Online Appendix

EC.1. Proofs

Proof of Lemma 1: Let us first provide detailed descriptions of the single-period model. First, given one-period contract (α, β), the agent of type $\mu \in \{h, l\}$ chooses the effort $e^\mu \geq 0$ that maximizes his payoff as follows:

$$\max_{e^\mu} \widetilde{\pi}^\mu(e^\mu; \alpha, \beta)\,,$$

where $\widetilde{\pi}^\mu(e^\mu; \alpha, \beta) := \alpha + \beta\mu e^\mu - \frac{1}{2}(e^\mu)^2$. Because $\widetilde{\pi}^\mu(e^\mu; \alpha, \beta)$ is concave in e^μ, we have $e^{\mu *} := \mu\beta$, which leads to $\widetilde{\pi}^\mu(e^{\mu *}; \alpha, \beta) = \alpha + \frac{1}{2}(\mu\beta)^2$.

Second, if the agent's type μ is *unknown* to the principal (i.e., the principal's belief on $\mu = h$ is $p \in (0,1)$), the principal chooses the contract (α, β) while ensuring that the agent accepts the contract as follows:

$$\max_{\alpha,\beta} \widetilde{\Pi}(\alpha, \beta; p) \tag{EC.1}$$

$$\text{s.t. } \widetilde{\pi}^h(e^{h*}; \alpha, \beta) \geq 0, \tag{EC.2}$$

$$\widetilde{\pi}^l(e^{l*}; \alpha, \beta) \geq 0\,, \tag{EC.3}$$

where $\widetilde{\Pi}(\alpha, \beta; p) := [phe^{h*} + (1-p)le^{l*}](1-\beta) - \alpha$ and $e^{\mu *} = \mu\beta$.

Third, if the agent's type μ is *known* to the principal (i.e., the principal's belief on $\mu = h$ is either $p = 1$ or $p = 0$), the principal's problem can be written as follows:

$$\max_{\alpha,\beta} \widetilde{\Pi}(\alpha, \beta; p) \tag{EC.4}$$

$$\text{s.t. } \widetilde{\pi}^\mu(e^{\mu *}; \alpha, \beta) \geq 0, \tag{EC.5}$$

where $\mu = h$ if $p = 1$ and $\mu = l$ if $p = 0$.

We now prove the statements of the lemma below.

(a) Suppose that $p \in (0,1)$. Note that the principal always wants (EC.3) to bind. Hence, the principal's payoff $\widetilde{\Pi}(\alpha, \beta; p)$ can be expressed as a function of β only and it is concave in β. Thus, we can obtain $\alpha^* = -\frac{1}{2}l^2(B(p))^2$ and $\beta^* = B(p)$ if $p \in (0,1)$, as clamed.

Therefore, if $p \in (0,1)$, then the high-type agent's equilibrium payoff is as follows:

$$\widetilde{\pi}^{h*}(p \in (0,1)) := \alpha^* + \frac{1}{2}h^2(\beta^*)^2 = \frac{1}{2}(h^2 - l^2)(B(p))^2\,, \tag{EC.6}$$

(b) Suppose that $p = 1$ or $p = 0$. Note that the principal always wants (EC.5) to bind. Hence, the principal's payoff $\widetilde{\Pi}(\alpha, \beta; p)$ can be expressed as a function of β only and it is concave in β. Thus, we can obtain $\alpha^* = -\frac{1}{2}h^2$ and $\beta^* = 1$ if $p = 1$ and $\alpha^* = -\frac{1}{2}l^2$ and $\beta^* = 1$ if $p = 0$, as clamed.

Recall from the first paragraph of the proof that the agent's optimal effort is $e^{\mu*} = \mu\beta$, which implies that it only depends on β. Therefore, the agent's first-best effort is achieved if and only if $\beta = 1$. ■

Proof of Lemma 2: Let $f(y_1^h) := \pi^h(y_1^h/h)$ for notational convenience. We first note that the high-type agent's expected payoff $f(\cdot)$ is a piecewise quadratic function of y_1^h; the first three terms in $\pi^h(y_1^h/h)$ is a quadratic function of y_1^h and $q_1(y_1^h)$ is a piecewise linear function of y_1^h. Specifically, the domain of $f(y_1^h)$ is partitioned into the four regions $I_1 := (-\infty, l^2\beta_1 - 2\sigma)$, $I_2 := [l^2\beta_1 - 2\sigma, l^2\beta_1)$, $I_3 := [l^2\beta_1, l^2\beta_1 + 2\sigma)$, and $I_4 := [l^2\beta_1 + 2\sigma, \infty)$, on each of which $f(y_1^h)$ is concave in y_1^h. Hence, taking the first derivative of $f(\cdot)$ on the first two regions I_1 and I_2 yields

$$\begin{aligned}
\lim_{y_1^h \uparrow l^2\beta_1 - 2\sigma} f'(y_1^h) &= \beta_1 - \frac{l^2\beta_1 - 2\sigma}{h^2} > \beta_1 - \frac{h^2\beta_1}{h^2} = 0 \\
\lim_{y_1^h \downarrow l^2\beta_1 - 2\sigma} f'(y_1^h) > \lim_{y_1^h \uparrow l^2\beta_1} f'(y_1^h) &= \beta_1 - \frac{l^2\beta_1}{h^2} + \frac{\delta}{4\sigma}(h^2 - l^2)B^2(p_1)\gamma \\
&> \beta_1 - \frac{h^2\beta_1}{h^2} + \frac{\delta}{4\sigma}(h^2 - l^2)B^2(p_1)\gamma \geq 0\,,
\end{aligned}$$

where we use the concavity of $f(y_1^h)$ and $h > l$. It thus follows that $y_1^{h*} \notin I_1$ and $y_1^{h*} \notin I_2$, which implies that $y_1^{h*} \geq l^2\beta_1$, i.e., $y_1^{h*} \in I_3$ or $y_1^{h*} \in I_4$.

Now, observe that the first derivatives of $f(y_1^h)$ on the two regions I_3 and I_4 are

$$\begin{aligned}
f'(y_1^h)\big|_{y_1^h \in I_3} &= \beta_1 - \frac{y_1^h}{h^2} - \frac{\delta}{4\sigma}(h^2 - l^2)B^2(p_1)\gamma \\
f'(y_1^h)\big|_{y_1^h \in I_4} &= \beta_1 - \frac{y_1^h}{h^2}\,,
\end{aligned}$$

each of which is equal to zero if and only if $y_1^h = y^M$ and $y_1^h = y^H$ respectively. Then because $f(y_1^h)$ is concave in y_1^h on each of the two regions I_3 and I_4, the maximizer of $f(y_1^h)$ on *each region* is characterized by the first order conditions $f'(y_1^h)\big|_{y_1^h \in I_3} = 0$ and $f'(y_1^h)\big|_{y_1^h \in I_4} = 0$ respectively with the possibility of a corner solution $y_1^{h*} = y^L = l^2\beta_1$ or $y_1^{h*} = l^2\beta_1 + 2\sigma$. Note that y_1^{h*} must be either the maximizer of $f(y_1^h)$ on I_3 or I_4. However, because $y^M \leq y^H$, we always have $f'(y^H)\big|_{y_1^h \in I_3} < 0$ and $f'(y^M)\big|_{y_1^h \in I_4} > 0$, which discards the possbility that $y_1^{h*} = l^2\beta_1 + 2\sigma$. Therefore, we can conclude that y_1^{h*} is always one of the three values y^H, y^M, or y^L.

Lastly, note that $y^M > y^L$ if and only if $f'(y^L)\big|_{y_1^h \in I_3} > 0$, which is a necessary condition for $y_1^{h*} = y^M$. Hence, we must have $y^H > y^M > y^L$ whenever y^M is a possbility for y_1^{h*}. ■

Proof of Proposition 1: First of all, we have already proved in Lemma 2 that y_1^{h*} must be y^H, y^M, or y^L which are all independent of α_1 and dependent only on β_1.

Second, we claim that $y_1^{h*} \neq y^M$ if $\sigma < \widehat{\sigma}$. We first note that if $\beta_1 < \underline{\beta}_1$, then we have $y^M < y^L = l^2\beta_1$. In the proof of Lemma 2, however, we already have shown that $y_1^{h*} \geq l^2\beta_1$, which implies

that $y_1^{h*} \neq y^M$ if $\beta_1 < \underline{\beta}_1$. We next consider the case when $\beta_1 \geq \underline{\beta}_1$, in which case $y^M > y^L = l^2\beta_1$. Observe that under the condition $\sigma < \widehat{\sigma}$, we always have

$$\underline{\beta}_1 > \frac{2\sigma}{h^2 - l^2}\,,$$

which implies that $y^H = h^2\beta_1 > l^2\beta_1 + 2\sigma$, i.e., y^H belongs to the *interior* of $I_4 = [l^2\beta_1 + 2\sigma, \infty)$ whenever $\beta_1 \geq \underline{\beta}_1$. Then because the function $\pi^h(y_1^h/h)$ is concave in y_1^h on I_4 and $\pi^{h'}(y^H/h)|_{I_4} = 0$, the value of y_1^h that maximizes $\pi^h(y_1^h/h)$ on the region I_4 is always y^H. In other words,

$$\max_{y_1^h \in I_4} \pi^h\left(\frac{y_1^h}{h}\right) = \pi^h\left(\frac{y^H}{h}\right), \tag{EC.7}$$

under the condition that $\sigma < \widehat{\sigma}$ and $\beta_1 > \underline{\beta}_1$. Then because $y_1^{h*} \geq l^2\beta_1$ once again by the proof of Lemma 2, the characterization of y_1^{h*} in this case only requires to examine the maximum of $\pi^h(y_1^h/h)$ on $y_1^h \in I_3 = [l^2\beta_1, l^2\beta_1 + 2\sigma)$ and compare it with $\pi^h(y^H/h)$. There are indeed two sub-cases to investigate: (i) $\beta_1 < \underline{\beta}_1 + \frac{2\sigma}{h^2 - l^2}$ and $\beta_1 \geq \underline{\beta}_1 + \frac{2\sigma}{h^2 - l^2}$. In case (i), it can be seen that $y^M < l^2\beta_1 + 2\sigma$, which implies that y^M belongs to the *interior* of I_3. Based on the facts that $\pi^h(y_1^h/h)$ is concave in y_1^h on I_3 and $\pi^{h'}(y^M/h)|_{I_3} = 0$, therefore, we have

$$\max_{y_1^h \in I_3} \pi^h\left(\frac{y_1^h}{h}\right) = \pi^h\left(\frac{y^M}{h}\right). \tag{EC.8}$$

Now, a series of algebra reveals that if $\beta_1 > \frac{1}{2}\underline{\beta}_1 + \frac{2\sigma}{h^2 - l^2}$, then we have

$$\pi^h\left(\frac{y^H}{h}\right) > \pi^h\left(\frac{y^M}{h}\right). \tag{EC.9}$$

Also, it is easy to see from direct calculations that $\sigma < \widehat{\sigma}$ if and only if $\underline{\beta}_1 > \frac{1}{2}\underline{\beta}_1 + \frac{2\sigma}{h^2 - l^2}$. Therefore, under the condition $\sigma < \widehat{\sigma}$ and $\beta_1 \geq \underline{\beta}_1$, we have $\pi^h(y^H/h) > \pi^h(y^M/h)$, which implies that $y_1^{h*} \neq y^M$. In case (ii), we note that $y^M \geq l^2\beta_1 + 2\sigma$. Then because $\pi^h(y_1^h/h)$ is concave in y_1^h on I_3 and $\pi^{h'}(y_1^h/h)|_{I_3} > 0$ for any $y_1^h < y^M$, it must be the case that

$$\max_{y_1^h \in I_3} \pi^h\left(\frac{y_1^h}{h}\right) < \pi^h\left(\frac{l^2\beta_1 + 2\sigma}{h}\right) \leq \max_{y_1^h \in I_4} \pi^h\left(\frac{y_1^h}{h}\right) = \pi^h\left(\frac{y^H}{h}\right),$$

which results in $y_1^{h*} \neq y^M$ once again. Therefore, we can conclude that $y_1^{h*} \neq y^M$ if $\sigma < \widehat{\sigma}$ as claimed.

Third, we will show that if $\sigma < \widehat{\sigma}$, we have $y_1^{h*} = y^L$ if $\beta_1 < \widehat{\beta}_1$ and $y_1^{h*} = y^H$ if $\beta_1 \geq \widehat{\beta}_1$. Observe that under the condition $\sigma < \widehat{\sigma}$, we always have

$$\frac{2\sigma}{h^2 - l^2} < \widehat{\beta}_1 < \underline{\beta}_1\,, \tag{EC.10}$$

which can be obtained from a series of calculations. Moreover, it can be easily seen that

$$\lim_{y_1^h \downarrow l^2\beta_1} \pi^{h'}\left(\frac{y_1^h}{h}\right) = \frac{h^2 - l^2}{h^2}(\beta_1 - \underline{\beta}_1)\,. \tag{EC.11}$$

Now, suppose that $\beta_1 < \widehat{\beta}_1$, which implies that $\beta_1 < \underline{\beta}_1$ by (EC.10), and hence by (EC.11), we have $\lim_{y_1^h \downarrow l^2\beta_1} \pi^{h'}(y_1^h/h) < 0$, which implies that $y^L = l^2\beta_1 = \arg\max_{y_1^h \in I_3} \pi^h(y_1^h/h)$ by the concavity of $\pi^h(y_1^h/h)$ on I_3. We examine two sub-cases: (i) $\beta_1 < \frac{2\sigma}{h^2-l^2}$ and (ii) $\beta_1 \geq \frac{2\sigma}{h^2-l^2}$. In case (i), we have $y^H = h^2\beta_1 < l^2\beta_1 + 2\sigma$, which implies that $\lim_{y_1^h \downarrow l^2\beta_1 + 2\sigma} \pi^{h'}(y_1^h/h) < 0$ because the function $\pi^h(y_1^h/h)$ is concave in y_1^h on $I_4 = [l^2\beta_1 + 2\sigma, \infty)$ and $\pi^{h'}(y_1^h/h)|_{I_4} < 0$ for all $y_1^h > y^H$. Hence, it must be the case $\pi^{h'}(y_1^h/h) < 0$ for all $y_1^h \geq l^2\beta_1$ by the (piecewise) concavity of $\pi^h(y_1^h/h)$. Combining this with the fact that $y_1^{h*} \geq y^L = l^2\beta_1$ (by the proof of Lemma 2), the function $\pi^h(y_1^h/h)$ must be maximized at $y_1^{h*} = y^L$. In case (ii), we have $y^H = h^2\beta_1 \geq l^2\beta_1 + 2\sigma$, in which case (EC.7) holds by the same argument. However, we have

$$\pi^h\left(\frac{y^L}{h}\right) - \pi^h\left(\frac{y^H}{h}\right) = -\frac{(h^2-l^2)^2}{2h^2}((\beta_1)^2 - (\widehat{\beta}_1)^2)\,, \tag{EC.12}$$

which must be strictly positive because $\beta_1 < \widehat{\beta}_1$. It thus follows that $y_1^{h*} = y^L$ in this case too.

Next, suppose that $\beta_1 \geq \widehat{\beta}_1$, which implies that $\beta_1 > \frac{2\sigma}{h^2-l^2}$ by (EC.10), and hence, the equation (EC.7) holds. Note that because the possibility $y^{h*} = y^M$ is crossed out under the condition $\sigma < \widehat{\sigma}$, we have $y^{h*} = y^L$ or $y^{h*} = y^H$. Also, the expression (EC.12) can never be strictly positive because $\beta_1 \geq \widehat{\beta}_1$. Therefore, we can conclude that $y^{h*} = y^H$ if $\beta_1 \geq \widehat{\beta}_1$.

Fourth, we will show that if $\sigma \geq \widehat{\sigma}$, we have $y_1^{h*} = y^L$ if $\beta_1 < \underline{\beta}_1$, $y_1^{h*} = y^M$ if $\underline{\beta}_1 \leq \beta_1 < \overline{\beta}_1$, and $y_1^{h*} = y^H$ if $\beta_1 \geq \overline{\beta}_1$.

Suppose first that $\beta_1 < \underline{\beta}_1$. Then we have already proved in the case of $\sigma < \widehat{\sigma}$ that $y_1^{h*} \neq y^M$ if $\beta_1 < \underline{\beta}_1$, and this result is independent of the value of σ. In other words, $y_1^{h*} = y^L$ or $y_1^{h*} = y^H$ in this case. Also, under the condition that $\sigma \geq \widehat{\sigma}$, we have $\widehat{\beta}_1 > \underline{\beta}_1$, which implies that $\beta_1 < \underline{\beta}_1 < \widehat{\beta}_1$. Then the argument used to prove $y_1^{h*} = y^L$ for the case when $\sigma < \widehat{\sigma}$ and $\beta_1 < \underline{\beta}_1$ applies.

Next, suppose that $\underline{\beta}_1 \leq \beta_1 < \overline{\beta}_1$. Note that $y^M > y^L = l^2\beta_1$ whenever $\beta_1 > \underline{\beta}_1$. Also, it is easy to see from a few calculations that $y^M < l^2\beta_1 + 2\sigma$ if $\beta_1 < \underline{\beta}_1 + \frac{2\sigma}{h^2-l^2}$, which is satisfied whenever $\beta_1 < \overline{\beta}_1$. Hence, it follows that the equation (EC.8) holds. Then a series of algebra can tell us that if $\beta_1 < \overline{\beta}_1$, then we have

$$\pi^h\left(\frac{y^H}{h}\right) < \pi^h\left(\frac{y^M}{h}\right),$$

which implies that $y_1^{h*} = y^M$.

Last, suppose that $\beta_1 \geq \overline{\beta}_1$. Then because $\overline{\beta}_1 > \frac{2\sigma}{h^2-l^2}$, it must be the case that $y^H > l^2 + 2\sigma$, and hence, (EC.7) holds. Also, because $\overline{\beta}_1 = \frac{1}{2}\underline{\beta}_1 + \frac{2\sigma}{h^2-l^2}$, the equation (EC.9) is valid by the same reason. Combining these two, we can conclude that $y_1^{h*} = y^H$ as desired. ■

Proof of Proposition 2: (a) Let $\sigma < \widehat{\sigma}$. Recall from Proposition 1 that $y_1^{h*}(\beta_1) = y^H = h^2\beta_1$ if $\beta_1 \geq \widehat{\beta}_1$ and $y_1^{h*}(\beta_1) = y^L = l^2\beta_1$ if $\beta_1 < \widehat{\beta}_1$. Also, it is easy to see that

$$\frac{2\widehat{\sigma}}{h^2-l^2} = \frac{1}{2}\widehat{\beta}_1\,.$$

Hence, if $\beta_1 \geq \widehat{\beta}_1$, then we have

$$\beta_1 \geq \widehat{\beta}_1 > \frac{1}{2}\widehat{\beta}_1 = \frac{2\widehat{\sigma}}{h^2 - l^2} > \frac{2\sigma}{h^2 - l^2}\,,$$

which implies that $h^2\beta_1 - l^2\beta_1 > 2\sigma$. It thus follows that $q_1(y_1^{h*}) = 0$ if $y_1^{h*} = h^2\beta_1$ and $q_1(y_1^{h*}) = 1$ if $y_1^{h*} = l^2\beta_1$. Therefore, we can write $\Pi(\beta_1) = V^H(\beta_1)$ if $\beta_1 \geq \widehat{\beta}_1$ and $\Pi(\beta_1) = V^L(\beta_1)$ if $\beta_1 < \widehat{\beta}_1$ where

$$\begin{aligned} V^H(\beta_1) :=\ & l^2\beta_1 - \frac{1}{2}l^2\beta_1^2 + \delta\frac{1}{2}[p_1h^2 + (1-p_1)l^2]B(p_1) \\ & - p_1(h^2 - l^2)\beta_1(\beta_1 - 1) + \delta\left[\frac{1}{2}(p_1h^2 + (1-p_1)l^2)(1 - B(p_1)) - p_1(1-\gamma)\frac{1}{2}(h^2 - l^2)B^2(p_1)\right], \end{aligned} \tag{EC.13}$$

$$V^L(\beta_1) := l^2\beta_1 - \frac{1}{2}l^2\beta_1^2 + \delta\frac{1}{2}[p_1h^2 + (1-p_1)l^2]B(p_1)\,. \tag{EC.14}$$

Note that we always have

$$\max_{\beta_1}\Pi(\beta_1) = \max\{\max_{\beta_1 \geq \widehat{\beta}_1}\Pi(\beta_1), \max_{\beta_1 < \widehat{\beta}_1}\Pi(\beta_1)\} = \max\{\max_{\beta_1 \geq \widehat{\beta}_1} V^H(\beta_1), \max_{\beta_1 < \widehat{\beta}_1} V^L(\beta_1)\}\,.$$

Next, observe that $\frac{1}{2}(p_1h^2 + (1-p_1)l^2)(1 - B(p_1)) - p_1(1-\gamma)\frac{1}{2}(h^2 - l^2)B^2(p_1) > 0$. Hence, we have $V^H(\beta_1) > V^L(\beta_1)$ for $0 \leq \beta_1 \leq 1$. Also, because $V^H(\beta_1) - V^L(\beta_1)$ is a quadratic function of β_1 and the product of its two roots is negative, it must have only one positive root, which implies that there exist some $\beta_1' > 1$ such that $V^H(\beta_1) > V^L(\beta_1)$ for $0 \leq \beta_1 < \beta_1'$ and $V^H(\beta_1) < V^L(\beta_1)$ for $\beta_1 > \beta_1'$. Moreover, because the function $V^L(\beta_1)$ is maximized at $\beta_1 = 1$, there uniquely exists β_1^{HL} such that $1 < \beta_1^{HL} < \beta_1'$ and $V^H(\beta^{HL}) = V^L(1)$. Last, note that the function $V^H(\beta_1)$ is maximized at $B(p_1) < 1$.

We now examine three cases. First, if $\widehat{\beta}_1 \leq B(p_1)$, then we have

$$\begin{aligned} \max_{\beta_1 \geq \widehat{\beta}_1}\Pi(\beta_1) = \max_{\beta_1 \geq \widehat{\beta}_1} V^H(\beta_1) &= V^H(B(p_1)) \\ &\geq V^H(\widehat{\beta}_1) > V^L(\widehat{\beta}_1) > \max_{\beta_1 < \widehat{\beta}_1} V^L(\beta_1) = \max_{\beta_1 < \widehat{\beta}_1}\Pi(\beta_1)\,, \end{aligned}$$

where we use the facts that $\widehat{\beta}_1 \leq B(p_1) < 1 < \beta_1'$. Hence, we have $\beta_1^* = B(p_1)$ and $y_1^{h*} = y^H$ if $\widehat{\beta}_1 \leq B(p_1)$. Second, if $B(p_1) < \widehat{\beta}_1 \leq \beta_1^{HL}$, then we have

$$\begin{aligned} \max_{\beta_1 \geq \widehat{\beta}_1}\Pi(\beta_1) = \max_{\beta_1 \geq \widehat{\beta}_1} V^H(\beta_1) &= V^H(\widehat{\beta}_1) \\ &\geq V^H(\beta_1^{HL}) = V^L(1) \geq \max_{\beta_1 < \widehat{\beta}_1} V^L(\beta_1) = \max_{\beta_1 < \widehat{\beta}_1}\Pi(\beta_1)\,, \end{aligned}$$

where we use the facts that $B(p_1) < \widehat{\beta}_1 \leq \beta_1^{HL}$ and the definition of β_1^{HL}. Thus, we have $\beta_1^* = \widehat{\beta}_1$ and $y_1^{h*} = y^H$ if $B(p_1) < \widehat{\beta}_1 \leq \beta_1^{HL}$. Lastly, if $\widehat{\beta}_1 > \beta_1^{HL}$, then we have

$$\begin{aligned} \max_{\beta_1 \geq \widehat{\beta}_1}\Pi(\beta_1) = \max_{\beta_1 \geq \widehat{\beta}_1} V^H(\beta_1) &= V^H(\widehat{\beta}_1) \\ &< V^H(\beta_1^{HL}) = V^L(1) = \max_{\beta_1 < \widehat{\beta}_1} V^L(\beta_1) = \max_{\beta_1 < \widehat{\beta}_1}\Pi(\beta_1)\,, \end{aligned}$$

where we use the facts that $\widehat{\beta}_1 > \beta_1^{HL} > 1 > B(p_1)$. Hence, we have $\beta_1^* = 1$ and $y_1^{h*} = y^L$ if $\widehat{\beta}_1 > \beta_1^{HL}$.

Finally, it can be shown that there exists some threshold $\underline{v}$ such that $\widehat{\beta}_1 > \beta_1^{HL}$ if $v < \underline{v}$ and $\widehat{\beta}_1 \le \beta_1^{HL}$ otherwise. Hence, we have proved the statement (a).

(b) We consider $v < \underline{v}$ and $v > \overline{v}$ respectively.

Case of $v < \underline{v}$: In this case, $\widehat{\beta}_1 > \beta_1^{HL}$ so that we have $y_1^{h*}(\beta_1^*) = y^L$ for any $\sigma < \widehat{\sigma}$ (followed by part (a) of this proposition). Now, let $\sigma \ge \widehat{\sigma}$. Then by Proposition 1, we have $y_1^{h*}(\beta_1) = y^H$ if $\beta_1 \ge \overline{\beta}_1$, $y_1^{h*}(\beta_1) = y^M$ if $\underline{\beta}_1 \le \beta_1 < \overline{\beta}_1$, and $y_1^{h*}(\beta_1) = y^L$ if $\beta_1 < \underline{\beta}_1$. Also, it is easy to see that

$$\overline{\beta}_1 > \frac{2\sigma}{h^2 - l^2}\,,$$

which implies that whenever $\beta_1 \ge \overline{\beta}_1$, we have

$$\beta_1 \ge \overline{\beta}_1 > \frac{2\sigma}{h^2 - l^2}\,,$$

from which it follows that $h^2\beta_1 - l^2\beta_1 > 2\sigma$ if $\beta_1 \ge \overline{\beta}_1$. Hence, similarly as in the proof for part (a), if $y_1^{h*}(\beta_1) = h^2\beta_1$, which is the case whenever $\beta_1 > \overline{\beta}_1$, then we always have $q_1(y_1^{h*}) = 0$; in other words, if β_1 is large enough so that the agent produces y^H on average, then it is always the case that the agent reveals his capability to the principal with probability one. Therefore, we can write $\Pi(\beta_1) = V^H(\beta_1)$ if $\beta_1 \ge \overline{\beta}_1$, $\Pi(\beta_1) = V^M(\beta_1)$ if $\underline{\beta}_1 \le \beta_1 < \overline{\beta}_1$, and $\Pi(\beta_1) = V^L(\beta_1)$ if $\beta_1 < \underline{\beta}_1$ where $V^H(\cdot)$ and $V^L(\cdot)$ are given in (EC.13) and (EC.14) respectively, and $V^M(\cdot)$ is given as follows:

$$V^M(\beta_1) := l^2\beta_1 - \frac{1}{2}l^2\beta_1^2 + \delta\frac{1}{2}[p_1h^2 + (1-p_1)l^2]\beta_2^*(p_1) - p_1(h^2 - l^2)(\beta_1 - \underline{\beta}_1)(\beta_1 - \tilde{\beta}_1) \tag{EC.15}$$

$$\tilde{\beta}_1 := 1 + \frac{\delta}{2\sigma}\left[\frac{1}{2}(h^2 + \frac{(1-p_1)}{p_1}l^2)(1 - \beta_2^*(p_1)) - (1-\gamma)\frac{1}{2}(h^2 - l^2)B^2(p_1)\right]. \tag{EC.16}$$

Note that we always have

$$\begin{aligned}\max_{\beta_1}\Pi(\beta_1) &= \max\{\max_{\beta_1 \ge \overline{\beta}_1}\Pi(\beta_1), \max_{\underline{\beta}_1 \le \beta_1 < \overline{\beta}_1}\Pi(\beta_1), \max_{\beta_1 < \underline{\beta}_1}\Pi(\beta_1)\},\\ &= \max\{\max_{\beta_1 \ge \overline{\beta}_1}V^H(\beta_1), \max_{\underline{\beta}_1 \le \beta_1 < \overline{\beta}_1}V^M(\beta_1), \max_{\beta_1 < \underline{\beta}_1}V^L(\beta_1)\}\,.\end{aligned}$$

Next, note from Proposition 1 that $\overline{\beta}_1 \ge \widehat{\beta}_1$ for any $\sigma \ge \widehat{\sigma}$ (the equality holds if and only if $\sigma = \widehat{\sigma}$), which implies that $\overline{\beta}_1 > \beta_1^{HL} > 1$ by our assumption $\widehat{\beta}_1 > \beta_1^{HL}$. Also, observe that the two functions $V^M(\beta_1)$ and $V^L(\beta_1)$ meet twice at $\beta_1 = \underline{\beta}_1$ and $\beta_1 = \tilde{\beta}_1 > 1$ respectively. Moreover, $V^M(\beta_1) > V^L(\beta_1)$ if $\underline{\beta}_1 < \beta_1 < \tilde{\beta}_1$ and the function $V^L(\beta_1)$ attains its unique maximum at $\beta_1 = 1$. We now examine two cases: First, if $\underline{\beta}_1 \le 1$, then we have

$$\max_{\beta_1 < \underline{\beta}_1} V^L(\beta_1) \le V^L(1) < V^M(1) \le \max_{\underline{\beta}_1 \le \beta_1 < \overline{\beta}_1} V^M(\beta_1)\,,$$

where the second inequality holds because $\underline{\beta}_1 \le 1 < \tilde{\beta}_1$ and the last inequality follows from the fact that $\underline{\beta}_1 \le 1 < \overline{\beta}_1$. Second, if $\underline{\beta}_1 > 1$, on the other hand, we have

$$\max_{\beta_1 < \underline{\beta}_1} V^L(\beta_1) = V^L(1)\,.$$

In either case, therefore, we can deduce that

$$\max\{\max_{\underline{\beta}_1 \le \beta_1 < \overline{\beta}_1} V^M(\beta_1), \max_{\beta_1 < \underline{\beta}_1} V^L(\beta_1)\} \ge V^L(1)\,.$$

Finally, because the function $V^H(\beta_1)$ attains its unique maximum at $B(p_1) < 1$, we have

$$\begin{aligned}\max_{\beta_1 \ge \overline{\beta}_1} \Pi(\beta_1) = \max_{\beta_1 \ge \overline{\beta}_1} V^H(\beta_1) &= V^H(\overline{\beta}_1) \\ &< V^H(\beta_1^{HL}) = V^L(1) \le \max\{\max_{\underline{\beta}_1 \le \beta_1 < \overline{\beta}_1} V^M(\beta_1), \max_{\beta_1 < \underline{\beta}_1} V^L(\beta_1)\}\,,\end{aligned}$$

which implies that $\beta_1^* < \overline{\beta}_1$, i.e., $y_1^{h*}(\beta_1^*) = y^L$ or y^M for any $\sigma \ge \widehat{\sigma}$ as desired.

Case of $v > \overline{v}$: In this case, $\widehat{\beta}_1 \le 1 < \beta_1^{HL}$ so that we have $y_1^{h*}(\beta_1^*) = y^H$ for any $\sigma < \widehat{\sigma}$ (followed by part (a) of this proposition). Now, let $\sigma \ge \widehat{\sigma}$. Then similarly as in the proof for part (b) of this proposition, we can write $\Pi(\beta_1) = V^H(\beta_1)$ if $\beta_1 \ge \overline{\beta}_1$, $\Pi(\beta_1) = V^M(\beta_1)$ if $\underline{\beta}_1 \le \beta_1 < \overline{\beta}_1$, and $\Pi(\beta_1) = V^L(\beta_1)$ if $\beta_1 < \underline{\beta}_1$ where $V^H(\cdot)$, $V^M(\cdot)$, and $V^L(\cdot)$ are given in (EC.13), (EC.16), and (EC.14) respectively.

Next, observe that the two functions $V^M(\beta_1)$ and $V^L(\beta_1)$ meet twice at $\beta_1 = \underline{\beta}_1$ and $\beta_1 = \tilde{\beta}_1 > 1$ respectively. Also, $V^M(\beta_1) > V^L(\beta_1)$ if $\underline{\beta}_1 < \beta_1 < \tilde{\beta}_1$ and the function $V^L(\beta_1)$ attains its unique maximum at $\beta_1 = 1$. Moreover, note from Proposition 1 that $\underline{\beta}_1 \le \widehat{\beta}_1$ for any $\sigma \ge \widehat{\sigma}$ (the equality holds if and only if $\sigma = \widehat{\sigma}$), which implies that $\underline{\beta}_1 \le 1$ by our assumption $\widehat{\beta}_1 \le 1$. We thus obtain

$$\max_{\beta_1 < \underline{\beta}_1} V^L(\beta_1) = V^L(\underline{\beta}_1) = V^M(\underline{\beta}_1) < \max_{\underline{\beta}_1 \le \beta_1 < \overline{\beta}_1} V^M(\beta_1)\,.$$

Therefore, it follows that $\beta_1^* \ge \underline{\beta}_1$, i.e., $y_1^{h*}(\beta_1^*) = y^H$ or y^M for any $\sigma \ge \widehat{\sigma}$ as desired. ■

Proof of Proposition 3: (a) We first examine a special case when $\sigma = 0$. In this case, the principal's expected payoffs under $\gamma = 0$ and $\gamma = 1$ can be written as follows:

$$\begin{aligned}\Pi(\beta_1, \gamma = 0) &= l^2\beta_1 - \frac{1}{2}l^2\beta_1^2 + \delta\frac{1}{2}[p_1h^2 + (1-p_1)l^2]B(p_1) \\ &\quad - p_1(h^2 - l^2)\beta_1(\beta_1 - 1) + \delta\frac{1}{2}(p_1h^2 + (1-p_1)l^2)(1 - B(p_1))^2\,, \qquad \text{(EC.17)} \\ \Pi(\beta_1 < \widehat{\beta}_1, \gamma = 1) &= l^2\beta_1 - \frac{1}{2}l^2\beta_1^2 + \delta\frac{1}{2}[p_1h^2 + (1-p_1)l^2]B(p_1)\,, \qquad \text{(EC.18)} \\ \Pi(\beta_1 \ge \widehat{\beta}_1, \gamma = 1) &= l^2\beta_1 - \frac{1}{2}l^2\beta_1^2 + \delta\frac{1}{2}[p_1h^2 + (1-p_1)l^2]B(p_1) \\ &\quad - p_1(h^2 - l^2)\beta_1(\beta_1 - 1) + \delta\frac{1}{2}(p_1h^2 + (1-p_1)l^2)(1 - B(p_1))\,, \qquad \text{(EC.19)}\end{aligned}$$

It is easy to see $\Pi(\beta_1, \gamma = 0) > \Pi(\beta_1 < \widehat{\beta}_1, \gamma = 1)$ for $\beta_1 \in [0,1]$. Because $\Pi(\beta_1 < \widehat{\beta}_1, \gamma = 1)$ achieves its optimal at $\beta_1 = 1$, $\Pi(\beta_1^*, \gamma = 0) > \Pi(\beta_1^* < \widehat{\beta}_1, \gamma = 1)$. Also, because $B(p_1) < 1$, we have $\Pi(\beta_1, \gamma = 0) < \Pi(\beta_1 \geq \widehat{\beta}_1, \gamma = 1)$ for any β_1, which implies that $\Pi(\beta_1^*, \gamma = 0) < \Pi(\beta_1^* \geq \widehat{\beta}_1, \gamma = 1)$. Furthermore, it can be seen from algebra that $\Pi(B(p_1), \gamma = 0) = \Pi(\widetilde{\beta}_1^0, \gamma = 1)$ where $\widetilde{\beta}_1^0 = B(p_1)(1 + \sqrt{\delta(1 - B(p_1))})$. Because $\Pi(\beta_1, \gamma = 0)$ achieves its optimal at $\beta_1 = B(p_1)$, it follows that $\Pi(\beta_1^*, \gamma = 0) > \Pi(\beta_1^*, \gamma = 1)$ if and only if $\widetilde{\beta}_1^0 < \widehat{\beta}_1$. Lastly, it can be shown from algebra that there exists some $\widehat{v}$ such that $\widetilde{\beta}_1^0 < \widehat{\beta}_1$ if and only if $v < \widehat{v}$, which proves the claim for the case of $\sigma = 0$.

We next examine the case of positive yet small σ. Let $\widehat{h} - \widehat{l} = \widehat{v}$ and $\underline{\sigma} = \frac{1}{4}\sqrt{\delta(\widehat{h}^2 - \widehat{l}^2)}\widehat{h}B(p_1; \widehat{h}, \widehat{l})$. If $\sigma < \underline{\sigma}$, then the principal's expected payoffs are given as (EC.17), (EC.18), and (EC.19). Hence, the above analysis applies exactly in the same way and we can complete the proof for the case when $\sigma < \underline{\sigma}$.

(b) Suppose that σ is sufficiently large. Then $\underline{\beta}_1$ can be made close to zero and $\overline{\beta}_1$ can be made arbitrarily large. Therefore, the principal's expected payoffs under $\gamma = 0$ and $\gamma = 1$ can be written as follows:

$$\Pi(\beta_1, \gamma = 0) = l^2\beta_1 - \frac{1}{2}l^2\beta_1^2 + \delta\frac{1}{2}[p_1h^2 + (1-p_1)l^2]B(p_1) - p_1(h^2 - l^2)\beta_1(\beta_1 - 1) \tag{EC.20}$$

$$\Pi(\beta_1, \gamma = 1) = l^2\beta_1 - \frac{1}{2}l^2\beta_1^2 + \delta\frac{1}{2}[p_1h^2 + (1-p_1)l^2]B(p_1) - p_1(h^2 - l^2)(\beta_1 - \underline{\beta}_1)(\beta_1 - \widetilde{\beta}_1)\,, \tag{EC.21}$$

where $\widetilde{\beta}_1 = 1 + \frac{\delta}{4\sigma}(h^2 + (\frac{1-p_1}{p_1})l^2)(1 - B(p_1))$. Then because $\underline{\beta}_1 > 0$ yet close to zero and $\widetilde{\beta}_1 > 1$, it follows that $\Pi(\beta_1^*, \gamma = 0) < \Pi(\beta_1^*, \gamma = 1)$ as claimed. ■

Proof of Lemma 3: We first note that if $\gamma = 0$, then we always have $y_1^{h*}(\beta_1^*, \gamma = 0) = y^H$ because (i) $\widehat{\beta}_1 = 0$ and $\underline{\beta}_1 = 0$ and (ii) $y^H = y^M$.

We next obtain the following chain of equality/inequality regarding the principal's expected payoff with $\gamma = 0$ as follows:

$$\begin{aligned}
\Pi(\beta_1, \gamma = 0) &= p_1(y_1^{h*} - \beta_1 y_1^{h*}) + (1-p_1)(y_1^{l*} - \beta_1 y_1^{l*}) + \frac{1}{2}l^2\beta_1^2 \\
&\quad + \delta\Big[q_1(y_1^{h*}) \cdot \frac{1}{2}(p_1h^2 + (1-p_1)l^2)B(p_1) \\
&\quad + (1 - q_1(y_1^{h*})) \cdot \big(p_1\big(\frac{1}{2}h^2 - \frac{1}{2}(h^2 - l^2)B^2(p_1)\big) + (1-p_1)\frac{1}{2}l^2\big)\Big] \\
&> p_1(y_1^{h*} - \beta_1 y_1^{h*}) + (1-p_1)(y_1^{l*} - \beta_1 y_1^{l*}) + \frac{1}{2}l^2\beta_1^2 + \delta\frac{1}{2}(p_1h^2 + (1-p_1)l^2)B(p_1) \\
&= -\frac{1}{2}\big(p_1(h^2 - l^2) + p_1h^2 + (1-p_1)l^2\big)\beta_1(\beta_1 - 2B(p_1)) + \delta\frac{1}{2}(p_1h^2 + (1-p_1)l^2)B(p_1)\,,
\end{aligned}$$

where the inequality follows from the fact that $y_1^{h*}(\beta_1^*, \gamma = 0) = y^H > y^L$ (so $q_1(y_1^{h*}) < 1$) and

$$\frac{1}{2}\big(p_1h^2 + (1-p_1)l^2 - p_1(h^2 - l^2)B^2(p_1)\big) > \frac{1}{2}(p_1h^2 + (1-p_1)l^2)B(p_1)\,.$$

The last equality holds because $y_1^{h*}(\beta_1^*, \gamma = 0) = y^H = h^2\beta_1$ and $y_1^{h*}(\beta_1^*, \gamma = 0) = y^L = l^2\beta_1$. Therefore, we obtain

$$\begin{aligned}\Pi(\beta_1 = B(p_1), \gamma = 0) &> \frac{1}{2}(p_1h^2 + (1-p_1)l^2)(1+\delta)B(p_1) \\ &> \frac{1}{2}l^2 + \delta\frac{1}{2}(p_1h^2 + (1-p_1)l^2)B(p_1)\,,\end{aligned}$$

where the last inequality holds because $l^2 < (p_1h^2 + (1-p_1)l^2)(1+\delta)B(p_1)$. However, the last Right-hand-side of the above chain of inequality is the principal's expected payoff when $y^{h*}(\beta_1, \gamma) = y^L$. In other words, any (β_1, γ) that yields $y^{h*}(\beta_1, \gamma) = y^L$ will result in the principal's payoff strictly less than the principal's payoff obtained from $(\beta_1 = B(p_1), \gamma = 0)$. It thus follows that y^L can never be the output resulting from the optimal (β_1^*, γ^*), i.e., $y^{h*}(\beta_1^*, \gamma^*)$ must be always either y^H or y^M as claimed. ■

Proof of Proposition 4: (a) If v is sufficiently large and and σ is sufficiently small, then we can make $\overline{\beta}_1 < B(p_1)$ for $\gamma = 1$. Because $y^{h*}(\beta_1, \gamma) = y^H$ if $\beta \geq \overline{\beta}_1$, this implies that $\beta_1^* = B(p_1)$ for $\gamma = 1$ and $y^{h*}(\beta_1 = B(p_1), \gamma = 1) = y^H$ under the given condition (v is sufficiently large and and σ is sufficiently small). Indeed, this implies that the principal can achieve the best possible payoff, hence dominating the payoffs obtained by any other strategy (β_1, γ) as desired.

(b) If v is sufficiently small and and σ is sufficiently large, then $\underline{\beta}_1$ is close to zero and $\overline{\beta}_1$ is sufficiently large for $\gamma = 1$. Because $y^{h*}(\beta_1, \gamma) = y^M$ if $\overline{\beta}_1 \leq \beta < \overline{\beta}_1$, this implies that $y^{h*}(\beta_1^*, \gamma) = y^M$ for $\gamma = 1$ under the given condition (v is sufficiently small and and σ is sufficiently large). Finally, if v is sufficiently small and and σ is sufficiently large, then y^M is close enough to y^H so that the principal will not have an incentive to lower down γ to induce y^H, which implies that $\gamma = 1$ will let the principal achieve the optimal expected payoff as claimed.

(c) Recall from Lemma 3 that $y^{h*}(\beta_1^*, \gamma^*) = y^H$ or $= y^M$. Also, based on the results in parts (a) and (b), we can deduce that, for a given v, we have $y^{h*}(\beta_1^*, \gamma) = y^H$ for sufficiently small σ and $y^{h*}(\beta_1^*, \gamma) = y^M$ for sufficiently large σ. Recall that $y^{h*}(\beta_1, \gamma) = y^M$ if $\beta < \overline{\beta}_1$ and $y^{h*}(\beta_1, \gamma) = y^H$ if $\beta \geq \overline{\beta}_1$. Here $\overline{\beta}_1$ strictly increases in σ for a given v. Therefore, there must exist some $\widetilde{\sigma}$ such that $y^{h*}(\beta_1^*, \gamma^*) = y^H$ if $\sigma \leq \widetilde{\sigma}$ and $y^{h*}(\beta_1^*, \gamma^*) = y^M$ if $\sigma > \widetilde{\sigma}$. ■