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Letter to the Editor—Constraining Relations Among Offsets in Synchronized Signal Networks

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Letters to the Editor

Constraining Relations Among Offsets In Synchronized Signal Networks

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Circuit constraints on the phasing of fixed-time synchronized signal networks are defined. A method, based on a graph-theoretic approach, for deriving a fundamental set of equations characterizing these constraints is presented.

The coordination strategy of a signalized traffic network has to comply with a set of constraints that are determined by the structure of the controlled-network circuits. The constraint equations are analogous to Kirchhoff's Voltage Law of electrical network theory and stem from a physical requirement of synchronization that a coordinated signal network has to obey. The necessity to observe these constraints, when attempting to optimize a signal-controlled traffic network, has been mentioned by several authors (e.g., references 1–3). The present paper gives an exact definition of the synchronization constraints, introduces two theorems concerning synchronized signal networks and presents a method for deriving a fundamental set of equations characterizing these constraints.

THE ELEMENTARY CIRCUIT

LET US EXAMINE the simplest network that contains a closed loop; the network formed by a pair of two-way traffic links connecting a pair of adjacent signalized intersections. Let S_i and S_j denote the signals at intersections i and j, respectively. Referring to Fig. 1, let

$g_i(g_j)$ = green time of $S_i(S_j)$ on link under study (effective value);
$r_i(r_j)$ = red time of $S_i(S_j)$ on link under study (effective value);
$x_a(x_b)$ = relative phase (offset) between $S_i(S_j)$ and $S_j(S_i)$, measured as the time from the starting point of a green of $S_i(S_j)$ to the next starting point of a green of $S_j(S_i)$. Let us adopt the convention $0 \leqq x < 1$.

All quantities are expressed in cycles. A quantity having the dimensions of time

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can always be expressed in cycles by division by the cycle time. Accordingly we have,

$$g_i + r_i = g_j + r_j = 1. \tag{1}$$

Figure 1 demonstrates clearly that the two offset variables in this case have to obey the relation

$$x_a + x_b = 1. \tag{2}$$

Equation (2) represents the synchronization law in its simplest form. It is based on the fact that synchronized signals have to switch their aspects at identical rates, i.e., with a common cycle time.

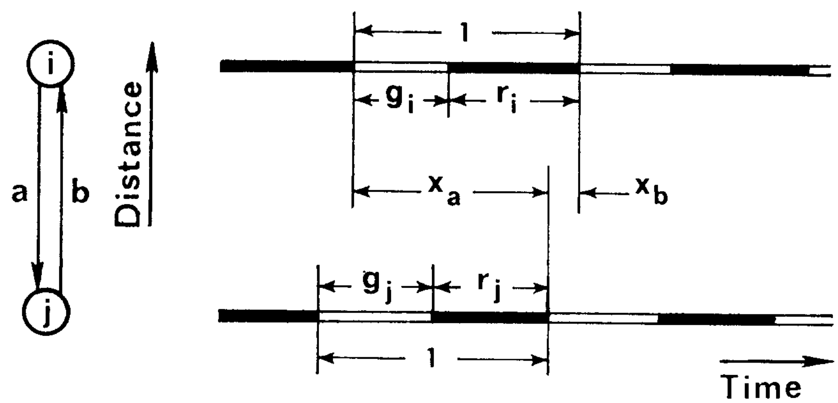

Fig. 1. Two-way traffic links connecting a pair of adjacent signalized intersections.

THE CONSTRAINT EQUATIONS

In the analysis of more complex networks, the demand of simultaneous operation is represented by a set of circuit constraint equations. It is assumed that the offset of each signal, relative to a reference point, is measured at a fixed point of its cycle. This point can be fixed, for instance, at the zero time of the green phase serving the largest volume of traffic at the intersection. Referring to Fig. 2, let us start a time measurement at the offset reference point of S_4 for example, and add together the relative phases along the links of the loop formed by signals (S_4, S_7, S_8, S_5, S_4). The result has to equal an integral number of cycles, since the act of measuring is terminated at the reference point of S_4 exactly after a whole number of cycles have been completed. In general, the algebraic sum of the offset variables along the links of each closed loop (circuit) of the signal network has to equal an integral number of cycles. The formal statement of this requirement constitutes the circuit constraint equations.

The formulation of the constraint equations is based on concepts from the theory of graphs.[4,5] Some of the basic definitions that are used are introduced here in a semiformal manner. The traffic-signal network is represented by its corresponding *linear graph G*. Concepts of *node* (or *intersection*) and *link* are replaced by *vertex* and *edge*, respectively. The present discussion is restricted to graphs that are *connected* and *directed*. Such graphs contain a *path* of connected edges between any two vertices of the graph and every edge has an *orientation* assigned by the ordering of its vertices.

Definition 1. The *degree* of a vertex is the number of edges incident at the vertex.

Definition 2. A *circuit* (or *loop*) is a connected graph or subgraph in which each vertex is of degree 2. A circuit with an orientation assigned by a cyclic ordering of vertices is an *oriented circuit.*

Definition 3. A *tree* is a connected subgraph of a connected graph that contains all the vertices of the graph but does not contain any circuits. An important property of a tree with ν vertices is that it contains exactly $\nu - 1$ edges.

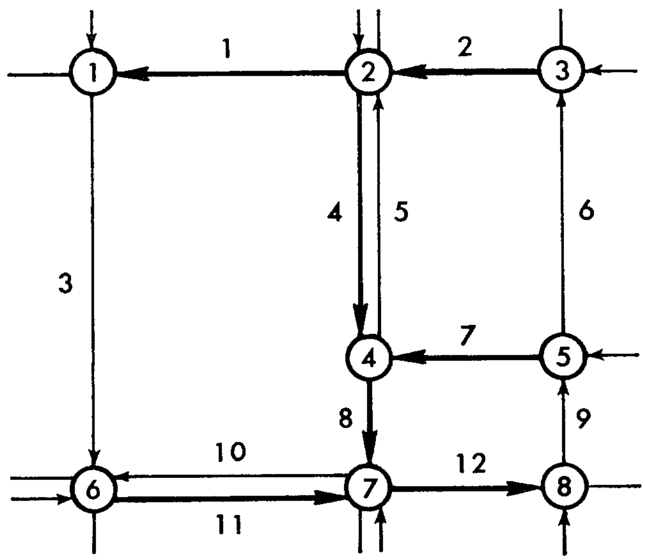

Fig. 2. Signal network containing closed loops. A tree is indicated by heavy lines.

Definition 4. An element of a tree is a *branch.* An element of the complement of a tree (a *cotree*) is a *chord.*

A connected graph of ν vertices and e edges contains $\nu - 1$ branches and $e - \nu + 1$ chords, with respect to a chosen tree of the graph. If we add one chord to a tree, the resulting graph is, of course, no longer a tree. The chord, and the path in the tree between the vertices of the chord, constitute a circuit. This, however, is a unique circuit and the only circuit of the resulting graph.

Definition 5. The *fundamental circuits* (f-circuits) of a connected directed graph G, for a chosen tree T, are the $e - \nu + 1$ circuits formed by each chord and its unique path in the tree between the vertices of the chord. The orientation of the f-circuit is chosen to agree with that of the defining chord.

The procedure for choosing a tree is very simple: Start with any edge of the network. It is either an element of some circuit or it is not. If it is an element of a circuit, remove the edge; its removal does not disconnect the network. If it is not an element of a circuit, leave it. Testing each edge in this way, and removing it, if it belongs to a circuit, we are left at each stage with a connected network. Finally all the circuits will be destroyed, and we shall have a connected network, containing all the vertices of the original network, without any circuits; that is, a tree.

Consider, for example, the connected directed graph G of the network shown in Fig. 2. A tree T is indicated by the heavy lines. The numbers of edges and vertices in G are twelve and eight, respectively. Therefore the number of chords is $e - \nu + 1 = 5$. The five f-circuits defined by these chords and their corresponding tree paths are as given in the accompanying table:

This tabulation includes only the f-circuits for the tree composed of edges (1, 2, 4, 7, 8, 11, 12). It can be established that any circuit in the graph can be made an f-circuit with respect to some tree (reference 5, p. 105).

TABLE I

f-Circuits of Graph in Fig. 2.

Chord	Tree Path	f-Circuit
(3)	(11, 8, 4, 1)	(3, 11, 8, 4, 1)
(5)	(4)	(5, 4)
(6)	(2, 4, 7)	(6, 2, 4, 7)
(9)	(7, 8, 12)	(9, 7, 8, 12)
(10)	(11)	(10, 11)

Definition 6. The *circuit matrix* $C = [c_{kl}]$, of a directed graph, with m rows (as the number of circuits) and e columns (as the number of edges) is defined by,

$c_{kl} = +1$ if edge l is in circuit k and the orientation of the circuit and the edge coincide,
$c_{kl} = -1$ if edge l is in circuit k and the orientations do not coincide, and
$c_{kl} = 0$ if edge l is not in circuit k.

CIRCUIT LAW. *The algebraic sum of offset variables associated with the edges of any circuit has an integer value*, i.e.,

$$\sum_{l=1}^{e} c_{kl}x_l = n_k \text{ for all circuits. } (k = 1, 2, \ldots, m) \quad (3)$$

x_l is the offset variable defined across edge l and n_k is an integer number associated with the constraint of circuit k. Each variable is added or subtracted, depending on whether the corresponding edge has a direction that agrees or disagrees with the circuit orientation.

In matrix form the constraint equations are written as follows,

$$\boldsymbol{Cx} = \boldsymbol{n}, \quad (4)$$

where $\boldsymbol{x}$ and $\boldsymbol{n}$ are the following column vectors:

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_l \\ \vdots \\ x_e \end{bmatrix}, \quad \boldsymbol{n} = \begin{bmatrix} n_1 \\ n_2 \\ \vdots \\ n_k \\ \vdots \\ n_m \end{bmatrix}. \quad (5)$$

THEOREM 1. *For a connected network, with ν vertices and e edges, exactly $e - \nu + 1$ constraint equations are linearly independent.*

This theorem follows directly from the theorem establishing that the rank of the circuit matrix C of a connected directed graph is $e - \nu + 1$ (reference 4, p. 92).

The set of independent constraint equations is written as follows:

$$\boldsymbol{C}_f \boldsymbol{x} = \boldsymbol{n}_f , \tag{6}$$

where C_f is the f-circuit matrix of the network; $\boldsymbol{x}$ is the vector of offsets defined across the edges of the network given in (5) and $\boldsymbol{n}_f$ is a vector of integer numbers associated with each of the f-circuit constraints.

To have a fundamental system, we must choose a tree T of the graph. Let the f-circuits defined by the tree be numbered in some arbitrary manner as $1, 2, \cdots, e - \nu + 1$. Let the columns of the f-circuit matrix be arranged as follows: The column corresponding to the chord that appears in circuit k is arranged as column k, where $k = 1, 2, \cdots, e - \nu + 1$. The columns corresponding to the branches are arranged in some arbitrary manner as columns $e - \nu + 2, e - \nu + 3, \cdots, e$. The matrix of f-circuits that follows takes the form

$$\boldsymbol{C}_f = \underset{(e - v + 1)}{\text{circuits}} \; \overset{\text{chords}}{\underset{(e - v + 1)}{[\boldsymbol{I}}} \;\Big|\; \overset{\text{branches}}{\underset{(v - 1)}{\boldsymbol{C}_{12}]}} , \tag{7}$$

where $\boldsymbol{I}$ is the identity matrix of order $e - \nu + 1$. The orientation for the circuits is taken so that the components of the f-circuit matrix corresponding to the defining chords are $+1$. Substituting (7) in (6) the fundamental circuit equations appear as

$$[\boldsymbol{I} \mid \boldsymbol{C}_{12}] \begin{bmatrix} \boldsymbol{x}_c \\ \hline \boldsymbol{x}_b \end{bmatrix} = \boldsymbol{n}_f , \tag{8}$$

where $\boldsymbol{x}_c$ and $\boldsymbol{x}_b$ represent, respectively, the $e - \nu + 1$ and $\nu - 1$ offset variables identified by the chords and branches that have been arranged according to the outlined agreement.

Example. The cotree T' indicated by the light lines in Fig. 2 defines the five f-circuits given in Table I. These f-circuits define a set of five corresponding fundamental equations. Following (8) we obtain,

$$\begin{array}{r} \text{circuits} \\ (3, 11, 8, 4, 1) \\ (5, 4) \\ (6, 2, 4, 7) \\ (9, 7, 8, 12) \\ (10, 11) \end{array}
\begin{array}{c} \overbrace{\begin{array}{ccccc} 3 & 5 & 6 & 9 & 10 \end{array}}^{\text{chords}} \qquad \overbrace{\begin{array}{ccccccc} 1 & 2 & 4 & 7 & 8 & 11 & 12 \end{array}}^{\text{branches}} \\
\left[\begin{array}{ccccc:ccccccc} 1 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}
\begin{bmatrix} x_3 \\ x_5 \\ x_6 \\ x_9 \\ x_{10} \\ \hline x_1 \\ x_2 \\ x_4 \\ x_7 \\ x_8 \\ x_{11} \\ x_{12} \end{bmatrix} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 \\ n_5 \end{bmatrix} .$$

THEOREM 2. *If T is any tree of a connected network, the offset variables across the chords of T (edges of T') are expressable as linear combinations modulo 1 of the offset variables across the branches of T.*

Proof. From equation (8) we derive the following expression:

$$\boldsymbol{x}_c = \boldsymbol{n}_f - \boldsymbol{C}_{12}\boldsymbol{x}_b\,. \tag{9}$$

Observing the adopted convention $0 \leqq x_k < 1$ we obtain,

$$\boldsymbol{x}_c = \{-\boldsymbol{C}_{12}\boldsymbol{x}_b\} \qquad (\text{mod } 1). \tag{10}$$

CONCLUSIONS

ONLY $\nu - 1$ offset variables can be assigned arbitrary values and the edges across which they are defined must constitute a tree pattern connecting all vertices of the network. The offsets across the remaining $e - \nu + 1$ edges are determined according to the circuit-structures that they form in conjunction with their corresponding tree branches.

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