Appendix

In all proofs that follow, we use $\phi(\cdot)$ and $\Phi(\cdot)$ to denote the standard Normal probability density function and cumulative distribution function respectively, with $\bar{\Phi}(\cdot) = 1 - \Phi(\cdot)$.

In the second period, observing n_1 reviews posted by previous product adopters, customers update their quality perceptions from prior belief $\hat{q}_0$ to posterior belief $\hat{q}_1$. $\hat{q}_1$ is normally distributed with a mean of q_1. According to DeGroot (2004), we have:

$$q_1 = \frac{\sigma_\mu^2}{n_1{\sigma_0}^2 + {\sigma_\mu}^2} q_0 + \frac{n_1{\sigma_0}^2}{n_1{\sigma_0}^2 + {\sigma_\mu}^2} R = \frac{1}{n_1\gamma + 1} q_0 + \frac{n_1\gamma}{n_1\gamma + 1} R,$$

where $q_0 = 0$ is the mean of prior belief, and $\gamma = {\sigma_0}^2/{\sigma_\mu}^2$. In the first period, q_1, the mean of the posterior belief is viewed as a random variable, as it depends on the unobservable realization of product quality μ and the noise in posted reviews. Specifically, if the product's quality realization is μ, the sample mean of n_1 reviews, R, follows $R \sim N(\mu, {\sigma_\mu}^2/n_1)$ so that $q_1|\mu \sim N(\frac{n_1\gamma}{n_1\gamma+1}\mu, (\frac{n_1\gamma}{n_1\gamma+1})^2 \frac{{\sigma_\mu}^2}{n_1})$. Therefore, since μ is an ex ante normal random variable, $\mu \sim N(q_0, {\sigma_0}^2)$, and we have:

$$E(q_1) = E(E[q_1|\mu]) = E(\frac{n_1\gamma}{n_1\gamma + 1}\mu) = q_0 = 0,$$

and

$$\begin{aligned} Var(q_1) &= E(Var[q_1|\mu]) + Var(E[q_1|\mu]) \\ &= (\frac{n_1\gamma}{n_1\gamma+1})^2 (\frac{{\sigma_\mu}^2}{n_1} + {\sigma_0}^2) = (\frac{n_1\gamma}{n_1\gamma+1})^2 \frac{{\sigma_\mu}^2(n_1\gamma+1)}{n_1} \\ &= \frac{n_1\gamma}{n_1\gamma+1}{\sigma_0}^2 \end{aligned}$$

Proof of Lemma 1

(i) Given any arbitrary price plan $\{p_1, p_2\}$, customers make purchasing decisions comparing the first-period profit and the second-period profit, which follows a threshold policy. Denote $\tau_{bp}(p_1, p_2)$ to be the threshold value. Derived from $\bar{v} - p_1 = \delta(\bar{v} + \rho(1 - \bar{v}) - p_2)$, we have: $\bar{v} = \frac{p_1 - \delta p_2 + \delta\rho}{1-\delta(1-\rho)}$. We then discuss several cases: when $p_1 < \bar{v} \leq 1$, that is $(1-p_1)\rho > p_2 - p_1$ and $(1-p_2)\delta \leq 1-p_1$, we have $\tau_{bp} = \frac{p_1 - \delta p_2 + \delta\rho}{1-\delta(1-\rho)}$; when $\bar{v} < p_1$ and $\bar{v} < 1$, a customer with preference higher than p_1 purchases the product in the first period, and the threshold τ_{bp} is denoted by p_1; when $\bar{v} > 1$, no sales occur in the first period, and the threshold value τ_{bp} is denoted by 1. Then in summary, we have:

$$\tau_{bp}(p_1, p_2) = \begin{cases} p_1 & (1-p_1)\rho \leq p_2 - p_1 \ and \ (1-p_2)\delta \leq 1 - p_1 \\ \frac{p_1 - \delta p_2 + \delta\rho}{1-\delta(1-\rho)} & (1-p_1)\rho > p_2 - p_1 \ and \ (1-p_2)\delta \leq 1 - p_1 \\ 1 & (1-p_2)\delta > 1 - p_1 \end{cases}$$

(ii) Customers with preference lower than $\tau_{bp}(p_1, p_2)$ choose to remain in the second period. Furthermore, customers purchase the product given nonnegative utilities, that is, $v + \rho(1 - \tau_{bp}(p_1, p_2)) - p_2 \geq 0$, thus we have $p_2 - \rho(1 - \tau_{bp}(p_1, p_2)) \leq v \leq \tau_{bp}(p_1, p_2)$.

Proof of Proposition 1

(i) According to the expression of $\tau_{bp}(p_1, p_2)$ in Lemma 1 and the firm's objective $\pi_{bp}(p_1, p_2) = p_1[1 - \tau_{bp}(p_1, p_2)]^+ + \lambda p_2[\tau_{bp}(p_1, p_2) + \rho(1 - \tau_{bp}(p_1, p_2)) - p_2]^+$, we discuss the following cases:

(1) When the threshold $\tau_{bp}(p_1, p_2)$ is $\frac{p_1 - \delta p_2 + \delta\rho}{1-\delta(1-\rho)}$, that is, the first-period market is partially covered. Moreover, the second-period market is not fully covered. Then the film's optimization is expressed by

$$\max_{p_1, p_2} \quad \pi_{bp(1)}(p_1, p_2) = p_1\left(1 - \frac{p_1 - \delta p_2 + \delta\rho}{1-\delta(1-\rho)}\right) + \lambda p_2\left(\frac{p_1 - \delta p_2 + \delta\rho}{1-\delta(1-\rho)} + \rho\left(1 - \frac{p_1 - \delta p_2 + \delta\rho}{1-\delta(1-\rho)}\right) - p_2\right)$$

$$\text{subject to} \begin{cases} p_1 \leq \frac{p_1 - \delta p_2 + \delta\rho}{1-\delta(1-\rho)} \\ 0 \leq \frac{p_1 - \delta p_2 + \delta\rho}{1-\delta(1-\rho)} + \rho\left(1 - \frac{p_1 - \delta p_2 + \delta\rho}{1-\delta(1-\rho)}\right) - p_2 \leq \frac{p_1 - \delta p_2 + \delta\rho}{1-\delta(1-\rho)} & \text{(from Lemma 1)} \\ \frac{p_1 - \delta p_2 + \delta\rho}{1-\delta(1-\rho)} \leq 1 \\ p_1, p_2 \geq 0 \end{cases}$$

The derivations of $\pi_{bp(1)}(p_1,p_2)$ with respect to p_1 and p_2 are given by

$$\frac{\partial \pi_{bp(1)}}{\partial p_1}=\frac{1-2p_1+\delta(-1+p_2)+\lambda(p_2-\rho p_2)}{1-\delta(1-\rho)}$$

and

$$\frac{\partial \pi_{bp(1)}}{\partial p_2}=\frac{\delta p_1+\lambda(\rho+p_1-\rho p_1-2p_2)}{1-\delta(1-\rho)}.$$

Let $\frac{\partial \pi_{bp(1)}}{\partial p_1}=0$ and $\frac{\partial \pi_{bp(1)}}{\partial p_2}=0$, we then get the interior solution: $p_1^*=\frac{\lambda(-2-\delta(-2+\rho)+\lambda\rho(-1+\rho))}{\delta^2-4\lambda+2\delta\lambda+\lambda^2-2\delta\rho\lambda-2\lambda^2\rho+\lambda^2\rho^2}$, $p_2^*=\frac{\delta(\delta-1)+\lambda(\delta-\rho-\delta\rho-1)}{\delta^2-4\lambda+2\delta\lambda+\lambda^2-2\delta\rho\lambda-2\lambda^2\rho+\lambda^2\rho^2}$ and $\pi_{bp(1)}=-\frac{\lambda(1-\delta+\lambda\rho)}{\delta^2-4\lambda+2\delta\lambda+\lambda^2-2\delta\rho\lambda-2\lambda^2\rho+\lambda^2\rho^2}$. Then we check constraints:

1) $p_1=\frac{\lambda(-2-\delta(-2+\rho)+\lambda\rho(-1+\rho))}{\delta^2-4\lambda+2\delta\lambda+\lambda^2-2\delta\rho\lambda-2\lambda^2\rho+\lambda^2\rho^2}\leq\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}=\frac{\delta^2+\delta(\lambda-2\lambda\rho)+\lambda(-2+\lambda(-1+\rho)\rho)}{\delta^2-4\lambda+2\delta\lambda+\lambda^2-2\delta\rho\lambda-2\lambda^2\rho+\lambda^2\rho^2}$;

2) $0\leq\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}+\rho(1-\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)})-p_2=\frac{\delta-\lambda(1+\rho)}{\delta^2-4\lambda+2\delta\lambda+\lambda^2-2\delta\rho\lambda-2\lambda^2\rho+\lambda^2\rho^2}\leq\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}=\frac{\delta^2+\delta(\lambda-2\lambda\rho)+\lambda(-2+\lambda(-1+\rho)\rho)}{\delta^2-4\lambda+2\delta\lambda+\lambda^2-2\delta\rho\lambda-2\lambda^2\rho+\lambda^2\rho^2}$;

3) $\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}=\frac{\delta^2+\delta(\lambda-2\lambda\rho)+\lambda(-2+\lambda(-1+\rho)\rho)}{\delta^2-4\lambda+2\delta\lambda+\lambda^2-2\delta\rho\lambda-2\lambda^2\rho+\lambda^2\rho^2}\leq 1$;

4) $p_1=\frac{\lambda(-2-\delta(-2+\rho)+\lambda\rho(-1+\rho))}{\delta^2-4\lambda+2\delta\lambda+\lambda^2-2\delta\rho\lambda-2\lambda^2\rho+\lambda^2\rho^2}\geq 0$;

5) $p_2=\frac{\delta(\delta-1)+\lambda(\delta-\rho-\delta\rho-1)}{\delta^2-4\lambda+2\delta\lambda+\lambda^2-2\delta\rho\lambda-2\lambda^2\rho+\lambda^2\rho^2}\geq 0$.

Accordingly, constraints hold if and only if (i) $\rho\leq 1+\delta$ and $\frac{\delta}{1+\rho}\leq\lambda\leq 1$; or (ii) $\rho>1+\delta$ and $\frac{\delta}{1+\rho}\leq\lambda\leq\frac{\delta}{-1+\rho}$, then we discuss the case when (i) $0\leq\lambda<\frac{\delta}{1+\rho}$; or (ii) $\rho>1+\delta$ and $\frac{\delta}{-1+\rho}<\lambda\leq 1$ as follows.

i) If $p_1^*=0$, let $\frac{\partial \pi_{bp(1)}(p_1,p_2)}{\partial p_2}=0$, then we have $p_2^*=\frac{\rho}{2}$.

Then we check constraints:

1) $\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}-p_1=\frac{\delta(1+\rho^2)}{2-2\delta(1-\rho)}\geq 0\Rightarrow p_1\leq\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}$ always holds;

2) $0\leq\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}+\rho(1-\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)})-p_2=\frac{\rho}{2-2\delta(1-\rho)}\leq\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}=\frac{\delta\rho}{2-2\delta(1-\rho)}\Rightarrow\delta=1$ or $\rho=0$ (excluded because $\rho>0$);

3) $1-\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}=\frac{2-2\delta+\delta\rho}{2-2\delta(1-\rho)}\geq 0\Rightarrow\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}\leq 1$ always holds;

4) $p_2=\frac{\rho}{2}>0$.

Accordingly, $p_1^*=0$ and $p_2^*=\frac{\rho}{2}$ are valid if and only if $\delta=1$, where $\pi_{bp(1)}=\frac{\lambda\rho}{4}$. Then if $\delta\neq 1$ we discuss boundary solutions as follows.

1) If $p_1^*=\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}$, we have $p_2^*=\rho$ and $\pi_{bp(1)}^*=0$.

2) If $\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}+\rho(1-\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)})-p_2=0$, we can have $p_2^*=\rho$ and $\pi_{bp(1)}^*=0\cdot\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}+\lambda p_2\cdot 0=0$.

3) If $\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}+\rho(1-\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)})-p_2=\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}$, we can have $p_2^*=\rho$ and thus $\pi_{bp(1)}^*=0$.

4) If $\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}=1$, we can have $p_2^*=\frac{\delta-1}{\delta}$ and thus $\pi_{bp(1)}^*=\frac{\lambda(\delta-1)}{\delta^2}<0$.

Thus in the case of $p_1^*=0$, we have

$$\begin{cases} p_2^*=\frac{\rho}{2},\ \pi_{bp(1)}^*=\frac{\lambda\rho}{4} \quad if\ \delta=1 \\ p_2^*=\rho,\ \pi_{bp(1)}^*=0 \quad if\ \delta\neq 1. \end{cases}$$

ii) If $p_2^*=0$, let $\frac{\partial\pi_{bp(1)}(p_1,p_2)}{\partial p_1}=0$, we have $p_1^*=\frac{1-\delta}{2}$, then we check constraints:

1) $\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}-p_1=\frac{\delta(1+\delta(-1+\rho)+\rho)}{2-2\delta(1-\rho)}\geq 0\Rightarrow p_1\leq\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}$ always holds;

2) $0\leq\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}+\rho(1-\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)})-p_2=\frac{1+\delta(-1+\rho)+\rho}{2-2\delta(1-\rho)}$ always holds, $\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}+\rho(1-\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)})-p_2>\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}$, the constraint $\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}+\rho(1-\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)})-p_2\leq\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}$ can not be satisfied;

Accordingly, there exists no feasible interior solution. Then we discuss boundary solutions as follows.

1) If $p_1^*=\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}$, we have $p_1^*=\frac{\rho}{\rho-1}$ and thus $\pi_{bp(1)}^*=-\frac{\rho}{(\rho-1)^2}<0$.

2) If $\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}+\rho(1-\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)})-p_2=0$, we can have $p_1^*=\frac{\rho}{\rho-1}$ and thus $\pi_{bp(1)}^*=-\frac{\rho}{(\rho-1)^2}<0$.

3) If $\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}+\rho(1-\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)})-p_2=\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}$, we can have $p_1^*=1-\delta$ and thus $\pi_{bp(1)}^*=0$.

4) If $\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}=1$, we have $p_1^*=1-\delta$ and $\pi_{bp(1)}^*=0$.

Thus in the case of $p_2^*=0$, we have $p_1^*=1-\delta$ and $\pi_{bp(1)}^*=0$.

iii) If $\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}=1$, then we have $p_1-\delta p_2+\delta-1=0$. Constructing the Lagrangian, we have

$$\begin{aligned}\pi_{bp(1)}(p_1,p_2)=&p_1(1-\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)})+\lambda p_2(\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}+\rho(1-\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)})-p_2)\\&+a(p_1-\delta p_2+\delta-1)\end{aligned}$$

The derivatives of $\pi_{bp(1)}$ with respect to a, p_1 and p_2 are given by

$$\frac{\partial\pi_{bp(1)}}{\partial a}=p_1-\delta p_2+\delta-1=0$$

$$\frac{\partial\pi_{bp(1)}}{\partial p_1}=\frac{1+a+a\delta(-1+\rho)-2p_1+\delta(-1+p_2)+\lambda(p_2-\rho p_2)}{1-\delta(1-\rho)}=0$$

$$\frac{\partial\pi_{bp(1)}}{\partial p_2}=\frac{-a\delta(1+\delta(-1+\rho))+\delta p_1+\lambda(\rho+p_1-\rho p_1-2p_2)}{1-\delta(1-\rho)}=0$$

Thus we have $p_1^*=1-\frac{\delta}{2}$, $p_2^*=\frac{1}{2}$ and $\pi_{bp(1)}^*=\frac{\lambda}{4}$. Then we check constraints:

1) $p_1=1-\frac{\delta}{2}\leq\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}=1$ and thus the constraint $p_1\leq\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}$ always holds;

2) $0\leq\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}+\rho(1-\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)})-p_2=\frac{1}{2}<\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}=1$ and thus the constraint $0\leq\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}+\rho(1-\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)})-p_2\leq\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}$ always holds;

3) $p_1=1-\frac{\delta}{2}\geq 0$;

4) $p_2=\frac{1}{2}>0$.

Accordingly, the above solution is valid satisfying all constraints. So $p_1^*=1-\frac{\delta}{2}$, $p_2^*=\frac{1}{2}$ and then $\pi_{bp(1)}^*=\frac{\lambda}{4}$.

iv) If $\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}=p_1$,we can obtain $(1-\rho)p_1-p_2+\rho=0$. Constructing the Lagrangian, we have

$$\begin{aligned}\pi_{bp(1)}(p_1,p_2)=&p_1(1-\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)})+\lambda p_2(\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}+\rho(1-\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)})-p_2)\\&+b((1-\rho)p_1-p_2+\rho)\end{aligned}$$

The derivatives of $\pi_{bp(1)}$ with respect to b, p_1 and p_2 are given by

$$\frac{\partial\pi_{bp(1)}}{\partial b}=(1-\rho)p_1-p_2+\rho=0$$

$$\frac{\partial \pi_{bp(1)}}{\partial p_1}=\frac{1-b(1+\delta(-1+\rho))(-1+\rho)-2p_1+\delta(-1+p_2)+\lambda(p_2-\rho p_2)}{1-\delta(1-\rho)}=0$$

$$\frac{\partial \pi_{bp(1)}}{\partial p_2}=\frac{\delta p_1+b(-1+\delta-\delta\rho)+\lambda(\rho+p_1(1-\rho)-2p_2)}{1-\delta(1-\rho)}=0$$

Thus we have $p_1^*=\frac{1}{2}$, $p_2^*=\frac{1+\rho}{2}$, and $\pi_{bp(1)}^*=\frac{1}{4}$. Then we check constraints:

1) $0=\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}+\rho(1-\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)})-p_2<\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}=\frac{1}{2}$ and thus the constraint $0\leq \frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}+\rho(1-\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)})-p_2\leq\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}$ always holds;

2) $\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}=\frac{1}{2}<1$ and thus the constraint $\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}\leq 1$ always holds;

3) $p_1=\frac{1}{2}>0$;

4) $p_2=\frac{1+\rho}{2}>0$.

Accordingly, the above solution is valid satisfying all constraints. Thus $p_1^*=\frac{1}{2}$, $p_2^*=\frac{1+\rho}{2}$, and $\pi_{bp(1)}^*=\frac{1}{4}$.

v) If $\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}+\rho(1-\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)})-p_2=0$, we have $(1-\rho)p_1-p_2+\rho=0$, and we can obtain $p_2=(1-\rho)p_1+\rho$. Thus the profit of the firm is expressed by $\pi_{bp(1)}(p_1,p_2)=p_1(1-\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)})$. Substituting $p_2=(1-\rho)p_1+\rho$ into $\pi_{bp(1)}(p_1,p_2)$, we have $\pi_{bp(1)}=p_1(1-p_1)$ and we can get the solution: $p_1^*=\frac{1}{2}$, $p_2^*=\frac{1+\rho}{2}$, and $\pi_{bp(1)}^*=\frac{1}{4}$. Then we check constraints:

1) $p_1=\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}=\frac{1}{2}$ and thus the constraint $p_1\leq\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}$ always holds;

2) $\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}+\rho(1-\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)})-p_2<\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}=\frac{1}{2}$;

3) $\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}=\frac{1}{2}<1$ and thus the constraint $\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}\leq 1$ always holds;

4) $p_1=\frac{1}{2}>0$;

5) $p_2=\frac{1+\rho}{2}>0$.

Accordingly, the above solution is valid satisfying all constraints. Thus $p_1^*=\frac{1}{2}$, $p_2^*=\frac{1+\rho}{2}$, and $\pi_{bp(1)}^*=\frac{1}{4}$.

vi) If $\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}+\rho(1-\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)})-p_2=\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}$, we can obtain $-\rho(-1+\delta+p_1)+(-1+\delta)p_2=0$. Constructing the Lagrangian, we have

$$\begin{aligned}\pi_{bp(1)}(p_1,p_2)=&p_1(1-\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)})+\lambda p_2(\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}+\rho(1-\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)})-p_2)\\&+c(-\rho(-1+\delta+p_1)+(-1+\delta)p_2)\end{aligned}$$

The derivatives of $\pi_{bp(1)}$ with respect to c, p_1 and p_2 are given by

$$\frac{\partial\pi_{bp(1)}}{\partial c}=-\rho(-1+\delta+p_1)+(-1+\delta)p_2=0$$

$$\frac{\partial\pi_{bp(1)}}{\partial p_1}=\frac{1-c\rho-2p_1+\lambda(p_2-\rho p_2)+\delta(-1+c\rho(1-\rho)+p_2)}{1-\delta(1-\rho)}=0$$

$$\frac{\partial\pi_{bp(1)}}{\partial p_2}=\frac{c(-1+\delta)(1+\delta(-1+\rho))+\delta p_1+\lambda(\rho+(1-\rho)p_1-2p_2)}{1-\delta(1-\rho)}=0$$

Thus we have $p_1^*=\frac{1-\delta}{2}$, $p_2^*=\frac{\rho}{2}$, and $\pi_{bp(1)}^*=\frac{1-\delta+\lambda\rho}{4}$. Then we check constraints:

1) $p_1=\frac{1-\delta}{2}<\frac{1}{2}=\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}$ and thus the constraint $p_1\le\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}$ always holds;

2) $0<\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}+\rho(1-\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)})-p_2=\frac{1}{2}$;

3) $p_1=\frac{1-\delta}{2}\ge 0$;

4) $p_2=\frac{\rho}{2}>0$.

Accordingly, the above solution is valid satisfying all constraints. So $p_1^*=\frac{1-\delta}{2}$, $p_2^*=\frac{\rho}{2}$, and $\pi_{bp(1)}^*=\frac{1-\delta+\lambda\rho}{4}$.

When $\rho>1+\delta$ and $\frac{\delta}{-1+\rho}<\lambda\le 1$, we have $\frac{1-\delta+\lambda\rho}{4}>\frac{1}{4}$ and $\frac{1-\delta+\lambda\rho}{4}\ge\frac{\lambda\rho}{4}$. When $\lambda<\frac{\delta}{1+\rho}$, we have $\frac{1}{4}>\frac{1-\delta+\lambda\rho}{4}$ and $\frac{1}{4}>\frac{\lambda\rho}{4}$. Then in summary, in the case (1) we have

$$\begin{cases}p_1^*=\frac{\lambda(-2-\delta(-2+\rho)+\lambda\rho(-1+\rho))}{\delta^2-4\lambda+2\delta\lambda+\lambda^2-2\delta\rho\lambda-2\lambda^2\rho+\lambda^2\rho^2},\ p_2^*=\frac{\delta(\delta-1)+\lambda(\delta-\rho-\delta\rho-1)}{\delta^2-4\lambda+2\delta\lambda+\lambda^2-2\delta\rho\lambda-2\lambda^2\rho+\lambda^2\rho^2},\\ \qquad\pi_{bp(1)}^*=-\frac{\lambda(1-\delta+\lambda\rho)}{\delta^2-4\lambda+2\delta\lambda+\lambda^2-2\delta\rho\lambda-2\lambda^2\rho+\lambda^2\rho^2},\ if\ (i)\ \rho\le 1+\delta\ and\ \frac{\delta}{1+\rho}\le\lambda\le 1;\\ \qquad or\ (ii)\rho>1+\delta\ and\ \frac{\delta}{1+\rho}\le\lambda\le\frac{\delta}{-1+\rho}\\ p_1^*=\frac{1-\delta}{2},\ p_2^*=\frac{\rho}{2},\ \pi_{bp(1)}^*=\frac{1-\delta+\lambda\rho}{4},\ if\ \rho>1+\delta\ and\ \frac{\delta}{-1+\rho}<\lambda\le 1\\ p_1^*=\frac{1}{2},\ p_2^*=\frac{1+\rho}{2},\ \pi_{bp(1)}^*=\frac{1}{4},\ if\ 0\le\lambda<\frac{\delta}{1+\rho}.\end{cases}$$

(2) When the threshold $\tau_{bp}(p_1,p_2)$ is $\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}$, that is, the first-period market is partially covered. Moreover, the second-period market is fully covered. Then the film's optimization is expressed by

$$\max_{p_1,p_2} \quad \pi_{bp(2)}(p_1,p_2)=p_1(1-\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)})+\lambda p_2\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}$$

$$\text{subject to}\begin{cases} p_1\le\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)} \\ \frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}+\rho(1-\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)})-p_2\ge\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)} \\ \frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}\le 1 \\ p_1,p_2\ge 0 \end{cases} \text{(from Lemma 1)}$$

The derivations of $\pi_{bp(2)}(p_1,p_2)$ with respect to p_1 and p_2 are given by

$$\frac{\partial\pi_{bp(2)}}{\partial p_1}=\frac{1-2p_1+\delta(-1+p_2)+\lambda p_2}{1+\delta(-1+\rho)}$$

$$\frac{\partial\pi_{bp(2)}}{\partial p_2}=\frac{\lambda p_1+\delta(p_1+\lambda(\rho-2p_2))}{1+\delta(-1+\rho)}$$

Let $\frac{\partial\pi_{bp(2)}}{\partial p_1}=0$ and $\frac{\partial\pi_{bp(2)}}{\partial p_2}=0$, we then get the interior solution: $p_1^*=-\frac{\lambda\delta(2-2\delta+\rho+\delta\rho)}{(\lambda-\delta)^2}$, $p_2^*=\frac{\delta^2-\lambda+\delta(-1+\lambda-2\lambda\rho)}{(\lambda-\delta)^2}$. But $p_1=-\frac{\lambda\delta(2-2\delta+\rho+\delta\rho)}{(\lambda-\delta)^2}<0$ contradicts with the constraint $p_1\ge 0$.

Thus there exists no feasible interior solution. Then we consider boundary solutions:

i) If $p_1^*=0$, let $\frac{\partial\pi_{bp}(p_1,p_2)}{\partial p_2}=0$, we have $p_2^*=\frac{\rho}{2}$ and $\pi_{bp(2)}^*=\frac{\lambda\delta\rho^2}{4+4\delta(-1+\rho)}$. Then we check constraints:

1) $p_1=0<\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}=\frac{\delta\rho}{2-2\delta(1-\rho)}$ and thus the constraint $p_1\le\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}$ holds;

2) $\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}+\rho(1-\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)})-p_2=\frac{\rho}{2-2\delta(1-\rho)}\ge\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}=\frac{\delta\rho}{2-2\delta(1-\rho)}$;

3) $\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}=\frac{\delta\rho}{2-2\delta(1-\rho)}<1$;

4) $p_2=\frac{\rho}{2}>0$.

Accordingly, the above solution is valid satisfying all constraints. So $p_1^*=0$, $p_2^*=\frac{\rho}{2}$ and $\pi_{bp(2)}^*=\frac{\lambda\delta\rho^2}{4+4\delta(-1+\rho)}$.

ii) If $p_2^* = 0$, let $\frac{\partial \pi_{bp}(p_1,p_2)}{\partial p_1} = 0$, we have $p_1^* = \frac{1-\rho}{2}$ and $\pi_{bp(2)}^* = \frac{(1-\rho)(1-2\delta+\rho)}{4+4\delta(-1+\rho)}$. Then we check constraints:

1) $p_1 = \frac{1-\rho}{2} \leq \frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)} = \frac{1+\rho(-1+2\delta)}{2-2\delta(1-\rho)}$;

2) $\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)} + \rho(1 - \frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}) - p_2 = \frac{1+\rho^2}{2-2\delta(1-\rho)} \geq \frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)} = \frac{1+\rho(-1+2\delta)}{2-2\delta(1-\rho)} \Rightarrow \rho \geq 2\delta - 1$;

3) $\frac{1-\rho}{2} \leq \frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)} = \frac{1+\rho(-1+2\delta)}{2-2\delta(1-\rho)} \leq 1 \Rightarrow \rho \geq 2\delta - 1$;

4) $p_1 = \frac{1-\rho}{2} \geq 0$ when $\rho \leq 1$.

Accordingly, $p_1^* = \frac{1-\rho}{2}$ and $p_2^* = 0$ are valid if and only if $2\delta - 1 \leq \rho \leq 1$ where $\pi_{bp(2)}^* = \frac{(1-\rho)(1-2\delta+\rho)}{4+4\delta(-1+\rho)}$. If $\rho > 1$ or $0 < \rho < 2\delta - 1$, we discuss boundary solutions as follows.

1) If $p_1^* = 0$, $\pi_{bp(2)}^* = 0$;

2) If $p_1 = \frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}$, we have $p_1^* = \frac{\rho}{\rho-1}$ and $\pi_{bp(2)}^* = -\frac{\rho}{(\rho-1)^2} < 0$.

3) If $\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)} + \rho(1 - \frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}) - p_2 = \frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}$, we have $p_1^* = 1 - \delta$ and $\pi_{bp(2)}^* = 0$.

4) If $\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)} = 1$, we have $\pi_{bp(2)}^* = p_1 * 0 + 0 * \frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)} = 0$.

Thus in the case of $p_2^* = 0$, we have

$$\begin{cases} p_1^* = 0 \ or \ 1-\delta, \ \pi_{bp(2)}^* = 0 \quad if \ \rho > 1 \ or \ 0 < \rho < 2\delta - 1 \\ p_1^* = \frac{1-\rho}{2}, \ \pi_{bp(2)}^* = \frac{\lambda\delta\rho^2}{4+4\delta(-1+\rho)} \quad if \ 2\delta - 1 \leq \rho \leq 1 \end{cases}$$

iii) If $\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)} = p_1$, then we obtain $(1-\rho)p_1 - p_2 + \rho = 0$. Constructing the Lagrangian, we have

$$\pi_{bp(2)}(p_1,p_2) = p_1(1 - \frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}) + \lambda p_2 \frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)} + a((1-\rho)p_1 - p_2 + \rho)$$

The derivatives of $\pi_{bp(2)}(p_1,p_2)$ with respect to a, p_1 and p_2 are given by

$$\frac{\partial \pi_{bp(2)}}{\partial a} = (1-\rho)p_1 - p_2 + \rho = 0$$

$$\frac{\partial \pi_{bp(2)}}{\partial p_1} = \frac{1 - a(1+\delta(-1+\rho))(-1+\rho) - 2p_1 + \delta(-1+p_2) + \lambda p_2}{1-\delta(1-\rho)} = 0$$

$$\frac{\partial \pi_{bp(2)}}{\partial p_2} = \frac{a(-1+\delta-\delta\rho) + \lambda p_1 + \delta(\lambda\rho + p_1 - 2\lambda p_2)}{1-\delta(1-\rho)} = 0$$

Thus we have $p_1^* = \frac{1+\rho}{2\rho}$ and $p_2^* = \frac{1+\rho^2}{2\rho}$, and then $\pi_{bp(2)}^* = \frac{(1+\rho)(-1+\lambda+\rho+\lambda\rho^2)}{4\rho^2}$. Then we check constraints:

1) $p_1 = \frac{1+\rho}{2\rho} = \frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)} = \frac{1+\rho}{2\rho}$ always holds;

2) $\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)} + \rho(1-\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}) - p_2 = 0 < \frac{1+\rho}{2\rho} = \frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}$, and thus the constraint $\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)} + \rho(1-\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}) - p_2 \geq \frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}$ doesn't hold;

Accordingly, the above solution can not satisfy all constraints and thus there exists no feasible interior solution. Then we discuss boundary solutions as follows.

1) If $p_1^* = 0$, we have $p_2^* = \rho$ and $\pi_{bp(2)}^* = 0\cdot(1-0) + p_2 \cdot 0 = 0$.

2) If $p_2^* = 0$, we have $p_1^* = \frac{\rho}{\rho-1}$ and $\pi_{bp(2)}^* = -\frac{\rho}{(\rho-1)^2} < 0$.

3) If $\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)} + \rho(1-\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}) - p_2 = \frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}$, we can have: $p_1^* = 0$, $p_2^* = \rho$. Thus $\pi_{bp(2)}^* = 0$.

4) If $\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)} = 1$, according to $\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)} + \rho(1-\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}) - p_2 \geq \frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}$, we get $p_2^* = 0$, and thus $p_1^* = 1$ where $\pi_{bp(2)}^* = p_1 \cdot (1-1) + 0 = 0$.

Thus in the case of $\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)} = p_1$, we have a) $p_1^* = 1$ and $p_2^* = 0$, or b) $p_1^* = 0$ and $p_2^* = \rho$, where $\pi_{bp(2)}^* = 0$.

iv) If $\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)} = 1$, then we can obtain $p_1 - \delta p_2 + \delta - 1 = 0$. Constructing the Lagrangian, we have

$$\pi_{bp(2)}(p_1,p_2) = p_1(1-\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}) + \lambda p_2 \frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)} + b(p_1-\delta p_2+\delta-1)$$

The derivatives of $\pi_{bp(2)}$ with respect to b, p_1 and p_2 are given by

$$\frac{\partial\pi_{bp(2)}}{\partial b} = p_1 - \delta p_2 + \delta - 1 = 0$$

$$\frac{\partial\pi_{bp(2)}}{\partial p_1} = \frac{-1+b+b\delta(-1+\rho)+\lambda p_2}{1-\delta(1-\rho)} = 0$$

$$\frac{\partial\pi_{bp(2)}}{\partial p_2} = \frac{\delta^2(b-b\rho)+\lambda p_1+\delta(1-b+\lambda\rho-2\lambda p_2)}{1-\delta(1-\rho)} = 0$$

According to the above group of equation, we have $p_1^*=1$, $p_2^*=1$ only when $\lambda=0$. We check the constraints:

1) $p_1=1=\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}$ always holds;

2) $\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}+\rho(1-\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)})-p_2=-1<0$ and thus the constraint $\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}+\rho(1-\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)})-p_2\geq 0$ does not hold.

Accordingly, the above solution can not satisfy all constraints and thus there exists no feasible interior solution. Then we consider about boundary cases.

1) If $p_1^*=0$, we have $p_2^*=\frac{-1+\delta}{\delta}$ and thus $\pi_{bp(2)}^*=\frac{-\lambda+\lambda\delta}{\delta}<0$.

2) If $p_2^*=0$, we have $p_1^*=1-\delta$, and thus $\pi_{bp(2)}^*=p_1(1-1)+0=0$.

3) If $\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}+\rho(1-\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)})-p_2=\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}$, we have $p_1^*=1-\delta$, $p_2^*=0$ and thus $\pi_{bp(2)}^*=p_1(1-1)+0=0$.

4) If $\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}=p_1$, according to $\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}+\rho(1-\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)})-p_2\geq\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}$, we have $p_1^*=1$, $p_2^*=0$, and thus $\pi_{bp(2)}^*$ is no more than 0.

Thus in the case of $\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}=1$, we have $p_1^*=1-\delta$ and $p_2^*=0$, $\pi_{bp(2)}^*=0$.

v) If $\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}+\rho(1-\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)})-p_2=\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}$, then we can obtain $-\rho(-1+\delta+p_1)+(-1+\delta)p_2=0$. Constructing the Lagrangian, we have

$$\pi_{bp(2)}(p_1,p_2)=p_1(1-\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)})+\lambda p_2\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}+c(-\rho(-1+\delta+p_1)+(-1+\delta)p_2)$$

The derivatives of $\pi_{bp(2)}(p_1,p_2)$ with respect to c, p_1 and p_2 are given by

$$\frac{\partial\pi_{bp(2)}}{\partial c}=-\rho(-1+\delta+2p_1)+(-1+\delta)p_2=0$$
$$\frac{\partial\pi_{bp(2)}}{\partial p_1}=\frac{1-c\rho-2p_1+\lambda p_2+\delta(-1-c(-1+\rho)\rho+p_2)}{1-\delta(1-\rho)}=0$$
$$\frac{\partial\pi_{bp(2)}}{\partial p_2}=\frac{c(-1+\delta)(1+\delta(-1+\rho))+\lambda p_1+\delta(\lambda\rho+p_1-2\lambda p_2)}{1-\delta(1-\rho)}=0$$

Thus we have $p_1^*=\frac{1-\delta}{2}$, $p_2^*=\frac{\rho}{2}$ and $\pi_{bp(2)}=\frac{1-\delta+\lambda\rho}{4}$. Then we check constraints:

1) $p_1 = \frac{1-\delta}{2} \leq \frac{1}{2} = \frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}$ and thus the constraint $p_1 \leq \frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}$ holds;

2) $\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)} + \rho(1 - \frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}) - p_2 = \frac{1}{2} = \frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}$, and thus the constraint $\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)} + \rho(1 - \frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}) - p_2 \geq \frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}$ holds;

3) $\frac{1-\delta}{2} \leq \frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)} = \frac{1}{2} < 1$;

4) $p_1 = \frac{1-\delta}{2} \geq 0$;

5) $p_2 = \frac{\rho}{2} > 0$.

Accordingly, the above solution is valid satisfying all constraints. So $p_1^* = \frac{1-\delta}{2}$, $p_2^* = \frac{\rho}{2}$ and $\pi_{bp(2)}^* = \frac{1-\delta+\lambda\rho}{4}$.

In addition, we can have $\frac{1-\delta+\lambda\rho}{4} \geq \frac{\lambda\delta\rho^2}{4+4\delta(-1+\rho)}$ and $\frac{1-\delta+\lambda\rho}{4} \geq \frac{(1-\rho)(1-2\delta+\rho)}{4+4\delta(-1+\rho)}$. Thus in the case (2), we obtain the solution: $p_1^* = \frac{1-\delta}{2}$, $p_2^* = \frac{\rho}{2}$ and $\pi_{bp(2)}^* = \frac{1-\delta+\lambda\rho}{4}$.

(3) When the threshold $\tau_{bp}(p_1, p_2)$ is 1, no sales occur in the first period, the film's optimization is expressed by

$$\max_{p_1,p_2} \quad \pi_{bp(3)}(p_1, p_2) = \lambda p_2(1 - p_2)$$

$$\text{subject to} \begin{cases} \frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)} \geq 1 & \text{(from Lemma 1)} \\ p_1, p_2 \geq 0 \end{cases}$$

According to the expression $\pi_{bp(3)} = p_2(1 - p_2)$, the firm can achieve its optimal profit at $p_2^* = \frac{1}{2}$ and then the profit is $\frac{\lambda}{4}$. Thus in the case (3), we obtain that the optimal profit is $\pi_{bp(3)}^* = \frac{\lambda}{4}$.

(4) When the threshold $\tau_{bp}(p_1, p_2)$ is p_1, that is, customers purchase in the first period given non-negative utility. Moreover, the the second-period market is not fully covered. Then the film's optimization is expressed by

$$\max_{p_1,p_2} \quad \pi_{bp(4)}(p_1, p_2) = p_1(1 - p_1) + \lambda p_2(p_1 + \rho(1 - p_1) - p_2)$$

$$\text{subject to} \begin{cases} \frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)} \le p_1 \le 1 \\ p_1, p_2 \ge 0 \end{cases} \text{(from Lemma 1)}$$

From the derivations of $\pi_{bp(4)}(p_1,p_2)$ with respect to p_1 and p_2, the solution can not satisfy all the above constraints and thus there exists no feasible interior solution. Then we consider boundary solutions as follows:

i) If $p_1^*=0$, then all customers purchase in the first period and thus the optimal profit is $\pi_{bp(4)}^*=0$.

ii) If $p_2^*=0$, then the firm's profit is expressed by $p_1(1-p_1)$, we can get $p_1^*=\frac{1}{2}$ and $\pi_{bp(4)}^*=\frac{1}{4}$, and $p_1-\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}=-\frac{\delta(1+\rho)}{2-2\delta(1-\rho)}\ge 0 \Rightarrow \delta=0$, so this solution is valid only when $\delta=0$. Then we consider about boundary solutions if $\delta\ne 0$.

1) If $p_1^*=0$, we have $\pi_{bp(4)}^*=0$;

2) If $\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}=p_1$, we have $p_1^*=\frac{\rho}{\rho-1}$ and thus $\pi_{bp(4)}^*=-\frac{\rho}{(\rho-1)^2}<0$.

3) If $p_1^*=1$, $\pi_{bp(4)}^*=0$.

Thus in the case of $p_2^*=0$, we have

$$\begin{cases} p_1^*=0 \; or \; 1, \; \pi_{bp(4)}^*=0 \quad if \; \delta\ne 0 \\ p_1^*=\frac{1}{2}, \; \pi_{bp(4)}^*=\frac{1}{4} \; if \; \delta=0 \end{cases}$$

iii) If $p_1^*=1$, the firm's profit is expressed by $\lambda p_2(1-p_2)$, then we can get $p_2^*=\frac{1}{2}$ and $\pi_{bp(4)}^*=\frac{\lambda}{4}$ and $\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}=\frac{2-\delta+2\delta\rho}{2-2\delta+2\delta\rho}\le p_1$ only when $\delta=0$, so this solution is valid only when $\delta=0$. Then we consider about boundary solutions if $\delta\ne 0$.

1) If $\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}=p_1$, we have $p_2^*=1$ and thus $\pi_{bp(4)}^*=0$.

2) If $p_2^*=0$, we have $\pi_{bp(4)}^*=0$.

Thus in the case of $p_1^*=1$, we have

$$\begin{cases} p_2^*=0 \; or \; 1, \; \pi_{bp(4)}^*=0 \quad if \; \delta\ne 0 \\ p_2^*=\frac{1}{2}, \; \pi_{bp(4)}^*=\frac{\lambda}{4} \quad if \; \delta=0 \end{cases}$$

iv) If $\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}=p_1$, we can obtain $(1-\rho)p_1-p_2+\rho=0$. Constructing the Lagrangian, we have

$$\pi_{bp(4)}=p_1(1-\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)})+\lambda p_2\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}+a((1-\rho)p_1-p_2+\rho)$$

The derivatives of $\pi_{bp(4)}$ with respect to a, p_1 and p_2 are given by

$$\frac{\partial\pi_{bp(4)}}{\partial a}=(1-\rho)p_1-p_2+\rho=0$$

$$\frac{\partial\pi_{bp(4)}}{\partial p_1}=1+a-a\rho-2p_1+\lambda p_2(1-\rho)=0$$

$$\frac{\partial\pi_{bp(4)}}{\partial p_2}=-a+\lambda(\rho(1-p_1)+p_1)-2\lambda p_2=0$$

According to the above expressions, we have $p_1^*=\frac{1}{2}$, $p_2^*=\frac{1+\rho}{2}$, and thus $\pi_{bp(4)}^*=\frac{1}{4}$. Then we check constraints:

1) $\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}=\frac{1}{2}=p_1<1$, thus the constraint $\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}\leq p_1\leq 1$ holds;

2) $p_1=\frac{1}{2}>0$;

3) $p_2=\frac{1+\rho}{2}>0$.

Accordingly, the solution is valid satisfying all constraints. So $p_1^*=\frac{1}{2}$, $p_2^*=\frac{1+\rho}{2}$, and thus $\pi_{bp(4)}^*=\frac{1}{4}$.

Thus in the case (4), we obtain $p_1^*=\frac{1}{2}$, $p_2^*=\frac{1+\rho}{2}$ and $\pi_{bp(4)}^*=\frac{1}{4}$.

(5) When the threshold $\tau_{bp}(p_1,p_2)$ is p_1, and the the second-period market is fully covered, the film's optimization is expressed by

$$\max_{p_1,p_2}\quad \pi_{bp(5)}(p_1,p_2)=p_1(1-p_1)+\lambda p_2p_1$$

$$\text{subject to}\begin{cases}\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}\leq p_1\leq 1\\ p_1+\rho(1-p_1)-p_2\geq p_1 \ \text{(from Lemma 1)}\\ p_1,p_2\geq 0\end{cases}$$

From the derivations of $\pi_{bp(5)}(p_1,p_2)$ with respect to p_1 and p_2, the solution can not satisfy all above constraints and thus there exists no feasible interior solution. Then we consider boundary solutions:

i) If $p_1^*=0$, the firm's profit can be expressed by $\pi_{bp(5)}^*=0$.

ii) If $p_2^*=0$, then the firm's profit is expressed by $p_1(1-p_1)$, we can get $p_1^*=\frac{1}{2}$ and thus $\pi_{bp(5)}^*=\frac{1}{4}$.

Then we check constraints:

1) $\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}\leq p_1 \Rightarrow \delta=0$;

2) $p_1+\rho(1-p_1)-p_2\geq p_1$ holds;

3) $p_1=\frac{1}{2}>0$;

Accordingly, the solution $p_1^*=\frac{1}{2}$ and $\pi_{bp(5)}^*=\frac{1}{4}$ is valid only if $\delta=0$ (excluded). Then we consider about boundary solutions if $\delta\neq 0$.

1) If $p_1^*=0$, we have $\pi_{bp(5)}^*=0$.

2) If $\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}=p_1$, we have $p_1^*=\frac{\rho}{\rho-1}$ and $\pi_{bp(5)}^*=-\frac{\rho}{(\rho-1)^2}<0$.

3) If $p_1^*=1$, we have $\pi_{bp(5)}^*=0$.

4) If $p_1+\rho(1-p_1)-p_2\geq p_1$, we have $p_1^*=1$ and then $\pi_{bp(5)}^*=0$.

Thus in the case of $p_2^*=0$, we have $p_1^*=0$ or $1,\pi_{bp(5)}^*=0$

iii) If $p_1^*=1$, then the firm's profit is expressed by $\pi_{bp(5)}=\lambda p_2$. According to the constraint $p_1+\rho(1-p_1)-p_2\geq p_1$, we can get that p_2^* must be 0, and thus $\pi_{bp(5)}^*=0$.

iv) If $\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}=p_1$, we can obtain $(1-\rho)p_1-p_2+\rho=0$. Constructing the Lagrangian, we have

$$\pi_{bp(5)}(p_1,p_2)=p_1(1-\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)})+\lambda p_2\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)}+a((1-\rho)p_1-p_2+\rho)$$

The derivatives of $\pi_{bp(5)}$ with respect to a, p_1 and p_2 are given by

$$\frac{\partial\pi_{bp(5)}}{\partial c}=(1-\rho)p_1-p_2+\rho=0$$

$$\frac{\partial \pi_{bp(5)}}{\partial p_1} = 1 + a - a\rho - 2p_1 + \lambda p_2 = 0$$
$$\frac{\partial \pi_{bp(5)}}{\partial p_2} = -a + \lambda p_1 = 0$$

According to the above expressions, we have $p_1^* = \frac{1+\lambda\rho}{2(1-\lambda+\lambda\rho)}$, $p_2^* = \frac{1+\rho-\lambda\rho+\lambda\rho^2}{2(1-\lambda+\lambda\rho)}$. Then we check constraints:

1) $\frac{p_1-\delta p_2+\delta\rho}{1-\delta(1-\rho)} = p_1 = \frac{1+\rho}{2\rho} \le 1 \Rightarrow \rho \ge 1$; or $0<\rho<1$ and $0<\lambda<\frac{1}{2-\rho}$;

2) $p_1 + \rho(1-p_1) - p_2 = 0 < p_1$, so the constraint $p_1 + \rho(1-p_1) - p_2 \ge p_1$ doesn't hold;

Accordingly, there exists no feasible solution meeting all constraints, then we discuss boundary solutions as follows.

1) If $p_1^* = 1$, according to the constraint $p_1 + \rho(1-p_1) - p_2 \ge p_1$, we have $p_2^* = 0$, and thus $\pi^*_{bp(5)} = 0$.

2) If $p_1 + \rho(1-p_1) - p_2 = p_1$, we have the solution: a) $p_1 = 1$ and $p_2 = 0$ when $\delta = 1$, or b) $p_1 = 0$ and $p_2 = \rho$. For solution a) $p_1 = 1$ and $p_2 = 0$ and solution b) $p_1 = 0$ and $p_2 = \rho$, we have $\pi^*_{bp(5)} = 0$.

Thus in the case (5), the firm's optimal profit is 0..

Discussing the above five cases, we have the following expression:

(1)

$$\pi^*_{bp(1)} = \begin{cases} -\frac{\lambda(1-\delta+\lambda\rho)}{\delta^2-4\lambda+2\delta\lambda+\lambda^2-2\delta\rho\lambda-2\lambda^2\rho+\lambda^2\rho^2},\ if\ (i)\ \rho \le 1+\delta\ and\ \frac{\delta}{1+\rho} \le \lambda \le 1; \\ \qquad or\ (ii)\rho > 1+\delta\ and\ \frac{\delta}{1+\rho} \le \lambda \le \frac{\delta}{-1+\rho} \\ \frac{1-\delta+\lambda\rho}{4},\ if\ \rho > 1+\delta\ and\ \frac{\delta}{-1+\rho} < \lambda \le 1 \\ \frac{1}{4},\ if\ 0 \le \lambda < \frac{\delta}{1+\rho}. \end{cases}$$

(2) $\pi^*_{bp(2)} = \frac{1-\delta+\lambda\rho}{4}$;

(3) $\pi^*_{bp(3)} = \frac{1}{4}$;

(4) $\pi^*_{bp(4)} = \frac{1}{4}$;

(5) $\pi^*_{bp(4)} = 0$;

Then we compare the profits in above five cases, where

When (i) $\rho \leq 1+\delta$ and $\frac{\delta}{1+\rho} \leq \lambda \leq 1$, or (ii) $\rho > 1+\delta$ *and* $\frac{\delta}{1+\rho} \leq \lambda \leq \frac{\delta}{-1+\rho}$, we have: $-\frac{\lambda(1-\delta+\lambda\rho)}{\delta^2-4\lambda+2\delta\lambda+\lambda^2-2\delta\rho\lambda-2\lambda^2\rho+\lambda^2\rho^2} - \frac{1-\delta+\lambda\rho}{4} \geq 0$ always holds and $-\frac{\lambda(1-\delta+\lambda\rho)}{\delta^2-4\lambda+2\delta\lambda+\lambda^2-2\delta\rho\lambda-2\lambda^2\rho+\lambda^2\rho^2} - \frac{1}{4} \geq 0$ always holds.

When $\rho > 1+\delta$ and $\frac{\delta}{-1+\rho} < \lambda \leq 1$, $\frac{1-\delta+\lambda\rho}{4} > \frac{1}{4}$ always holds.

When $0 \leq \lambda < \frac{\delta}{1+\rho}$, $\frac{1}{4} > \frac{1-\delta+\lambda\rho}{4}$ always holds.

In summary, we have

$$\begin{cases} p_1^* = \frac{\lambda(-2-\delta(-2+\rho)+\lambda\rho(-1+\rho))}{\delta^2-4\lambda+2\delta\lambda+\lambda^2-2\delta\rho\lambda-2\lambda^2\rho+\lambda^2\rho^2},\ p_2^* = \frac{\delta(\delta-1)+\lambda(\delta-\rho-\delta\rho-1)}{\delta^2-4\lambda+2\delta\lambda+\lambda^2-2\delta\rho\lambda-2\lambda^2\rho+\lambda^2\rho^2}, \\ \qquad \pi^*_{bp} = -\frac{\lambda(1-\delta+\lambda\rho)}{\delta^2-4\lambda+2\delta\lambda+\lambda^2-2\delta\rho\lambda-2\lambda^2\rho+\lambda^2\rho^2},\ if\ (i)\ \rho \leq 1+\delta\ and\ \frac{\delta}{1+\rho} \leq \lambda \leq 1; \\ \qquad or\ (ii) \rho > 1+\delta\ and\ \frac{\delta}{1+\rho} \leq \lambda \leq \frac{\delta}{-1+\rho} \\ p_1^* = \frac{1-\delta}{2},\ p_2^* = \frac{\rho}{2},\ \pi^*_{bp} = \frac{1-\delta+\lambda\rho}{4},\ if\ \rho > 1+\delta\ and\ \frac{\delta}{-1+\rho} < \lambda \leq 1 \\ p_1^* = \frac{1}{2},\ p_2^* = \frac{1+\rho}{2},\ \pi^*_{bp} = \frac{1}{4},\ if\ 0 \leq \lambda < \frac{\delta}{1+\rho}. \end{cases}$$

(ii) For the above optimal price plan, we have that:

(1) In the interval $\rho \leq 1+\delta$ and $\frac{\delta}{1+\rho} \leq \lambda \leq 1$; or $\rho > 1+\delta$ and $\frac{\delta}{1+\rho} \leq \lambda \leq \frac{\delta}{-1+\rho}$, we have: when $0 < \rho < 1-\delta$ and $\frac{\delta}{1-\rho} < \lambda \leq 1$, $p_1^* > p_2^*$;

(2) In the interval $\rho > 1+\delta$ and$\frac{\delta}{-1+\rho} < \lambda \leq 1$, p_1^* is always lower than p_2^*;

(3) In the interval $0 \leq \lambda < \frac{\delta}{1+\rho}$, p_1^* is always lower than p_2^*;

In summary, we have: $p_1^* > p_2^*$ if and only if: $0 < \rho < 1-\delta$ and $\frac{\delta}{1-\rho} < \lambda \leq 1$.

Proof of Lemma 2

(i) Given any arbitrary price plan $\{p_1, p_2\}$, for a customer with preference v, we define a function $\Delta(v;\xi) = u_1(v) - u_2(v;\xi)$, where $1-\xi$ customers choose to make purchasing decisions in the first period, u_1 is a customer's expected utility if purchasing in the first period and u_2 is a customer's expected utility if purchasing in the second period. Then this customer makes purchasing decisions in the first period if $\Delta(v;\xi)\geq 0$. Specifically,

$$u_1(v)=v-p_1,$$

$$u_2(v;\xi)=\delta\int_{p_2-v-\rho(1-\xi)}^{+\infty}(v+q_1+\rho(1-\xi)-p_2)f(q_1;\xi)dq_1.$$

Suppose A is constant. Then we have

$$\begin{aligned}\Delta(v;\xi)&=v-p_1-\delta\int_{p_2-v-\rho(1-\xi)}^{+\infty}(v+q_1+\rho(1-\xi)-p_2)f(q_1;\xi)dq_1\\&=v-p_1-\delta\int_{p_2-v-\rho(1-\xi)}^{A}(v+q_1+\rho(1-\xi)-p_2)f(q_1;\xi)dq_1\\&-\delta\int_{A}^{+\infty}(v+q_1+\rho(1-\xi)-p_2)f(q_1;\xi)dq_1\end{aligned}$$

And we have:

$$\begin{aligned}\frac{d}{dv}\int_{p_2-v-\rho(1-\xi)}^{A}(v+q_1+\rho(1-\xi)-p_2)f(q_1;\xi)dq_1&=\int_{p_2-v-\rho(1-\xi)}^{A}f(q_1;\xi)dq_1\\&+(v+A+\rho(1-\xi)-p_2)\frac{dA}{dv}\\&-(v+p_2-v-\rho(1-\xi)+\rho(1-\xi)-p_2)\frac{d(p_2-v-\rho(1-\xi}{dv}\\&=\int_{p_2-v-\rho(1-\xi)}^{A}f(q_1;\xi)dq_1,\end{aligned}$$

and

$$\frac{d}{dv}\int_{A}^{+\infty}(v+q_1+\rho(1-\xi)-p_2)f(q_1;\xi)dq_1=\int_{A}^{+\infty}f(q_1;\xi)dq_1.$$

Thus

$$\begin{aligned}\frac{d\Delta(v;\xi)}{dv}&=1-\delta(\int_{p_2-v-\rho(1-\xi)}^{A}f(q_1;\xi)dq_1+\int_{A}^{+\infty}f(q_1;\xi)dq_1)\\&=1-\delta\int_{p_2-v-\rho(1-\xi)}^{+\infty}f(q_1;\xi)dq_1>1-\delta>0.\end{aligned}$$

Since the above monotonicity holds for any arbitrary ξ customers, any pure-strategy equilibrium must admit a first-period threshold structure. We call this threshold $\tau_p(p1,p2)$; the customer purchases if $v\geq\tau_p(p1,p2)$, $\tau_p(p1,p2)=\theta\in[0,1)$.

The indifference equation is:

$$\theta - p_1 = \delta \int_{p_2-\theta-\rho(1-\theta)}^{+\infty} (\theta + q_1 + \rho(1-\theta) - p_2) f(q_1;\theta) dq_1. \tag{1}$$

Next we will show that the above indifference equation has exactly only one solution $\theta \in [p_1, 1]$.

Since q_1 is normally distributed with zero mean and standard deviation $\sigma_1(\theta)$, and $f(q_1;\theta) = \frac{1}{\sqrt{2\pi}\sigma_1(\theta)} e^{-\frac{q_1^2}{2\sigma_1(\theta)^2}}$, we have:

$$\begin{aligned}
\int_{p_2-\theta-\rho(1-\theta)}^{+\infty} f(q_1;\theta) dq_1 &= \int_{-\infty}^{\theta+\rho(1-\theta)-p_2} f(q_1;\theta) dq_1 \\
&= \int_{-\infty}^{\theta+\rho(1-\theta)-p_2} \frac{1}{\sqrt{2\pi}\sigma_1(\theta)} e^{-\frac{q_1^2}{2\sigma_1(\theta)^2}} dq_1 \\
&= \int_{-\infty}^{\theta+\rho(1-\theta)-p_2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{q_1}{\sigma_1(\theta)})^2} d(\frac{q_1}{\sigma_1(\theta)}) \\
&= \int_{-\infty}^{\frac{\theta+\rho(1-\theta)-p_2}{\sigma_1(\theta)}} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy \\
&= \Phi(\frac{\theta+\rho(1-\theta)-p_2}{\sigma_1(\theta)}),
\end{aligned}$$

and

$$\begin{aligned}
\int_{p_2-\theta-\rho(1-\theta)}^{+\infty} q_1 f(q_1;\theta)dq_1 &= \int_{p_2-\theta-\rho(1-\theta)}^{+\infty} q_1 \frac{1}{\sqrt{2\pi}\sigma_1(\theta)} e^{-\frac{q_1^2}{2\sigma_1(\theta)^2}} dq_1 \\
&= \sigma_1(\theta) \int_{p_2-\theta-\rho(1-\theta)}^{+\infty} \frac{q_1}{\sigma_1(\theta)} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{q_1}{\sigma_1(\theta)})^2} d(\frac{q_1}{\sigma_1(\theta)}) \\
&= \sigma_1(\theta) \int_{\frac{p_2-\theta-\rho(1-\theta)}{\sigma(\theta)}}^{+\infty} y \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy \\
&= \sigma_1(\theta) \int_{\frac{p_2-\theta-\rho(1-\theta)}{\sigma(\theta)}}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} d\frac{y^2}{2} \\
&= \sigma_1(\theta) \int_{(\frac{p_2-\theta-\rho(1-\theta)}{\sigma(\theta)})^2}^{+\infty} \frac{1}{2\sqrt{2\pi}} e^{-\frac{z}{2}} dz \\
&= \ \sigma_1(\theta) \int_{(\frac{p_2-\theta-\rho(1-\theta)}{\sigma(\theta)})^2}^{+\infty} \frac{-1}{\sqrt{2\pi}} de^{-\frac{z}{2}} \\
&= \sigma_1(\theta) \frac{1}{\sqrt{2\pi}} e^{-\frac{(\frac{p_2-\theta-\rho(1-\theta)}{\sigma(\theta)})^2}{2}} \\
&= \sigma_1(\theta)\phi(\frac{\theta+\rho(1-\theta)-p_2}{\sigma(\theta)}).
\end{aligned}$$

Since

$$\begin{aligned}
&\delta \int_{p_2-\theta-\rho(1-\theta)}^{+\infty} (\theta+q_1+\rho(1-\theta)-p_2) f(q_1;\theta)dq_1 \\
&= \delta \int_{p_2-\theta-\rho(1-\theta)}^{+\infty} (\theta+\rho(1-\theta)-p_2) f(q_1;\theta)dq_1 + \delta \int_{p_2-\theta-\rho(1-\theta)}^{+\infty} q_1 f(q_1;\theta)dq_1 \\
&= \delta(\theta+\rho(1-\theta)-p_2)\Phi(\frac{\theta+\rho(1-\theta)-p_2}{\sigma_1(\theta)}) + \delta\sigma_1(\theta)\phi(\frac{\theta+\rho(1-\theta)-p_2}{\sigma_1(\theta)})
\end{aligned}$$

taking the derivative of the right-hand side of the indifference equation with respect to θ, we have:

$$\begin{aligned}\frac{dRHS}{d\theta} =&\delta(\theta+\rho(1-\theta)-p_2)\phi(\frac{\theta+\rho(1-\theta)-p_2}{\sigma_1(\theta)})\frac{\partial(\frac{\theta+\rho(1-\theta)-p_2}{\sigma_1(\theta)})}{\partial\theta}+\delta(1-\rho)\Phi(\frac{\theta+\rho(1-\theta)-p_2}{\sigma_1(\theta)})\\ &+\delta\sigma_1'(\theta)\phi(\frac{\theta+\rho(1-\theta)-p_2}{\sigma_1(\theta)})\\ &-\delta\sigma_1(\theta)\frac{\theta+\rho(1-\theta)-p_2}{\sigma_1(\theta)}\phi(\frac{\theta+\rho(1-\theta)-p_2}{\sigma_1(\theta)})\frac{\partial(\frac{\theta+\rho(1-\theta)-p_2}{\sigma_1(\theta)})}{\partial\theta}(\text{because}\frac{d\phi(X)}{dX}=-X\phi(X))\\ &=\delta(1-\rho)\Phi(\frac{\theta+\rho(1-\theta)-p_2}{\sigma_1(\theta)})+\delta\sigma_1'(\theta)\phi(\frac{\theta+\rho(1-\theta)-p_2}{\sigma_1(\theta)})\\ &<1\ (\text{because } \sigma_1(\theta)=\sigma_0\sqrt{(1-\theta)\gamma} \text{ and } \sigma_1'(\theta)<0)\end{aligned}$$

Taking the derivative of the left-hand side of the indifference equation, $\frac{d}{d\theta}(\theta-p_1)=1$. Then we have $\frac{dLHS}{d\theta}-\frac{dRHS}{d\theta}>0$. .

We now characterize the condition under which no sale occurs in the first period (i.e., $\tau_p(p1,p2)=1$). To do so, we consider a customer with the highest expected first-period utility, that is $1-p_1$. If she delays, no customer purchases in the first period, no reviews are posted, no externality effects are generated, and the customers' expected utility in the second period is $\delta(1-p_2)$. Thus, there are no first-period sales iff $1-p_1-\delta(1-p_2)<0$, equivalently, $LHS(\theta)-RHS(\theta)<0$ when $\theta=1$.

When $\theta=p_1$, $LHS(\theta)=\theta-p_1=0$ and $RHS(\theta)=\delta\int_{p_2-\theta-\rho(1-\theta)}^{+\infty}(\theta+q_1+\rho(1-\theta)-p_2)f(q_1;\theta)dq_1\geq 0$ because when $q_1\geq p_2-\theta-\rho(1-\theta)$, $\theta+q_1+\rho(1-\theta)-p_2\geq 0$.

Therefore, there exists a unique solution $\theta\in[p_1,1]$ satisfying $LHS(\theta)-RHS(\theta)=0$.

If $1-p_1\geq\delta(1-p_2)$, then we have $\tau_p(p1,p2)=\theta\in[p_1,1]$. Rewriting the indifference equation as:

$$\theta-p_1-RHS(\theta)=0.$$

Using the Implicit Functions Theorem, from

$$\frac{d\theta}{d\delta}-(\frac{\partial RHS(\theta)}{\partial\delta}+\frac{dRHS(\theta)}{d\theta}\frac{d\theta}{d\delta})=0,$$

we have

$$\frac{d\theta}{d\delta}=\frac{\frac{\partial RHS(\theta)}{\partial\delta}}{1-\frac{dRHS(\theta)}{d\theta}}.$$

Since

$$\frac{\partial RHS(\theta)}{\partial\delta}=(\theta+\rho(1-\theta)-p_2)\Phi(\frac{\theta+\rho(1-\theta)-p_2}{\sigma_1(\theta)})+\sigma_1(\theta)\phi(\frac{\theta+\rho(1-\theta)-p_2}{\sigma_1(\theta)})>0$$

and from the previous proof we obtain $\frac{dRHS(\theta)}{d\theta}<1$, we have $\frac{d\theta}{d\delta}>0$. Similarly, from

$$\frac{d\theta}{d\rho}-(\frac{\partial RHS(\theta)}{\partial\rho}+\frac{dRHS(\theta)}{d\theta}\frac{d\theta}{d\rho})=0,$$

we have

$$\frac{d\theta}{d\rho}=\frac{\frac{\partial RHS(\theta)}{\partial\rho}}{1-\frac{dRHS(\theta)}{d\theta}}.$$

Since

$$\begin{aligned}\frac{\partial RHS(\theta)}{\partial\rho}=&\delta(1-\theta)\Phi(\frac{\theta+\rho(1-\theta)-p_2}{\sigma_1(\theta)})+\delta(\theta+\rho(1-\theta)-p_2)\phi(\frac{\theta+\rho(1-\theta)-p_2}{\sigma_1(\theta)})(1-\theta)\\&-\delta\sigma_1(\frac{\theta+\rho(1-\theta)-p_2}{\sigma_1(\theta)})(\theta)\phi(\frac{\theta+\rho(1-\theta)-p_2}{\sigma_1(\theta)})(1-\theta)\\=&\delta(1-\theta)\Phi(\frac{\theta+\rho(1-\theta)-p_2}{\sigma_1(\theta)})>0,\end{aligned}$$

we have $\frac{d\theta}{d\rho}>0$. From

$$\frac{d\theta}{d\sigma_0}-(\frac{\partial RHS(\theta)}{\partial\sigma_0}+\frac{dRHS(\theta)}{d\theta}\frac{d\theta}{d\sigma_0})=0,$$

we have

$$\frac{d\theta}{d\sigma_0} = \frac{\frac{\partial RHS(\theta)}{\partial \sigma_0}}{1 - \frac{dRHS(\theta)}{d\theta}}.$$

Since

$$\begin{aligned}\frac{\partial RHS(\theta)}{\partial \sigma_0} =& \delta(\theta+\rho(1-\theta)-p_2)\phi(\frac{\theta+\rho(1-\theta)-p_2}{\sigma_1(\theta)})\frac{\partial(\frac{\theta+\rho(1-\theta)-p_2}{\sigma_1(\theta)})}{\partial\sigma_0} + \delta\frac{\partial\sigma_1(\theta)}{\partial\sigma_0}\phi(\frac{\theta+\rho(1-\theta)-p_2}{\sigma_1(\theta)}) \\ &- \delta\sigma_1(\theta)\phi(\frac{\theta+\rho(1-\theta)-p_2}{\sigma_1(\theta)})(\frac{\theta+\rho(1-\theta)-p_2}{\sigma_1(\theta)})\frac{\partial(\frac{\theta+\rho(1-\theta)-p_2}{\sigma_1(\theta)})}{\partial\sigma_0} \\ =& \delta\frac{\partial\sigma_1(\theta)}{\partial\sigma_0}\phi(\frac{\theta+\rho(1-\theta)-p_2}{\sigma_1(\theta)}) > 0\end{aligned}$$

Therefore, $\frac{\partial\theta}{\partial\sigma_0} > 0$, which means that the first-period sales is decreasing in σ_0. In a similar manner, it can also be proved that $\frac{\partial\theta}{\partial\sigma_\mu} < 0$, and since $\gamma = {\sigma_0}^2/{\sigma_\mu}^2$, the two latter results imply that $\frac{\partial\theta}{\partial\gamma} > 0$.

(ii) In the second period, the remaining customers satisfy $v < \tau_p(p1, p2)$ and make purchasing decision only if $v - p_2 + q_1 + \rho(1 - \tau_p(p1, p2)) \geq 0$.

Proof of Proposition 2

(i) We first prove that it can never be optimal when no sale occurs in the first period. Note that among all price plans when no first-period sale occurs, the firm's profit function is expressed by $\lambda p_2(1-p_2)$. The firm achieves the highest profit $\frac{\lambda}{4}$ at $p_2 = \frac{1}{2}$. We next prove this policy is dominated by the pricing policy of $\{\frac{\lambda}{2}, \frac{1}{2}\}$. Given the price plan $\{\frac{\lambda}{2}, \frac{1}{2}\}$, the first-period profit is $\pi_1 = \frac{\lambda}{2}(1-\theta)$, where θ is the threshold value and $\theta \in [\frac{1}{2}, 1]$. Denoting the expected second-period profit for the firm by π_2, we will show that $\pi_1 + \pi_2 > \frac{\lambda}{4}$. Equivalently, we will show $\pi_2 = \frac{\lambda}{2}E(s_2) > \frac{\lambda}{4} - \pi_1 = \frac{\lambda}{2}(\theta - \frac{1}{2})$, where s_2 denotes sales that occur in the second period. Then it is equivalent to prove that $E(s_2) > \theta - \frac{1}{2}$, where

$$s_2 = \begin{cases} 0 & q_1 \leq \frac{1}{2} - \theta - \rho(1-\theta) \\ \theta + q_1 + \rho(1-\theta) - \frac{1}{2} & \frac{1}{2} - \theta - \rho(1-\theta) < q_1 \leq \frac{1}{2} - \rho(1-\theta) \\ \theta & q_1 > \frac{1}{2} - \rho(1-\theta) \end{cases}$$

If $\rho \in (0, \frac{1}{2}]$, since $\theta \in [\frac{1}{2}, 1]$ and $2\theta - 1 \geq 0$, we have $\frac{1}{2} - \theta - \rho(1-\theta) \leq \theta + \rho(1-\theta) - \frac{1}{2}$.

Thus we draw the following expression.

$$\begin{aligned}
E(s_2) =& \int_{\frac{1}{2}-\theta-\rho(1-\theta)}^{\frac{1}{2}-\rho(1-\theta)} (\theta + q_1 + \rho(1-\theta) - \frac{1}{2}) f(q_1;\theta) dq_1 + \int_{\frac{1}{2}-\rho(1-\theta)}^{\infty} \theta f(q_1;\theta) dq_1 \\
=& \int_{\frac{1}{2}-\theta-\rho(1-\theta)}^{\frac{1}{2}-\rho(1-\theta)} (\theta + q_1 - \frac{1}{2}) f(q_1;\theta) dq_1 + \int_{\frac{1}{2}-\theta-\rho(1-\theta)}^{\frac{1}{2}-\rho(1-\theta)} \rho(1-\theta) f(q_1;\theta) dq_1 + \int_{\frac{1}{2}-\rho(1-\theta)}^{\infty} \theta f(q_1;\theta) dq_1 \\
\geq& \int_{\frac{1}{2}-\theta-\rho(1-\theta)}^{\frac{1}{2}-\rho(1-\theta)} (\theta + q_1 - \frac{1}{2}) f(q_1;\theta) dq_1 + \int_{\theta+\rho(1-\theta)-\frac{1}{2}}^{\frac{1}{2}-\rho(1-\theta)} \rho(1-\theta) f(q_1;\theta) dq_1 + \int_{\frac{1}{2}-\rho(1-\theta)}^{\infty} \theta f(q_1;\theta) dq_1
\end{aligned}$$

Define

$$\begin{aligned}
E(s_m) =& \int_{\frac{1}{2}-\theta-\rho(1-\theta)}^{\frac{1}{2}-\rho(1-\theta)} (\theta + q_1 - \frac{1}{2}) f(q_1;\theta) dq_1 + \int_{\theta+\rho(1-\theta)-\frac{1}{2}}^{\frac{1}{2}-\rho(1-\theta)} \rho(1-\theta) f(q_1;\theta) dq_1 + \int_{\frac{1}{2}-\rho(1-\theta)}^{\infty} \theta f(q_1;\theta) dq_1 \\
=& \int_{\frac{1}{2}-\theta-\rho(1-\theta)}^{\theta+\rho(1-\theta)-\frac{1}{2}} (\theta + q_1 - \frac{1}{2}) f(q_1;\theta) dq_1 + \int_{\theta+\rho(1-\theta)-\frac{1}{2}}^{\frac{1}{2}-\rho(1-\theta)} \Big((2\theta-1) - \theta + q_1 + \frac{1}{2} + \rho(1-\theta)\Big) f(q_1;\theta) dq_1 \\
&+ \int_{\frac{1}{2}-\rho(1-\theta)}^{\infty} \Big((2\theta-1) + (-\theta+1)\Big) f(q_1;\theta) dq_1 \\
=& [\int_{\frac{1}{2}-\theta-\rho(1-\theta)}^{\theta+\rho(1-\theta)-\frac{1}{2}} (\theta + q_1 - \frac{1}{2}) f(q_1;\theta) dq_1 + \int_{\theta+\rho(1-\theta)-\frac{1}{2}}^{\infty} (2\theta-1) f(q_1;\theta) dq_1] \\
&+ [\int_{\theta+\rho(1-\theta)-\frac{1}{2}}^{\frac{1}{2}-\rho(1-\theta)} \Big(-\theta + q_1 + \frac{1}{2} + \rho(1-\theta)\Big) f(q_1;\theta) dq_1 + \int_{\frac{1}{2}-\rho(1-\theta)}^{\infty} (-\theta+1) f(q_1;\theta) dq_1] \\
:=& E(s_a) + E(s_b),
\end{aligned}$$

where

$$s_a(q_1) := \begin{cases} 0 & q_1 \leq \frac{1}{2} - \theta - \rho(1-\theta) \\ \theta + q_1 - \frac{1}{2} & \frac{1}{2} - \theta - \rho(1-\theta) < q_1 \leq \theta + \rho(1-\theta) - \frac{1}{2} \\ 2\theta - 1 & q_1 > \theta + \rho(1-\theta) - \frac{1}{2} \end{cases}$$

and

$$s_b(q_1) := \begin{cases} 0 & q_1 \leq \theta + \rho(1-\theta) - \frac{1}{2} \\ -\theta + q_1 + \frac{1}{2} + \rho(1-\theta) & \theta + \rho(1-\theta) - \frac{1}{2} < q_1 \leq \frac{1}{2} - \rho(1-\theta) \\ -\theta + 1 & q_1 > \frac{1}{2} - \rho(1-\theta) \end{cases}$$

We have:

$$\int_{-\infty}^{+\infty} s_a(q_1)f(q_1;\theta)dq_1 = \int_{\theta+\rho(1-\theta)-\frac{1}{2}}^{\frac{1}{2}-\rho(1-\theta)} (\theta-\frac{1}{2})f(q_1;\theta)dq_1 + \int_{\frac{1}{2}-\rho(1-\theta)}^{+\infty} (-\theta+1)f(q_1;\theta)dq_1$$
$$= \theta - \frac{1}{2}$$

Moreover, according to the expression of $s_b(q_1)$, we have that in the interval of $q_1 \in (\theta + \rho(1-\theta) - \frac{1}{2}, \frac{1}{2} - \rho(1-\theta)]$, $-\theta + q_1 + \frac{1}{2} + \rho(1-\theta) > 2\rho(1-\theta) \geq 0$. Thus $s_b(q_1)$ is a nonnegative function of q_1 (and positive for some values of q_1), it follows that $\int_{-\infty}^{+\infty} s_b(q_1)f(q_1;\theta)dq_1 > 0$. Therefore, we have $E(s_m) > \theta - \frac{1}{2}$ and then $E(s_2) > \theta - \frac{1}{2}$ when $\rho \in [0, \frac{1}{2}]$. Since

$$E(s_2) = \int_{\frac{1}{2}-\theta-\rho(1-\theta)}^{\frac{1}{2}-\rho(1-\theta)} (\theta + q_1 + \rho(1-\theta) - \frac{1}{2})f(q_1;\theta)dq_1 + \int_{\frac{1}{2}-\rho(1-\theta)}^{\infty} \theta f(q_1;\theta)dq_1$$
$$= (\theta+\rho(1-\theta)-\frac{1}{2})[\Phi(\frac{\frac{1}{2}-\rho(1-\theta)}{\sigma(\theta)}) - \Phi(\frac{\frac{1}{2}-\theta-\rho(1-\theta)}{\sigma(\theta)})] + \theta\Phi(\frac{\rho(1-\theta)-\frac{1}{2}}{\sigma(\theta)})$$
$$+ \sigma(\theta)[\phi(\frac{\frac{1}{2}-\rho(1-\theta)}{\sigma(\theta)}) - \phi(\frac{\frac{1}{2}-\theta-\rho(1-\theta)}{\sigma(\theta)})],$$

we take the derivative of $E(s_2)$ with respect to ρ, where

$$\frac{\partial E(s_2)}{\partial \rho} = (1-\theta)[\Phi(\frac{\frac{1}{2}-\rho(1-\theta)}{\sigma(\theta)}) - \Phi(\frac{\frac{1}{2}-\theta-\rho(1-\theta)}{\sigma(\theta)})] + \theta\frac{1-\theta}{\sigma(\theta)}\phi(\frac{\rho(1-\theta)-\frac{1}{2}}{\sigma(\theta)})$$
$$+ \frac{1-\theta}{\sigma(\theta)}(\theta+\rho(1-\theta)-\frac{1}{2})[\phi(\frac{\frac{1}{2}-\rho(1-\theta)}{\sigma(\theta)}) - \phi(\frac{\frac{1}{2}-\theta-\rho(1-\theta)}{\sigma(\theta)})]$$
$$- \frac{1-\theta}{\sigma(\theta)}(\frac{1}{2}-\rho(1-\theta))\phi(\frac{\frac{1}{2}-\rho(1-\theta)}{\sigma(\theta)}) + \frac{1-\theta}{\sigma(\theta)}(\frac{1}{2}-\theta-\rho(1-\theta))\phi(\frac{\frac{1}{2}-\theta-\rho(1-\theta)}{\sigma(\theta)})$$
$$= (1-\theta)[\Phi(\frac{\frac{1}{2}-\rho(1-\theta)}{\sigma(\theta)}) - \Phi(\frac{\frac{1}{2}-\theta-\rho(1-\theta)}{\sigma(\theta)})] > 0.$$

From the above equation, it follows that $E(s_2)$ is a monotonic increasing function of ρ. Therefore, in the interval $\rho \in (\frac{1}{2}, +\infty)$, $E(s_2)$ is strictly higher than $\theta - \frac{1}{2}$. So it is always optimal for the firm to have sales in both periods.

(ii) Based on the above result, we restrict our attention to policies $\{p_1, p_2\}$ resulting in $\tau_p(p_1, p_2) = \theta \in [p_1, 1)$.

Denoting the second-period expected utility by $w_2(v,\theta)$, we have

$$\begin{aligned} w_2(v,\theta) &= \int_{p_2-v-\rho(1-\theta)}^{\infty} (v+q_1+\rho(1-\theta)-p_2) f(q_1;\theta) dq_1 \\ &= \int_{-\infty}^{+\infty} w(q_1) f(q_1;\theta) dq_1 \end{aligned}$$

where $w(q_1)$ is expressed by

$$w(q_1) = \begin{cases} 0 & q_1 < p_2 - v - \rho(1-\theta) \\ v+q_1+\rho(1-\theta)-p_2 & q_1 \geq p_2 - v - \rho(1-\theta) \end{cases}$$

Since $w(q_1)$ is nonnegative (and positive for some values of q_1), convex, and increasing in q_1, the integral $\int_{-\infty}^{+\infty} w(q_1) f(q_1;\theta) dq_1$ is strictly increasing in the preposterior variance (see Papanastasiou and Savva, 2017, p. 935). Referring to the expression of q_1 ($q_1 \sim N(0, n_1\gamma\sigma_0^2/(n_1\gamma+1))$), the variance of q_1 is strictly increasing in the first-period sale. So the customer's second-period expected utility is strictly increasing in the first-period sales. Since a positive number of customers purchase in the first period, for any $\gamma > 0$ we have

$$\delta \int_{p_2-v-\rho(1-\theta)}^{\infty} (v+q_1+\rho(1-\theta)-p_2) f(q_1;\theta) dq_1 > \delta(v-p_2)$$

that is, customer's second-period expected utility in the presence of social learning ($\gamma > 0$) is higher than that in the case of no social learning ($\gamma \to 0$). Correspondingly, we obtain that:

$$\theta - p_1 = \delta \int_{p_2-v-\rho(1-\theta)}^{\infty} (v+q_1+\rho(1-\theta)-p_2) f(q_1;\theta) dq_1 > \delta(\theta - p_2)$$

At the extreme case of $\delta = 1$ and $\rho = 0$, substituting $\delta = 1$ and $\rho = 0$ into the above inequality shows, we have:

$$\theta - p_1 > \delta(\theta + \rho(1-\theta) - p_2) \Leftrightarrow \theta - p_1 > \theta - p_2$$

Thus we have: $\theta - p_1^* > \theta - p_2^*$, that is, $p_1^* < p_2^*$. Since the price plan is upper-semicontinuous in $\delta \in [0,1]$ and $\rho \in [0,+\infty)$ (Papanastasiou and Savva, 2017). This implies the existence of the threshold $\Delta_1(\gamma,\rho,\lambda) \in [0,1]$ such that $p_1^* < p_2^*$ for $\delta \geq \Delta_1(\gamma,\rho,\lambda)$.

Proof of Proposition 3

Customers' purchasing decisions follow a threshold policy; the customer with a preference higher than $\bar{v}$ will buy in the first period. Then the firm's second-period profit is $\lambda p_2 \min[\bar{v}, \bar{v}+\rho(1-\bar{v})-p_2]^+$. We discuss the following cases:

(1) If $p_2 > \rho(1-\bar{v})$, the second-period market is not fully covered and the firm's second-period profit is $p_2(\bar{v}+\rho(1-\bar{v})-p_2)$, then we have: $p_2^*(\bar{v}) = \frac{\bar{v}+\rho(1-\bar{v})}{2}$, with the optimal value $\frac{(\bar{v}+\rho(1-\bar{v}))^2}{4}$. Plugging $p_2^*(\bar{v})$ into $p_2 > \rho(1-\bar{v})$, we have $\frac{\rho}{1+\rho} < \bar{v} \leq 1$.

(2) If $p_2 \leq \rho(1-\bar{v})$, the second-period market is fully covered and the firm's profit is $p_2\bar{v}$, then we have: $p_2^*(\bar{v}) = \rho(1-\bar{v})$, with the optimal value $\rho\bar{v}(1-\bar{v})$.

Since $\frac{(\bar{v}+\rho(1-\bar{v}))^2}{4} \geq \rho\bar{v}(1-\bar{v})$, we obtain the optimal second-period price as follows:

$$p_2^*(\bar{v}) = \begin{cases} \frac{\bar{v}+\rho(1-\bar{v})}{2} & \frac{\rho}{1+\rho} < \bar{v} \leq 1 \\ \rho(1-\bar{v}) & 0 \leq \bar{v} \leq \frac{\rho}{1+\rho} \end{cases}$$

Accordingly, when the second-period market is not fully covered, $\frac{\rho}{1+\rho} < \bar{v} \leq 1$, the optimal second-period price is $\frac{\bar{v}+\rho(1-\bar{v})}{2}$. Deriving from $\bar{v} - p_1 = \delta(\bar{v}+\rho(1-\bar{v}) - \frac{\bar{v}+\rho(1-\bar{v})}{2})$, we have $\bar{v} = \frac{2p_1+\delta\rho}{2-\delta(1-\rho)}$. If $p_1 \leq \frac{2p_1+\delta\rho}{2-\delta(1-\rho)} \leq 1 \Leftrightarrow (\rho-1)p_1 \leq \rho$ and $p_1 \leq \frac{2-\delta}{2}$, we obtain $\tau_{bd} = \frac{2p_1+\delta\rho}{2-\delta(1-\rho)}$; if $\frac{2p_1+\delta\rho}{2-\delta(1-\rho)} < p_1 \leq 1 \Leftrightarrow (\rho-1)p_1 > \rho$, we obtain $\tau_{bd} = p_1$; if $\frac{2p_1+\delta\rho}{2-\delta(1-\rho)} > 1 \Leftrightarrow p_1 > \frac{2-\delta}{2}$, no sales occur in the first period, and thus $\tau_{bd} = 1$.

When the second-period market is fully covered, $0 \leq \bar{v} \leq \frac{\rho}{1+\rho}$, the optimal second-period price is $\rho(1-\bar{v})$. Deriving from $\bar{v} - p_1 = \delta(1+\rho(1-\bar{v}) - \rho(1-\bar{v}))$, we have $\bar{v} = \frac{p_1}{1-\delta}$. Note that $\frac{p_1}{1-\delta}$ is larger than p_1. If $\frac{p_1}{1-\delta} \leq 1 \Leftrightarrow p_1 \leq 1-\delta$, $\tau_{bd} = \frac{p_1}{1-\delta}$. Else, if $p_1 > 1-\delta$, we obtain $\tau_{bd} = 1$, which means no sales occur in the first period.

(i) Next based on the above analysis, we first derive the optimal price plan and optimal profit of the firm.

(1) For a partial-covered second-period market, a fraction of remaining customers make purchasing decisions in the second period.

When $\tau_{bd} = \frac{2p_1+\delta\rho}{2-\delta(1-\rho)}$, the firm's optimization problem is:

$$\max_{p_1}\ \pi_{bd}(p_1) = p_1(1-\frac{2p_1+\delta\rho}{2-\delta(1-\rho)}) + \lambda\frac{(\frac{2p_1+\delta\rho}{2-\delta(1-\rho)}+\rho(1-\frac{2p_1+\delta\rho}{2-\delta(1-\rho)}))^2}{4}$$

$$\text{subject to}\begin{cases} p_1 \leq \frac{2p_1+\delta\rho}{2-\delta(1-\rho)} \leq 1 \text{ (from } p_1 \leq\ \bar{v} \leq 1 \\ 0 \leq \frac{\rho+p_1-\rho p_1}{2-\delta(1-\rho)} \leq \frac{2p_1+\delta\rho}{2-\delta(1-\rho)} (\text{from} \frac{\rho}{1+\rho} < \bar{v}) \\ p_1 \geq 0 \end{cases}$$

Rewriting the above expression, we can obtain

$$\max_{p_1}\ \pi_{bd}(p_1) = \frac{(\lambda-4+2\delta-2\lambda\rho-2\delta\rho+\lambda\rho^2)p_1^2+(4+2\delta(-2+\rho)+\delta^2(1-\rho)-2\lambda\rho(-1+\rho))p_1+\rho^2}{(2+\delta(-1+\rho))^2}$$

$$\text{subject to}\begin{cases} p_1 \leq \frac{2p_1+\delta\rho}{2-\delta(1-\rho)} \leq 1 \\ 0 \leq \frac{\rho+p_1-\rho p_1}{2-\delta(1-\rho)} \leq \frac{2p_1+\delta\rho}{2-\delta(1-\rho)} \\ p_1 \geq 0 \end{cases}$$

Now we check the non-negativity of the coefficient of p_1 and we obtain as follows:

(a) When $\rho > 2+\sqrt{5}$ and $\frac{4-2\delta+2\delta\rho}{1-2\rho+\rho^2} \leq \lambda \leq 1$; or $3 < \rho < 2+\sqrt{5}$ and $0 \leq \delta \leq \frac{-3-2\rho+\rho^2}{-2+2\rho}$ and $\frac{4-2\delta+2\delta\rho}{1-2\rho+\rho^2} \leq \lambda \leq 1$, we have $\lambda-4+2\delta-2\lambda\rho-2\delta\rho+\lambda\rho^2 \geq 0$.

For $\lambda-4+2\delta-2\lambda\rho-2\delta\rho+\lambda\rho^2 \geq 0$, the optimal p_1^* is at the endpoints. From $p_1 \leq \frac{2p_1+\delta\rho}{2-\delta(1-\rho)} \leq 1$, we obtain that p_1^* is no more than 1. Given $p_1 = 1$, we have $\pi_{bd}^* = \frac{\lambda-2\delta+\delta^2(1-\rho)}{(2+\delta(-1+\rho))^2}$. Furthermore, from $0 \leq \frac{\rho+p_1-\rho p_1}{2-\delta(1-\rho)} \leq \frac{2p_1+\delta\rho}{2-\delta(1-\rho)}$, we obtain p_1^* is at least $\frac{\rho-\delta\rho}{1+\rho}$. Given $p_1^* = \frac{\rho-\delta\rho}{1+\rho}$, we have $\pi_{bd}^* = \frac{\rho(1-\delta+\lambda\rho)}{(1+\rho)^2}$. Accordingly, in the case when $-3+2\delta-2\rho-2\delta\rho+\rho^2 \geq 0$, $\pi_{bd}^* \leq \max\{\frac{\lambda-2\delta+\delta^2(1-\rho)}{(2+\delta(-1+\rho))^2}, \frac{\rho(1-\delta+\lambda\rho)}{(1+\rho)^2}\}$.

(b) Else, when $0 < \rho \leq 3$; or $3 < \rho < 2+\sqrt{5}$ and $0 \leq \delta \leq \frac{-3-2\rho+\rho^2}{-2+2\rho}$ and $0 \leq \lambda < \frac{4-2\delta+2\delta\rho}{1-2\rho+\rho^2}$; or $3 < \rho < 2+\sqrt{5}$ and $\frac{-3-2\rho+\rho^2}{-2+2\rho} < \delta \leq 1$ and $\frac{4-2\delta+2\delta\rho}{1-2\rho+\rho^2} \leq \lambda \leq 1$; or $\rho > 2+\sqrt{5}$ and $0 \leq \lambda < \frac{4-2\delta+2\delta\rho}{1-2\rho+\rho^2}$, we have $\lambda-4+2\delta-2\lambda\rho-2\delta\rho+\lambda\rho^2 < 0$.

For $\lambda - 4 + 2\delta - 2\lambda\rho - 2\delta\rho + \lambda\rho^2 < 0$, we first calculate the derivative of $\pi_{bd}(p_1)$ with respect to p_1, which is given by

$$\frac{d\pi_{bd}(p_1)}{dp_1} = \frac{4 + \delta^2(1-\rho) + 2\lambda(-1+\rho)(\rho(1-p_1) - p_1) - 8p_1 + \delta(-4 + 2\rho + 4p_1 - 4\rho p_1)}{(2 - \delta(1-\rho))^2}$$

Setting $\frac{d\pi_{bd}(p_1)}{dp_1} = 0$, we get $p_1 = \frac{4+2\delta(-2+\rho)-\delta^2(-1+\rho)+2\lambda\rho(1-\rho)}{8+4\delta(-1+\rho)-2\lambda(-1+\rho)^2}$. Then we have $\frac{2p_1+\delta\rho}{2-\delta(1-\rho)} = \frac{2+\lambda\rho-\lambda\rho^2-\delta(1-2\rho)}{3-2\delta(1-\rho)+2\rho-\rho^2}$, $p_2 = \frac{\frac{2p_1+\delta\rho}{2-\delta(1-\rho)}+\rho(1-\frac{2p_1+\delta\rho}{2-\delta(1-\rho)})}{2} = \frac{\delta(-1+\rho)+2(1+\rho)}{8+4\delta(-1+\rho)+4\rho-2\lambda(-1+\rho)^2}$ and $\pi_{bd} = \frac{4-4\delta+\delta^2+4\lambda\rho}{16+8\delta(-1+\rho)-4\lambda(-1+\rho)^2}$. Checking the constraints, we have:

$$\begin{cases} \frac{4+2\delta(-2+\rho)-\delta^2(-1+\rho)+2\lambda\rho(1-\rho)}{6+4\delta(-1+\rho)+4\rho-2\rho^2} \leq \frac{2+\lambda\rho-\lambda\rho^2-\delta(1-2\rho)}{3-2\delta(1-\rho)+2\rho-\rho^2} \leq 1 \\ 0 \leq \frac{\delta(-1+\rho)+2(1+\rho)}{8+4\delta(-1+\rho)+4\rho-2\lambda(-1+\rho)^2} \leq \frac{2+\lambda\rho-\lambda\rho^2-\delta(1-2\rho)}{3-2\delta(1-\rho)+2\rho-\rho^2} \\ \frac{4+2\delta(-2+\rho)-\delta^2(-1+\rho)+2\lambda\rho(1-\rho)}{6+4\delta(-1+\rho)+4\rho-2\rho^2} \geq 0 \\ \frac{\delta(-1+\rho)+2(1+\rho)}{8+4\delta(-1+\rho)+4\rho-2\lambda(-1+\rho)^2} \geq 0 \end{cases}$$

Thus we can obtain: only when 1) $\rho \leq 1$; or 2) $1 < \rho < \frac{3+\sqrt{17}}{4}$ and $\frac{-2+2\rho^2}{-1+3\rho} \leq \delta \leq 1$; or 3) $1 < \rho < \frac{3+\sqrt{17}}{4}$ and $\frac{-2+2\rho}{-1+3\rho} < \delta < \frac{-2+2\rho^2}{-1+3\rho}$ and $0 \leq \lambda \leq \frac{2-\delta-2\rho+3\delta\rho}{-2\rho+2\rho^2}$; or 4) $\rho \geq \frac{3+\sqrt{17}}{4}$ and $\frac{-2+2\rho}{-1+3\rho} < \delta \leq 1$ and $0 \leq \lambda \leq \frac{2-\delta-2\rho+3\delta\rho}{-2\rho+2\rho^2}$ can the solution be valid satisfying all constraints. Next we consider boundary solutions when 1) $1 < \rho < \frac{3+\sqrt{17}}{4}$ and $0 < \delta < \frac{-2+2\rho^2}{-1+3\rho}$; or 2) $1 < \rho < \frac{3+\sqrt{17}}{4}$ and $\frac{-2+2\rho}{-1+3\rho} < \delta < \frac{-2+2\rho^2}{-1+3\rho}$ and $\frac{2-\delta-2\rho+3\delta\rho}{-2\rho+2\rho^2} < \lambda \leq 1$; or 3) $\rho \geq \frac{3+\sqrt{17}}{4}$ and $\frac{-2+2\rho}{-1+3\rho} < \delta \leq 1$ and $\frac{2-\delta-2\rho+3\delta\rho}{-2\rho+2\rho^2} < \lambda \leq 1$; or; 4) $\rho \geq \frac{3+\sqrt{17}}{4}$ and $0 < \delta \leq \frac{-2+2\rho}{-1+3\rho}$ and $0 \leq \lambda \leq \frac{2-\delta-2\rho+3\delta\rho}{-2\rho+2\rho^2}$.

i) If $p_1^* = 0$, $\tau_{bd} = \frac{\delta\rho}{2-\delta(1-\rho)}$ and $p_2^* = \frac{\rho}{2-\delta(1-\rho)}$. Thus $\pi_{bd}^* = \frac{\lambda\rho^2}{(2-\delta(1-\rho))^2}$. Then we check constraints:

1) $p_1 = 0 \leq \frac{\delta\rho}{2-\delta(1-\rho)} < 1$;

2) $\frac{\rho+p_1-\rho p_1}{2-\delta(1-\rho)} = \frac{\rho}{2-\delta(1-\rho)} \leq \frac{\delta\rho}{2-\delta(1-\rho)} \Rightarrow \delta = 1$;

3) $p_2 = \frac{\rho}{2-\delta(1-\rho)} > 0$.

There exists the interior solution only when $\delta = 1$, and we have $p_1^* = 0$, $p_2^* = \frac{\rho}{1+\rho}$, $\pi_{bd}^* = \frac{\lambda\rho^2}{(1+\rho)^2}$. Then we further discuss boundary solutions when $\delta < 1$.

1) If $\frac{\rho+p_1-\rho p_1}{2-\delta(1-\rho)}=0$, then no customer purchases in the second period and thus the firm's profit is $p_1(1-\tau_{bd}(p_1))=0$.

2) If $\frac{\rho+p_1-\rho p_1}{2-\delta(1-\rho)}=\frac{2p_1+\delta\rho}{2-\delta(1-\rho)}$, the solution $p_1^*=0$ only holds if $\rho=0$ or $\delta=1$, which is not within the feasible region. Thus we consider about the boundary solution.

a) If $p_1^*=\frac{2p_1+\delta\rho}{2-\delta(1-\rho)}$, then all customers purchase in the first period and the profit is 0.

b) If $\frac{\rho+p_1-\rho p_1}{2-\delta(1-\rho)}=0$, no customer purchases in the second period and thus the firm's profit is $p_1(1-\tau_{bd}(p_1))=0$.

Thus in the case of $p_1^*=0$, the firm's optimal profit is $\pi_{bd}^*=0$ when $\delta<1$; or $p_2^*=\frac{\rho}{1+\rho}$, $\pi_{bd}^*=\frac{\lambda\rho^2}{(1+\rho)^2}$ when $\delta=1$.

ii) If $p_1=\frac{2p_1+\delta\rho}{2-\delta(1-\rho)}$, we obtain $p_1=\frac{\rho}{\rho-1}$. Then $\bar{v}=\frac{2\frac{\rho}{\rho-1}+\delta\rho}{2-\delta(1-\rho)}=\frac{\rho}{\rho-1}$and $p_2=0$, with the profit $-\frac{\rho}{(\rho-1)^2}<0$, which will be surely dominated.

iii) If $\frac{2p_1+\delta\rho}{2-\delta(1-\rho)}=1$, we can have $p_1=1-\frac{\delta}{2}$. Then we can obtain $p_2=\frac{1}{2}$ with the profit $\frac{\lambda}{4}$. Then we check constraints:

1) $p_1\leq\frac{2p_1+\delta\rho}{2-\delta(1-\rho)}=1$;

2) $0<\frac{\rho+p_1-\rho p_1}{2-\delta(1-\rho)}=\frac{1}{2}<\frac{2p_1+\delta\rho}{2-\delta(1-\rho)}=1$;

3) $p_1=1-\frac{\delta}{2}>0$;

4) $p_2=\frac{1}{2}>0$.

Accordingly, the above solution is valid satisfying all constraints. Thus $p_1^*=1-\frac{\delta}{2}$, $p_2^*=\frac{1}{2}$ and thus $\pi_{bd}^*=\frac{\lambda}{4}$.

iv) If $\frac{\rho+p_1-\rho p_1}{2-\delta(1-\rho)}=\frac{2p_1+\delta\rho}{2-\delta(1-\rho)}$, we can obtain $p_1+\rho(-1+\delta+p_1)=0$. Then we have $p_1=\frac{\rho-\delta\rho}{1+\rho}$ and obtain $p_2=\frac{\rho}{\rho+1}$with the profit$\frac{\rho(1+\lambda\rho-\delta)}{(\rho+1)^2}$. Then we check constraints:

1) $p_1=\frac{\rho-\delta\rho}{1+\rho}\leq\frac{2p_1+\delta\rho}{2-\delta(1-\rho)}=\frac{\rho}{\rho+1}\leq 1$;

2) $0\leq\frac{\rho+p_1-\rho p_1}{2-\delta(1-\rho)}=\frac{\rho-\delta\rho}{1+\rho}$;

3) $p_1=\frac{\rho-\delta\rho}{1+\rho}\geq 0$;

4) $p_2 = \frac{\rho}{\rho+1} > 0$

Accordingly, the above solution is valid satisfying all constraints. So $p_1^* = \frac{\rho-\delta\rho}{1+\rho}$, $p_2^* = \frac{\rho}{\rho+1}$, and thus $\pi_{bd}^* = \frac{\rho(1+\lambda\rho-\delta)}{(\rho+1)^2}$.

Comparing i), ii), iii) and iv), we have $\frac{\rho(1+\lambda\rho-\delta)}{(1+\rho)^2} \geq \frac{\lambda\rho^2}{(1+\rho)^2}$ and $\frac{\rho(1+\lambda\rho-\delta)}{(\rho+1)^2} > \frac{\lambda}{4}$.

In summary, when 1) $\rho \leq 1$; or 2) $1 < \rho < \frac{3+\sqrt{17}}{4}$ and $\frac{-2+2\rho^2}{-1+3\rho} \leq \delta \leq 1$; or 3) $1 < \rho < \frac{3+\sqrt{17}}{4}$ and $\frac{-2+2\rho}{-1+3\rho} < \delta < \frac{-2+2\rho^2}{-1+3\rho}$ and $0 \leq \lambda \leq \frac{2-\delta-2\rho+3\delta\rho}{-2\rho+2\rho^2}$; or 4) $\rho \geq \frac{3+\sqrt{17}}{4}$ and $\frac{-2+2\rho}{-1+3\rho} < \delta \leq 1$ and $0 \leq \lambda \leq \frac{2-\delta-2\rho+3\delta\rho}{-2\rho+2\rho^2}$ (we denote it by condition 1), $p_1^* = \frac{4+2\delta(-2+\rho)-\delta^2(-1+\rho)+2\lambda\rho(1-\rho)}{8+4\delta(-1+\rho)-2\lambda(-1+\rho)^2}$, $p_2^* = \frac{\delta(-1+\rho)+2(1+\rho)}{8+4\delta(-1+\rho)+4\rho-2\lambda(-1+\rho)^2}$; else (we denote it condition 2), $p_1^* = 0$, $p_2^* = \frac{\rho}{1+\rho}$ when $\delta = 1$, or $\pi_{bd} = 0$ when $\delta < 1$.

The optimal profit is expressed as follows:

$$\pi_{bd}^* = \begin{cases} \frac{4-4\delta+\delta^2+4\lambda\rho}{16+8\delta(-1+\rho)-4\lambda(-1+\rho)^2}, \ condition\ 1 \\ \frac{\lambda\rho^2}{(1+\rho)^2}\ condition\ 2\ and\ \delta = 1 \\ 0\ condition\ 2\ and\ \delta < 1 \end{cases}$$

(2) For the full second-period market, all remaining customers make purchasing decisions in the second period and the demand is $\tau_{bd}(p_1)$. Recalling the previous analysis, if $\frac{p_1}{1-\delta} \leq 1$, $\tau_{bd} = \frac{p_1}{1-\delta}$ and the optimal second-period price is $\rho(1 - \frac{p_1}{1-\delta})$. Else, if $\frac{p_1}{1-\delta} > 1$, we obtain $\tau_{bd} = 1$, which means no sales occur in the first period, the second-period price is zero and the profit is zero, which is dominated.

Given $\tau_{bd}(p_1) = \frac{p_1}{1-\delta}$, the optimization is expressed by

$$\max_{p_1}\ \pi_{bd}(p_1) = p_1(1 - \frac{p_1}{1-\delta}) + \lambda\rho(1 - \frac{p_1}{1-\delta})\frac{p_1}{1-\delta}$$

$$\text{subject to} \begin{cases} 0 \leq \frac{p_1}{1-\delta} \leq 1 \\ p_1, p_2 \geq 0 \end{cases}$$

Rewriting the above expression, we can obtain

$$\max_{p_1}\ \pi_{bd}(p_1) = \frac{p_1(-1+\delta-\lambda\rho)(-1+\delta+p_1)}{(1-\delta)^2}$$

$$\text{subject to} \begin{cases} p_1 \leq 1-\delta \\ p_1, p_2 \geq 0 \end{cases}$$

The deviative of $\pi_{bd}(p_1)$ with respect to p_1 is given by

$$\frac{d\pi_{bd}(p_1)}{dp_1} = \frac{(-1+\delta-\lambda\rho)(-1+\delta+2p_1)}{(1-\delta)^2}$$

Let $\frac{d\pi_{bd}(p_1)}{dp_1} = 0$, we get $p_1 = \frac{1-\delta}{2}$. Then $p_2 = \rho(1-\frac{p_1}{1-\delta}) = \frac{\rho}{2}$ and the profit is$\frac{1-\delta+\rho}{4}$. Then we check constraints:

1) $p_1 \leq 1-\delta$ always holds;

2) $p_1 \geq 0$;

3) $p_2 > 0$.

The above solution is valid satisfying all constraints. So In this case, we have: $p_1^* = \frac{1-\delta}{2}$, $p_2^* = \frac{\rho}{2}$ and $\pi_{bd}^* = \frac{1-\delta+\lambda\rho}{4}$.

Comparing the cases in (1)(a) and (2), we have $\frac{1-\delta+\lambda\rho}{4} > \max\{\frac{\lambda-2\delta+\delta^2(1-\rho)}{(2+\delta(-1+\rho))^2}, \frac{\rho(1-\delta+\lambda\rho)}{(1+\rho)^2}\}$ when $\rho > 2+\sqrt{5}$ and $\frac{4-2\delta+2\delta\rho}{1-2\rho+\rho^2} \leq \lambda \leq 1$; or $3 < \rho < 2+\sqrt{5}$ and $0 \leq \delta \leq \frac{-3-2\rho+\rho^2}{-2+2\rho}$ and $\frac{4-2\delta+2\delta\rho}{1-2\rho+\rho^2} \leq \lambda \leq 1$, we have $\lambda - 4 + 2\delta - 2\lambda\rho - 2\delta\rho + \lambda\rho^2 \geq 0$. So we only need to compare (1)(b) and (2).

Then we obtain: when $0 < \rho \leq \bar{\rho}$; or $\bar{\rho} < \rho < \frac{-2+\delta}{-2+2\delta}$ and $0 \leq \lambda \leq \frac{1-\delta-\rho+3\delta\rho}{2\rho(-1+\rho)} - \frac{1}{2}\sqrt{\frac{-1+2\delta-\delta^2+\rho+2\delta\rho+\delta^2\rho}{(-1+\rho)\rho^2}}$, $p_1^* = \frac{4+2\delta(-2+\rho)-\delta^2(-1+\rho)+2\lambda\rho(1-\rho)}{6+4\delta(-1+\rho)+4\rho-2\rho^2}$, ${p_2}^* = \frac{\delta(-1+\rho)+2(1+\rho)}{8+4\delta(-1+\rho)+4\rho-2\lambda(-1+\rho)^2}$; otherwise when $\bar{\rho} < \rho < \frac{-2+\delta}{-2+2\delta}$ and $\frac{1-\delta-\rho+3\delta\rho}{2\rho(-1+\rho)} - \frac{1}{2}\sqrt{\frac{-1+2\delta-\delta^2+\rho+2\delta\rho+\delta^2\rho}{(-1+\rho)\rho^2}} < \lambda \leq 1$; or $\rho \geq \frac{-2+\delta}{-2+2\delta}$, $p_1^* = \frac{1-\delta}{2}$, $p_2^* = \frac{\rho}{2}$. Note that $\bar{\rho}$ is the second root of $1+\delta-\delta^2+(-1+2\delta+2\delta^2)\rho+(-1-3\delta)\rho^2+\rho^3 = 0$ and numerically $1 < \bar{\rho} < \frac{3}{2}$.

(ii) We then compare p_1^* and p_2^*.

(1) In the interval of $0 < \rho \leq \bar{\rho}$; or $\bar{\rho} < \rho < \frac{-2+\delta}{-2+2\delta}$ and $0 \leq \lambda \leq \frac{1-\delta-\rho+3\delta\rho}{2\rho(-1+\rho)} - \frac{1}{2}\sqrt{\frac{-1+2\delta-\delta^2+\rho+2\delta\rho+\delta^2\rho}{(-1+\rho)\rho^2}}$, $p_1^* > p_2^*$ if and only if: $0 < \rho \leq \frac{2-3\delta+\delta^2}{2-\delta+\delta^2}$; or $\frac{2-3\delta+\delta^2}{2-\delta+\delta^2} < \rho < \frac{\delta-\delta^2}{4} + \frac{\sqrt{16-24\delta+9\delta^2-2\delta^3+\delta^4}}{4}$ and $\frac{2-3\delta+\delta^2-2\rho+\delta\rho-\delta^2\rho}{-2\rho+2\rho^2} < \lambda \leq 1$.

(2) In the interval of $\bar{\rho} < \rho < \frac{-2+\delta}{-2+2\delta}$ and $\frac{1-\delta-\rho+3\delta\rho}{2\rho(-1+\rho)} - \frac{1}{2}\sqrt{\frac{-1+2\delta-\delta^2+\rho+2\delta\rho+\delta^2\rho}{(-1+\rho)\rho^2}} < \lambda \leq 1$; or $\rho \geq \frac{-2+\delta}{-2+2\delta}$, $p_1^* = \frac{1-\delta}{2}$ is always lower than $p_2^* = \frac{\rho}{2}$.

Proof of Lemma 3

(i) In the second period, the firm faces customers with valuations uniformly distributed in $[0, \tau_d]$. If $q_1 \leq -\tau_d - \rho(1-\tau_d)$, then no sales occur in the second period at any possible price $p_2 \geq 0$. Without loss of generality, we assume in this case $p_2^* = 0$ with zero profit. Then we analyze the case when $q_1 > -\tau_d - \rho(1-\tau_d)$. In this case the firm's expected second-period profit function is

$$\pi_2(p_2) = p_2 \min\left[\tau_d + q_1 + \rho(1-\tau_d) - p_2, \tau_d\right]^+$$

When $p_2 < q_1 + \rho(1-\tau_d)$, $\pi_2(p_2) = p_2\tau_d < [q_1 + \rho(1-\tau_d)]\tau_d$.

When $p_2 \geq q_1 + \rho(1-\tau_d)$, $\pi_2(p_2) = p_2[\tau_d + q_1 + \rho(1-\tau_d) - p_2]$. If $\frac{\tau_d+q_1+\rho(1-\tau_d)}{2} \geq q_1 + \rho(1-\tau_d) > -\tau_d$ (equivalently, $-\tau_d - \rho(1-\tau_d) < q_1 \leq \tau_d - \rho(1-\tau_d)$), its interior solution is optimal; $p_2 = \frac{\tau_d+q_1+\rho(1-\tau_d)}{2}$ with the objective value $(\frac{\tau_d+q_1+\rho(1-\tau_d)}{2})^2$. If $\frac{\tau_d+q_1+\rho(1-\tau_d)}{2} < q_1 + \rho(1-\tau_d)$ (equivalently, $q_1 > \tau_d - \rho(1-\tau_d)$), its boundary solution is optimal; $p_2^* = q_1 + \rho(1-\tau_d)$ with the objective value $(q_1 + \rho(1-\tau_d))\tau_d$. Summarizing the above cases, we can obtain the results in the lemma.

(ii) Customers in the second period purchase products only when their utilities are nonnegative, that is, $p_2^*(\tau_d, q_1) - q_1 - \rho(1-\tau_d) \leq v \leq \tau_d$.

Proof of Lemma 4

First we show that any pure-strategy purchasing equilibrium must follow a first-period threshold policy by contradiction. Given p_1, a fraction of $1-\chi$ customers purchase in the first period, and the firm's optimal second-period price is $p_2^*(\chi, q_1)$. If the purchasing equilibrium is not of a threshold type, i.e. there exists some customer with v_l who purchases in the first period, whereas another customer with v_h $(> v_l)$ does not. Defining

$$\Delta_i(\chi) = v_i - p_1 - \delta E[v_i + q_1 + \rho(1-\chi) - p_2^*(\chi, q_1)]^+, \ i = h, l,$$

we have $\Delta_l(\chi) \geq 0$, $\Delta_h(\chi) < 0$ so that $\Delta_h(\chi) - \Delta_l(\chi) < 0$. Define the sets $Q^l = \{q_1 : v_l + q_1 + \rho(1-\chi) - p_2^*(\chi, q_1) \geq 0\}$ and $Q^h = \{q_1 : v_h + q_1 + \rho(1-\chi) - p_2^*(\chi, q_1) \geq 0\}$. We have $Q^l \subseteq Q^h$. Define $Q^c = \{q : q_1 \in Q^h, q_1 \notin Q^l\}$ and note that at any $q_1 \in Q^c$, $v_l + q_1 + \rho(1-\chi) - p_2^*(\chi, q_1) < 0 \Rightarrow q_1 + \rho(1-\chi) - p_2^*(\chi, q_1) < -v_l$, and thus we get

$$v_h + q_1 + \rho(1-\chi) - p_2^*(\chi, q_1) < v_h - v_l.$$

In addition, we can obtain that:

$$\begin{aligned}\int_{q_1 \in Q^h} (v_h + q_1 + \rho(1-\chi) - p_2^*(\chi, q_1))F(q_1;\chi) = &\int_{q_1 \in Q^l} (v_h + q_1 + \rho(1-\chi) - p_2^*(\chi, q_1))F(q_1;\chi) \\ &+ \int_{q_1 \in Q^c} (v_h + q_1 + \rho(1-\chi) - p_2^*(\chi, q_1))F(q_1;\chi)\end{aligned}$$

We have

$$
\begin{aligned}
\Delta_h(\chi)-\Delta_l(\chi) &= v_h-p_1-\delta\int_{q_1\in Q^h}(v_h+q_1+\rho(1-\chi)-p_2^*(\chi,q_1))F(q_1;\chi)\\
&-(v_l-p_1-\delta\int_{q_1\in Q^l}(v_l+q_1+\rho(1-\chi)-p_2^*(\chi,q_1))F(q_1;\chi))\\
&=(v_h-v_l)-\delta\int_{q_1\in Q^l}(v_h+q_1+\rho(1-\chi)-p_2^*(\chi,q_1))F(q_1;\chi)\\
&-\delta\int_{q_1\in Q^c}(v_h+q_1+\rho(1-\chi)-p_2^*(\chi,q_1))F(q_1;\chi)\\
&+\delta\int_{q_1\in Q^l}(v_l+q_1+\rho(1-\chi)-p_2^*(\chi,q_1))F(q_1;\chi)\\
&=(v_h-v_l)-\delta\int_{q_1\in Q^l}(v_h-v_l)F(q_1;\chi)-\delta\int_{q_1\in Q^c}(v_h+q_1+\rho(1-\chi)-p_2^*(\chi,q_1))F(q_1;\chi)\\
&>(v_h-v_l)-\delta\int_{q_1\in Q^l}(v_h-v_l)F(q_1;\chi)-\delta\int_{q_1\in Q^c}(v_h-v_l)F(q_1;\chi)\\
&\text{(because for } q_1\in Q^c,\ v_h+q_1+\rho(1-\chi)-p_2^*(\chi,q_1)<v_h-v_l)\\
&=(v_h-v_l)-\delta\int_{q_1\in Q^h}(v_h-v_l)F(q_1;\chi)\\
&>0.
\end{aligned}
$$

This leads to a contradiction to the assumption. Therefore, the purchasing equilibrium is characterized by a threshold policy.

Next we establish the condition under which no sales occur in the first period. No sales occur in the first period if the customer with the highest valuation, 1, prefers to delay. That is, $1-p_1<\delta(1-p_2)$. When all customers choose to wait until the second period, $\tau_d(p_1)=1$. Thus there exists no social learning or externality and then the profit is $p_2(1-p_2)$. We can get that $p_2=\frac{1}{2}$. Subsituting $p_2=\frac{1}{2}$into $1-p_1<\delta(1-p_2)$, we obtain that $p_1>1-\frac{\delta}{2}$.

When there exist first-period sales, the threshold value w satisfies the following equations:

$$
w-p_1=\delta\int_{-w-\rho(1-w)}^{+\infty}(w+q_1+\rho(1-w)-p_2^*(q_1,w))f(q_1;w)dq_1.
$$

Proof of Proposition 4

(i) When no sales occur in the first period, $q_1 = 0$ and $\bar{v}_d = 1$. According to Lemma 3, $p_2^* = \frac{1}{2}$. Then the expected second-period utility is $\delta(1 - p_2^*) = \frac{\delta}{2}$, which is greater than the highest utility in the first period, $1 - p_1^*$, to guarantee no sales in the first period. So we obtain $p_1^* > 1 - \frac{\delta}{2}$ and the firm's expected profit is $\pi = \frac{\lambda}{4}$. We will show that there exists an alternative pricing plan, which does not induce adoption inertia and performs better. Suppose the firm announces a first-period price $p_1 = \frac{\lambda}{2}$. Since $\frac{\lambda}{2} \leq 1 - \frac{\delta}{2}$ for any $\delta \in [0,1]$, there exist sales in the first period. Accordingly, the threshold is some $\tau_d \in [p_1, 1]$. Suppose that the firm maintains $p_2 = \frac{1}{2}$. We next show that this policy performs better than adoption inertia. When no sales occur in the first period and $p_2 = \frac{1}{2}$, the second-period sale is $s_{no} = \frac{1}{2}$. We denote the total sales when sales occur in the first period as s, where

$$s = \begin{cases} 1 - \tau_d & q_1 \leq -\tau_d - \rho(1-\tau_d) + \frac{1}{2} \\ \frac{1}{2} + q_1 + \rho(1-\tau_d) & -\tau_d - \rho(1-\tau_d) + \frac{1}{2} < q_1 \leq -\rho(1-\tau_d) + \frac{1}{2} \\ 1 & q_1 > -\rho(1-\tau_d) + \frac{1}{2} \end{cases}$$

The expected sales are given by

$$\begin{aligned} E(s) = & \int_{-\infty}^{-\tau_d-\rho(1-\tau_d)+\frac{1}{2}} (1-\tau_d) f(q_1;\tau_d) dq_1 + \int_{-\tau_d-\rho(1-\tau_d)+\frac{1}{2}}^{-\rho(1-\tau_d)+\frac{1}{2}} (\frac{1}{2} + q_1 + \rho(1-\tau_d)) f(q_1;\tau_d) dq_1 \\ & + \int_{-\rho(1-\tau_d)+\frac{1}{2}}^{+\infty} f(q_1;\tau_d) dq_1 \\ \geq & \int_{-\infty}^{-\tau_d-\rho(1-\tau_d)+\frac{1}{2}} (1-\tau_d) f(q_1;\tau_d) dq_1 + \int_{-\tau_d-\rho(1-\tau_d)+\frac{1}{2}}^{\tau_d+\rho(1-\tau_d)-\frac{1}{2}} (\frac{1}{2} + q_1) f(q_1;\tau_d) dq_1 \\ & + \int_{\tau_d+\rho(1-\tau_d)-\frac{1}{2}}^{-\rho(1-\tau_d)+\frac{1}{2}} (\frac{1}{2} + q_1 + \rho(1-\tau_d)) f(q_1;\tau_d) dq_1 + \int_{-\rho(1-\tau_d)+\frac{1}{2}}^{+\infty} f(q_1;\tau_d) dq_1 \end{aligned}$$

For $\rho \in (0, \frac{1}{2}]$, since $\tau_d \in [\frac{1}{2}, 1]$, we have $\tau_d + \rho(1-\tau_d) - \frac{1}{2} \geq 0$ and $\tau_d + \rho(1-\tau_d) - \frac{1}{2} \leq -\rho(1-\tau_d) + \frac{1}{2}$. Thus we draw the following expression. We define

$$E(s_n) = \int_{-\infty}^{-\tau_d-\rho(1-\tau_d)+\frac{1}{2}} (1-\tau_d) f(q_1;\tau_d) dq_1 + \int_{-\tau_d-\rho(1-\tau_d)+\frac{1}{2}}^{\tau_d+\rho(1-\tau_d)-\frac{1}{2}} (\frac{1}{2} + q_1) f(q_1;\tau_d) dq_1$$

$$
\begin{aligned}
&+\int_{\tau_d+\rho(1-\tau_d)-\frac{1}{2}}^{-\rho(1-\tau_d)+\frac{1}{2}}(\frac{1}{2}+q_1+\rho(1-\tau_d))f(q_1;\tau_d)dq_1+\int_{-\rho(1-\tau_d)+\frac{1}{2}}^{+\infty}f(q_1;\tau_d)dq_1\\
=&\Big(\int_{-\tau_d-\rho(1-\tau_d)+\frac{1}{2}}^{\tau_d+\rho(1-\tau_d)-\frac{1}{2}}(\frac{1}{2}+q_1)f(q_1;\tau_d)dq_1+\int_{-\tau_d-\rho(1-\tau_d)+\frac{1}{2}}^{+\infty}f(q_1;\tau_d)dq_1\Big)\\
&+\Big(\int_{-\infty}^{\tau_d+\rho(1-\tau_d)-\frac{1}{2}}(1-\tau_d)f(q_1;\tau_d)dq_1+\int_{\tau_d+\rho(1-\tau_d)-\frac{1}{2}}^{-\rho(1-\tau_d)+\frac{1}{2}}(\frac{1}{2}+q_1+\rho(1-\tau_d)-1)f(q_1;\tau_d)dq_1\Big)\\
=&E(s_c)+E(s_d),
\end{aligned}
$$

where we define

$$
s_c=\begin{cases}0 & q_1\le -\tau_d-\rho(1-\tau_d)+\frac{1}{2}\\ \frac{1}{2}+q_1 & -\tau_d-\rho(1-\tau_d)+\frac{1}{2}<q_1\le \tau_d+\rho(1-\tau_d)-\frac{1}{2}\\ 1 & q_1>\tau_d+\rho(1-\tau_d)-\frac{1}{2}\end{cases}
$$

and

$$
s_d=\begin{cases}1-\tau_d & q_1\le -\tau_d-\rho(1-\tau_d)+\frac{1}{2}\\ q_1+\rho(1-\tau_d)-\frac{1}{2} & \tau_d+\rho(1-\tau_d)-\frac{1}{2}<q_1\le -\rho(1-\tau_d)+\frac{1}{2}\\ 0 & q_1>-\rho(1-\tau_d)+\frac{1}{2}\end{cases}
$$

We have:

$$
\begin{aligned}
E(s_c)&=\int_{-\infty}^{+\infty}s_c f(q_1;\tau_d)dq_1\\
&=\int_{-\tau_d-\rho(1-\tau_d)+\frac{1}{2}}^{\tau_d+\rho(1-\tau_d)-\frac{1}{2}}\frac{1}{2}f(q_1;\tau_d)dq_1+\int_{\tau_d+\rho(1-\tau_d)-\frac{1}{2}}^{+\infty}f(q_1;\tau_d)dq_1\\
&=\frac{1}{2}
\end{aligned}
$$

In addition, when $q_1 = \tau_d + \rho(1-\tau_d) - \frac{1}{2}$, we have $q_1 + \rho(1-\tau_d) - \frac{1}{2} = \tau_d + 2\rho(1-\tau_d) - 1 \leq 0$. Then we obtain:

$$\begin{aligned}
E(s_d) =& \int_{-\infty}^{\tau_d+\rho(1-\tau_d)-\frac{1}{2}} (1-\tau_d) f(q_1;\tau_d) dq_1 + \int_{\tau_d+\rho(1-\tau_d)-\frac{1}{2}}^{-\rho(1-\tau_d)+\frac{1}{2}} \Big(q_1 + \rho(1-\tau_d) - \frac{1}{2}\Big) f(q_1;\tau_d) dq_1 \\
>& \int_{-\infty}^{\tau_d+\rho(1-\tau_d)-\frac{1}{2}} (1-\tau_d) f(q_1;\tau_d) dq_1 + \int_{\tau_d+\rho(1-\tau_d)-\frac{1}{2}}^{+\infty} \Big(\tau_d + 2\rho(1-\tau_d) - 1\Big) f(q_1;\tau_d) dq_1 \\
>& \int_{-\infty}^{0} (1-\tau_d) f(q_1;\tau_d) dq_1 + \int_{0}^{+\infty} \Big(\tau_d + 2\rho(1-\tau_d) - 1\Big) f(q_1;\tau_d) dq_1 \\
>& \frac{1}{2}(1-\tau_d + \tau_d + 2\rho(1-\tau_d) - 1) \\
>& 0
\end{aligned}$$

Therefore, $E(s) > E(s_c) + E(s_d) > \frac{1}{2}$. The profit when $p_1 = p_2 = \frac{\lambda}{2}$ is greater than $\frac{\lambda}{4}$. So it is not optimal for the firm to induce the situation when no sales occur in the first period when $\rho \in (0, \frac{1}{2}]$. Rewriting $E(s_2)$, we have

$$\begin{aligned}
E(s_2) =& \int_{-\infty}^{-\tau_d-\rho(1-\tau_d)+\frac{1}{2}} (1-\tau_d) f(q_1;\tau_d) dq_1 + \int_{-\tau_d-\rho(1-\tau_d)+\frac{1}{2}}^{-\rho(1-\tau_d)+\frac{1}{2}} (\frac{1}{2} + q_1 + \rho(1-\tau_d)) f(q_1;\tau_d) dq_1 \\
&+ \int_{-\rho(1-\tau_d)+\frac{1}{2}}^{+\infty} f(q_1;\tau_d) dq_1 \\
=&(1-\tau_d)\Phi(\frac{\frac{1}{2}-\tau_d-\rho(1-\tau_d)}{\sigma_1(\tau_d)}) + (\frac{1}{2}+\rho(1-\tau_d))[\Phi(\frac{\tau_d+\rho(1-\tau_d)-\frac{1}{2}}{\sigma_1(\tau_d)}) - \Phi(\frac{\rho(1-\tau_d)-\frac{1}{2}}{\sigma_1(\tau_d)})] \\
&+ \Phi(\frac{\rho(1-\tau_d)-\frac{1}{2}}{\sigma_1(\tau_d)}) + \sigma_1(\tau_d)[\phi(\frac{\tau_d+\rho(1-\tau_d)-\frac{1}{2}}{\sigma_1(\tau_d)}) - \phi(\frac{\rho(1-\tau_d)-\frac{1}{2}}{\sigma_1(\tau_d)})]
\end{aligned}$$

Next, we take the derivative of $E(s_2)$ with respect to ρ, where

$$\begin{aligned}
\frac{\partial E(s_2)}{\partial \rho} =&(1-\tau_d)\frac{\tau_d-1}{\sigma_1(\tau_d)}\phi(\frac{\frac{1}{2}-\tau_d-\rho(1-\tau_d)}{\sigma_1(\tau_d)}) + (1-\tau_d)[\Phi(\frac{\tau_d+\rho(1-\tau_d)-\frac{1}{2}}{\sigma_1(\tau_d)}) - \Phi(\frac{\rho(1-\tau_d)-\frac{1}{2}}{\sigma_1(\tau_d)})] \\
&+ (\frac{1}{2}+\rho(1-\tau_d))\frac{1-\tau_d}{\sigma_1(\tau_d)}[\phi(\frac{\tau_d+\rho(1-\tau_d)-\frac{1}{2}}{\sigma_1(\tau_d)}) - \phi(\frac{\rho(1-\tau_d)-\frac{1}{2}}{\sigma_1(\tau_d)})] + \frac{1-\tau_d}{\sigma_1(\tau_d)}\phi(\frac{\rho(1-\tau_d)-\frac{1}{2}}{\sigma_1(\tau_d)}) \\
&- \sigma_1(\tau_d)\frac{1-\tau_d}{\sigma_1(\tau_d)}(\frac{\tau_d+\rho(1-\tau_d)-\frac{1}{2}}{\sigma_1(\tau_d)})\phi(\frac{\tau_d+\rho(1-\tau_d)-\frac{1}{2}}{\sigma_1(\tau_d)}) \\
&+ \sigma_1(\tau_d)\frac{1-\tau_d}{\sigma_1(\tau_d)}(\frac{\rho(1-\tau_d)-\frac{1}{2}}{\sigma_1(\tau_d)})\phi(\frac{\rho(1-\tau_d)-\frac{1}{2}}{\sigma_1(\tau_d)}) \\
=&(1-\tau_d)[\Phi(\frac{\tau_d+\rho(1-\tau_d)-\frac{1}{2}}{\sigma_1(\tau_d)}) - \Phi(\frac{\rho(1-\tau_d)-\frac{1}{2}}{\sigma_1(\tau_d)})] > 0
\end{aligned}$$

From the above equation, it follows that $E(s)$ is a monotonic increasing function about ρ. Therefore, in the interval $\rho \in (\frac{1}{2}, +\infty)$, $E(s)$ is strictly higher than $\frac{1}{2}$. Then it is always optimal for the firm to have sales in both periods.

(ii) Substituting p_1 and $\tau_d(p_1)$ into the result of Lemma 3, we obtain:

$$p_2^* = \begin{cases} 0 & q_1 \leq -\tau_d(p_1^*) - \rho(1-\tau_d(p_1^*)) \\ \frac{\tau_d(p_1^*)+q_1+\rho(1-\tau_d(p_1^*))}{2} & -\tau_d(p_1^*) - \rho(1-\tau_d(p_1^*)) < q_1 \leq \tau_d(p_1^*) - \rho(1-\tau_d(p_1^*)) \\ q_1 + \rho(1-\tau_d(p_1^*)) & q_1 > \tau_d(p_1^*) - \rho(1-\tau_d(p_1^*)) \end{cases}$$

Proof of Proposition 5

Using the optimal profit proved in Proposition 1 and Proposition 3, there are several cases:

(1) When the optimal profit under preannounced pricing is $\pi_{bp}^* = -\frac{\lambda(1-\delta+\lambda\rho)}{\delta^2-4\lambda+2\delta\lambda+\lambda^2-2\delta\rho\lambda-2\lambda^2\rho+\lambda^2\rho^2}$ and the optimal profit under dynamic pricing is $\pi_{bd}^* = \frac{4-4\delta+\delta^2+4\lambda\rho}{16+8\delta(-1+\rho)-4\lambda(-1+\rho)^2}$, in the intersection of this optimal strategy, we have $\pi_{bp}^* > \pi_{bd}^*$;

(2) When the optimal profit under preannounced pricing is $\pi_{bp}^* = -\frac{\lambda(1-\delta+\lambda\rho)}{\delta^2-4\lambda+2\delta\lambda+\lambda^2-2\delta\rho\lambda-2\lambda^2\rho+\lambda^2\rho^2}$ and the optimal profit under dynamic pricing is $\pi_{bd}^* = \frac{1-\delta+\lambda\rho}{4}$, in the intersection of this optimal strategy, we have $\pi_{bp}^* > \pi_{bd}^*$;

(2) When the optimal profit under preannounced pricing is $\pi_{bp}^* = \frac{1-\delta+\lambda\rho}{4}$ and the optimal profit under dynamic pricing is $\pi_{bd}^* = \frac{1-\delta+\lambda\rho}{4}$, in the intersection of this optimal strategy, we have $\pi_{bp}^* = \pi_{bd}^*$;

(3) When the optimal profit under preannounced pricing is $\pi_{bp}^* = \frac{1}{4}$ and the optimal profit under dynamic pricing is $\pi_{bd}^* = \frac{1-\delta+\lambda\rho}{4}$, in the intersection of this optimal strategy, we have $\pi_{bp}^* > \pi_{bd}^*$;

(4) When the optimal profit under preannounced pricing is $\pi_{bp}^* = \frac{1}{4}$ and the optimal profit under dynamic pricing is $\pi_{bd}^* = \frac{4-4\delta+\delta^2+4\lambda\rho}{16+8\delta(-1+\rho)-4\lambda(-1+\rho)^2}$, in the intersection of this optimal strategy, we have $\pi_{bp}^* > \pi_{bd}^*$.

Therefore, we have that when $\rho > 1+\delta$ and $\frac{\delta}{-1+\rho} < \lambda \leq 1$, the firm's optimal profit under preannounced pricing equals to that under dynamic pricing; else, the firm's optimal profit under preannounced pricing is higher than that under dynamic pricing.

Proof of Proposition 6

Let γ be an arbitrary value $k > 0$ and consider the extreme case $\delta = 0$. Let $\{p_1^*, p_2^*\}$ be the optimal preannounced price plan. Consider a dynamic price plan $\{p_1, p_2\}$, where $p_1 = p_1^*$, $p_2 = p_2^*$ if $q_1 \geq p_2^* - p_1^* - \rho(1-p_1^*)$ and $p_2 = p_2^* - \epsilon$ if $q_1 < p_2^* - p_1^* - \rho(1-p_1^*)$. ϵ is small and positive. Since $\delta = 0$, under both price plans, customers with valutions $v \geq p_1^*$ will purchase in the first period and remaining customers in the second period with valuations uniformly distributed in $[0, p_1^*]$ make purchasing decisions if $v + q_1 + \rho(1-p_1^*) - p_2 \geq 0$. Then the second-period sale is $\min[p_1^* - p_2 + \rho(1-p_1^*) + q_1, p_1^*]^+$. At any realization of $q_1 \geq p_2^* - p_1^* - \rho(1-p_1^*)$, the two price plans achieve identical first-period profits, identical reviews, and identical second-period profits. However, the firm can achieve a higher expected profit under dynamic pricing at least some realizations of $q_1 < p_2^* - p_1^* - \rho(1-p_1^*)$ because there exists no second-period sales under preannounced pricing strategy while the firm can achieve a positive second-period sale when setting $p_2 = p_2^* - \epsilon$ under dynamic pricing strategy. Then this dynamic price plan $\{p_1, p_2\}$ performs better than the optimal preannounced pricing policy. Therefore, we obtain $\pi_d^*|_{\gamma=k,\delta=0} - \pi_p^*|_{\gamma=k,\delta=0} > 0$. Note that π_d^* and π_p^* is both continuous for any $\delta \in [0,1]$ and $\rho \in [0, +\infty)$, then refer to the maximum theorem (see Sundaram 1996, Theorem 9.14), we get the existence of the threshold stated in the proposition.