Transportation Science

Publication details, including instructions for authors and subscription information:
http://pubsonline.informs.org

Letter to the Editor—Locating Stepping-Stone Paths in Distribution Problems Via the Predecessor Index Method

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(Received, February 1969)

Locating Stepping-Stone Paths in Distribution Problems Via the Predecessor Index Method

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This paper presents an explicit procedure for finding improving cycles or paths in the distribution model. The procedure developed may be incorporated in the row-column sum method. When this is done, both the row and column numbers and the predecessor index numbers may be determined simultaneously, a fact which contributes to the rapidity of the method. The chief difference between our method and other labeling procedures derives from the fact that the ordinary procedures are implemented in a 'dual' framework and are concerned with identifying a flow augmenting path thdt maintains dual feasibility.

This paper presents an explicit procedure for finding improving cycles or paths in the distribution model. The procedure for assigning predecessor index numbers is closely related to the row-column sum method (*see* reference 2 vol. II) for assigning dual prices in a distribution problem tableau. Both procedures may be performed simultaneously, a fact that contributes to the rapidity of the method.

The predecessor index method may be classified as a 'labeling procedure' of the kind developed by FORD AND FULKERSON.[3] The chief difference between our method and other labeling procedures derives from the fact that the ordinary procedures are implemented in a 'dual' framework and are concerned with identifying a flow augmenting path that maintains dual feasibility. Our approach is implemented in a 'primal' framework, and is concerned with identifying a cost-improving path that maintains primal feasibility. One consequence of this difference is that only one lable, arbitrarily placed, rather than a set of labels placed in precisely specified rows, is used to start the process. A second consequence is that the path is traced in two directions, rather than one. Finally, the path is determined only by a subset, rather than all of the cells that are traversed in tracing back to the starting label.

The labeling of the predecessor index method corresponds to the type of scheme that one would naturally employ to identify a cycle in a graph, and thus, in this sense, involves nothing particularly novel. Nevertheless, we have not found an exposition of this extremely useful procedure in any of the texts or journal articles we have encountered that deal directly or indirectly with transportation models, and feel for this reason that the method deserves being brought to the attention of those interested in solving such models.

The predecessor index method is useful not only for computer programming codes but also for hand computation. The authors have found it extremely useful in solving large problems by hand when it is not possible to observe the entire tableau at once. Furthermore, it is our belief that the method is more efficient than the search procedures currently being used for this purpose, although we have not at present conducted extensive numerical comparisons.

PROCEDURE FOR SPECIFYING THE PREDECESSOR INDEX NUMBERS

ASSUME WE have a basic feasible solution for the problem:

Minimize $\sum_{i\epsilon I, j\epsilon J} c_{ij}x_{ij}$,

subject to

$$\sum_{j\epsilon J} x_{ij} = a_i, \qquad i \epsilon I = \{1, 2, \cdots, m\},$$
$$\sum_{i\epsilon I} x_{ij} = b_j, \qquad j \epsilon J = \{1, 2, \cdots, n\},$$
$$x_{ij} \geqq 0, \qquad i \epsilon I, j \epsilon J.$$

Let N denote the basic cells in the distribution tableau for this solution. (Our definition of 'basic solution' is such that degeneracy will still yield the full number $m + n - 1$ of basic cells, some of which will, however, contain zero x_{ij} values.) Then define a set of row predecessor indexes, r_i, $i \epsilon I$, and column predecessor indexes, k_j, $j \epsilon j$, according to the following instructions:

1. Set some row or column predecessor index equal to zero. Suppose row p is chosen, then $r_p = 0$. Set the column predecessor indexes of the basic cells in row p equal to p (i.e., if $M = \{j \mid (p, j) \epsilon N\}$ then $k_j = p$, $j \epsilon M$).

2. After performing instruction 1 either all of the row and column predecessor index numbers have been determined or there is at least one basic

cell, call it $(i, j) \in N$, such that exactly one of its row and column predecessor indexes has been determined. If $r_i(k_j)$ has been determined, set $k_j = i$ $(r_i = j)$.

3. Continue executing instruction 2 until all the predecessor index numbers have been determined.

USES OF PREDECESSOR INDEXES IN LOCATING STEPPING-STONE PATHS FOR NONBASIC CELLS

SUPPOSE WE want to determine the stepping-stone path for a nonbasic cell (i^*, j^*) with respect to a basic feasible solution to a distribution problem. Assume row and column predecessor index numbers have been dedetermined according to the above procedure.

In finding such a path, the fundamental operation consists in identifying *backward paths* for row i^* and column j^*. We define the backward path for $i^*(j^*)$ to be the sequence $H = \{h_1, h_2, \cdots, h_u\}$ $(T = \{t_1, t_2, \cdots, t_v\})$, where $h_1 = i^*(t_1 = j^*)$ and each h_{s+1} and t_{q+1} is determined uniquely from h_s and t_q respectively, by the rule: if h_s or t_q is the row index i (column index j), then h_{s+1} or t_{q+1} is set equal to the row predecessor index r_i (column predecessor index k_j). The indexes h_u and t_v are both equal to the row or column index whose predecessor index is equal to zero. Thus, according to the procedure specified in the preceding section, we would have $t_v = h_u = p$ if $r_p = 0$.

It is important to keep in mind that each index of a backward path specifically refers to either a row or a column index, but not both. It is convenient to acknowledge this reference by defining *flagged backward paths* $H' = \{h_1', h_2', \cdots, h_u'\}$ $(t' = \{t_1', t_2', \cdots, t_v'\})$ where $h_r' = h_r(t_s' = t_s)$ if $h_r(t_s)$ represents a row index and $h_r' = -h_r(t_s' = -t_s)$ if $h_r(t_s)$ represents a column index. Note that exactly one of every two successive indexes of the flagged backward path is negative.

Corresponding to either backward path H or T, there is identified a set of *associated basic cells* $B(H)$ or $B(T)$. In particular, consider H; for $r = 1, \cdots, u - 1$, if h_r' is positive, then the rth associated basic cell of $B(H)$ is (h_r, h_{r+1}) and if h_r' is negative, then the rth associated basic cell of $B(H)$ is (h_{r+1}, h_r).

Since we have determined the backward paths H and T, a stepping-stone path is easily found for cell (i^*, j^*) by considering the following two cases:

Case 1. H and T intersect only in their last components [i.e., $h_u' = t_v'$ and $h_p' \neq t_q'$ for all $p \leqq u$, $q \leqq v$ such that $(p, q) \neq (u, v)$].

In this case a stepping-stone path for cell (i^*, j^*) is specified by the associated basic cells in H and T, namely $B(H)$ and $B(T)$.

Case 2. H and T intersect in components other than the last components.

For this case, let r and s be such that $H^0 = \{h_1, \cdots, h_r\}$ and $T^0 = \{t_1, \cdots, t_s\}$ intersect only in their last components. The stepping-stone path for cell (i^*, j^*) is then specified by $B(H^0)$ and $B(T^0)$.

The nature of two intersecting backward paths H and T implies that $u - r = v - s$ and $h'_{r+k} = t'_{s+k}$ for all k satisfying $0 \leqq k \leqq u - r$. Thus, one way to determine r and s is to identify the largest $k \geqq 0$ such that $h'_{u-k} = t'_{v-k}$. (NOTE: Perhaps the most efficient way to find r and s is to flag each predecessor index in the path H while H is being traced. The indexes r and s are then determined as soon as a flagged index is reached when tracing T.)

EXAMPLES

BEFORE PRESENTING a justification for the above, let us consider two examples that illustrate the two cases.

In both of the examples, we shall be using as our distribution problem the one presented in Table I. The r_i and k_j numbers around the rim of the table represent the predecessor indexes found by executing the instructions in the first section, where we arbitrarily let $r_1 = 0$. The R_i and K_j numbers around the rim of the table represent the row-column numbers where we arbitrarily let $R_1 = 0$. [NOTE: The row-column numbers are determined by setting $R_i + K_j = c_{ij}$ for each basic cell (i, j). Because of the linear dependence associated with the transportation condition, $\sum a_i = \sum b_j$, any one R_i or K_j may be chosen arbitrarily, and the remaining ones are then determined by the condition $R_i + K_j = c_{ij}$ on all basic cells (i, j). The solution (x_{ij}) is optimal for a minimizing problem if it is feasible and if $R_i + K_j \leqq c_{ij}$ for every cell (i, j). Since in Table I, the solution indicated is feasible and $R_i + K_j \leqq c_{ij}$ for each cell, the solution is optimal.]

For the first example we shall establish a stepping-stone path for cell (1, 4). Following the procedure specified in the last section, we first identify the backward paths $H = \{h_1, h_2, \cdots, h_u\}$ and $T = \{t_1, t_2, \cdots, t_v\}$ where $h_1 = 1$ and $t_1 = 4$. For this particular example, it may be verified that $H = \{1\}$ and $T = \{4, 3, 2, 2, 3, 1\}$. Since $B(H) = \phi$ and $B(T) = \{(3, 4), (3, 2), (2, 2), (2, 3), (1, 3)\}$ the stepping-stone path for cell (1, 4) is given by $B(T)$.

This illustrates that if we use the predecessor index method in finding improving paths in the row-column sum method, and first determine the 'incoming' cell (i, j) and next the predecessor index numbers such that $r_i = 0$ $(k_j = 0)$, then the stepping-stone path for cell (i, j) will be completely specified by $B(T)$ $[B(H)]$. However, the fastest procedure is to simultaneously determine the row and column numbers and the row and column predecessor index numbers, since this requires only one pass through the basic cells to determine both sets. To see this, consider Table I. We arbitrarily let both $R_1 = 0$ and $r_1 = 0$. Then to determine some K_j and k_j required us to first find a basic cell in row 1, say cell $(1, j)$. Once cell $(1, j)$ had been found, we could immediately determine that $k_j = 1$ and $K_j = -R_1 + c_{ij}$ or since $R_1 = 0$, $K_j = c_{ij}$.

To illustrate Case 2 in the second section, determine the stepping-

TABLE I

$K_1 = 1 \quad K_2 = 5 \quad K_3 = 3 \quad K_4 = 5$
$k_1 = 1 \quad k_2 = 2 \quad k_3 = 1 \quad k_4 = 3$

		$O_i \backslash D_j$	D_1	D_2'	D_3	D_4	Total
$R_1 = 0$	$r_1 = 0$	O_1	$\boxed{1}$ 11	$\boxed{6}$	$\boxed{3}$ 9	$\boxed{5}$	20
$R_2 = -2$	$r_2 = 3$	O_2	$\boxed{7}$	$\boxed{3}$ 2	$\boxed{1}$ 8	$\boxed{6}$	10
$R_3 = -1$	$r_3 = 2$	O_3	$\boxed{9}$	$\boxed{4}$ 11	$\boxed{5}$	$\boxed{4}$ 14	25
		Total	11	13	17	14	55

stone path for cell (2, 4). In this example $H = \{2, 3, 1\}$ and $T = \{4, 3, 2, 2, 3, 1\}$. From $H' = \{2, -3, 1\}$ and $T' = \{-4, 3, -2, 2, -3, 1\}$ we define $H^0 = \{2\}$ and $T^0 = \{-4, 3, -2, 2\}$. Consequently, $B(T^0) = \{(3, 4), (3, 2), (2, 2)\}$ determines the stepping-stone path for cell (2, 4) since $B(H^0) = \phi$.

JUSTIFICATION

SINCE A basic feasible solution is the groundwork from which the predecessor index numbers are defined, it follows directly that it is possible to define them according to the procedure given in the section, "Procedure." Furthermore, it is apparent from their definition that starting with any row (column) index and proceeding to the column (row) index specified by its predecessor index will ultimately lead to the row p (i.e., row or column index whose predecessor index is zero) since the predecessor indexes were specified by observing the location of the last step starting at row p. Thus, for any nonbasic cell (i, j) there will always exist backward paths $H = \{h_1, h_2, \cdots, h_u\}$ and $T = \{t_1, t_2, \cdots, t_v\}$ where $h_1 = i$, $t_1 = j$, and $h_u = t_v = p$. Therefore the relation of H and T must satisfy one of the two cases discussed earlier.

In Case 1, $B(H)$ and $B(T)$ clearly specify a set of basic cells that form a stepping-stone path for cell (i, j). This may be verified by noting that $B(H)$ provides a stepping-stone path from cell (i, j) to the basic cell (p, h_{u-1}) starting from the basic cell (i, h_2) while $B(T)$ provides a stepping-stone path from cell (i, j) to some basic cell (p, t_{v-1}) starting from the basic cell (t_2, j). Since $h_{u-1} \neq t_{v-1}$ by assumption, $B(H)$ and $B(T)$ form the appropriate stepping-stone path.

In Case 2, once r and s are found, the case reduces to that of the prceeding one. Consequently, the assertions concerning this case follow directly from the above paragraph.

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(Received May, 1969)