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Letters to the Editor

On Vertex Addends in Minimax Location Problems

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This letter gives a useful interpretation of vertex addends in minimax location problems. Also, Handler's solution method for this problem is extended to the case when addends are present.

Let $G(V, E)$ be a finite undirected connected graph with vertex-set V and edge-set E consisting of rectifiable arcs. A 'point' of G is either a vertex or an interior point of an edge. Thus the (symmetric) shortest-path distance $d(x, y)$, between two points x and y of G, is well defined. Associated with each vertex v is a non-negative 'addend' $a(v)$. Define the (asymmetric) function $\bar{d}(x, v)$ as

$$\bar{d}(x, v) = d(x, v) + a(v), \tag{1}$$

and let

$$L(x) = \max\{\bar{d}(x, v): v \epsilon V\}. \tag{2}$$

A vertex v_0 that minimizes $L(x)$ over V is called a 'vertex center' (VC), and a point x that minimizes $L(x)$ over all points is called an 'absolute center' (AC). In this note, we show how to determine a VC and AC by solving a related problem without addends.

For finding these 'centers,' in the general network case, GOLDMAN[1] has proposed a graph reduction method, while HANDLER[2] has given a simple solution algorithm for the case of a tree graph with all of whose vertex addends $a(v)$ are zero. Recently, by first finding the VC and then AC, HALFIN[3] obtained an efficient method for this problem. However, its explanation seems cumbersome. The development stated in this paper, though not necessarily more efficient than Halfin's, can give a more straight-forward and intuitive understanding than previously available. It has stimulated an extension of Handler's method to the case when (nonzero) addends are present.

THEORETICAL ANALYSIS

WE BEGIN BY disposing of the somewhat special case in which G has a dominant vertex, i.e., a vertex v such that

$$a(v) \geqq d(v, v') + a(v') = \bar{d}(v, v') \tag{3}$$

for all $v' \in V$, $v' \neq v$.

It is readily seen that G can have at most one dominant vertex. Also, v is dominant if and only if

$$L(v) = \bar{d}(v, v) = a(v). \tag{4}$$

LEMMA. *A dominant vertex is an* AC, *in fact the unique* AC.

Proof. For the proof, let v be a dominant vertex and let x be any point of G, then

$$L(x) \geqq \bar{d}(x, v) = d(x, v) + a(v) \geqq a(v) = L(v), \tag{5}$$

so v is an AC. Since equality holds only if $d(x, v) = 0$, v is the only AC.

For the remainder of the analysis, it is convenient to enlarge the graph $G = (V, E)$ to a graph $G^* = (V^*, E^*)$ defined as follows. Let $V = \{v_1, \cdots, v_m\}$; then we introduce a set $V' = \{v_1', \cdots, v_m'\}$ of new vertices and set $V^* = V \cup V'$. We also introduce a set $E' = \{e_1', \cdots, e_m'\}$ of new edges $e_i' = (v_i, v_i')$ with respective lengths $a(v_i)$, and put $E^* = E \cup E'$. The shortest-path distance $d(x, y)$ on G is readily and uniquely extended to a corresponding function on G^*, and so for $x \in G^*$ we may in analogy with reference (1) define

$$g'(x) = \max\{d(x, v) : v \in V^*\}. \tag{6}$$

A point $x^* \in G^*$ which minimizes $g'(x)$ will be called an AC of G^*. Determining an AC of G^* is an 'addend-free' minimax location problem; it will be shown below to be equivalent to the problem (with addends) of determining an AC of G.

THEOREM. *Let x^* be an* AC *of G^*. If $x^* \in G$, then x^* is an* AC *of G and $L(x^*) = g'(x^*)$. If $x^* \in G^* - G$, specifically if $x^* \in e_i' - G$, then v_i is a dominant vertex of G and hence the unique* AC *of G.*

Proof. For the first part of the proof, consider any $x \in G$. Since each $a(v_i) \geqq 0$, we have

$$\begin{aligned} g'(x) &= \max\{\max_i d(x, v_i), \max_i d(x, v_i')\} \\ &= \max\{\max_i d(x, v_i), \max_i [d(x, v_i) + a(v_i)]\} \\ &= \max_i [d(x, v_i) + a(v_i)] = L(x). \end{aligned} \tag{7}$$

In particular, if $x^* \in G$ then for any $x \in G$

$$L(x) = g(x) \geqq g(x^*) = L(x^*), \tag{8}$$

which shows that x^* is an AC of G.

For the second part of the proof, assume $x^* \epsilon e_i' - G$. If G consists of a single vertex, the theorem is certainly true; hence assume G has at least two vertices, and let

$$c_i = \max_{j \neq i}\{d(v_i, v_j) + a(v_j)\}. \tag{9}$$

For any $x \epsilon e_i'$, with $d(v_i', x) = s$ where $0 \leqq s \leqq a(v_i)$, we have

$$g'(x) = \max\{s, a(v_i) - s + c_i\}, \tag{10}$$

and thus

$$g'(x^*) = \min\{\max[s, a(v_i) - s + c_i]:0 \leqq s \leqq a(v_i)\}. \tag{11}$$

If $c_i > a(v_i)$, the minimum occurs uniquely for $s = a(v_i)$, corresponding to $x^* = v_i$ and contradicting the assumption $x^* \epsilon e_i' - G$. Hence $a(v_i) \geqq c_i$, implying that v_i is a dominant vertex of G and completing the proof.

INTERPRETATION

The theorem actually states that, from the point of view of finding the AC, though G^* (addend-free) is a graph larger than G (with addends), they have the same topological structure. Thus, the addend $a(v)$ associated with vertex v is equivalent to a new edge extending from v to a new vertex v' with edge length $d(v, v') = a(v)$. In particular, if G is acyclic (i.e., a tree), then the same is true for G^*, so that Handler's method[2] is available.

Indeed, in order to find the AC of G by this method, we don't need formally to construct the graph G^*. We may first check the maximal vertex addend $a(v')$ to see whether v' is a dominant vertex. If it is, stop; v' is the AC. Otherwise, we choose any point $x \epsilon G$ (choosing a vertex is advised) and find a vertex $v_1 \epsilon V$ such that $\bar{d}(x, v_1) = L(x)$, then find a vertex $v_2 \epsilon V$ such that

$$\begin{aligned} H(v_1, v_2) &= a(v_1) + d(v_1, v_2) + a(v_2) \\ &= a(v_1) + \bar{d}(v_1, v_2) = a(v_1) + L(v_1). \end{aligned} \tag{12}$$

Then the AC is on the path from v_1 to v_2 at the distance of $[a(v_2) - a(v_1) + d(v_1, v_2)]/2$ from v_1. Note that this quantity is very similar in form to the d^* defined in reference 3, and essentially is obtained by subtracting $a(v_1)$ from $H(v_1, v_2)/2$, i.e., from $d(v_1', v_2')/2$.

Both Goldman[1] and Halfin's[3] method reduce the graph by converting edges into vertices with addends, while, in order to employ Handler's method,[2] we enlarge the graph by converting vertices with addends into new edges. Thus, a comparison of their efficiency becomes desirable. The core problem of Handler's algorithm (and hence ours) is to find the longest distance $L(x)$ from a point x on the tree. To do this, we need to find the path length $d(x, v)$ from x to any vertex v, and hence the path length $d(v', v)$, where v' is any of the two nodes of the edge containing x. Given the connection matrix and the length of each edge, though the existing algorithm—e.g., Dijkstra's procedure[4]—can be suitably simplified for the tree case, the operations included are not too simple compared with Halfin's. So that, despite the present development's ease of understanding, it is not necessarily more efficient than the graph-reduction approach.

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