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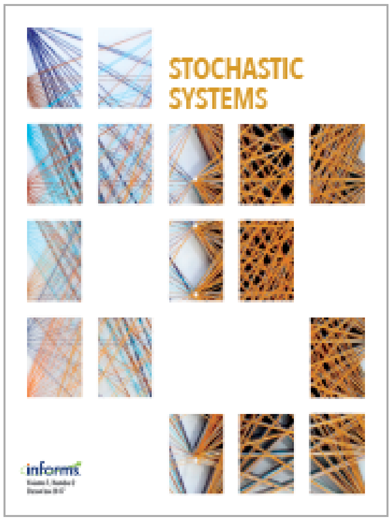

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Long Time Limits of Fluid Models for Many-Server Queues with Abandonment via Nonlinear Volterra Equations

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Abstract. In this paper, we study the fluid limit of many-server queues with abandonment (with traffic intensity $\lambda \in [1, \infty)$) via a class of nonlinear Volterra equations. For a broad class of service time distributions, we establish the asymptotic behavior of the solutions to the class of nonlinear Volterra equations, which in turn implies the large time behavior of the fluid limit of the many-server queues with abandonment.

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Keywords: many-server queues • $G_t/GI/N + GI$ queue • fluid model equations • abandonment • nonlinear Volterra equations

1. Introduction

In recent years, there has been tremendous attention given to the study of many-server queueing systems with customer abandonment due to its applications for telephone contact centers or (generally) customer contact centers (Garnett et al. 2002, Gans et al. 2003, Brown et al. 2005), patient flows in hospitals (Brown et al. 2005, Green 2006, Armony et al. 2015, Shi et al. 2015), and enzymatic processing networks in biology, where reneging seeks to model the phenomenon of dilution (Mather et al. 2010). A basic model, also known as the $G_t/GI/N + GI$ queue, describes a service system with N parallel identical servers, and customers (jobs) arrive with a (possibly) time-dependent arrival rate, require independent and identically distributed (i.i.d.) service times drawn from a general distribution, and have i.i.d. patience times drawn from another general distribution. Arriving customers enter service immediately if there is an idle server available, else they join the back of the queue. A customer at the head of the queue starts service once a server becomes available. Once a customer completes service, it departs the system. Customers are assumed to abandon from the system if their time spent waiting in queue reaches their patience times. The arrival process, service, and patience times are assumed to be mutually independent. The service discipline is first-come-first-serve (FCFS) and nonidling; that is, no server will idle whenever there is a customer in queue. Important system performance measures of interest include the stationary waiting time and queue distributions. When the customers' arrivals follow Poisson and the service times follow exponential distribution, but the patience times follow a general distribution, explicit formulas for the scaled steady-state distributions were obtained in Baccelli and Hebuterne (1981), and their asymptotics as N, the number of servers, goes to infinity, were studied in Zeltyn and Mandelbaum (2005). However, statistical analysis of real call centers has shown that both service times and patience times are typically not exponentially distributed (Brown et al. 2005, Zeltyn and Mandelbaum 2005), and the exact analysis of the scaled steady-state distributions is typically not feasible. Instead, an asymptotic analysis, as the number of servers goes to infinity, is desired.

In Kang and Ramanan (2010), the state descriptor of the system with N identical servers includes a pair of measure-valued processes, the "potential queue measure" process, $\eta^{(N)}$, the "age measure" process, $\nu^{(N)}$, and the total count process, $X^{(N)}$. Here, $\eta^{(N)}$ keeps track of the waiting times of customers in the "potential" queue that includes, not only those customers in queue, but also those customers who may have already entered service (and possibly departed the system), but for whom the time since entry into the system has not yet exceeded its patience time; $\nu^{(N)}$ keeps track of the amounts of time that customers currently receiving service have been in service; and $X^{(N)}$ represents the total number of customers in the system (including those in service and those in queue). Under suitable conditions on the service and patience distributions, it was shown in Kang and Ramanan (2010) that when the average arrival rate or traffc intensity converges to $\lambda > 0$, the rescaled state descriptor

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$(X^{(N)}, \nu^{(N)}, \eta^{(N)})$ converges to a deterministic limit $(\bar{X}, \bar{\nu}, \bar{\eta})$ that is the unique solution to the (measure-valued) fluid model equations in Definition 1.

In AKKR (Atar et al. 2023), the long time behavior of $(\bar{X}, \bar{\nu}, \bar{\eta})$, parameterized by the constant arrival density $\lambda \in [0, \infty)$, is studied under the assumption that the fluid model equations admit a unique invariant state (equivalently, fixed point). When $\lambda < 1$ (subcritical regime), the convergence of $(\bar{X}_t, \bar{\nu}_t, \bar{\eta}_t)$, as $t \to \infty$, is established under the assumption that the patience time distribution G^r has finite mean and the service time distribution G^s has finite mean as well. On the other hand, when $\lambda = 1$ (critical regime) and $\lambda > 1$ (supercritical regime), only the convergence of $(\bar{\nu}_t, \bar{\eta}_t)$ is established under additional assumption that the hazard rate function h^s of the service time distribution G^s is either decreasing or bounded away from zero and infinity. The proof techniques are different under the two sets of assumptions on the service time distribution G^s. When the hazard rate function h^s is decreasing, a reformulation of the dynamics in terms of a certain renewal equation is used, in conjunction with recursive asymptotic estimates. When the hazard rate function h^s is bounded away from zero and infinity, the proof uses an extended relative entropy functional as a Lyapunov function. Note that, in AKKR, the convergence of $\bar{X}(t)$ for $\lambda \geq 1$ is not obtained, and the convergence results there do not cover distributions like lognormal, Weibull, or gamma distributions because the hazard rate functions of these distributions are neither decreasing in general nor bounded either away from zero or from infinity. It was discussed in Brown et al. (2005) that the service duration of customers at a small call center for one of Israel's banks actually follows a lognormal distribution. This observation is also confirmed in Cogdill and Monticino (2007) for service times of customers in retail banks. Therefore, it is imperative to establish the convergence of $(\bar{X}_t, \bar{\nu}_t, \bar{\eta}_t)$ for a more broad class of distributions than those in AKKR in the critical or supercritical regime $\lambda \geq 1$. This is the main goal of this paper.

The main tool used here is different from the ones used in AKKR. We study the process $\bar{X}$ directly as the solution to a nonlinear Volterra equation (see Lemma 1). This enables us to use certain integrability and comparison results on solutions to a certain class of nonlinear Volterra equations in Engler (1986), a sensitive analysis on solutions to a certain class of nonlinear Volterra equations in Miller (1971), and another comparison result on solutions to a certain class of nonlinear Volterra equations in Sato (1953). We establish the convergence of $\bar{X}$ and hence the convergence of $\bar{\nu}$ in Theorem 1 for $\lambda \geq 1$ under Assumption 2 on the service time distribution G^s, the initial state $\bar{\nu}_0$, and the patience time distribution G^r. Note that Assumption 2, (a)–(d), is very mild, and Assumption 2, (1)–(3), on the service time distribution G^s only comes in when the arrival rate $\lambda \in [1, 1+\alpha]$ for some α that depends only on G^s (see (30)). For lognormal, Weibull, or gamma distributions, the value α is typically small as discussed in Remark 5. Therefore, when the system is slightly overloaded ($\lambda > 1+\alpha$), the convergence of $(\bar{X}_t, \bar{\nu}_t, \bar{\eta}_t)$ holds just under Assumption 2, (a)–(d). Even when $\lambda \in [1, 1+\alpha]$, Assumption 2, (1)–(3), is more general than the ones in AKKR (see Remark 3). Also, the assumption used in AKKR that the fluid equations admit a unique invariant state is only needed here for the critical region ($\lambda = 1$). For a future investigation, it would be interesting to see if the convergence of $(\bar{X}_t, \bar{\nu}_t, \bar{\eta}_t)$ can be established just under Assumption 2, (a)–(d), for all $\lambda \geq 1$.

1.1. Notation and Terminology

The following notation will be used throughout the paper. $\mathbb{N}$ is the set of strictly positive integers, $\mathbb{R}$ is set of real numbers, and $\mathbb{R}_+$ is the set of nonnegative real numbers. For $a, b \in \mathbb{R}$, $a \vee b$ denotes the maximum of a and b, $a \wedge b$ the minimum of a and b, and the shorthand a^+ is used for $a \vee 0$. Given a set B, $\mathbf{1}_B$ denotes the indicator function of the set B (that is, $\mathbf{1}_B(x) = 1$ if $x \in B$ and $\mathbf{1}_B(x) = 0$ otherwise). The constant function $f \equiv 1$ will be represented by the symbol $\mathbf{1}$. Given a nondecreasing, right continuous function f having left limits on $\mathbb{R}_+$, f^{-1} denotes the inverse function of f in the sense that

$$f^{-1}(y) = \inf\{x \geq 0 : f(x) \geq y\},$$

with the convention that infimum over an empty set is ∞. For any finite measure μ on $[0, H)$, we define

$$\bar{F}^{\mu}(x) \doteq \mu[0, x], \quad x \in [0, H). \tag{1}$$

The space of Radon measures on a Polish space E, endowed with the Borel σ-algebra, is denoted by $\mathcal{M}(E)$, whereas $\mathcal{M}_F(E)$ is the subspace of finite nonnegative measures in $\mathcal{M}(E)$. Here $\mathcal{M}_F(E)$ is equipped with the usual weak topology; that is, a sequence $\{\mu_n\}$ in $\mathcal{M}_F(E)$ is said to converge weakly to μ if and only if for every bounded and continuous function ϕ on E,

$$\int_E \phi(x)\mu_n(dx) \to \int_E \phi(x)\mu(dx), \text{ as } n \to \infty.$$

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The symbol δ_x will be used to denote the measure with unit mass at the point x. When E is an interval, say $[0,H)$, for notational conciseness, we will often write $\mathcal{M}_F[0,H)$ instead of $\mathcal{M}_F([0,H))$. For any Borel measurable function $f:[0,H)\to\mathbb{R}$ that is integrable with respect to $\xi\in\mathcal{M}[0,H)$, we often use the short-hand notation

$$\langle f,\xi\rangle \doteq \int_{[0,H)} f(x)\,\xi(dx).$$

Let $\mathcal{I}_0(\mathbb{R}_+)$ be the set of nondecreasing, right continuous functions f having left limits on $\mathbb{R}_+$ with $f(0)=0$. Let $\mathcal{C}(\mathbb{R}_+)$ be the set of continuous functions on $\mathbb{R}_+$, $\mathcal{C}_b(\mathbb{R}_+)$ be the subset of $\mathcal{C}(\mathbb{R}_+)$ of functions that are bounded, and $\mathcal{C}[a,b]$ be the set of continuous functions on $[a, b]$. For two functions f and g defined on $[0,\infty)$, let $f\star g$ denote the convolution of f and g, that is, for each $t\in\mathbb{R}_+$,

$$f\star g(t)=\int_0^t f(t-s)g(s)ds.$$

2. Fluid Model Equations and Nonlinear Volterra Equations

2.1. Fluid Model Equations and Invariant States

We now state the fluid model equations considered in Kang and Ramanan (2010) as a fluid analog of the $G_t/GI/N+GI$ queues and the associated invariant states.

For a cumulative distribution function G on $\mathbb{R}_+$ with density g, the right end of the support H of g is defined as $H\doteq\sup\{x\in\mathbb{R}_+ : g(x)>0\}$ and then the hazard rate function h on $\mathbb{R}_+$ is defined as $h(x)=g(x)/\bar{G}(x)$, where $\bar{G}(x)\doteq 1-G(x)$. Note that when $x\ge H$ if $H<\infty$, $g(x)=\bar{G}(x)=0$. In this case, $h(x)$ is interpreted as zero.

In this paper, we always impose the following mild density assumption on the patience and service time distribution functions G^r and G^s and first moment assumption on the service time distribution function G^s. Without loss of generality, we can normalize the service time distribution so that its mean equals one. This is the reason why we call $\lambda=1$ the critical regime because, in this regime, the arrival rate equals to the mean rate of service completion.

Assumption 1. *The patience time distribution function G^r has density g^r on $[0,H^r)$, where H^r is the right end of the support of g^r. The service time distribution function G^s has density g^s on $[0,H^s)$, where H^s is the right end of the support of g^s and*

$$\int_0^\infty xg^r(x)dx=\int_0^\infty \bar{G}^r(x)dx<\infty,\ \int_0^\infty xg^s(x)dx=\int_0^\infty \bar{G}^s(x)dx=1. \tag{2}$$

Define the following space of feasible input data for the fluid model equations stated in Definition 1:

$$\mathcal{S}_0\doteq\left\{\begin{array}{c}(e,x,\nu,\eta)\in\mathcal{I}_0(\mathbb{R}_+)\times\mathbb{R}_+\times\mathcal{M}_F[0,H^s)\times\mathcal{M}_F[0,H^r):\\ 1-\langle\mathbf{1},\nu\rangle=[1-x]^+,\ [x-1]^+\le\langle\mathbf{1},\eta\rangle\end{array}\right\}.$$

For any $(\bar{E},\bar{X}(0),\bar{\nu}_0,\bar{\eta}_0)\in\mathcal{S}_0$, $\bar{E}$ represents the arrival process for the amount of mass (limiting fraction of customers) arriving to the system since time 0, $\bar{X}(0)$ represents the initial amount of mass in the system including those in service and those waiting in queue at time 0, $\bar{\nu}_0(dx)$ represents the amount of mass in service at time 0 whose elapsed service time (time spent in service) lies in the range $[x,x+dx)$ at time 0, and $\bar{\eta}_0(dx)$ represents the amount of mass in the system by time 0 whose elapsed patience time lies in the range $[x,x+dx)$ at time 0. The first restriction that $1-\langle\mathbf{1},\bar{\nu}_0\rangle=[1-\bar{X}(0)]^+$ simply reflects the nonidling condition. Note that $\langle\mathbf{1},\bar{\eta}_0\rangle$ represents the total mass that have arrived to the system by time 0 and whose patient times have not been reached by time 0. This mass not only includes those waiting in queue at time 0, which is $[\bar{X}(0)-1]^+$, but also includes those who have entered service by time 0. This naturally leads to the second restriction that $[\bar{X}(0)-1]^+\le\langle\mathbf{1},\bar{\eta}_0\rangle$.

Definition 1 (Fluid Model Equations). The càdlàg function $(\bar{X},\bar{\nu},\bar{\eta})$ defined on $\mathbb{R}_+$ such that for each $t\in\mathbb{R}_+$, $\bar{X}(t)\in\mathbb{R}_+$, $\bar{\nu}_t\in\mathcal{M}_F[0,H^s)$, and $\bar{\eta}_t\in\mathcal{M}_F[0,H^r)$ is said to solve the fluid model equations associated with $(\bar{E},\bar{X}(0),\bar{\nu}_0,\bar{\eta}_0)\in\mathcal{S}_0$ and the hazard rate functions $h^r=g^r/\bar{G}^r$ and $h^s=g^s/\bar{G}^s$, if and only if for every $t\in\mathbb{R}_+$,

$$\int_0^t\langle h^r,\bar{\eta}_u\rangle du<\infty,\qquad \int_0^t\langle h^s,\bar{\nu}_u\rangle du<\infty, \tag{3}$$

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and the following relations are satisfied: For every f and $\tilde{f} \in \mathcal{C}_b(\mathbb{R}_+)$,

$$\int_{[0,H^s)} f(x)\,\bar{\nu}_t(dx) = \int_{[0,H^s)} f(x+t)\frac{\bar{G}^s(x+t)}{\bar{G}^s(x)}\,\bar{\nu}_0(dx) + \int_0^t f(t-u)\bar{G}^s(t-u)\,d\bar{K}(u), \tag{4}$$

where

$$\bar{K}(t) = \langle \mathbf{1}, \bar{\nu}_t\rangle - \langle \mathbf{1}, \bar{\nu}_0\rangle + \int_0^t \langle h^s, \bar{\nu}_u\rangle\,du; \tag{5}$$

$$\int_{[0,H^r)} \tilde{f}(x)\,\bar{\eta}_t(dx) = \int_{[0,H^r)} \tilde{f}(x+t)\frac{\bar{G}^r(x+t)}{\bar{G}^r(x)}\,\bar{\eta}_0(dx) + \int_0^t \tilde{f}(t-u)\bar{G}^r(t-u)\,d\bar{E}(u); \tag{6}$$

$$\bar{Q}(t) = \bar{X}(t) - \langle \mathbf{1}, \bar{\nu}_t\rangle; \tag{7}$$

$$\bar{R}(t) = \int_0^t \left(\int_0^{\bar{Q}(w)} h^r((\bar{F}^{\bar{\eta}_w})^{-1}(y))dy\right)dw, \tag{8}$$

where we recall from (2) that $\bar{F}^{\bar{\eta}_w}(x) \doteq \bar{\eta}_w[0,x]$;

$$\bar{X}(t) = \bar{X}(0) + \bar{E}(t) - \int_0^t \langle h^s, \bar{\nu}_u\rangle\,du - \bar{R}(t); \tag{9}$$

and the nonidling condition

$$1 - \langle \mathbf{1}, \bar{\nu}_t\rangle = [1 - \bar{X}(t)]^+. \tag{10}$$

From the definition of the fluid model equations, we obtain the following two additional balance equations: From (7) and (10),

$$\bar{Q}(t) = [\bar{X}(t) - 1]^+, \tag{11}$$

and from (5), (7), and (9),

$$\bar{Q}(0) + \bar{E}(t) = \bar{Q}(t) + \bar{K}(t) + \bar{R}(t). \tag{12}$$

Remark 1. Note that (4) and (6) are required to be satisfied only for bounded continuous functions in Definition 1. But by using a standard approximation argument, namely representing indicators of finite open intervals in $\mathbb{R}_+$ as monotone limits of continuous functions with compact support and appealing to the monotone class theorem, it follows that both equations in fact hold for any bounded measurable or nonnegative measurable f and $\tilde{f}$, respectively. In particular, these equations hold with $f = h^s$ in (4) and $\tilde{f} = h^r$ in (6). The latter fact is used several times in this paper.

We now give an informal, intuitive explanation for the form of the fluid equations. Recall that $\bar{E}(t)$ represents the cumulative arrival of the amount of mass (limiting fraction of customers) arriving to the system in the time interval [0, t]. Note that $\bar{\nu}_u(dx)$ represents the amount of mass in service at time u whose elapsed service time lies in the range $[x, x+dx)$, and $h^s(x)$ represents the fraction of mass with elapsed service time x that would complete service while having elapsed service time in $[x, x+dx)$. Hence, in (3), $\langle h^s, \bar{\nu}_u\rangle$ represents the departure rate of mass from the system due to service completion at time u, and its integral, $\int_0^t \langle h^s, \bar{\nu}_u\rangle du$, is the cumulative departure due to service completion in the interval [0, t]. On the other hand, $\bar{\eta}_u(dx)$ represents the amount of mass at time u whose elapsed patience time (time elapsed since arrival before its patience exhausted) lies in the range $[x, x+dx)$, and $h^r(x)$ represents the fraction of mass with elapsed patience time x that would exhaust patience while having elapsed patience time in $[x, x+dx)$. Hence, in (3), $\langle h^r, \bar{\eta}_u\rangle$ represents the departure rate of mass from the system due to patience exhaustion at time u, and its integral, $\int_0^t \langle h^r, \bar{\eta}_u\rangle du$, is the cumulative departure due to patience exhaustion in the interval [0, t]. Because of the FCFS nature of the system, the fluid queue at time w contains all the mass in $\bar{\eta}_w$ that is to the left of $(\bar{F}^{\bar{\eta}_w})^{-1}(\bar{Q}(w))$ and all the mass to the right of $(\bar{F}^{\bar{\eta}_w})^{-1}(\bar{Q}(w))$ has entered service before time w. Roughly speaking, given any $y \in [0, \bar{Q}(w)]$, there is a mass of dy customers in the queue whose elapsed patience time at w is $(\bar{F}^{\bar{\eta}_w})^{-1}(y)$ and the mean abandonment rate of customers with this elapsed patience time is $h^r(\bar{F}^{\bar{\eta}_w})^{-1}(y))$. Thus, the total actual abandonment that has occurred in the interval [0, t], denoted by $\bar{R}(t)$, is represented by the integral, as specified in (8). Next, $\bar{Q}(t)$ represents the total mass in queue

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(awaiting service) at time t, $\bar{X}(t)$ represents the total mass in the system at time t including both those in service and those in queue, and $\bar{K}(t)$ represents the cumulative mass of entry into service. Then, Equations (5), (7), and (9) are simply mass conservation equations, and (10) represents a nonidling condition that ensures that no server can idle when there is work in the queue. Finally, Equations (4) and (6) govern the evolution of $\bar{\nu}$ and $\bar{\eta}$, respectively. In particular, if $x > t$, the mass $\bar{\nu}_t(dx)$ is coming from initial mass $\bar{\nu}_0(d(x-t))$ conditioning its service has not completed by time t with the fraction $\bar{G}(x)/\bar{G}(x-t)$, and if $x \le t$, the mass $\bar{\nu}_t(dx)$ is coming from the fraction $\bar{G}(t-u)$ of arrived mass $d\bar{K}(u)$ with service time exceeding $t-u$ for all $u \in [0,t]$. These lead to the two terms on the right-hand side of (4). Equation (6) is exactly analogous, but with h^r and the cumulative arrivals $\bar{E}$ into the system in place of h^s and K, respectively.

Let ν_* be a Borel probability measure on $[0,H^s)$ and η_* be a Borel finite nonnegative measure defined on $[0,H^r)$ as follows:

$$\nu_*[0,x] \doteq \int_0^x \bar{G}^s(y)\,dy, \qquad x \in [0,H^s), \tag{13}$$

$$\eta_*[0,x] \doteq \int_0^x \bar{G}^r(y)\,dy, \qquad x \in [0,H^r). \tag{14}$$

Note that ν_* and η_* are well defined due to Assumption 1. For $\lambda \ge 1$, define the set $\mathcal{X}_\lambda$ as follows:

$$\mathcal{X}_\lambda \doteq \left\{ x \in [1,\infty) : G^r((\bar{F}^{\lambda\eta_*})^{-1}((x-1)^+)) = \frac{\lambda-1}{\lambda} \right\}, \tag{15}$$

where recall by (2) that $\bar{F}^{\lambda\eta_*}(x) = \lambda\eta_*[0,x]$, and let

$$x_l^\lambda \doteq \inf\{x \in [1,\infty) : x \in \mathcal{X}_\lambda\} \qquad \text{and} \qquad x_r^\lambda \doteq \sup\{x \in [1,\infty) : x \in \mathcal{X}_\lambda\}.$$

By (14), the map $x \to \eta_*[0,x]$ is continuous and strictly increasing on $[0,H^r)$, and therefore $(\bar{F}^{\lambda\eta_*})^{-1}$ is continuous and strictly increasing on its domain for each $\lambda \ge 1$. Because G^r is also continuous, we have $\mathcal{X}_\lambda = [x_l^\lambda, x_r^\lambda]$ is nonempty for each $\lambda \ge 1$. For $\lambda \ge 1$, let $\mathcal{I}_\lambda \subset [1,\infty) \times \mathcal{M}_F[0,H^s) \times \mathcal{M}_F[0,H^r)$ be the invariant manifold for the fluid model equations, defined by

$$\mathcal{I}_\lambda \doteq \{(x^*, \nu_*, \lambda\eta_*) : x^* \in \mathcal{X}_\lambda\}. \tag{16}$$

Each element in $\mathcal{I}_\lambda$ is called an invariant state for the fluid model equations.

Remark 2. For each $(x^*, \nu_*, \lambda\eta_*) \in \mathcal{I}_\lambda$, let $(\bar{X}, \bar{\nu}, \bar{\eta})$ be a solution to the fluid model equations associated with $(\lambda\mathbf{1}, x^*, \nu_*, \lambda\eta_*) \in \mathcal{S}_0$. Note that, by theorem 4.6 of Kang and Ramanan (2010), the solutions to the fluid model equations associated with $(\lambda\mathbf{1}, x^*, \nu_*, \lambda\eta_*) \in \mathcal{S}_0$ are unique, and it is easy to check that the constant function $(x^*, \nu_*, \lambda\eta_*)$ is a solution to the fluid model equations associated with $(\lambda\mathbf{1}, x^*, \nu_*, \lambda\eta_*) \in \mathcal{S}_0$. It follows that $(\bar{X}(t), \bar{\nu}_t, \bar{\eta}_t) = (x^*, \nu_*, \lambda\eta_*)$, $t \ge 0$. This is the reason why we call each element in $\mathcal{I}_\lambda$ an invariant state and $\mathcal{I}_\lambda$ the invariant manifold. Note that, for each $\lambda \ge 1$, each invariant state $(x^*, \nu_*, \lambda\eta_*)$ depends only on the arrival rate λ, the service time distribution function G^s, and the patience time distribution function G^r, and the invariant manifold $\mathcal{I}_\lambda$ has infinitely many invariant states if $x_l^\lambda < x_r^\lambda$. But if the equation $G^r(x) = (\lambda-1)/\lambda$ has a unique solution, then $x_l^\lambda = x_r^\lambda$ and hence $\mathcal{I}_\lambda$ has a single invariant state.

2.2. Nonlinear Volterra Integral Equation

In this section, we show that the function $\bar{X}$ in a solution $(\bar{X}, \bar{\nu}, \bar{\eta})$ to the fluid model equations associated with initial data $(\bar{E}, \bar{X}(0), \bar{\nu}_0, \bar{\eta}_0) \in \mathcal{S}_0$ satisfies a nonlinear Volterra integral equation.

Lemma 1. *Let $(\bar{X}, \bar{\nu}, \bar{\eta})$ be a solution to the fluid model equations associated with an initial data $(\bar{E}, \bar{X}(0), \bar{\nu}_0, \bar{\eta}_0) \in \mathcal{S}_0$. Then the nonnegative function $\bar{X}$ satisfies the following nonlinear Volterra equation:*

$$\bar{X}(t) + \int_0^t (\bar{X}(u) \wedge 1) r^s(t-u)\,du + \int_0^t B(u, \bar{X}(u))\,du = \xi(t) + (\xi \star r^s)(t), \tag{17}$$

where $r^s(\cdot)$ denotes the renewal density associated with g^s, for each $t \in \mathbb{R}_+$ and $x \in \mathbb{R}$,

$$B(t,x) \doteq \int_0^{[x-1]^+} h^r((\bar{F}^{\bar{\eta}_t})^{-1}(u))\,du, \tag{18}$$

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and

$$\xi(t) \doteq \bar X(0) - \bar Q(0)G^s(t) + \int_0^t \bar G^s(t-u)d\bar E(u) - \int_{[0,H^s)} \frac{G^s(x+t)-G^s(x)}{\bar G^s(x)}\,\bar\nu_0(dx). \tag{19}$$

Proof. Fix $(\bar E, \bar X(0), \bar\nu_0, \bar\eta_0) \in \mathcal{S}_0$ and a solution $(\bar X, \bar\nu, \bar\eta)$ to the fluid model equations associated with $(\bar E, \bar X(0), \bar\nu_0, \bar\eta_0)$. By (4), Remark 1, an application of changing the order of integration and an application of integration by parts, and (12), for each $t \in \mathbb{R}_+$,

$$\begin{aligned}\int_0^t \langle h^s, \bar\nu_u\rangle\, du &= \int_0^t \left(\int_{[0,H^s)} \frac{g^s(x+u)}{\bar G^s(x)}\,\bar\nu_0(dx)\right)du + \int_0^t\int_0^u g^s(u-w)\,d\bar K(w)du \\ &= \int_{[0,H^s)} \frac{G^s(x+t)-G^s(x)}{\bar G^s(x)}\,\bar\nu_0(dx) + \int_0^t \bar K(u)g^s(t-u)du \\ &= \int_{[0,H^s)} \frac{G^s(x+t)-G^s(x)}{\bar G^s(x)}\,\bar\nu_0(dx) + \int_0^t (\bar Q(0)+\bar E(u)-\bar Q(u)-\bar R(u))g^s(t-u)du.\end{aligned}$$

From the above display and (9), we can see that for each $t \in \mathbb{R}_+$,

$$\begin{aligned}\bar X(t) = \bar X(0) + \bar E(t) &- \int_{[0,H^s)} \frac{G^s(x+t)-G^s(x)}{\bar G^s(x)}\,\bar\nu_0(dx) \\ &- \int_0^t (\bar Q(0)+\bar E(u))g^s(t-u)du + \int_0^t (\bar Q(u)+\bar R(u))g^s(t-u)du - \bar R(t).\end{aligned} \tag{20}$$

By (8) and an application of integration by parts on the left-hand side of the display below, it follows that for each $t \in \mathbb{R}_+$,

$$\bar R(t) - \int_0^t \bar R(u)g^s(t-u)du = \int_0^t \bar G^s(t-u)\left(\int_0^{\bar Q(u)} h^r((\bar F^{\bar\eta_u})^{-1}(w))dw\right)du. \tag{21}$$

Then (20), (1), (21), (11), and (18) together imply that, for each $t \in \mathbb{R}_+$,

$$\bar X(t) = \xi(t) + \int_0^t [\bar X(u)-1]^+ g^s(t-u)du - \int_0^t \bar G^s(t-u)B(u,\bar X(u))du.$$

Note that for each $t \in \mathbb{R}_+$,

$$\bar X(t) = [\bar X(t)-1]^+ + \bar X(t)\wedge 1.$$

For each $t \in \mathbb{R}_+$, define

$$f(t) \doteq \xi(t) - \int_0^t (\bar X(u)\wedge 1)g^s(t-u)du - \int_0^t \bar G^s(t-u)B(u,\bar X(u))du. \tag{22}$$

It follows from the above three displays that, for each $t \in \mathbb{R}_+$,

$$\bar X(t) = f(t) + \int_0^t \bar X(u)g^s(t-u)du. \tag{23}$$

Recall that $r^s(\cdot)$ denotes the renewal density associated with g^s and satisfies that, for each $t \in \mathbb{R}_+$,

$$r^s(t) = g^s(t) + (g^s \star r^s)(t), \tag{24}$$

where recall that $g^s \star r^s$ stands for the convolution of g^s and r^s (proposition 5.2.7 of Asmussen (2003)). Moreover, because for each $t \in \mathbb{R}_+$,

$$(\bar G^s)'(t) + (\bar G^s \star r^s)'(t) = -g^s(t) + r^s(t) - (g^s \star r^s)(t) = 0,$$

then it follows that, for each $t \in \mathbb{R}_+$

$$\bar G^s(t) + (\bar G^s \star r^s)(t) = \bar G^s(0) + (\bar G^s \star r^s)(0) = 1. \tag{25}$$

By applying theorem 5.2.4 of Asmussen (2003) to (23), for each $t \in \mathbb{R}_+$,

$$\bar X(t) = f(t) + (f \star r^s)(t). \tag{26}$$

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This, (22), (24), and (25) together imply that, for each $t \in \mathbb{R}_+$,

$$\begin{aligned}\bar{X}(t) &= [\xi(t) + (\xi \star r^s)(t)] - [(\bar{X} \wedge 1) \star g^s(t) + ((\bar{X} \wedge 1) \star g^s \star r^s)(t)] - [(B(\cdot, \bar{X}(\cdot)) \star \bar{G}^s(t) + ((B(\cdot, \bar{X}(\cdot)) \star \bar{G}^s \star r^s)(t)]\\ &= [\xi(t) + (\xi \star r^s)(t)] - [(\bar{X} \wedge 1) \star (g^s + g^s \star r^s)(t)] - [(B(\cdot, \bar{X}(\cdot)) \star (\bar{G}^s + \bar{G}^s \star r^s)(t)]\\ &= [\xi(t) + (\xi \star r^s)(t)] - (\bar{X} \wedge 1) \star r^s(t) - (B(\cdot, \bar{X}(\cdot)) \star 1)(t), \end{aligned} \tag{27}$$

which implies (17). □

3. Assumptions and Main Result

We now state the main result of the paper under the following assumption on the initial data $(\bar{E}, \bar{\nu}_0)$ and the service time distribution G^s and patience time distribution G^r.

Assumption 2. *The following conditions are assumed to hold:*

(a) *The arrival process* $\bar{E}$ *is absolutely continuous with the constant derivative* $\lambda \in [1, \infty)$.

(b) *The service time distribution* G^s *has finite second moment, that is,*

$$\int_0^\infty t^2 g^s(t)dt < \infty.$$

(c) *The initial data* $\bar{\nu}_0$ *satisfies that*

$$\int_{[0,H^s)} \frac{\int_x^\infty \bar{G}^s(u)du}{\bar{G}^s(x)} \bar{\nu}_0(dx) < \infty, \int_{[0,H^s)} \frac{g^s(x+t)}{\bar{G}^s(x)} \bar{\nu}_0(dx) \longrightarrow 0 \text{ as } t \longrightarrow \infty. \tag{28}$$

(d) *The density* g^s *of* G^s *is such that its renewal density* r^s *is differentiable and*

$$\int_0^\infty |(r^s)'(t)|dt < \infty. \tag{29}$$

In addition, if $1 \le \lambda \le 1 + \alpha$, *where*

$$\alpha \doteq -\int_0^\infty (r^s)'(t)\mathbf{1}_{\{(r^s)'(t)<0\}}dt \ge 0, \tag{30}$$

one of the following conditions holds:

(1) *The renewal density* r^s *satisfies*

$$\int_0^\infty |(r^s)'(t)|dt < r^s(0). \tag{31}$$

(2) *The renewal density* r^s *is nondecreasing and if* $\lambda = 1$, $r^s(0) > 0$.

(3) *The quantities* $\varepsilon_h \doteq \inf_{t\in\mathbb{R}_+} h^s(t) > 0$ *and* $c_h \doteq \sup_{t\in\mathbb{R}_+} h^s(t) < \infty$ *if* $\lambda > 1$.

Lastly, if $\lambda = 1$, *the patience time distribution* G^r *satisfies that the equation* $G^r(x) = 0$ *has a unique solution* $x = 0$.

Remark 3. The finite second moment assumption in Assumption 2(b) is to ensure that certain properties involving the renewal density r^s hold, which are stated in Lemma 2. The two conditions in (28) are assumed for technical reasons to ensure the convergence of certain quantities involving $\bar{\nu}_0$. The two conditions in (28) and Assumption 2(b) actually hold automatically under Assumption 2(3). In fact, if $\sup_{t\in\mathbb{R}_+} h^s(t) = c_h < \infty$, then

$$\int_{[0,H^s)} \frac{g^s(x+t)}{\bar{G}^s(x)} \bar{\nu}_0(dx) \le c_h \int_{[0,H^s)} \frac{\bar{G}^s(x+t)}{\bar{G}^s(x)} \bar{\nu}_0(dx) \longrightarrow 0 \text{ as } t \longrightarrow \infty.$$

On the other hand, if $\inf_{t\in\mathbb{R}_+} h^s(t) = \varepsilon_h > 0$, then

$$\int_{[0,H^s)} \frac{\int_x^\infty \bar{G}^s(u)du}{\bar{G}^s(x)} \bar{\nu}_0(dx) \le \frac{1}{\varepsilon_h} \int_{[0,H^s)} \frac{\int_x^\infty g^s(u)du}{\bar{G}^s(x)} \bar{\nu}_0(dx) \le \frac{1}{\varepsilon_h} < \infty,$$

and

$$\int_0^\infty t^2 g^s(t)dt = 2\int_0^\infty t\bar{G}^s(t)dt \le \frac{2}{\varepsilon_h} \int_0^\infty t g^s(t)dt = \frac{2}{\varepsilon_h} < \infty.$$

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The two conditions in (28) also hold under Assumption 2(b) if $\bar{\nu}_0$ is supported on $[0, M]$ for some $M < H^s$. This is because, under Assumption 2(b), $r^s(t) \to 1$ as $t \to \infty$ (see the proof of (1) of Lemma 2). Because $g^s(t) = r^s(t) - (r^s \star g^s)(t)$ for each $t \in \mathbb{R}_+$ and $(r^s \star g^s)(t) \to 1$ as $t \to \infty$, it follows that $g^s(t) \to 0$ as $t \to \infty$. Hence, the second condition in (28) follows from a simple application of the dominated convergence theorem. The first condition in (28) holds trivially when $\bar{\nu}_0$ is supported on $[0, M]$ for some $M < H^s$.

Remark 4. When the hazard rate function h^s is decreasing on $\mathbb{R}_+$, then by theorem 3 of Brown (1980) and Alexandrov's theorem (Niculescu and Persson 2005, p. 172), the renewal density r^s is nonincreasing on $\mathbb{R}_+$ and then $(r^s)'(t) \le 0$ for all $t \in \mathbb{R}_+$. Then

$$\int_0^\infty |(r^s)'(t)|dt = -\int_0^\infty (r^s)'(t)dt = r^s(0) - r^s(\infty) = r^s(0) - 1 < r^s(0),$$

which shows that Assumption 2(1) holds automatically. Note that there are distributions that do not satisfy (31) but satisfy Assumption 2(2). For example, if G^s follows a Gamma distribution with shape 2 and rate 2, then the renewal density $r^s(t) = 1 - e^{-4t}$ (see (2.13) of Barlow and Proschan (1965)), and then r^s is increasing and satisfies Assumption 2(2). On the other hand, because $r^s(0) = 0$, then (31) does not hold for this G^s. This G^s also does not satisfy Assumption 2(3). Another example of distributions with increasing renewal density is the generalized Erlang distribution with order 2. Assumption 2(3) is the same as the one assumed in AKKR (see assumption 3.1 therein). When $\lambda = 1$, g^s is assumed to satisfy Assumption 2(1) or Assumption 2(2) only. Also note that Assumption 2, (1)–(3), does not cover distributions such as gamma distribution with shape 3 and rate 3 because its renewal density (see the discussion after theorem 3.2.7 of Barlow and Proschan (1965)) is $r^s(t) = 1 + 2e^{-9t/2}\cos(2\pi/3 + t3\sqrt{3}/2)$, as illustrated in Figure 1, which oscillates slightly as $t \to \infty$ and $r^s(0) = 0$.

Remark 5. For certain distributions, the constant α in (30) is actually very small. In this case, as long as the system is reasonably overloaded (that is, $\lambda > 1 + \alpha$), we only need Assumption 2, (a)–(d), for the main results to hold. For example, when the renewal density is nondecreasing, then $\alpha = 0$. For a gamma distribution with shape 3 and rate 3, its renewal density is $r^s(t) = 1 + 2e^{-9t/2}\cos(2\pi/3 + t3\sqrt{3}/2)$ and $\alpha = 4 \times 3^{-1/2}(1 + e^{\sqrt{3}\pi})^{-1} \approx 0.01$. For a gamma distribution with shape 6 and rate 6, its renewal density (see the discussion after theorem 3.2.7 of Barlow and Proschan (1965)) is $r^s(t) = 1 + e^{-3t}\cos(\pi/3 + 3\sqrt{3}t) + e^{-9t}\cos(2\pi/3 + 3\sqrt{3}t) - e^{-12t}$, as illustrated in Figure 2, and $\alpha \approx 0.1333$. For distributions with no explicit expressions of their renewal density functions, such as log-normal distribution and Weibull distribution, numerical approximation of α is needed. For example, the renewal density function of the log-normal distribution with parameters $\mu = -1/8$ and $\sigma = 0.5$ is numerically computed and graphed in Figure 3 with $\alpha \approx 0.07$, and the renewal density function of the Weibull distribution with parameters $\lambda = 1/\Gamma(1.5)$ and $k = 2$ is numerically computed and graphed in Figure 4 with $\alpha \approx 0.03$.

Remark 6. When $\lambda = 1$, the assumption on the patience time distribution G^r in Assumption 2 implies that the fluid model equations admit a unique invariant state.

Now we state the main theorem of this paper.

Theorem 1. *Suppose that Assumption 2 holds. For any* $(\lambda\mathbf{1}, \bar{X}(0), \bar{\nu}_0, \bar{\eta}_0) \in \mathcal{S}_0$, *the solution* $(\bar{X}(t), \bar{\nu}_t, \bar{\eta}_t)$ *converges to an invariant state* $(x^*, \nu_*, \lambda\eta_*) \in \mathcal{I}_\lambda$ *as* $t \to \infty$.

4. Proof of Theorem 1

The proof of Theorem 1 is divided into the two mutually exclusive cases $\lambda > 1$ and $\lambda = 1$. The first case is established in Section 4.2, and the second case is established in Section 4.3. We shall focus on the convergence of $(\bar{X}(t), \bar{\nu}_t)$, as $t \to \infty$, because the convergence of $\bar{\eta}_t$ to $\lambda\eta_*$, as $t \to \infty$, is established in lemma 4.1 of AKKR and is included here for completeness.

Proposition 1. *Fix* $\lambda \ge 1$ *and, given any* $\bar{\eta}_0 \in \mathcal{M}_F[0, H^r)$, *let* $\bar{\eta} = (\bar{\eta}_t)_{t\ge 0}$ *be the solution to* (6). *Then* $\bar{\eta}_t \Rightarrow \lambda\eta_*$ *as* $t \to \infty$.

Proof. Fix $\tilde{f} \in \mathcal{C}_b(\mathbb{R}_+)$. In view of (6), the boundedness of $\tilde{f}$, the finiteness of the measure η_0, the dominated convergence theorem, and the fact that $\bar{G}^r(x + t)/\bar{G}^r(x) \to 0$ for every $x \in [0, H^r)$, as $t \to \infty$, together imply that the first term on the right-hand side of (6) vanishes. On the other hand, because the mean patience time $\int_0^\infty \bar{G}^r(u)du$ is finite by Assumption 1, the dominated convergence theorem shows that the last term on the right-hand side of (6) converges to $\langle \tilde{f}, \lambda\eta_* \rangle$. This concludes the proof that $\bar{\eta}_t \Rightarrow \lambda\eta_*$ as $t \to \infty$. $\square$

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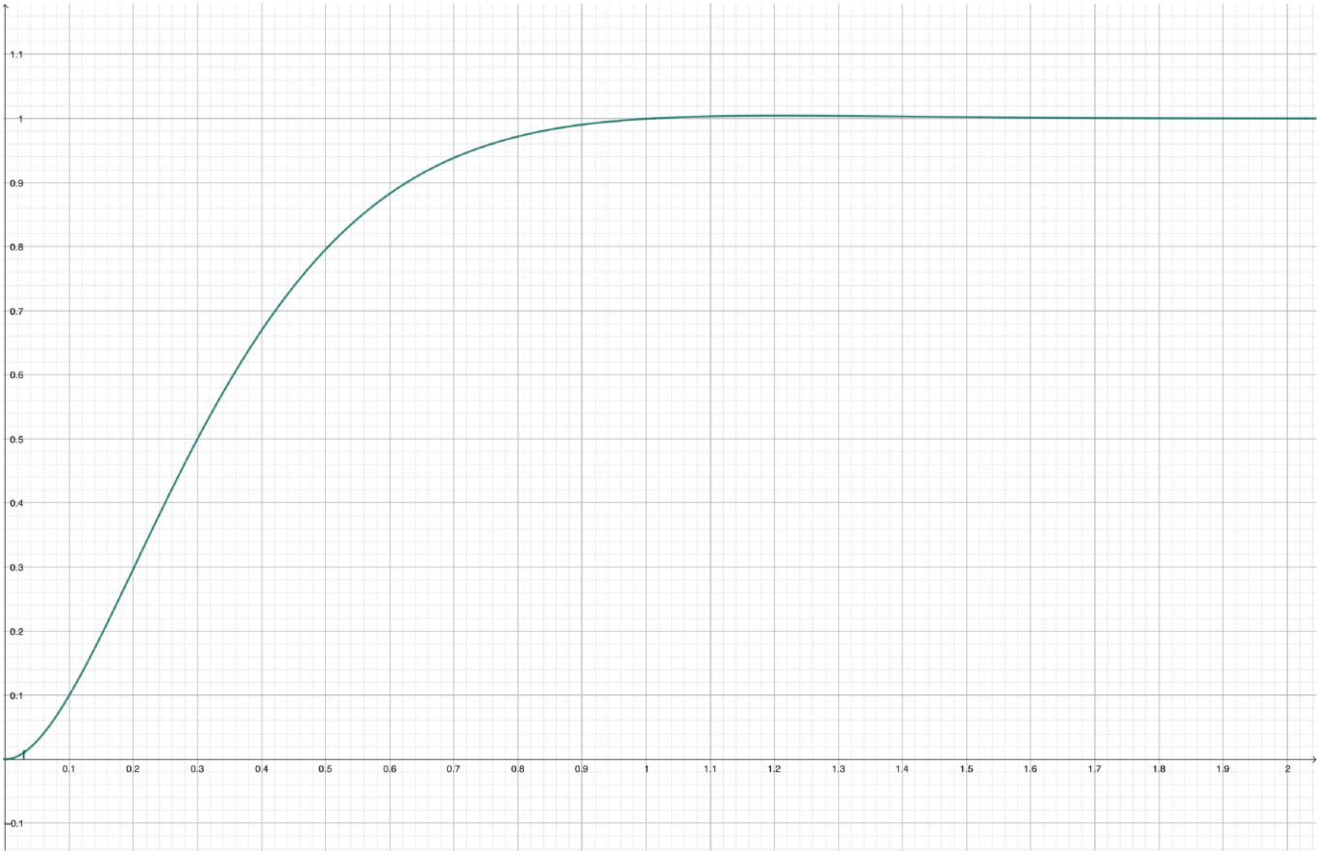
Figure 1. Graph of $r^s(t) = 1 + 2e^{-9t/2}\cos(2\pi/3 + t3\sqrt{3}/2)$



4.1. Some Crucial Estimates

We first discuss some crucial consequences of Assumption 2 in Lemmas 2–5.

Lemma 2. *Suppose that Conditions* (a)–(d) *of Assumption* 2 *hold. Then the following hold.*

1. $r^s(t) \to 1$ *as* $t \to \infty$;
2. $\int_0^\infty |r^s(t) - 1| dt < \infty$;
3. $\int_0^\infty \left[\int_0^t \left(\int_{[0,H^s)} \frac{\bar{G}^s(x+u)}{\bar{G}^s(x)} \bar{\nu}_0(dx)\right) |(r^s)'(t-u)| du\right] dt < \infty.$

Proof. Condition (1) and Condition (2) follow directly from (1) and the discussions in Stone (1966) under the finite second moment assumption on G^s in Assumption 2(b). For Condition (3), note that

$$\int_0^\infty \left[\int_0^t \left(\int_{[0,H^s)} \frac{\bar{G}^s(x+u)}{\bar{G}^s(x)} \bar{\nu}_0(dx)\right) |(r^s)'(t-u)| du\right] dt$$

$$= \int_0^\infty \left[\int_u^\infty \left(\int_{[0,H^s)} \frac{\bar{G}^s(x+u)}{\bar{G}^s(x)} \bar{\nu}_0(dx)\right) |(r^s)'(t-u)| dt\right] du$$

$$= \int_0^\infty |(r^s)'(t)| dt \int_0^\infty \left(\int_{[0,H^s)} \frac{\bar{G}^s(x+u)}{\bar{G}^s(x)} \bar{\nu}_0(dx)\right) du < \infty,$$

where the first equality follows from the exchange of the order of integration, the second equality follows from a change of variables in the inner integral, and the last inequality follows from (28) in Assumption 2(c) and the integrability of $|(r^s)'|$ on $\mathbb{R}_+$ assumed in Assumption 2(d). □

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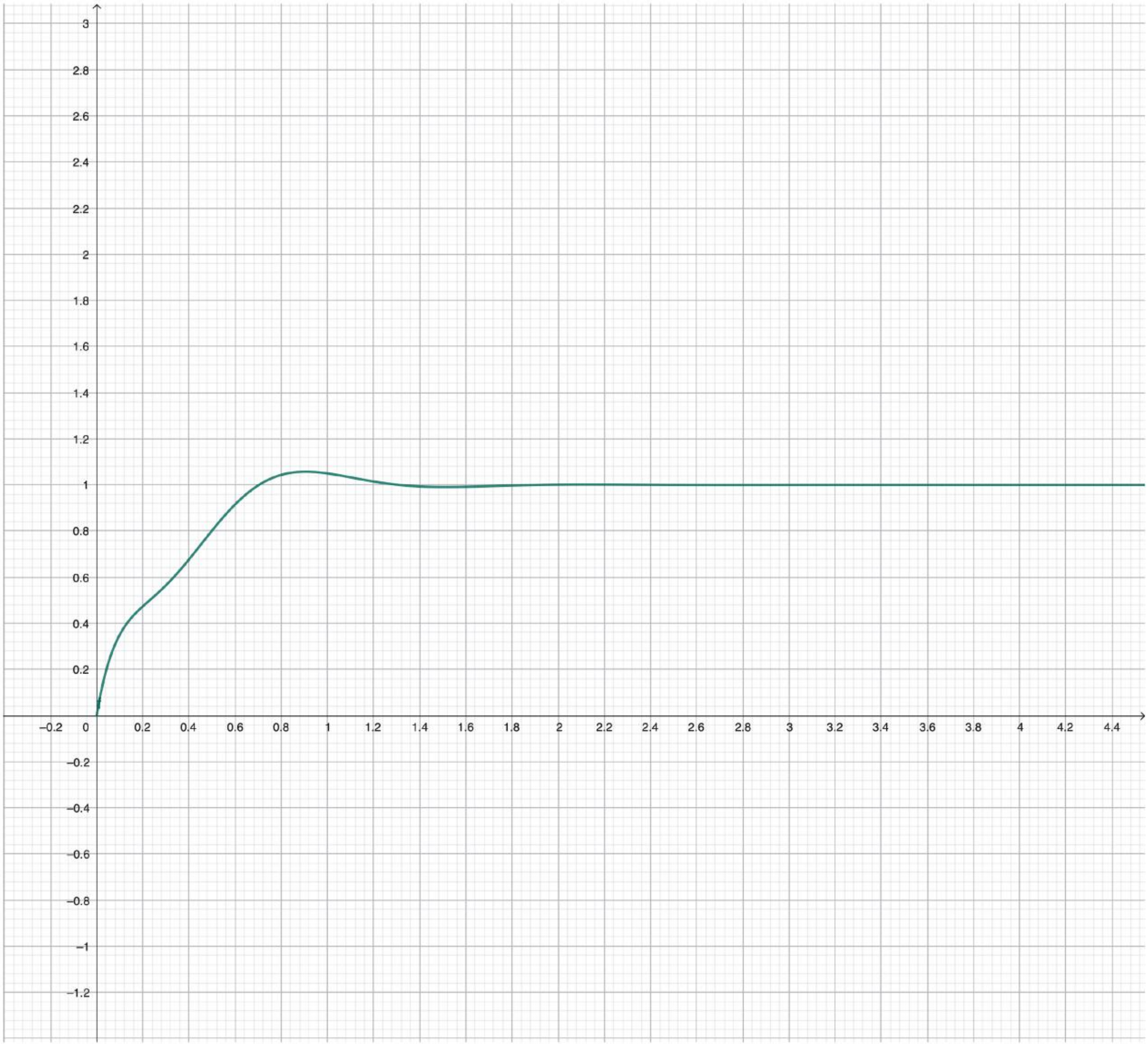
Figure 2. Graph of $r^s(t) = 1 + e^{-3t}\cos(\pi/3 + 3\sqrt{3}t) + e^{-9t}\cos(2\pi/3 + 3\sqrt{3}t) - e^{-12t}$



Lemma 3. *Suppose that Conditions* (a)–(d) *of Assumption* 2 *hold. Recall* ξ *from* (1). *For each* $t \in \mathbb{R}_+$, *let*

$$f_\xi(t) \doteq \xi(t) + (\xi \star r^s)(t). \tag{32}$$

Then f_ξ *is differentiable on* $\mathbb{R}_+$ *and its derivative* $(f_\xi)'$ *satisfies that*

$$(f_\xi)'(t) \longrightarrow \lambda \text{ as } t \longrightarrow \infty.$$

Proof. It follows from (1) that ξ is differentiable and, for each $t \in \mathbb{R}_+$,

$$\xi'(t) = -\bar{Q}(0)g^s(t) + \lambda \bar{G}^s(t) - \int_{[0,H^s)} \frac{g^s(x+t)}{\bar{G}^s(x)}\, \bar{\nu}_0(dx). \tag{33}$$

Then the differentiability of f_ξ follows from the differentiability of ξ and, for each $t \in \mathbb{R}_+$,

$$(f_\xi)'(t) = \xi'(t) + \xi(0)r^s(t) + (\xi' \star r^s)(t). \tag{34}$$

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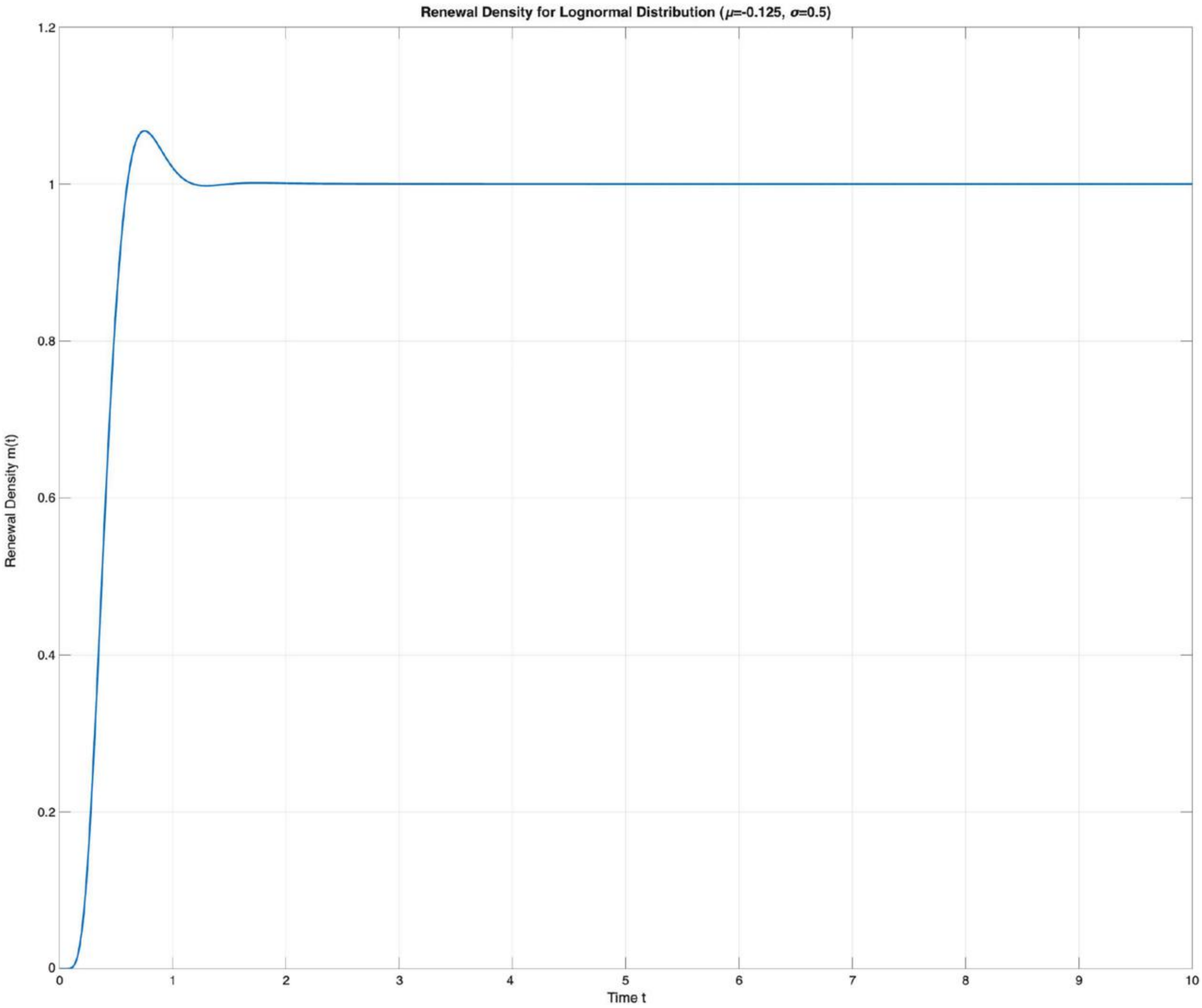
Figure 3. Graph of the Renewal Density Function of the Log-Normal Distribution with Parameters $\mu = -1/8$ and $\sigma = 0.5$



By the representation of ξ' in (33), we have that, for each $t \in \mathbb{R}_+$,

$$
\begin{aligned}
(\xi' \star r^s)(t) &= -\bar{Q}(0) g^s \star r^s(t) + \lambda \bar{G}^s \star r^s(t) - \int_0^t \int_{[0,H^s)} \frac{g^s(x+t-u)}{\bar{G}^s(x)} \bar{\nu}_0(dx) r^s(u) du \\
&= -\bar{Q}(0)(r^s(t) - g^s(t)) + \lambda G^s(t) - \int_0^t \int_{[0,H^s)} \frac{g^s(x+t-u)}{\bar{G}^s(x)} \bar{\nu}_0(dx) r^s(u) du. \qquad (35)
\end{aligned}
$$

Note that by exchanging the order of integration, we have that, for all $t \in \mathbb{R}_+$,

$$
\int_0^t \int_{[0,H^s)} \frac{g^s(x+t-u)}{\bar{G}^s(x)} \bar{\nu}_0(dx) r^s(u) du = \int_{[0,H^s)} \frac{\int_0^t g^s(x+t-u) r^s(u) du}{\bar{G}^s(x)} \bar{\nu}_0(dx).
$$

By applying a change of variables and then the integration by parts, we have that, for each $t \in \mathbb{R}_+$ and $x \in [0, H^s)$,

$$
\begin{aligned}
&\int_0^t g^s(x+t-u) r^s(u) du \\
&= \int_0^t r^s(t-u) g^s(x+u) du \\
&= -\bar{G}^s(x+t) r^s(0) + \bar{G}^s(x) r^s(t) - \int_0^t \bar{G}^s(x+u) (r^s)'(t-u) du.
\end{aligned}
$$

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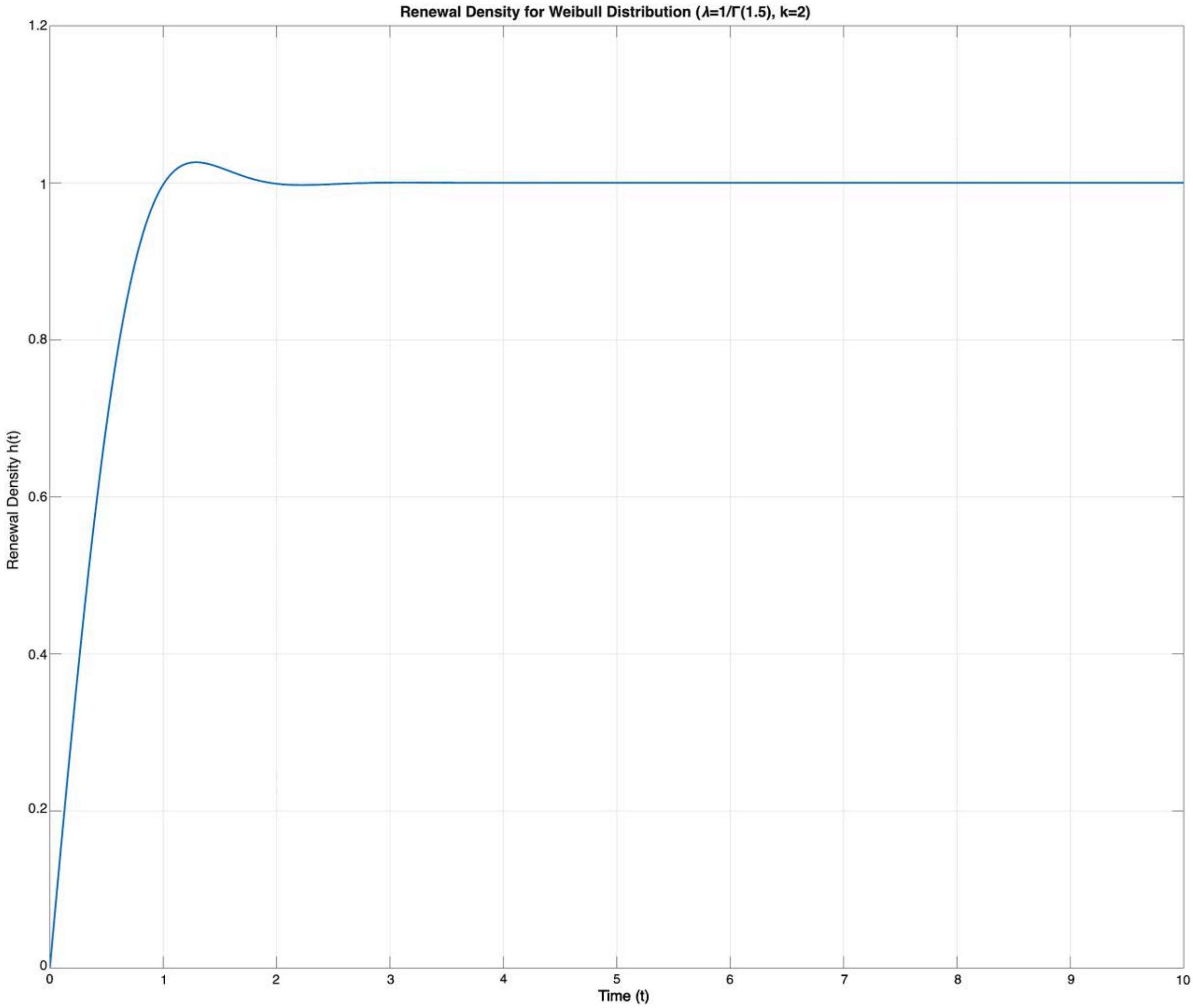
Figure 4. Graph of the Renewal Density Function of the Weibull Distribution with Parameters $\lambda = 1/\Gamma(1.5)$ and $k = 2$



Combining the above two displays, we have that, for all $t \in \mathbb{R}_+$,

$$\begin{aligned}
&\int_0^t \int_{[0,H^s)} \frac{g^s(x+t-u)}{\bar{G}^s(x)}\,\bar{\nu}_0(dx) r^s(u)du \\
&= -r^s(0)\int_{[0,H^s)} \frac{\bar{G}^s(x+t)}{\bar{G}^s(x)}\,\bar{\nu}_0(dx) + \langle \mathbf{1}, \bar{\nu}_0\rangle r^s(t) - \int_0^t \left(\int_{[0,H^s)} \frac{\bar{G}^s(x+u)}{\bar{G}^s(x)}\,\bar{\nu}_0(dx)\right)(r^s)'(t-u)du.
\end{aligned} \tag{36}$$

Combining the above displays with (33), we have that, for all $t \in \mathbb{R}_+$,

$$\begin{aligned}
(f_\xi)'(t) &= -\bar{Q}(0)g^s(t) + \lambda \bar{G}^s(t) - \int_{[0,H^s)} \frac{g^s(x+t)}{\bar{G}^s(x)}\,\bar{\nu}_0(dx) \\
&\quad + \bar{X}(0)r^s(t) - \bar{Q}(0)(r^s(t) - g^s(t)) + \lambda G^s(t) \\
&\quad + r^s(0)\int_{[0,H^s)} \frac{\bar{G}^s(x+t)}{\bar{G}^s(x)}\,\bar{\nu}_0(dx) - \langle \mathbf{1}, \bar{\nu}_0\rangle r^s(t) \\
&\quad + \int_0^t \left(\int_{[0,H^s)} \frac{\bar{G}^s(x+u)}{\bar{G}^s(x)}\,\bar{\nu}_0(dx)\right)(r^s)'(t-u)du \\
&= \lambda - \int_{[0,H^s)} \frac{g^s(x+t)}{\bar{G}^s(x)}\,\bar{\nu}_0(dx) + r^s(0)\int_{[0,H^s)} \frac{\bar{G}^s(x+t)}{\bar{G}^s(x)}\,\bar{\nu}_0(dx) \\
&\quad + \int_0^t \left(\int_{[0,H^s)} \frac{\bar{G}^s(x+u)}{\bar{G}^s(x)}\,\bar{\nu}_0(dx)\right)(r^s)'(t-u)du.
\end{aligned} \tag{37}$$

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Note that

$$\int_{[0,H^s)} \frac{g^s(x+t)}{\bar{G}^s(x)}\,\bar{\nu}_0(dx) \to 0 \text{ and } \int_{[0,H^s)} \frac{\bar{G}^s(x+t)}{\bar{G}^s(x)}\,\bar{\nu}_0(dx) \to 0 \text{ as } t \to \infty,$$

where the first convergence is assumed in Assumption 2(c) (see (28)), and the second convergence follows from an application of dominated convergence theorem because the integrand $\bar{G}^s(x+t)/\bar{G}^s(x)$ is bounded by one and $\bar{G}^s(x+t)/\bar{G}^s(x) \to 0$ as $t \to \infty$. By an application of change of variables and then an application of dominated convergence theorem using the fact that $\int_0^\infty |(r^s)'(t)|dt < \infty$, we have that

$$\begin{aligned}
&\int_0^t \left(\int_{[0,H^s)} \frac{\bar{G}^s(x+u)}{\bar{G}^s(x)}\,\bar{\nu}_0(dx)\right)(r^s)'(t-u)du \\
&= \int_0^t \left(\int_{[0,H^s)} \frac{\bar{G}^s(x+t-u)}{\bar{G}^s(x)}\,\bar{\nu}_0(dx)\right)(r^s)'(u)du \to 0 \text{ as } t \to \infty.
\end{aligned}$$

It follows that $(f_\xi)'(t) \to \lambda$ as $t \to \infty$. □

Lemma 4. *Suppose that Conditions* (a)–(c) *of Assumption* 2 *hold and let* $x^* \in \mathcal{X}_\lambda$. *Recall* $B(u,x)$ *in* (18). *Define a function* f^* *on* $\mathbb{R}_+$ *as*

$$f^*(t) \doteq x^* + \int_0^t r^s(u)du + \int_0^t B(u,x^*)du,\ t \in \mathbb{R}_+. \tag{38}$$

Then

$$(f^*)'(t) \to \lambda \text{ as } t \to \infty. \tag{39}$$

Proof. Fix $x^* \in \mathcal{X}_\lambda$. For the function f^* defined in (38), it is clear that f^* is differentiable on $\mathbb{R}_+$ with derivative

$$(f^*)'(t) = r^s(t) + B(t,x^*),\ t \in \mathbb{R}_+. \tag{40}$$

Recall that $B(t,x^*) = \int_0^{[x^*-1]^+} h^r((\bar{F}^{\bar{\eta}_t})^{-1}(u))du$ from (18). We define a function $\bar{\chi}(\cdot)$ on $\mathbb{R}_+$ as

$$\bar{\chi}(t) \doteq (\bar{F}^{\bar{\eta}_t})^{-1}([x^*-1]^+), \quad t \in \mathbb{R}_+. \tag{41}$$

It follows from an application of change of variables, (6), and Remark 1 that, for all $t \in \mathbb{R}_+$,

$$\begin{aligned}
B(t,x^*) &= \int_{[0,H^r)} \mathbf{1}_{[0,\chi(t)]}(x)h^r(x)\bar{\eta}_t(dx) \\
&= \int_{[0,H^r)} \mathbf{1}_{[0,\chi(t)]}(x+t)\frac{g^r(x+t)}{\bar{G}^r(x)}\,\bar{\eta}_0(dx) + \lambda\int_0^t \mathbf{1}_{[0,\chi(t)]}(t-u)g^r(t-u)\,du.
\end{aligned}$$

Note that, by Proposition 1, $\bar{\eta}_t \Rightarrow \lambda\eta_*$ as $t \to \infty$. It follows that $\bar{\chi}(t) \to (\bar{F}^{\lambda\eta_*})^{-1}([x^*-1]^+) < \infty$ as $t \to \infty$. Then the above display implies that, for all t sufficiently large,

$$B(t,x^*) = \lambda G^r(\bar{\chi}(t)). \tag{42}$$

Because $[x^*-1]^+ = \bar{\eta}_t[0,\bar{\chi}(t)]$, we have that, for all t sufficiently large,

$$\begin{aligned}
[x^*-1]^+ &= \int_{[0,H^r)} \mathbf{1}_{[0,\bar{\chi}(t)]}(x+t)\frac{\bar{G}^r(x+t)}{\bar{G}^r(x)}\,\bar{\eta}_0(dx) + \lambda\int_0^t \mathbf{1}_{[0,\bar{\chi}(t)]}(t-u)\bar{G}^r(t-u)\,du \\
&= \lambda\int_0^t \mathbf{1}_{[0,\bar{\chi}(t)]}(u)\bar{G}^r(u)\,du = \lambda\eta_*[0,\bar{\chi}(t)].
\end{aligned}$$

This, in particular, implies that, for all t sufficiently large,

$$\bar{\chi}(t) = (\bar{F}^{\lambda\eta_*})^{-1}([x^*-1]^+),$$

and then

$$\lambda G^r((\bar{F}^{\lambda\eta_*})^{-1}([x^*-1]^+)) = B(t,x^*). \tag{43}$$

It follows from Lemma 2(1) and the fact that $x^* \in \mathcal{X}_\lambda$ that $(f^*)'(t) \to \lambda$ as $t \to \infty$. □

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Lemma 5. *Suppose that Conditions* (a)–(d) *of Assumption* 2 *hold. Then the functions* f_ξ *and* f^* *stated in Lemma* 3 *and Lemma* 4*, respectively, satisfy that*

$$\int_0^\infty |(f_\xi)'(t) - (f^*)'(t)|dt < \infty. \tag{44}$$

Proof. Note that by (37) and (40), for each $t \in \mathbb{R}_+$,

$$\begin{aligned}(f_\xi)'(t) - (f^*)'(t) &= \lambda - \int_{[0,H^s)} \frac{g^s(x+t)}{\bar{G}^s(x)}\,\bar{\nu}_0(dx) + r^s(0)\int_{[0,H^s)} \frac{\bar{G}^s(x+t)}{\bar{G}^s(x)}\,\bar{\nu}_0(dx) \\ &\quad + \int_0^t \left(\int_{[0,H^s)} \frac{\bar{G}^s(x+u)}{\bar{G}^s(x)}\,\bar{\nu}_0(dx)\right)(r^s)'(t-u)du - r^s(t) - B(t,x^*) \\ &= -\int_{[0,H^s)} \frac{g^s(x+t)}{\bar{G}^s(x)}\,\bar{\nu}_0(dx) + r^s(0)\int_{[0,H^s)} \frac{\bar{G}^s(x+t)}{\bar{G}^s(x)}\,\bar{\nu}_0(dx) \\ &\quad + (1 - r^s(t)) + \int_0^t \left(\int_{[0,H^s)} \frac{\bar{G}^s(x+u)}{\bar{G}^s(x)}\,\bar{\nu}_0(dx)\right)(r^s)'(t-u)du + \lambda - 1 - B(t,x^*).\end{aligned}$$

Note that

$$\int_0^\infty \left(\int_{[0,H^s)} \frac{g^s(x+t)}{\bar{G}^s(x)}\,\bar{\nu}_0(dx)\right)dt = \langle \mathbf{1}, \bar{\nu}_0\rangle \le 1.$$

By Assumption 2(c), the initial data $\bar{\nu}_0$ satisfies (28), then

$$\int_0^\infty \int_{[0,H^s)} \frac{\bar{G}^s(x+t)}{\bar{G}^s(x)}\,\bar{\nu}_0(dx)dt = \int_{[0,H^s)} \frac{\int_0^\infty \bar{G}^s(x+t)dt}{\bar{G}^s(x)}\,\bar{\nu}_0(dx) < \infty. \tag{45}$$

It follows that

$$\int_0^\infty \left(r^s(0)\int_{[0,H^s)} \frac{\bar{G}^s(x+t)}{\bar{G}^s(x)}\,\bar{\nu}_0(dx)\right)dt = r^s(0)\int_0^\infty \int_{[0,H^s)} \frac{\bar{G}^s(x+t)}{\bar{G}^s(x)}\,\bar{\nu}_0(dx)dt < \infty.$$

By Lemma 2(2),

$$\int_0^\infty |r^s(t) - 1|dt < \infty.$$

Also, by Lemma 2(3),

$$\int_0^\infty \left[\int_0^t \left(\int_{[0,H^s)} \frac{\bar{G}^s(x+u)}{\bar{G}^s(x)}\,\bar{\nu}_0(dx)\right)|(r^s)'(t-u)|du\right]dt < \infty.$$

Lastly, because $x^* \ge 1$ satisfies

$$1 + \lambda G^r((\bar{F}^{\lambda\eta_*})^{-1}([x^*-1]^+)) = \lambda,$$

and for all t sufficiently large,

$$\lambda G^r((\bar{F}^{\lambda\eta_*})^{-1}([x^*-1]^+)) = B(t,x^*),$$

as shown in the proof of Lemma 4. This shows that, for all t sufficiently large, $B(t,x^*) = \lambda - 1$, and then

$$\int_0^\infty |\lambda - 1 - B(t,x^*)|dt < \infty.$$

Then it follows that (44) holds. □

4.2. Long Time Limit of $(\bar{X}, \bar{\nu})$ When $\lambda > 1$

In this section, we consider the case when $\lambda > 1$.

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Lemma 6. *Suppose that Assumption 2 holds. If $\lambda > 1 + \alpha$, then*

$$\text{there exists } T > 0 \text{ such that } \bar{X}(t) \geq 1 \text{ for all } t \geq T. \tag{46}$$

If $1 < \lambda \leq 1 + \alpha$ and the density g^s of G^s satisfies Assumption 2(2) or Assumption 2(3), then Assertion (46) also holds.

Proof. Recall that $\bar{X}$ satisfies the following nonlinear Volterra equation:

$$\bar{X}(t) + \int_0^t (\bar{X}(u) \wedge 1) r^s(t-u)du + \int_0^t B(u, \bar{X}(u))du = f_\xi(t),\ t \in \mathbb{R}_+. \tag{47}$$

Now define the function $\bar{Y}$ such that $\bar{Y}(t) \doteq \bar{X}(t) - 1$ for each $t \in \mathbb{R}_+$. Then it follows that for each $t \in \mathbb{R}_+$, $\bar{X}(t) \wedge 1 = (\bar{Y}(t) + 1) \wedge 1 = \bar{Y}(t) \wedge 0 + 1$ and

$$B(t, \bar{X}(t)) = \int_0^{[\bar{X}(t)-1]^+} h^r((\bar{F}^{\bar{\eta}_t})^{-1}(u))du = \int_0^{[\bar{Y}(t)]^+} h^r((\bar{F}^{\bar{\eta}_t})^{-1}(u))du.$$

For each $t \in \mathbb{R}_+$ and $x \in \mathbb{R}$, define

$$\tilde{B}(t, x) \doteq \int_0^{[x]^+} h^r((\bar{F}^{\bar{\eta}_t})^{-1}(u))du. \tag{48}$$

Then $\bar{Y}$ satisfies the following modified nonlinear Volterra equation: For each $t \in \mathbb{R}_+$,

$$\bar{Y}(t) + \int_0^t (\bar{Y}(u) \wedge 0) r^s(t-u)du + \int_0^t \tilde{B}(u, \bar{Y}(u))du = f_\xi(t) - 1 - \int_0^t r^s(u)du.$$

For each $T \in \mathbb{R}_+$, consider $\bar{Y}^{[T]}$, the time-shifted $\bar{Y}$ by T, as

$$\bar{Y}^{[T]}(t) \doteq \bar{Y}(T+t),\ t \in \mathbb{R}_+.$$

Then, it is readily checked that $\bar{Y}^{[T]}$ is a solution to the following nonlinear Volterra equation:

$$\bar{Y}^{[T]}(t) + \int_0^t (\bar{Y}^{[T]}(u) \wedge 0) r^s(t-u)du + \int_0^t \tilde{B}(T+u, \bar{Y}^{[T]}(u))du = \tilde{f}^{[T]}(t),$$

where

$$\tilde{f}^{[T]}(t) \doteq f_\xi(T+t) - 1 - \int_0^{T+t} r^s(u)du - \int_0^T (\bar{Y}(u) \wedge 0) r^s(T+t-u)du - \int_0^T \tilde{B}(u, \bar{Y}(u))du.$$

Note that

$$\begin{aligned}\tilde{f}^{[T]}(0) &= f_\xi(T) - 1 - \int_0^T r^s(u)du - \int_0^T (\bar{Y}(u) \wedge 0) r^s(T-u)du - \int_0^T \tilde{B}(u, \bar{Y}(u))du \\ &= \bar{Y}(T),\end{aligned}$$

and for almost every (a.e.) $t \in \mathbb{R}_+$,

$$(\tilde{f}^{[T]})'(t) = (f_\xi)'(T+t) - r^s(T+t) - \int_0^T (\bar{Y}(u) \wedge 0)(r^s)'(T+t-u)du.$$

We first consider the case that $\lambda > 1 + \alpha$. Because the function $\bar{X}$ is nonnegative, then $-(\bar{Y}(t) \wedge 0) \in [0, 1]$ for each $t \in \mathbb{R}_+$. Recall α in (30). For all $T, t \in \mathbb{R}_+$,

$$\begin{aligned}&-\int_0^T (\bar{Y}(u) \wedge 0)(r^s)'(T+t-u)du \\ &\geq \int_0^T -(\bar{Y}(u) \wedge 0)(r^s)'(T+t-u)\mathbf{1}_{\{(r^s)'(T+t-u)<0\}}du \\ &\geq \int_0^T (r^s)'(T+t-u)\mathbf{1}_{\{(r^s)'(T+t-u)<0\}}du \\ &= \int_t^{T+t} (r^s)'(u)\mathbf{1}_{\{(r^s)'(u)<0\}}du \geq -\alpha.\end{aligned}$$

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Because $\lambda > 1+\alpha$, there exists $\varepsilon > 0$ such that $\lambda - 1 > \alpha + \varepsilon$. By Lemma 2(1) and Lemma 3, there exists $\bar{T} > 0$ such that $(\tilde{f}^{[T]})'(t) > \lambda - 1 - \alpha > \varepsilon$ for all $T > \bar{T}$ and all $t \in \mathbb{R}_+$. We may assume that there exists $T > \bar{T}$ such that $\bar{Y}(T) \geq 0$, because, otherwise, for all $T \geq \bar{T}$, $\bar{Y}(T) < 0$, and then $\bar{X}(T) < 1$, $\bar{Q}(T) = 0$ by (11) and $\bar{R}'(T) = 0$ by (8) and $\langle \mathbf{1}, \bar{\nu}_T\rangle = \bar{X}(T) < 1$ by (10). It follows from (12) that, for all $T \geq \bar{T}$, $\bar{Q}(0) + \bar{E}(T) = \bar{K}(T) + \bar{R}(T)$, which implies that $\bar{K}'(T) = \bar{E}'(T) = \lambda > 1$ for all $T \geq \bar{T}$. But this together with (4) implies that $\langle \mathbf{1}, \bar{\nu}_T\rangle \to \lambda > 1$ as $T \to \infty$, which contradicts that $\langle \mathbf{1}, \bar{\nu}_T\rangle = \bar{X}(T) < 1$ for all $T \geq \bar{T}$. Now fix a $T > \bar{T}$ such that $\tilde{f}^{[T]}(0) = \bar{Y}(T) \geq 0$. For all $0 \leq s \leq t < \infty$ and $u \in \mathbb{R}$, let

$$\tilde{g}(t,s,u) \doteq r^s(t-s)(u \wedge 0) + \tilde{B}(T+s,u).$$

Then $\tilde{g}(t,t,0) = 0$ and $\partial_t \tilde{g}(t,s,u) = (r^s)'(t-s)(u \wedge 0)$. It follows that for every bounded, measurable, nonnegative function w and for each $s,t \in \mathbb{R}_+$ with $s \leq t$, $\partial_t \tilde{g}(t,s,w(s)) = (r^s)'(t-s)(w(s) \wedge 0) = 0$, and then

$$\tilde{g}(t,t,0) + \int_0^t \partial_t \tilde{g}(t,s,w(s))ds = 0 < (\tilde{f}^{[T]})'(t),\ t \in \mathbb{R}_+.$$

By corollary 5.2 of Engler (1986), we have that $\bar{Y}^{[T]}(t) \geq 0$ for all $t \in \mathbb{R}_+$, that is, $\bar{X}(t) \geq 1$ for all $t \geq T$.

We next consider the case that $1 < \lambda \leq 1 + \alpha$. In this case, the density g^s of G^s is assumed to satisfy Assumption 2(2) or Assumption 2(3). If g^s satisfies Assumption 2(3), then the conclusion holds by theorem 3.2(2) of AKKR (see (3.1) therein). Now, let us assume that g^s satisfies Assumption 2(2). Because the renewal density r^s is assumed to be nondecreasing on $\mathbb{R}_+$ under Assumption 2(2), then $(r^s)'(t) \geq 0$ for each $t \in \mathbb{R}_+$ and hence for all $T, t \in \mathbb{R}_+$,

$$-\int_0^T (\bar{Y}(u) \wedge 0)(r^s)'(T+t-u)du \geq 0.$$

Then we also have that there exists $\bar{T} > 0$ such that $(\tilde{f}^{[T]})'(t) > 0$ for all $T \geq \bar{T}$ and all $t \in \mathbb{R}_+$. Thus, corollary 5.2 of Engler (1986) can still be applied to show that $\bar{Y}^{[T]}(t) \geq 0$ for all $t \in \mathbb{R}_+$, that is, $\bar{X}(t) \geq 1$ for all $t \geq T$. □

We next establish a supporting lemma. For r^s that satisfies Assumption 2, let $M > 0$ such that $\int_0^\infty |(r^s)'(t)|dt < M$. Consider

$$\tilde{r}^s(t) \doteq r^s(t) + aM,\ t \in \mathbb{R}_+, \tag{49}$$

where

$$a \doteq \begin{cases} 0, & \text{if Assumption 2(1) holds,} \\ 1, & \text{otherwise.} \end{cases}$$

Then $\bar{X}$ satisfies the following nonlinear Volterra equation:

$$\begin{aligned} \bar{X}(t) + \int_0^t (\bar{X}(u) \wedge 1)\tilde{r}^s(t-u)du + \int_0^t B(u, \bar{X}(u))du \\ = f_\xi(t) + aM \int_0^t (\bar{X}(u) \wedge 1)du. \end{aligned} \tag{50}$$

Note that

$$\int_0^\infty |(\tilde{r}^s)'(t)|dt = \int_0^\infty |(r^s)'(t)|dt < r^s(0) + aM = \tilde{r}^s(0). \tag{51}$$

For each $0 \leq s \leq t < \infty$ and $u \in \mathbb{R}$, define

$$\tilde{q}(t,s,u) \doteq \tilde{r}^s(t-s)(u \wedge 1) + B(s,u). \tag{52}$$

Lemma 7. *For the kernel $\tilde{r}^s$ that satisfies (51) and the function $\tilde{q}$ defined in (52), let x_1, x_2 be two solutions to the following equation:*

$$x(t) + \int_0^t \tilde{q}(t,u,x(u))du = f(t), \tag{53}$$

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with absolutely continuous right-hand side f_1, f_2, respectively. Then there exists $\varepsilon \in (0,1]$ such that

$$\int_0^\infty |x_1'(t) - x_2'(t)|dt \le \left(\frac{2}{\varepsilon} - 1\right)|f_1(0) - f_2(0)| + \frac{2}{\varepsilon}\int_0^\infty |f_1'(t) - f_2'(t)|dt. \tag{54}$$

Proof. Because $\tilde{r}^s$ satisfies (51), then there exists $\varepsilon \in (0,1]$ such that

$$\int_0^\infty |(\tilde{r}^s)'(t)|dt \le (1-\varepsilon)\tilde{r}^s(0), \tag{55}$$

and from the definition of $\tilde{q}$ in (52), we have that, for each $0 \le s \le t < \infty$ and $u \in \mathbb{R}$,

$$\partial_t \tilde{q}(t,s,u) = (\tilde{r}^s)'(t-s)(u \wedge 1).$$

It follows that for all $u, v \in \mathbb{R}$ and for a.e. $s \in [0,\infty)$,

$$\begin{aligned}\int_s^\infty |\partial_t \tilde{q}(t,s,u) - \partial_t \tilde{q}(t,s,v)|dt &= \left(\int_s^\infty |(\tilde{r}^s)'(t-s)(u \wedge 1 - v \wedge 1)|dt\right)\\ &= \left(\int_0^\infty |(\tilde{r}^s)'(t)|dt\right)|u \wedge 1 - v \wedge 1|.\end{aligned}$$

On the other hand, for all $u, v \in \mathbb{R}$ and for a.e. $s \in [0,\infty)$,

$$\begin{aligned}&[\tilde{q}(s,s,u) - \tilde{q}(s,s,v)]sign(u-v)\\ &\quad = \tilde{r}^s(0)(u \wedge 1 - v \wedge 1)sign(u-v) + (B(s,u) - B(s,v))sign(u-v)\\ &\quad = \tilde{r}^s(0)|u \wedge 1 - v \wedge 1| + |B(s,u) - B(s,v)|.\end{aligned}$$

Combining the above two displays with (55), we have that, for all $u, v \in \mathbb{R}$ and for a.e. $s \in [0,\infty)$,

$$\begin{aligned}&(1-\varepsilon)[\tilde{q}(s,s,u) - \tilde{q}(s,s,v)]sign(u-v)\\ &\quad \ge \int_s^\infty |\partial_t \tilde{q}(t,s,u) - \partial_t \tilde{q}(t,s,v)|dt.\end{aligned} \tag{56}$$

Then (54) follows from theorem 4.1 of Engler (1986) with $T = \infty$. □

Now we prove the main result of this section.

Theorem 2. *Suppose that Assumption 2 holds and $\lambda > 1$. The solution $\bar{X}$ to the nonlinear Volterra Equation (50) satisfies*

$$\int_0^\infty |\bar{X}'(t)|dt < \infty. \tag{57}$$

Proof. Fix $x^* \in \mathcal{X}_\lambda$. Let $x_2(t) = x^*, t \in \mathbb{R}_+$, be the constant function on $\mathbb{R}_+$. It is clear that the function x_2 satisfies (53) with

$$f_2(t) = f^*(t) + aMt,\ t \in \mathbb{R}_+.$$

By (50), $x_1 = \bar{X}$ also satisfies (53) with

$$f_1(t) = f_\xi(t) + aM\int_0^t (\bar{X}(u) \wedge 1)du,\ t \in \mathbb{R}_+,$$

and then it follows from (54) in Lemma 7 that there exists $\varepsilon \in (0,1]$ such that

$$\int_0^\infty |\bar{X}'(t)|dt \le \left(\frac{2}{\varepsilon} - 1\right)|\xi(0) - x^*| + \frac{2}{\varepsilon}\int_0^\infty |f_1'(t) - f_2'(t)|dt. \tag{58}$$

Note that for each $t \in \mathbb{R}_+$,

$$f_1'(t) = f_\xi'(t) + aM(\bar{X}(t) \wedge 1) \text{ and } f_2'(t) = (f^*)'(t) + aM.$$

Then for each $T > 0$,

$$\int_0^T |f_1'(t) - f_2'(t)|dt \le \int_0^T |(f_\xi)'(t) - (f^*)'(t)|dt + aM\int_0^T |\bar{X}(t) \wedge 1 - 1|dt.$$

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Note that by Lemma 6, if $\lambda > 1+\alpha$ or if $1 < \lambda \le 1+\alpha$ and the density g^s of G^s satisfies Assumption 2(2) or Assumption 2(3),

$$\int_0^\infty |\bar{X}(t)\wedge 1 - 1|dt < \infty, \tag{59}$$

and then by (44),

$$\int_0^\infty |f_1'(t)-f_2'(t)|dt \le \int_0^\infty |(f_\xi)'(t)-(f^*)'(t)|dt + M\int_0^\infty |\bar{X}(t)\wedge 1 - 1|dt < \infty.$$

If $1 < \lambda \le 1+\alpha$ and the density g^s of G^s satisfies Assumption 2(1), then $a = 0$, and then by (44),

$$\int_0^\infty |f_1'(t)-f_2'(t)|dt \le \int_0^\infty |(f_\xi)'(t)-(f^*)'(t)|dt < \infty.$$

Thus, (57) follows from (58). □

Corollary 1. *Suppose that Assumption 2 holds and* $\lambda > 1$. *Let* $(\bar{X},\bar{\nu},\bar{\eta})$ *be a solution to the fluid model equations associated with* $(\lambda\mathbf{1},\bar{X}(0),\bar{\nu}_0,\bar{\eta}_0)\in\mathcal{S}_0$. *Then, as* $t\to\infty$, $\bar{X}(t)\to x^*\in\mathcal{X}_\lambda$, $\bar{X}'(t)\to 0$ *and* $\bar{\nu}_t \Rightarrow \nu_*$.

Proof. Because $\bar{X}$ is a solution to the nonlinear Volterra Equation (50), then by Theorem 2,

$$\int_0^\infty |\bar{X}'(t)|dt < \infty.$$

It follows that there exists x^* such that $\bar{X}(t)\to x^*$ as $t\to\infty$. Then

$$B(t,\bar{X}(t)) \to \lambda G^r((\bar{F}^{\lambda\eta_*})^{-1}([x^*-1]^+)) \text{ as } t\to\infty.$$

Because for each $t\in\mathbb{R}_+$,

$$\bar{X}(t) = \xi(t) + \int_0^t [\bar{X}(u)-1]^+ g^s(t-u)du - \int_0^t \bar{G}^s(t-u)B(u,\bar{X}(u))du,$$

and $\xi(t)\to\lambda$ as $t\to\infty$, then it follows that

$$x^* = \lambda + [x^*-1]^+ - \lambda G^r((\bar{F}^{\lambda\eta_*})^{-1}([x^*-1]^+)),$$

that is, $x^*\in\mathcal{X}_\lambda$. Note that $x^* > 1$ because $\lambda > 1$.

We next show that $\bar{X}'(t)\to 0$ as $t\to\infty$. Note that by differentiating both sides of (27), for each $t\in\mathbb{R}_+$,

$$\begin{aligned}\bar{X}'(t) = &-(\bar{X}(t)\wedge 1)r^s(0) - \int_0^t (\bar{X}(u)\wedge 1)(r^s)'(t-u)du - B(t,\bar{X}(t))\\ &+ \xi'(t) + \xi(0)r^s(t) + (\xi'\star r^s)(t).\end{aligned}$$

Because $\bar{X}(t)\to x^*$ as $t\to\infty$, then $(\bar{X}(t)\wedge 1)r^s(0)\to(x^*\wedge 1)r^s(0) = r^s(0)$ as $t\to\infty$. By using a similar argument that yields (42), we can also show that for all t sufficiently large,

$$B(t,\bar{X}(t)) = \lambda G^r((\bar{F}^{\lambda\eta_*})^{-1}([\bar{X}(t)-1]^+)).$$

Then it follows that

$$B(t,\bar{X}(t)) \to \lambda G^r((\bar{F}^{\lambda\eta_*})^{-1}([x^*-1]^+)) \text{ as } t\to\infty.$$

By taking the limits on both sides of (33) and using (28), we see that $\xi'(t)\to 0$ as $t\to\infty$. Note that $\xi(0)r^s(t)\to\bar{X}(0)$ due to Lemma 2(1). Moreover, by letting $T = 0$ in (35) and using (36),

$$\begin{aligned}(\xi'\star r^s)(t) = &-\bar{Q}(0)(r^s(t)-g^s(t)) + \lambda G^s(t) + r^s(0)\int_{[0,H^s)} \frac{\bar{G}^s(x+t)}{\bar{G}^s(x)}\,\bar{\nu}_0(dx) - \langle\mathbf{1},\bar{\nu}_0\rangle r^s(t)\\ &+\int_0^t\left(\int_{[0,H^s)}\frac{\bar{G}^s(x+u)}{\bar{G}^s(x)}\,\bar{\nu}_0(dx)\right)(r^s)'(t-u)du.\end{aligned} \tag{60}$$

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Because $\int_{[0,H^s)} \frac{\bar{G}^s(x+t)}{\bar{G}^s(x)} \bar{\nu}_0(dx) \longrightarrow 0$ as $t \longrightarrow \infty$, this and the fact that $\int_0^\infty |(r^s)'(t)|dt < \infty$ together imply that

$$\int_0^t \left(\int_{[0,H^s)} \frac{\bar{G}^s(x+u)}{\bar{G}^s(x)} \bar{\nu}_0(dx) \right) (r^s)'(t-u)du \longrightarrow 0 \text{ as } t \longrightarrow \infty,$$

and then

$$(\xi' \star r^s)(t) \longrightarrow \lambda - \langle \mathbf{1}, \bar{\nu}_0 \rangle - \bar{Q}(0) \text{ as } t \longrightarrow \infty.$$

Note that

$$\int_0^t (\bar{X}(u) \wedge 1 - x^* \wedge 1)(r^s)'(t-u)du \longrightarrow 0 \text{ as } t \longrightarrow \infty,$$

and

$$\int_0^t (x^* \wedge 1)(r^s)'(t-u)du = (x^* \wedge 1)(r^s(t) - r^s(0)),$$

and thus it follows that

$$\int_0^t (\bar{X}(u) \wedge 1)(r^s)'(t-u)du \longrightarrow (x^* \wedge 1)(1 - r^s(0)) = 1 - r^s(0) \text{ as } t \longrightarrow \infty.$$

Combining all the convergence results derived above, we see that $\bar{X}'(t) \longrightarrow 0$ as $t \longrightarrow \infty$.

At last, we show that $\bar{\nu}_t \Rightarrow \nu_*$ as $t \longrightarrow \infty$. It follows from (9), (8), (11), and (18) that, for each $t \in \mathbb{R}_+$,

$$\bar{X}'(t) = \lambda - \langle h^s, \bar{\nu}_t \rangle - B(t, \bar{X}(t)).$$

By taking the limits on both sides of the above display, we have that, as $t \longrightarrow \infty$,

$$\langle h^s, \bar{\nu}_t \rangle \longrightarrow \lambda - \lambda G^r((\bar{F}^{\lambda \eta_*})^{-1}([x^* - 1]^+)) = x^* \wedge 1 = 1.$$

It follows from theorem 2.6 of AKKR that the auxiliary process $\bar{K}$ in (4) has a derivative $\bar{K}'$ such that for a.e. $t \in \mathbb{R}_+$,

$$\bar{K}'(t) = \begin{cases} \lambda & \text{if } \langle \mathbf{1}, \bar{\nu}_t \rangle < 1, \\ \langle h^s, \bar{\nu}_t \rangle & \text{if } \langle \mathbf{1}, \bar{\nu}_t \rangle = 1. \end{cases} \tag{61}$$

Because $\lambda > 1$ and $\bar{X}(t) \longrightarrow x^* > 1$, there exists $T > 0$ such that $\bar{X}(t) \geq 1$ and then $\langle \mathbf{1}, \bar{\nu}_t \rangle = 1$ for all $t \geq T$. It follows that $\bar{K}'(t) \longrightarrow 1$ as $t \longrightarrow \infty$. Then for each $f \in \mathcal{C}_b(\mathbb{R}_+)$, as $t \longrightarrow \infty$,

$$\int_0^t f(t-u)\bar{G}^s(t-u)\, d\bar{K}(u) = \int_0^t f(t-u)\bar{G}^s(t-u)\bar{K}'(u)du \longrightarrow \int_0^\infty f(u)\bar{G}^s(u)du.$$

Then by (4), for each $f \in \mathcal{C}_b(\mathbb{R}_+)$, as $t \longrightarrow \infty$,

$$\int_{[0,H^s)} f(x)\bar{\nu}_t(dx) \longrightarrow \int_0^\infty f(u)\bar{G}^s(u)du,$$

that is, $\bar{\nu}_t \Rightarrow \nu_*$. □

4.3. Long Time Limit of $(\bar{X}, \bar{\nu})$ When $\lambda = 1$

In this section, we consider the case when $\lambda = 1$; hence, we fix $\lambda = 1$. Note that, by Assumption 2, when $\lambda = 1$, $\mathcal{X}_\lambda = \{1\}$, Assumption 2(a)–(d), and Assumption 2(1)–(2) hold.

Lemma 8. *Let R be a solution to the following equation:*

$$R(t) = \int_0^t (r^s(0)((-R(u)) \vee 0) - B(u, 1 + R(u)))du + (f_\xi - f^*)(t). \tag{62}$$

Then we have that $R(t) \longrightarrow 0$ as $t \longrightarrow \infty$.

Proof. We prove this lemma by contradiction. Suppose that $R(t)$ does not converge to zero as $t \longrightarrow \infty$. Then either $\limsup_{t \to \infty} R(t) > 0$ or $\liminf_{t \to \infty} R(t) < 0$.

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We first consider the case that $\limsup_{t\to\infty} R(t) > 0$. Let $\varepsilon > 0$ be such that $\limsup_{t\to\infty} R(t) > \varepsilon$. We claim that there exists a sequence $\{t_n\} \subset \mathbb{R}_+$ such that $R(t_n) > \varepsilon$, $R'(t_n) \ge 0$ and $t_n \to \infty$ as $n \to \infty$. If the claim does not hold, then there exists a $T \in \mathbb{R}_+$ such that for all $t > T$, either $R(t) \le \varepsilon$ or $R'(t) < 0$. If there exists $t_0 > T$ such that $R(t_0) \le \varepsilon$, then $R(t) \le \varepsilon$ for all $t > t_0$, because otherwise, there exists $t > t_0$ such that $R(t) > \varepsilon$, let $s \doteq \sup\{u < t : R(u) \le \varepsilon\} < t$. Then for all $u \in (s,t)$, $R(u) > \varepsilon$ and then $R'(u) < 0$. It follows that $R(t) = R(s) + \int_s^t R'(u)du \le R(s) \le \varepsilon$, which is a contradiction. But $R(t) \le \varepsilon$ for all $t > t_0$ implies that $\limsup_{t\to\infty} R(t) \le \varepsilon$; this is again a contradiction. Therefore, there does not exist $t_0 > T$ such that $R(t_0) \le \varepsilon$; that is, for all $u \in (T,\infty)$, $R(u) > \varepsilon$, and then $R'(u) < 0$. This implies that R is decreasing on (T,∞), $\lim_{t\to\infty} R(t)$ exists, and $\lim_{t\to\infty} R(t) \ge \varepsilon$. Therefore, $\lim_{t\to\infty} R'(t) = 0$. By (62), for all $t > T$,

$$\begin{aligned} R'(t) &= r^s(0)(-R(t)) \vee 0 - B(t, 1 + R(t)) + (f_\xi - f^*)'(t) \\ &= -B(t, 1 + R(t)) + (f_\xi - f^*)'(t). \end{aligned}$$

From this, the fact that $\lim_{t\to\infty} R(t) \ge \varepsilon$, Lemma 3, and Lemma 4, we see that

$$0 \le -\lim_{t\to\infty} B(t, 1 + \varepsilon/2) = -\lambda G^r((\bar{F}^{\lambda\eta_*})^{-1}([1 + \varepsilon/2 - 1]^+)) \le 0.$$

This implies that $\lambda G^r((\bar{F}^{\lambda\eta_*})^{-1}([1 + \varepsilon/2 - 1]^+)) = 0$ and then $1 + \varepsilon/2 \in \mathcal{X}_\lambda$, which contradicts the fact that $\mathcal{X}_\lambda = \{1\}$. This shows that the claim in fact holds.

It follows from the claim and (62) that

$$0 \le R'(t_n) = -B(t_n, 1 + R(t_n)) + ((f_\xi)'(t_n) - (f^*)'(t_n)).$$

From Lemma 3 and Lemma 4, we see that $(f_\xi)'(t_n) - (f^*)'(t_n) \to 0$ as $n \to \infty$. On the other hand, because $B(u,x)$ is nondecreasing in x, then

$$-B(t_n, 1 + R(t_n)) \le -B(t_n, 1 + \varepsilon) \le 0.$$

It follows that

$$\lim_{n\to\infty} B(t_n, 1 + \varepsilon) = 0.$$

By (43), for all t_n large enough,

$$\lambda G^r((\bar{F}^{\lambda\eta_*})^{-1}(\varepsilon)) = \lambda G^r((\bar{F}^{\lambda\eta_*})^{-1}([x^* + \varepsilon - 1]^+)) = B(t_n, x^* + \varepsilon) = B(t_n, 1 + \varepsilon).$$

It follows that $\lambda G^r((\bar{F}^{\lambda\eta_*})^{-1}(\varepsilon)) = 0$ and then $1 + \varepsilon \in \mathcal{X}_\lambda$, which contradicts the fact that $\mathcal{X}_\lambda = \{1\}$.

We next consider the case that $\liminf_{t\to\infty} R(t) < 0$. Let $\varepsilon > 0$ be such that $\liminf_{t\to\infty} R(t) < -\varepsilon$. We claim that there exists a sequence $\{t_n\} \subset \mathbb{R}_+$ such that $R(t_n) < -\varepsilon$, $R'(t_n) \le 0$ and $t_n \to \infty$ as $n \to \infty$. If the claim does not hold, then there exists a $T \in \mathbb{R}_+$ such that, for all $t > T$, either $R(t) \ge -\varepsilon$ or $R'(t) > 0$. If there exists $t_0 > T$ such that $R(t_0) \ge -\varepsilon$, then $R(t) \ge -\varepsilon$ for all $t > t_0$ because otherwise, there exists $t > t_0$ such that $R(t) < -\varepsilon$, let $s \doteq \sup\{u < t : R(u) \ge -\varepsilon\} < t$. Then for all $u \in (s,t)$, $R(u) < -\varepsilon$, and then $R'(u) > 0$. It follows that $R(t) = R(s) + \int_s^t R'(u)du > R(s) \ge -\varepsilon$, which is a contradiction. But $R(t) \ge -\varepsilon$ for all $t > t_0$ implies that $\liminf_{t\to\infty} R(t) \ge -\varepsilon$; this is again a contradiction. Therefore, there does not exist $t_0 > T$ such that $R(t_0) \ge -\varepsilon$; that is, for all $u \in (T,\infty)$, $R(u) < -\varepsilon$ and then $R'(u) > 0$. This implies that R is increasing on (T,∞), $\lim_{t\to\infty} R(t)$ exists, and $\lim_{t\to\infty} R(t) \le -\varepsilon$. Therefore, $\lim_{t\to\infty} R'(t) = 0$. By (62), for all $t > T$,

$$R'(t) = -r^s(0)R(t) + (f_\xi - f^*)'(t).$$

From Lemma 3 and Lemma 4, we have that

$$(f_\xi - f^*)'(t) \to 0 \text{ as } t \to \infty. \tag{63}$$

From the above two displays and the fact that $\lim_{t\to\infty} R'(t) = 0$, we see that $\lim_{t\to\infty} R(t) = 0$, which contradicts that $\lim_{t\to\infty} R(t) \le -\varepsilon$. Then the claim that there exists a sequence $\{t_n\} \subset \mathbb{R}_+$ such that $R(t_n) < -\varepsilon$, $R'(t_n) \le 0$ and $t_n \to \infty$ as $n \to \infty$ holds. Then $\limsup_{n\to\infty} R'(t_n) \le 0$. By (62), for all $n \ge 1$,

$$R'(t_n) = -r^s(0)R(t_n) + (f_\xi - f^*)'(t_n).$$

Note that

$$\limsup_{n\to\infty} (-r^s(0)R(t_n) + (f_\xi - f^*)'(t_n)) = -r^s(0) \liminf_{n\to\infty} R(t_n) \ge r^s(0)\varepsilon > 0,$$

and then $\limsup_{n\to\infty} R'(t_n) > 0$. This is a contradiction.

Because both cases lead to contradictions, this proves that $R(t) \to 0$ as $t \to \infty$. □

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Lemma 9. *Suppose that* $\lambda = 1$ *and the density* g^s *of* G^s *satisfies Assumption* 2(2), *then* $\liminf_{t\to\infty}\bar{X}(t) \geq 1$.

Proof. By (17), $\bar{X}$ is a solution to the following equation:

$$\bar{X}(t) = \int_0^t (-r^s(t-u)(\bar{X}(u) \wedge 1) - B(u, \bar{X}(u)))du + f_\xi(t),\ t \in \mathbb{R}_+. \tag{64}$$

Consider the function $\bar{Y}$ on $\mathbb{R}_+$ defined by $\bar{Y}(t) \doteq \bar{X}(t) - 1$, $t \in \mathbb{R}_+$. Let $\bar{X}^*(t) = 1$, $t \in \mathbb{R}_+$, be the constant function on $\mathbb{R}_+$. Then $\bar{X}^*$ satisfies (64) with f_ξ replaced by f^* in (38) with $x^* = 1$. For each $0 \leq u \leq t < \infty$ and $v \in \mathbb{R}$, let

$$k(t,u,v) \doteq r^s(t-u)((-v) \vee 0) - B(u, 1+v).$$

It follows from (64) that

$$\bar{Y}(t) = \int_0^t k(t,u,\bar{Y}(u))du + (f_\xi - f^*)(t),\ t \in \mathbb{R}_+. \tag{65}$$

For each $\varepsilon > 0$, $0 \leq u \leq t < \infty$ and $v \in \mathbb{R}$, let

$$\bar{k}^\varepsilon(t,u,v) \doteq r^s(0)((-v) \vee 0) - B(u, 1+v) - \varepsilon.$$

Note that, because r^s is nondescreasing and bounded below by $r^s(0)$ by Assumption 2(2), for $0 \leq u \leq t < \bar{t} < \infty$, $-\infty < v < \bar{v} < \infty$, and $\varepsilon > 0$,

$$k(t,u,v) - \bar{k}^\varepsilon(t,u,v) = (r^s(t-u) - r^s(0))((-v) \vee 0) + \varepsilon > 0,$$

and

$$\begin{aligned} k(\bar{t},u,\bar{v}) - k(t,u,\bar{v}) &= (r^s(\bar{t}-u) - r^s(t-u))((-\bar{v}) \vee 0) \\ &\geq 0 = \bar{k}^\varepsilon(\bar{t},u,v) - \bar{k}^\varepsilon(t,u,v). \end{aligned}$$

For each $\varepsilon > 0$, let R_ε be a solution to the following equation:

$$R_\varepsilon(t) = \int_0^t \bar{k}^\varepsilon(t,u,R_\varepsilon(u))du + (f_\xi - f^*)(t).$$

By applying theorem 4 of Sato (1953), we have that $R_\varepsilon(t) \leq \bar{Y}(t)$ for all $t \in \mathbb{R}_+$ and $\varepsilon > 0$. By theorem 4.2 in chapter 2 of Miller (1971), we have that there exists a sequence $\varepsilon_n > 0$, $n \in \mathbb{N}$ such that $\varepsilon_n \to 0$ and $R_{\varepsilon_n} \to R$ uniformly on compact sets as $n \to \infty$, where R is a solution to the Equation (62). It follows that $\bar{X}(t) - 1 = \bar{Y}(t) \geq R(t)$ for all $t \in \mathbb{R}_+$, and then by Lemma 8, we have that $\liminf_{t\to\infty}\bar{X}(t) \geq 1$. □

Theorem 3. *Suppose that* $\lambda = 1$, $\mathcal{X}_1 = \{1\}$ *and the density* g^s *of* G^s *satisfies Assumption* 2(1) *or Assumption* 2(2); *then,* $\bar{X}(t) \to 1$, $\bar{X}'(t) \to 0$ *and* $\bar{\nu}_t \Rightarrow \nu_*$, *as* $t \to \infty$.

Proof. When the density g^s of G^s satisfies Assumption 2(1) and $a = 0$ in (49), then by the same argument as in Theorem 2, $\int_0^\infty |\bar{X}'(t)|dt < \infty$. By the same argument as in Corollary 1, there exists x^* such that $\bar{X}(t) \to x^*$ as $t \to \infty$ and $x^* \in \mathcal{X}_1$. It follows that $x^* = 1$. When g^s of G^s satisfies Assumption 2(2), by Lemma 9, to show that $\bar{X}(t) \to 1$, we just need to show that

$$\limsup_{t\to\infty}\bar{X}(t) \leq 1.$$

We prove this inequality by contradiction. Suppose not, there exists an $\varepsilon > 0$ such that $\limsup_{t\to\infty}\bar{X}(t) > 1 + \varepsilon$. We claim that there exists a sequence $\{t_n\} \subset \mathbb{R}_+$, such that $\bar{X}(t_n) > 1 + \varepsilon$, $\bar{X}'(t_n) \geq 0$, and $t_n \to \infty$ as $n \to \infty$. If the claim does not hold, then there exists a $T \in \mathbb{R}_+$ such that, for all $t > T$, either $\bar{X}(t) \leq 1 + \varepsilon$ or $\bar{X}'(t) < 0$. If there exists $t_0 > T$ such that $\bar{X}(t_0) \leq 1 + \varepsilon$, then $\bar{X}(t) \leq 1 + \varepsilon$ for all $t > t_0$ because otherwise, there exists $t > t_0$ such that $\bar{X}(t) > 1 + \varepsilon$, let $s \doteq \sup\{u < t : \bar{X}(u) \leq 1 + \varepsilon\} < t$. Then for all $u \in (s,t)$, $\bar{X}(u) > 1 + \varepsilon$, and then $\bar{X}'(u) < 0$. It follows that $\bar{X}(t) = \bar{X}(s) + \int_s^t \bar{X}'(u)du < \bar{X}(s) \leq 1 + \varepsilon$, which is a contradiction. But $\bar{X}(t) \leq 1 + \varepsilon$ for all $t > t_0$ implies that $\limsup_{t\to\infty}\bar{X}(t) \leq 1 + \varepsilon$; this is again a contradiction to $\limsup_{t\to\infty}\bar{X}(t) > 1 + \varepsilon$. Therefore, there does not exist $t_0 > T$ such that $\bar{X}(t_0) \leq 1 + \varepsilon$; that is, for all $u \in (T, \infty)$, $\bar{X}(u) > 1 + \varepsilon$, and then $\bar{X}'(u) < 0$. This implies that $\bar{X}$ is decreasing on

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(T,∞), $\lim_{t\to\infty}\bar{X}(t)$ exists, and $\lim_{t\to\infty}\bar{X}(t)\geq 1+\varepsilon$. Therefore, $\lim_{t\to\infty}\bar{X}'(t)=0$. By (64), for all $t>0$ and $T'>T$,

$$
\begin{aligned}
&\bar{X}(T'+t)\\
&=-\int_0^{T'+t} r^s(u)\Big(\bar{X}(T'+t-u)\wedge 1\Big)du-\int_0^{T'+t}B(u,\bar{X}(u))du+f_\xi(T'+t)\\
&=-\int_0^{t} r^s(u)\Big(\bar{X}(T'+t-u)\wedge 1\Big)du-\int_t^{T'+t} r^s(u)\Big(\bar{X}(T'+t-u)\wedge 1\Big)du\\
&\quad-\int_0^{T'+t}B(u,\bar{X}(u))du+f_\xi(T'+t)\\
&=-\int_0^{t} r^s(u)\Big(\bar{X}(T'+t-u)\wedge 1\Big)du-\int_0^{T'} r^s(T'+t-u)\Big(\bar{X}(u)\wedge 1\Big)du\\
&\quad-\int_0^{T'+t}B(u,\bar{X}(u))du+f_\xi(T'+t)\\
&=-\int_0^{t} r^s(u)du-\int_0^{T'} r^s(T'+t-u)\Big(\bar{X}(u)\wedge 1\Big)du\\
&\quad-\int_0^{T'+t}B(u,\bar{X}(u))du+f_\xi(T'+t),
\end{aligned}
$$

where the last equality follows from the fact that $\bar{X}(T'+t-u)\wedge 1=1$ for all $u\leq t$ because $\bar{X}(u)>1+\varepsilon$ for all $u>T$. It follows that, for all $t>0$ and $T'>T$,

$$
\begin{aligned}
\bar{X}'(T'+t)=-r^s(t)-\int_0^{T'}(r^s)'(T'+t-u)\Big(\bar{X}(u)\wedge 1\Big)du\\
-B(T'+t,\bar{X}(T'+t))+(f_\xi)'(T'+t).
\end{aligned}
$$

By Property (1) of Lemma 2 and Lemma 3, we have that $-r^s(t)+(f_\xi)'(T'+t)\to 0$ as $t\to\infty$. Note that $B(T'+t,\bar{X}(T'+t))\geq B(T'+t,1+\varepsilon)$ for all $t>0$ and $T'>T$. Because $\bar{X}$ is nonnegative, then $0\leq\bar{X}(t)\wedge 1\leq 1$ for all $t\in\mathbb{R}_+$. We see that

$$
\left|\int_0^{T'}(r^s)'(T'+t-u)(\bar{X}(u)\wedge 1)du\right|\leq\int_0^{T'}|(r^s)'(T'+t-u)|du=\int_t^{T'+t}|(r^s)'(u)|du.
$$

Then, by Assumption 2(d), we have that

$$
\int_0^{T'}(r^s)'(T'+t-u)(\bar{X}(u)\wedge 1)du\to 0\text{ as }t\to\infty.
$$

It follows that

$$
0\leq\lim_{t\to\infty}B(t,1+\varepsilon)=\lambda G^r((\bar{F}^{\lambda\eta_*})^{-1}([1+\varepsilon-1]^+))\leq\lim_{t\to\infty}B(t,\bar{X}(t))=0.
$$

This implies that $\lambda G^r((\bar{F}^{\lambda\eta_*})^{-1}([1+\varepsilon-1]^+))=0$ and then $1+\varepsilon\in\mathcal{X}_\lambda$, which contradicts the fact that $\mathcal{X}_\lambda=\{1\}$. This shows that the claim in fact holds. Note that along the sequence $\{t_n\}$, we have by (64) that, for all $n\in\mathbb{N}$,

$$
\begin{aligned}
\bar{X}'(t_n)&=-r^s(0)(\bar{X}(t_n)\wedge 1)-\int_0^{t_n}(r^s)'(t_n-u)(\bar{X}(u)\wedge 1)du-B(t_n,\bar{X}(t_n))+(f_\xi)'(t_n)\\
&=-r^s(0)-\int_0^{t_n}(r^s)'(t_n-u)(\bar{X}(u)\wedge 1)du-B(t_n,\bar{X}(t_n))+(f_\xi)'(t_n)\\
&=-r^s(t_n)-\int_0^{t_n}(r^s)'(t_n-u)(\bar{X}(u)\wedge 1-1)du-B(t_n,\bar{X}(t_n))+(f_\xi)'(t_n).
\end{aligned}
$$

Note that $(f_\xi)'(t_n)-r^s(t_n)\to 0$ as $n\to\infty$ by Lemma 3 and Property (1) of Lemma 2. Because $\mathcal{X}_\lambda=\{1\}$, we have that

$$
\liminf_{n\to\infty}B(t_n,\bar{X}(t_n))\geq\liminf_{n\to\infty}B(t_n,1+\varepsilon)=\lambda G^r((\bar{F}^{\lambda\eta_*})^{-1}([1+\varepsilon-1]^+))>0.
$$

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Because $\liminf_{t\to\infty}\bar{X}(t)\geq 1$, then $\bar{X}(t)\wedge 1-1\to 0$ as $t\to\infty$. This and the fact that $\int_0^\infty |(r^s)'|(t)dt<\infty$ by Assumption 2(d) together imply that

$$\int_0^{t_n}(r^s)'(t_n-u)(\bar{X}(u)\wedge 1-1)du=\int_0^{t_n}(r^s)'(u)(\bar{X}(t_n-u)\wedge 1-1)du\to 0 \text{ as } n\to\infty.$$

Note that, because $\bar{X}'(t_n)\geq 0$ for all $n\geq 1$, then $B(t_n,\bar{X}(t_n))\leq B(t_n,\bar{X}(t_n))+\bar{X}'(t_n)$ for all $n\geq 1$. Note that $B(t_n,\bar{X}(t_n))+\bar{X}'(t_n)\to 0$ as $n\to\infty$. It follows that

$$\limsup_{n\to\infty} B(t_n,\bar{X}(t_n))\leq 0,$$

which contradicts $\liminf_{n\to\infty}B(t_n,\bar{X}(t_n))>0$ derived above. Therefore, we proved that $\limsup_{t\to\infty}\bar{X}(t)\leq 1$ and hence $\bar{X}(t)\to 1$ as $t\to\infty$. The convergence results of $\bar{X}'(t)$ to zero and $\bar{\nu}_t$ to ν_* follow the same argument as in Corollary 1. □

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