{\alpha^{**}}$ for some nonnegative constants $A$, $B$, and $C$, that is, $(1 - A)h_{\alpha_0} = Bh_{\alpha^*} + Ch_{\alpha^{**}}$. Now, considering the asymptotics (8.4), (8.5), and (8.6), we immediately get $B = C = 0$ and $A = 1$. Hence, $\mu$ is the Dirac measure at $\alpha_0$, and $h_{\alpha_0}$ is minimal. The cases $\alpha_0 = \alpha^*$ and $\alpha_0 = \alpha^{**}$ are treated similarly. □

**Proof of Theorem 3.** This is a direct consequence of Lemma 13, Remark 4, and Proposition 9. □

## 9. From Assumption (1.4) to the General Case

We stated and proved Theorems 1, 2, and 3 under Assumption (1.4). In this section, we generalize these theorems without assuming (1.4). To achieve this, we apply transformations to the $x-$axis, $y-$axis, and time $t$ in order to reduce the problem to a process $\tilde{Z}$ that satisfies (1.4).

**Proposition 10** (Space-Time Dilatation). *Let $(Z_t)_{t\geq 0}$ be a degenerate reflected Brownian motion with parameters*

$$\Sigma = \begin{pmatrix} \sigma_1^2 & -\sigma_1\sigma_2 \\ -\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}, \mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, R = \begin{pmatrix} 1 & r_2 \\ r_1 & 1 \end{pmatrix}.$$

*Then, the process*

$$(\tilde{Z}_t)_{t\geq 0} := \left( \left( \frac{\mu_1}{\sigma_1} + \frac{\mu_2}{\sigma_2} \right) \begin{pmatrix} \frac{1}{\sigma_1} & 0 \\ 0 & \frac{1}{\sigma_2} \end{pmatrix} Z \left( \frac{t}{\left( \frac{\mu_1}{\sigma_1} + \frac{\mu_2}{\sigma_2} \right)^2} \right) \right)_{t\geq 0}$$

*is a degenerate reflected Brownian motion with parameters*

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \tilde{\mu} := \frac{1}{\left( \frac{\mu_1}{\sigma_1} + \frac{\mu_2}{\sigma_2} \right)} \begin{pmatrix} \frac{\mu_1}{\sigma_1} \\ \frac{\mu_2}{\sigma_2} \end{pmatrix}, \tilde{R} := \begin{pmatrix} 1 & r_2 \frac{\sigma_2}{\sigma_1} \\ r_1 \frac{\sigma_1}{\sigma_2} & 1 \end{pmatrix}.$$

*Furthermore, $\tilde{Z}$ satisfies (1.2) to (1.4).*

**Proof.** This is a direct consequence of Definition 1 applying the corresponding transformation to (2.1). □

**Theorem 10** (Harmonic Functions and Martin Boundary: General Case). *Suppose that (1.2) and (1.3). Let $\alpha^*, \alpha^{**}$ and $(\tilde{h}_\alpha)_{\alpha\in[\alpha^*,\alpha^{**}]}$ be the angles and the harmonic functions in Theorem 2 for the degenerate reflected Brownian motion of parameters $\left( \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \tilde{\mu}, \tilde{R} \right)$ (using the notation of Proposition 10). Then, the family of minimal harmonic functions for the initial process is given by*

$$(x_0, y_0) \longmapsto \tilde{h}_\alpha \left( \left( \frac{\mu_1}{\sigma_1} + \frac{\mu_2}{\sigma_2} \right) \left( \frac{x_0}{\sigma_1}, \frac{y_0}{\sigma_2} \right) \right), \qquad \alpha \in [\alpha^*, \alpha^{**}]. \tag{9.1}$$

*Furthermore, the Martin boundary remains homeomorphic to $[\alpha^*, \alpha^{**}]$ and is minimal.*

**Proof.** Let $\psi : \mathbb{R}^2 \longmapsto \mathbb{R}^2$ be the map defined by $\psi(z) = (\frac{\mu_1}{\sigma_1} + \frac{\mu_2}{\sigma_2}) \begin{pmatrix} \frac{1}{\sigma_1} & 0 \\ 0 & \frac{1}{\sigma_2} \end{pmatrix} z$. We denote by $G(z_0, \cdot)$ (resp. $\tilde{G}(\tilde{z}_0, \cdot)$) the Green's measure associated with $(Z_t)_{t\geq 0}$ (resp. with $(\tilde{Z}_t)_{t\geq 0}$) and $g^{z_0}(z)$ (resp. $\tilde{g}^{\tilde{z}_0}(\tilde{z})$) the corresponding Green's functions, where $\tilde{z}_0 = \psi(z_0)$. Note that

$$\begin{aligned} G(z_0, A) &= \int_0^{+\infty} \mathbb{P}_{z_0}(Z_t \in A) dt \\ &= \int_0^{+\infty} \mathbb{P}_{z_0}\left( Z \left( \frac{u}{\left( \frac{\mu_1}{\sigma_1} + \frac{\mu_2}{\sigma_2} \right)^2} \right) \in A \right) \frac{du}{\left( \frac{\mu_1}{\sigma_1} + \frac{\mu_2}{\sigma_2} \right)^2} \\ &= \int_0^{+\infty} \mathbb{P}_{\tilde{z}_0}(\tilde{Z}_u \in \psi(A)) \frac{du}{\left( \frac{\mu_1}{\sigma_1} + \frac{\mu_2}{\sigma_2} \right)^2}. \end{aligned}$$

Furthermore,

$$\mathbb{P}_{z_0}(\tilde{Z}_u \in \psi(A)) = \int_{\psi(A)} \mathbb{P}_{\tilde{z}_0}(\tilde{Z}_u = u)du = \int_A \mathbb{P}_{\tilde{z}_0}(\tilde{Z}_u = \psi(v))|Jac(\psi)|dv.$$

Therefore, the following holds for all $z_0, a \in \mathbb{R}_+^2$:

$$g^{z_0}(a) = \frac{1}{\sigma_1\sigma_2}\tilde{g}^{\tilde{z}_0}(\psi(a)). \tag{9.2}$$

Then,

$$g^{z_0}(r\cos(\alpha), r\sin(\alpha)) = \frac{1}{\sigma_1\sigma_2}\tilde{g}^{\psi(z_0)}(\tilde{r}\cos(\tilde{\alpha}), \tilde{r}\sin(\tilde{\alpha})) \tag{9.3}$$

with $\tilde{\alpha} = \arctan\left(\frac{\sigma_2}{\sigma_1}\tan(\alpha)\right)$. The conclusion follows from relation (9.3). □

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