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Letters to the Editor

Polar and Rectangular Road Networks for Circular Cities

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TAN[1] considered a circular city consisting of a circular central business district W of radius a surrounded by a contiguous concentric annular residential area H of outer radius b. He obtained average distances between random independent points in W and H when various types of routes were followed. He pointed out that if $b/a < 2.6$ the average distance by the shortest route with a rectangular grid road network was less than that with any of the routing systems on a polar (ring/radial) network that he considered. EINHORN[2] pointed out, however, that the 'radial-arc' routing system on a polar network (not considered by Tan) gives smaller average distances than rectangular routing for all values of b/a.

If the object is to compare the rectangular and polar networks in respect of average distance traveled, then the shortest routes should be compared in each case. Neither Tan nor Einhorn, however, considered the shortest-distance routing system on a polar network. It was pointed out by FAIRTHORNE[3] that this is achieved by following a radial-arc route if the radii through the two ends of the journey make an angle less than 2 radians with each other, and a radial route otherwise. The average distance may be shown to be

$$\bar{d} = 2(b^2 + ab + a^2)/3\,(b + a) + (2\pi - 2)a/3\pi.$$

This is less than the value of $\bar{d}$ for the rectangular route for all values of b/a, confirming Einhorn's conclusion that the polar network gives shorter average distances than the rectangular network.

The main purpose of this note is to point out that this conclusion may be extended to any city possessing circular symmetry in which origins and destinations of journeys are distributed independently. To show this we consider the average distance apart of a random point P_1 on the circumference of a circle of radius r_1 and an independent random point P_2 on the circumference of a concentric circle of radius r_2 $(>r_1)$. Fairthorne[3] gave the average distance in a straight line, from which it follows that the average distance on a rectangular grid is

$$d_g(r_1, r_2) = \frac{8(r_1 + r_2)}{\pi^2} E\left(\frac{2\sqrt{r_1 r_2}}{r_1 + r_2}\right),$$

where E is the complete elliptic integral of the second kind. The average distance by the shortest route on a polar network is

$$d_p(r_1, r_2) = \frac{\pi - 2}{\pi} r_1 + r_2.$$

It is found that $d_g(r_1, r_2) > d_p(r_1, r_2)$ for all values of r_1 and r_2. The value of $d_g(r_1, r_2)/d_p(r_1, r_2)$ is 1.27 when $r_1 = 0$, decreases to a minimum of 1.14 when $r_1 = 0.6r_2$, and then increases to 1.19 when $r_1 = r_2$.

Now suppose that A and B are points independently distributed with circular symmetry about the same center O. Let us take P_1 to be whichever of A and B is nearer to O, and P_2 to be the other. The (independent) distributions of OA and OB determine the joint distribution of r_1 and r_2, and the mean distances between A and B by rectangular and polar routes, $\bar{d}_g$ and $\bar{d}_p$, may be obtained by averaging $d_g(r_1, r_2)$ and $d_p(r_1, r_2)$ over this joint distribution. Since the inequality

$$1.14d_p(r_1, r_2) \leqq d_g(r_1, r_2) \leqq 1.27d_p(r_1, r_2)$$

holds for every value of r_1 and r_2, it follows that

$$1.14\bar{d}_p \leqq \bar{d}_g \leqq 1.27\bar{d}_p.$$

Thus for any city with circular symmetry and independent origins and destinations the mean length of a journey on a rectangular network is between 14 and 27 per cent longer than on a polar network.

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