Transportation Science

Publication details, including instructions for authors and subscription information:
http://pubsonline.informs.org

Letter to the Editor—A Discrete-Time Model for a Traffic Intersection

J. D. Griffiths, R. H. Thomas,

Letters to the Editor

A Discrete-Time Model for a Traffic Intersection

J. D. GRIFFITHS AND R. H. THOMAS

University of Wales Institute of Science and Technology, Cardiff, Great Britain

Ideas introduced by OLIVER AND BISBEE *are used in the consideration of a more realistic model of a T-junction. Results are shown to be in close agreement with those of Tanner, but are obtained by a simpler analysis.*

The problem of delay to vehicles on the minor road at a T-junction, where the major-road traffic has priority has been considered by a number of authors (BECKMANN, MCGUIRE, AND WINSTEN,[1] OLIVER AND BISBEE,[2] TANNER,[3] HAWKES,[4,5,6] for example). The major-road traffic gives rise alternately to gaps and blocks (as viewed from the minor road), a gap being defined as a time-interval during which minor-road vehicles can enter the junction, and a block being other-wise. Essentially, the problem is to find the statistics of the minor-road queue formed by vehicles waiting for gaps to occur in the major-road traffic.

In Tanner's model,[3] major-road gaps were distributed exponentially and blocks were distributed arbitrarily. Minor-road vehicles arrived at random and a minimum time was required for the leading vehicle to enter the junction.

In contrast to Tanner, Oliver and Bisbee[2] assumed both arbitrary gap and block distributions. Again, arrivals on the minor road were at random and the leading minor-road vehicle required a minimum time to enter the junction. However, it was assumed that not more than one vehicle could cross in a gap. This somewhat unrealistic assumption was thought to be reasonable in the case of relatively high major-road traffic flow.

It is shown here that Oliver and Bisbee's results are not satisfactory even for high major-road traffic-flow rates. For this reason the Oliver and Bisbee approach is extended. It is assumed that more than one vehicle may depart per gap, such departures only occurring at certain specified times in the gap. The use of this assumption leads to an easy solution of a realistic T-junction problem of the type solved by Tanner.

In the next section we shall summarize briefly the Oliver and Bisbee results.

OLIVER AND BISBEE MODEL

For purposes of comparison with Tanner,[3] we shall use a notation different from that used by Oliver and Bisbee.[2]

Traffic on the major road generates gaps and blocks in the usual manner. A gap is defined as an intervehicle spacing greater than or equal to a critical time T. It is assumed that at most one vehicle enters the junction during a gap, any entry occurring at the beginning of the gap. The distributions of gaps and blocks are arbitrary. Minor-road departures are allowed at the beginning of gaps; for ease of reference these possible departure points are denoted by 'A.'

Oliver and Bisbee used a Markov chain embedded at the points 'A' and derived the related stationary probability generating function $\Pi(z)$ for the minor-road queue. In fact they showed that

$$\Pi(z) = \pi_0 k_0(1-z)/[K(z)-z], \tag{1}$$

where π_0 is the probability that the minor-road queue is empty at 'A' whether or not a departure has occurred, k_0 is the probability that there are no minor-road arrivals in time 'AA,' and $K(z)$ is the probability generating function for minor-road arrivals in time 'AA.'

Also

$$\pi_0 k_0 = 1 - q_2\xi(t), \tag{2}$$

where q_2 is the mean arrival rate on the minor road, and $\xi(t)$ is the first moment of the 'AA' time interval distribution.

The associated mean queue at the 'A' points is given by

$$\bar{n} = q_2^2\xi(t^2)/2[1-q_2\xi(t)], \tag{3}$$

where $\xi(t^2)$ is the second moment of the 'AA' time interval distribution.

It should be noted that (3) is the mean queue at beginnings of gaps and not necessarily at departure instants of vehicles. Modification of the Oliver-Bisbee result gives

$$\begin{matrix}\text{mean queue at}\\ \text{departure instants}\end{matrix} = q_2^2\xi(t^2)/2q_2\xi(t)[1-q_2\xi(t)]. \tag{4}$$

COMPARISON WITH TANNER MODEL

Tanner's work[3] was concerned with conditions at random instants, it being observed that the expressions for mean queue size at random and departure instants were identical. It appears reasonable, therefore, to compare result (4) (relating to mean queue size at departure instants), rather than the Oliver and Bisbee result (3), with that of Tanner. For convenience, the comparison shown in Table I refers to mean waiting times.

In the table, q_1 is the mean arrival rate of major-road vehicles, T is the minimum intervehicle spacing accepted by a minor-road vehicle, and β_2 is as defined by Tanner.[3] Also, the gaps and blocks are those arising as a result of negative exponential intervehicle spacings.

It is observed that the modified Oliver and Bisbee formula (4) is not in good agreement with Tanner's results for any of the assumed major-road flow rates. This is in apparent disagreement with Oliver and Bisbee's claim of increasing accuracy with high major-road flow rates. The explanation of this lies in the fact that saturation occurs at lower major-road flow rates in the Oliver and Bisbee case than in the more realistic Tanner model.

TABLE I

Comparison of Mean Waiting Time (sec) for Tanner Model (Upper Result) with that Derived from (4) (Lower Result), for Various Values of q_1 and q_2 (Veh/Sec), with $T = 6$ sec, $\beta_2 = 3$ sec

q_1/q_2	0.02	0.04	0.06	0.08
0.05	1.20	1.43	1.71	2.06
	45.64	(a)	(a)	(a)
0.10	2.60	3.07	3.66	4.44
	19.23	45.07	(a)	(a)
0.15	4.42	5.33	6.57	8.37
	15.47	30.22	643.45	(a)
0.20	6.83	8.57	11.27	15.99
	15.87	31.55	2673.59	(a)
0.25	10.08	13.55	20.05	36.67
	18.59	42.15	(a)	(a)
0.30	14.64	21.95	41.67	282.63
	23.74	73.25	(a)	(a)
0.35	21.32	38.72	173.45	(a)
	32.50	259.76	(a)	(a)

(a) Refers to saturated queue.

The results of Oliver and Bisbee's analysis are now used to solve a more realistic problem.

THE PROPOSED MODEL

IT IS ASSUMED that during a gap more than one vehicle may leave the minor-road queue. However, in no circumstances may vehicles depart other than at certain discrete instants (called 'A' points). These instants are at the beginning of a gap and subsequently at intervals of size β_2 within the gap. The constant β_2 is the time taken for the second vehicle in the queue to move up to the head of the queue. This concept of move-up time has been used by other authors, Tanner[3] for example. However, the present work differs from Tanner's work for those cases when minor-road vehicles arrive to find an empty queue: here, such vehicles are required to wait until the next discrete time point. This compulsory wait makes some allowance for drivers coming to a halt, spending time on gear changes, etc. (c.f., HAWKES[6]). In contrast, Tanner assumes no time is lost in such activities, although his model could be adapted to take account of this effect.

The gaps between major-road vehicles are taken to be distributed as the negative exponential, the blocks being distributed arbitrarily. The gap probability density is taken to be

$$q_1 \exp(-q_1 t). \qquad (0 \leqq t < \infty)$$

It is assumed that the minimum intervehicle spacing on the major road accepted by a minor-road vehicle is T. This time is included as part of the block, as in Tanner.[8] The probability density of the arbitrary block distribution is $f(t)$ $(0 \leqq t < \infty)$.

Then it is easy to show that the probability density for the time interval between successive 'A' points is given by

$$g(t) = \delta(t - \beta_2) \exp(-q_1\beta_2) + \int_0^\infty \int_0^{\beta_2} \delta(t - x - y) q_1 \exp(-q_1 x) f(y)\, dx\, dy, \quad (0 \leqq t < \infty) \tag{5}$$

where $\delta(t)$ is the usual Dirac delta function.* The first term arises when the adjacent 'A' points occur in the same gap, the second when they are the last and first 'A' points of successive gaps.

The time intervals between successive pairs of 'A' points are independently distributed, the independence arising directly as a result of the 'forgetfulness' property of the negative exponential distribution assumed for the gap.

The reasoning is then as follows. The 'A' points of the present model are essentially of the same type as the 'A' points of the Oliver and Bisbee problem: in both cases they are the instants at which minor-road departures are permitted. Thus, we find $\xi(t)$ and $\xi(t^2)$ from (5) and substitute in the modified Oliver and Bisbee formula (4). This gives immediately the solution relating to the present model, namely the mean queue at departure instants when vehicles may only leave at discrete time-points.

After some reduction, the first and second moments are found to be

$$\xi(t) = [1 - \exp(-q_1\beta_2)]Y, \tag{6}$$

$$\xi(t^2) = 2[1 - (1 + q_1\beta_2)\exp(-q_1\beta_2)]Y/q_1 + [1 - \exp(-q_1\beta_2)]\xi(y^2), \tag{7}$$

where

$$Y = \xi(y) + 1/q_1 .$$

Substitution from (6) and (7) into (4) gives the mean queue

$$\bar{n} = \frac{q_2[\{2[1 - (1 + q_1\beta_2)\exp(-q_1\beta_2)]Y/q_1\} + [1 - \exp(-q_1\beta_2)]\xi(y^2)]}{2Y[1 - \exp(-q_1\beta_2)][1 - q_2Y\{1 - \exp(-q_1\beta_2)\}]}. \tag{8}$$

Dividing by the mean minor-road arrival rate q_2, we have for the mean waiting time

* $\delta(t)$ is defined by $\delta(t) = 0$ $(t \neq 0)$, $\int_{-\infty}^{\infty} \delta(t) = 1$.

for minor-road vehicles:

$$\bar{w}_2 = \frac{\{[1-(1+q_1\beta_2)\exp(-q_1\beta_2)]/q_1[1-\exp(-q_1\beta_2)]\} + \xi(y^2)/2Y}{1-q_2Y[1-\exp(-q_1\beta_2)]}. \tag{9}$$

For purposes of comparison with Tanner we rewrite (9) in the form

$$\bar{w}_2 = \frac{\xi(y^2)/2Y + q_2Y[1-(1+q_1\beta_2)\exp(-q_1\beta_2)]/q_1}{1-q_2Y[1-\exp(-q_1\beta_2)]} + \frac{1-(1+q_1\beta_2)\exp(-q_1\beta_2)}{q_1[1-\exp(-q_1\beta_2)]}. \tag{10}$$

The first term is the result given previously by Tanner,* so that the second term represents the correction required as a result of the more realistic assumption relating to driver behavior at an empty minor-road queue. For most practical cases the second term is of the order of $\beta_2/2$. However, as $q_1 \to \infty$ the correction term tends to zero, as would be expected intuitively. Also, it is interesting to note that this correction term is independent of the block distribution, a somewhat surprising result.

The result (10) has been derived with little difficulty, in contrast to the rather complicated analysis used by other authors. Recently, Hawkes[6] has suggested more general models than that proposed here; but the advantage of the argument introduced here lies in its simplicity and possible application to even more complicated problems.

REFERENCES

1. M. J. Beckmann, C. B. McGuire, and C. B. Winsten, *Studies in the Economics of Transportation*, Chap. 1, Yale University Press, New Haven, 1956.
2. R. M. Oliver and E. F. Bisbee, "Queuing for Gaps in High Flow Traffic Streams," *Opns. Res.* **10,** 105–114 (1962).
3. J. C. Tanner, "A Theoretical Analysis of Delays at an Uncontrolled Intersection," *Biometrika* **49,** 163–170 (1962).
4. A. G. Hawkes, "Queuing for Gaps in Traffic," *Biometrika* **52,** 79–85 (1965).
5. ———, "Delay at Traffic Intersections," *J. Roy. Stat. Soc. B.* **28,** 202–212 (1966).
6. ———, "Gap-acceptance in road Traffic," *J. Appl. Prob.* **5,** 84–92 (1968).
7. T. L. Saaty, *Elements of Queuing Theory*, McGraw-Hill, New York (1961).

(Received, September 1969)

* Tanner's result refers to conditions at random instants. The present result relates to conditions at departure instants. However the use of an argument similar to that of Khintchine (*see* Saaty[7]) shows that the derived result also applies at random instants.