1. Proof of Theorem 1

Let $[X]$ denote the density of the random variable X, and $[Y|X]$ that of the conditional distribution $Y|X$. Given prior distribution (7)-(9), the joint distribution is:

$$\begin{aligned}
&[\mathbf{Y}^-,\mathbf{Y}^+,\mathbf{u},\phi,b] \\
&=[\phi][b|\phi][\mathbf{u}|\phi][\mathbf{Y}^-|\mathbf{u},\phi,b][\mathbf{Y}^+|\mathbf{u},\phi,b] \\
&\propto \phi^{\frac{2|\mathbb{J}|+n+q}{2}-1}\exp\{-\tfrac{\phi}{2}[a+\lambda_b(b-b^{prior})^2+\lambda_u(\mathbf{u}-\mathbf{u}^{prior})^T(\mathbf{u}-\mathbf{u}^{prior}) \\
&\qquad +(\mathbf{Y}^- -\mathbf{X}\mathbf{u}+b\mathbf{1})^T(\mathbf{Y}^- -\mathbf{X}\mathbf{u}+b\mathbf{1})+(\mathbf{Y}^+ -\mathbf{X}\mathbf{u}-b\mathbf{1})^T(\mathbf{Y}^+ -\mathbf{X}\mathbf{u}-b\mathbf{1})]\}.
\end{aligned}$$

Then, the marginal distribution of parameters $\mathbf{u}$, b and $(\mathbf{u},b)$ is derived as

$$\begin{aligned}
[\mathbf{u},b|\mathbf{Y}^-,\mathbf{Y}^+,\phi] \propto \exp\big\{&-\tfrac{\phi}{2}[\lambda_b(b-b^{prior})^2+\lambda_u(\mathbf{u}-\mathbf{u}^{prior})^T(\mathbf{u}-\mathbf{u}^{prior}) \\
&-2b\mathbf{1}^T(\mathbf{Y}^+-\mathbf{Y}^-)-2\mathbf{u}^T\mathbf{X}^T(\mathbf{Y}^++\mathbf{Y}^-) \\
&+2b^2\mathbf{1}^T\mathbf{1}+2\mathbf{u}^T\mathbf{X}^T\mathbf{X}\mathbf{u}]\big\},
\end{aligned}$$

and

$$\begin{aligned}
[\mathbf{u}|\mathbf{Y}^-,\mathbf{Y}^+,\phi,b] &\propto \exp\big\{-\tfrac{\phi}{2}[\lambda_u(\mathbf{u}-\mathbf{u}^{prior})^T(\mathbf{u}-\mathbf{u}^{prior})-2\mathbf{u}\mathbf{X}^T(\mathbf{Y}^++\mathbf{Y}^-)+2\mathbf{u}^T\mathbf{X}^T\mathbf{X}\mathbf{u}]\big\} \\
&\propto \exp\big\{-\tfrac{\phi}{2}[\mathbf{u}^T(\lambda_u\mathbf{I}+2\mathbf{X}^T\mathbf{X})\mathbf{u}-2\mathbf{u}^T[\mathbf{X}^T(\mathbf{Y}^++\mathbf{Y}^-)+\lambda_u\mathbf{u}^{prior}]]\big\} \\
&\propto (2\pi)^{-\frac{n-1}{2}}\phi^{\frac{n-1}{2}}|\mathbf{S}_u|^{-\frac{1}{2}}\exp\big\{-\tfrac{\phi}{2}(\mathbf{u}-\mathbf{m}_u)^T\mathbf{S}_u^{-1}(\mathbf{u}-\mathbf{m}_u)\big\},
\end{aligned}$$

$$\begin{aligned}
[b|\mathbf{Y}^-,\mathbf{Y}^+,\phi,\mathbf{u}] &\propto \exp\big\{-\tfrac{\phi}{2}[\lambda_b(b-b^{prior})^2-2b\mathbf{1}^T(\mathbf{Y}^+-\mathbf{Y}^-)+2b^2\mathbf{1}^T\mathbf{1}]\big\} \\
&\propto \exp\big\{-\tfrac{\phi}{2}[b^2(\lambda_b+2\mathbf{1}^T\mathbf{1})-2b[\mathbf{1}^T(\mathbf{Y}^+-\mathbf{Y}^-)+\lambda_b b^{prior}]]\big\} \\
&\propto (2\pi)^{-\frac{1}{2}}\phi^{\frac{1}{2}}s_b^{-\frac{1}{2}}\exp\left[-\tfrac{\phi}{2s_b}(b-m_b)^2\right],
\end{aligned}$$

where

$$\mathbf{m}_u=\left(\lambda_{\mathbf{u}}\mathbf{I}+2\mathbf{X}^T\mathbf{X}\right)^{-1}\left[\mathbf{X}^T\left(\mathbf{Y}^-+\mathbf{Y}^+\right)+\lambda_u\mathbf{u}^{prior}\right], \mathbf{S}_u=\left(\lambda_{\mathbf{u}}\mathbf{I}+2\mathbf{X}^T\mathbf{X}\right)^{-1},$$

$$m_b=\frac{\mathbf{1}^T\left(\mathbf{Y}^+-\mathbf{Y}^-\right)+\lambda_b b^{prior}}{\lambda_b+2\mathbf{1}^T\mathbf{1}}, s_b=\frac{1}{\lambda_b+2\mathbf{1}^T\mathbf{1}}.$$

It is obvious that $\mathbf{u}$ and b are conditionally independent, i.e.

$$[b|\mathbf{Y}^-,\mathbf{Y}^+,\phi]=\frac{[\mathbf{u},b|\mathbf{Y}^-,\mathbf{Y}^+,\phi]}{[\mathbf{u}|\mathbf{Y}^-,\mathbf{Y}^+,\phi,b]}=[b|\mathbf{Y}^-,\mathbf{Y}^+,\phi,\mathbf{u}],$$

$$[\mathbf{u}|\mathbf{Y}^-,\mathbf{Y}^+,\phi]=\frac{[\mathbf{u},b|\mathbf{Y}^-,\mathbf{Y}^+,\phi]}{[b|\mathbf{Y}^-,\mathbf{Y}^+,\phi,\mathbf{u}]}=[\mathbf{u}|\mathbf{Y}^-,\mathbf{Y}^+,\phi,b],$$

and

$$\mathbf{u}\,\big|\mathbf{Y}^-,\mathbf{Y}^+,\phi\ \sim N_{n-1}\left(\mathbf{m}_u,\phi^{-1}\mathbf{S}_u\right),$$

$$b\,\big|\mathbf{Y}^-,\mathbf{Y}^+,\phi\ \sim N\left(m_b,\phi^{-1}s_b\right).$$

Consequently, the posterior of ϕ is

$$\begin{aligned}
&[\phi|\mathbf{Y}^-,\mathbf{Y}^+] \\
&=\tfrac{[\mathbf{Y}^-,\mathbf{Y}^+,\mathbf{u},\phi,b]}{[\mathbf{u},b|\mathbf{Y}^-,\mathbf{Y}^+,\phi]} \\
&\propto \phi^{\frac{2|\mathbb{J}|+q}{2}-1}\exp\left[-\tfrac{\phi}{2}\left(a+\mathbf{Y}^{-T}\mathbf{Y}^-+\mathbf{Y}^{+T}\mathbf{Y}^++\lambda_u\mathbf{u}^{prior\,T}\mathbf{u}^{prior}+\lambda_b(b^{prior})^2-\mathbf{m}_u^T\mathbf{S}_u^{-1}\mathbf{m}_u-m_b^2/s_b\right)\right]
\end{aligned}$$

Then

$$\phi \,\big|\mathbf{Y}^-,\mathbf{Y}^+ \;\sim G\left(\frac{2|\mathbb{J}|+q}{2}, \frac{a+\mathbf{Y}^{-T}\mathbf{Y}^- + \mathbf{Y}^{+T}\mathbf{Y}^+ + \lambda_u \mathbf{u}^{prior T}\mathbf{u}^{prior} + \lambda_b (b^{prior})^2 - \mathbf{m}_u^T\mathbf{S}_u^{-1}\mathbf{m}_u - m_b^2/s_b}{2}\right),$$

And, the posterior joint density of $\mathbf{u}$ and ϕ is

$$\begin{aligned}
&[\mathbf{u},\phi|\mathbf{Y}^-,\mathbf{Y}^+] \\
&= [\phi|\mathbf{Y}^-,\mathbf{Y}^+][\mathbf{u}|\mathbf{Y}^-,\mathbf{Y}^+,\phi] \\
&\propto \phi^{\frac{2|\mathbb{J}|+q+n-1}{2}-1}\exp\Big[-\tfrac{\phi}{2}\Big(a+\mathbf{Y}^{-T}\mathbf{Y}^- + \mathbf{Y}^{+T}\mathbf{Y}^+ + \lambda_u \mathbf{u}^{prior T}\mathbf{u}^{prior} + \lambda_b (b^{prior})^2 \\
&\qquad\qquad -\mathbf{m}_u^T\mathbf{S}_u^{-1}\mathbf{m}_u - m_b^2/s_b + (\mathbf{u}-\mathbf{m}_u)^T\mathbf{S}_u^{-1}(\mathbf{u}-\mathbf{m}_u)\Big)\Big]
\end{aligned}$$

The posterior marginal density function of $\mathbf{u}$ is

$$\begin{aligned}
&[\mathbf{u}|\mathbf{Y}^-,\mathbf{Y}^+] \\
&= \int_0^{+\infty}[\mathbf{u},\phi|\mathbf{Y}^-,\mathbf{Y}^+]d\phi \\
&\propto \Big[a+\mathbf{Y}^{-T}\mathbf{Y}^- + \mathbf{Y}^{+T}\mathbf{Y}^+ + \lambda_u \mathbf{u}^{prior T}\mathbf{u}^{prior} + \lambda_b (b^{prior})^2 \\
&\qquad -\mathbf{m}_u^T\mathbf{S}_u^{-1}\mathbf{m}_u - m_b^2/s_b + (\mathbf{u}-\mathbf{m}_u)^T\mathbf{S}_u^{-1}(\mathbf{u}-\mathbf{m}_u)\Big]^{-\frac{2|\mathbb{J}|+q+n-1}{2}} \\
&\propto \Big[1+\tfrac{1}{2|\mathbb{J}|+q}(\mathbf{u}-\mathbf{m}_u)^T(s^2\mathbf{S}_u)^{-1}(\mathbf{u}-\mathbf{m}_u)\Big]^{-\frac{2|\mathbb{J}|+q+n-1}{2}},
\end{aligned}$$

It follows that the posterior distribution of $\mathbf{u}$ is a multivariate t distribution, i.e.

$$\mathbf{u}\,\big|\mathbf{Y}^-,\mathbf{Y}^+ \;\sim T_{n-1}\left(\mathbf{m}_u, s^2\mathbf{S}_u, 2|\mathbb{J}|+q\right),$$

where

$$s^2 = \frac{a+\mathbf{Y}^{-T}\mathbf{Y}^- + \mathbf{Y}^{+T}\mathbf{Y}^+ + \lambda_u \mathbf{u}^{prior T}\mathbf{u}^{prior} + \lambda_b ((b^{prior}))^2 - \mathbf{m}_u^T\mathbf{S}_u^{-1}\mathbf{m}_u - m_b^2/s_b}{2|\mathbb{J}|+q},$$

The posterior distribution of b is similarly derived:

$$b\,\big|\mathbf{Y}^-,\mathbf{Y}^+ \;\sim T_1\left(m_b, s^2 s_b, 2|\mathbb{J}|+q\right).$$

2. Proof of Corollary 1

First, the following equation is proved.

$$\begin{aligned}
||\mathbf{Y}^- - \mathbf{X}\mathbf{u} + b\mathbf{1}||^2 + ||\mathbf{Y}^+ - \mathbf{X}\mathbf{u} - b\mathbf{1}||^2 &= ||\mathbf{Y}^- - \mathbf{X}\widehat{\mathbf{m}}_\mathbf{u} + \widehat{m}_b\mathbf{1}||^2 + ||\mathbf{Y}^+ - \mathbf{X}\widehat{\mathbf{m}}_\mathbf{u} - \widehat{m}_b\mathbf{1}||^2 \\
&\quad +2||\mathbf{X}\mathbf{u} - \mathbf{X}\widehat{\mathbf{m}}_\mathbf{u}||^2 + 2||b\mathbf{1} - \widehat{m}_b\mathbf{1}||^2.
\end{aligned} \tag{1}$$

In accordance to Theorem 2, if the prior distributions are noninformative, i.e., $q\to 0$, $a\to 0, \lambda_M \to 0$ and $\lambda_b \to 0$, it is obvious that

$$\lim_{\lambda_u\to 0}\mathbf{m}_u = \lim_{\lambda_u\to 0}\left(\lambda_\mathbf{u}\mathbf{I} + 2\mathbf{X}^T\mathbf{X}\right)^{-1}\left[\mathbf{X}^T\left(\mathbf{Y}^- + \mathbf{Y}^+\right) + \lambda_u\mathbf{u}^{prior}\right] = \left(2\mathbf{X}^T\mathbf{X}\right)^{-1}\left[\mathbf{X}^T\left(\mathbf{Y}^- + \mathbf{Y}^+\right)\right] = \widehat{\mathbf{m}}_\mathbf{u},$$

$$\lim_{\lambda_b\to 0} m_b = \lim_{\lambda_u\to 0}\frac{\mathbf{1}^T\left(\mathbf{Y}^+ - \mathbf{Y}^-\right) + \lambda_b b^{prior}}{\lambda_b + 2\mathbf{1}^T\mathbf{1}} = \frac{\mathbf{1}^T\left(\mathbf{Y}^+ - \mathbf{Y}^-\right)}{2\mathbf{1}^T\mathbf{1}} = \widehat{m}_b.$$

Furthermore, the following limits are proved

$$\begin{aligned}
&\lim_{a,\lambda_u,\lambda_b\to 0}\left(a+\mathbf{Y}^{-T}\mathbf{Y}^{-}+\mathbf{Y}^{+T}\mathbf{Y}^{+}+\lambda_u\mathbf{u}^{prior^T}\mathbf{u}^{prior}+\lambda_b(b^{prior})^2-\mathbf{m}_u^T\mathbf{S}_u^{-1}\mathbf{m}_u-m_b^2/s_b\right)\\
&=\mathbf{Y}^{-T}\mathbf{Y}^{-}+\mathbf{Y}^{+T}\mathbf{Y}^{+}-\widehat{\mathbf{m}}_{\mathbf{u}}^T2\mathbf{X}^T\mathbf{X}\widehat{\mathbf{m}}_{\mathbf{u}}-\widehat{m}_b^2\times 2\mathbf{1}^T\mathbf{1}\\
&=||\mathbf{Y}^{-}-\mathbf{X}\widehat{\mathbf{m}}_{\mathbf{u}}+\widehat{m}_b\mathbf{1}||^2+||\mathbf{Y}^{+}-\mathbf{X}\widehat{\mathbf{m}}_{\mathbf{u}}-\widehat{m}_b\mathbf{1}||^2+2||\mathbf{X}\widehat{\mathbf{m}}_u||^2+2||\widehat{m}_b\mathbf{1}||^2-\widehat{\mathbf{m}}_{\mathbf{u}}^T2\mathbf{X}^T\mathbf{X}\widehat{\mathbf{m}}_{\mathbf{u}}-\widehat{m}_b^2\times 2\mathbf{1}^T\mathbf{1}\\
&=||\mathbf{Y}^{-}-\mathbf{X}\widehat{\mathbf{m}}_{\mathbf{u}}+\widehat{m}_b\mathbf{1}||^2+||\mathbf{Y}^{+}-\mathbf{X}\widehat{\mathbf{m}}_{\mathbf{u}}-\widehat{m}_b\mathbf{1}||^2,
\end{aligned}$$

$$\lim_{q\to 0}(2|\mathbb{J}|+q)=2|\mathbb{J}|.$$

Thus, Corollary 2 is true.

3. Proof of Corollary 2

In accordance to Corollary 2, if the prior distribution is noninformative ($q\to 0$, $a\to 0$, $\lambda_{\mathbf{u}}\to 0$ and $\lambda_b\to 0$) and all the judgments have been elicited by the DM ($|\mathbb{J}|=n(n-1)/2$), then the posterior mean of b after observing the IMPCM $\tilde{R}$ is computed as

$$\widehat{m}_b=\left(2\mathbf{1}^T\mathbf{1}\right)^{-1}\left[\mathbf{1}^T\left(\mathbf{Y}^{+}-\mathbf{Y}^{-}\right)\right]=\frac{\sum_{(i,j)\in J}\ln r_{ij}^{+}-\ln r_{ij}^{-}}{2|\mathbb{J}|}=\frac{1}{2}\times\frac{\sum_{(i,j)\in J}\Delta_{ij}}{|\mathbb{J}|},$$

where $\Delta_{ij}=\ln r_{ij}^{+}-\ln r_{ij}^{-}$. Considering the definition of IMPCM, $\Delta_{ij}=\Delta_{ji}$, we obtain

$$\widehat{m}_b=\frac{1}{2}\times\frac{\sum_{(i,j)\in J}\Delta_{ij}}{|\mathbb{J}|}=\frac{1}{2}\times\frac{\sum_{i=1}^{n}\sum_{j=1}^{n}\Delta_{ij}}{n(n-1)}=\frac{1}{2}UI(\tilde{R}).$$

4. Proof of Corollary 3

For a complete IMPCM, according to the construction of $\mathbf{X}$, we obtain

$$\mathbf{X}^T\mathbf{X}=\begin{pmatrix} n-1 & -1 & \cdots & -1\\ -1 & n-1 & \cdots & -1\\ \vdots & \vdots & \ddots & \vdots\\ -1 & -1 & \cdots & n-1\end{pmatrix}_{(n-1)\times(n-1)}.$$

whose inverse is

$$(\mathbf{X}^T\mathbf{X})^{-1}=\begin{pmatrix} \frac{2}{n} & \frac{1}{n} & \cdots & \frac{1}{n}\\ \frac{1}{n} & \frac{2}{n} & \cdots & \frac{1}{n}\\ \vdots & \vdots & \ddots & \vdots\\ \frac{1}{n} & \frac{1}{n} & \cdots & \frac{2}{n}\end{pmatrix}_{(n-1)\times(n-1)}=\frac{1}{n}(\mathbf{I}+\mathbf{1}).$$

Therefore, the posterior mean of $\mathbf{u}$ after observing the MPCM R under noninformative priors is:

$$\left(2\mathbf{X}^T\mathbf{X}\right)^{-1}[\mathbf{X}^T(\mathbf{Y}^{-}+\mathbf{Y}^{+})]=\left(\mathbf{X}^T\mathbf{X}\right)^{-1}[\mathbf{X}^T\frac{\mathbf{Y}^{-}+\mathbf{Y}^{+}}{2}],$$

i.e.

$$\mu_i=\frac{\sum_{j=1}^{n}\ln(r_{ij})+\sum_{i=1}^{n-1}\sum_{j=1}^{n}\ln(r_{ij})}{n},\quad i=1,2,\cdots,n-1,$$

$$\mu_n=0=\frac{-\sum_{i=1}^{n-1}\sum_{j=1}^{n}\ln(r_{ij})+\sum_{i=1}^{n-1}\sum_{j=1}^{n}\ln(r_{ij})}{n}.$$

In accordance to the definition of MPCM, $\ln(r_{ij}) = -\ln(r_{ji})$, so

$$-\sum_{i=1}^{n-1}\sum_{j=1}^{n}\ln(r_{ij}) = \sum_{j=1}^{n}\ln(r_{nj}).$$

Thus,

$$w_i = \frac{\sqrt[n]{\prod_{j=1}^{n} r_{ij}} \times \sqrt[n]{\prod_{i=1}^{n-1}\prod_{j=1}^{n} r_{ij}}}{\sum_{i=1}^{n}\left(\sqrt[n]{\prod_{j=1}^{n} r_{ij}} \times \sqrt[n]{\prod_{i=1}^{n-1}\prod_{j=1}^{n} r_{ij}}\right)} = \frac{\sqrt[n]{\prod_{j=1}^{n} r_{ij}}}{\sum_{i=1}^{n}\left(\sqrt[n]{\prod_{j=1}^{n} r_{ij}}\right)}, \quad i = 1, 2, \cdots, n.$$

From Corollary 2, it follows that:

$$\begin{aligned}
\overline{\sigma^2} &= \tfrac{\left(\mathbf{Y}^- - \mathbf{X}\widehat{\mathbf{m}}_{\mathbf{u}} + \widehat{m}_b\mathbf{1}\right)^T\left(\mathbf{Y}^- - \mathbf{X}\widehat{\mathbf{m}}_{\mathbf{u}} + \widehat{m}_b\mathbf{1}\right) + \left(\mathbf{Y}^+ - \mathbf{X}\widehat{m}_{\mathbf{u}} - \widehat{\mathbf{m}}_b\mathbf{1}\right)^T\left(\mathbf{Y}^+ - \mathbf{X}\widehat{\mathbf{m}}_{\mathbf{u}} - \widehat{m}_b\mathbf{1}\right)}{2|\mathbb{J}|-2} \\
&\overset{\mathbf{Y}^- = \mathbf{Y}^+, \widehat{m}_b = 0}{=\!=\!=\!=\!=} \tfrac{\sum_{(i,j)\in J}\left[\ln\left(r_{ij}\right) - \ln(w_i) + \ln\left(w_j\right)\right]^2}{|\mathbb{J}|-1} \\
&= \tfrac{\sum_{i,j=1,i<j}\left[\ln\left(r_{ij}\right) - \ln(w_i) + \ln\left(w_j\right)\right]^2}{|\mathbb{J}|-1} \\
&= \tfrac{\frac{(n-1)(n-2)}{2}}{|\mathbb{J}|-1}\mathrm{GCI}(\mathbf{R}) \\
&\overset{|\mathbb{J}| = n(n-1)/2}{=\!=\!=\!=\!=} \tfrac{|\mathbb{J}|-n+1}{|\mathbb{J}|-1}\mathrm{GCI}(\mathbf{R}) \\
&\propto \mathrm{GCI}(\mathbf{R}).
\end{aligned}$$

5. Proof of Theorem 2

From Theorem 2, it follows that

$$\overline{\sigma^2} = E\left(\sigma^2\middle|\mathbf{Y}^-, \mathbf{Y}^+\right) = \frac{a + \mathbf{Y}^{-T}\mathbf{Y}^- + \mathbf{Y}^{+T}\mathbf{Y}^+ + \lambda_u \mathbf{u}^{prior^T}\mathbf{u}^{prior} + \lambda_b (b^{prior})^2 - \mathbf{m}_u^T\mathbf{S}_u^{-1}\mathbf{m}_u - m_b^2/s_b}{2|\mathbb{J}| + q - 2}.$$

According to (1),

$$\mathbf{Y}^{-T}\mathbf{Y}^- + \mathbf{Y}^{+T}\mathbf{Y}^+ = ||\mathbf{Y}^- - \mathbf{X}\widehat{\mathbf{m}}_{\mathbf{u}} + \widehat{m}_b\mathbf{1}||^2 + ||\mathbf{Y}^+ - \mathbf{X}\widehat{\mathbf{m}}_{\mathbf{u}} - \widehat{m}_b\mathbf{1}||^2 + 2||\mathbf{X}\widehat{\mathbf{m}}_u||^2 + 2||\widehat{m}_b\mathbf{1}||^2,$$

then, the following result is obtained:

$$\begin{aligned}
\overline{\sigma^2} &= \tfrac{1}{2|\mathbb{J}|+q-2}\ [a + ||\mathbf{Y}^- - \mathbf{X}\widehat{\mathbf{m}}_{\mathbf{u}} + \widehat{m}_b\mathbf{1}||^2 + ||\mathbf{Y}^+ - \mathbf{X}\widehat{\mathbf{m}}_{\mathbf{u}} - \widehat{m}_b\mathbf{1}||^2 \\
&\qquad + 2||\mathbf{X}\widehat{\mathbf{m}}_u||^2 + 2||\widehat{m}_b\mathbf{1}||^2 + \lambda_u \mathbf{u}^{prior^T}\mathbf{u}^{prior} + \lambda_b (b^{prior})^2 - \mathbf{m}_u^T\mathbf{S}_u^{-1}\mathbf{m}_u - m_b^2/s_b\Big]. \\
&= \tfrac{1}{2|\mathbb{J}|+q-2}\ [||\mathbf{Y}^- - \mathbf{X}\widehat{\mathbf{m}}_{\mathbf{u}} + \widehat{m}_b\mathbf{1}||^2 + ||\mathbf{Y}^+ - \mathbf{X}\widehat{\mathbf{m}}_{\mathbf{u}} - \widehat{m}_b\mathbf{1}||^2 + C],
\end{aligned}$$

where $C = a + 2||\mathbf{X}\widehat{\mathbf{m}}_u||^2 + 2||\widehat{m}_b\mathbf{1}||^2 + \lambda_u \mathbf{u}^{prior^T}\mathbf{u}^{prior} + \lambda_b (b^{prior})^2 - \mathbf{m}_u^T\mathbf{S}_u^{-1}\mathbf{m}_u - m_b^2/s_b$. Furthermore,

$$\begin{aligned}
\mathbf{m_u}^T\mathbf{S}_u^{-1}\mathbf{m_u} &= (\lambda_u \mathbf{u}^{prior^T} + 2\widehat{\mathbf{m}}_u^T\mathbf{X}^T\mathbf{X})(2\mathbf{X}^T\mathbf{X} + \lambda_u\mathbf{I})^{-1}(2\mathbf{X}^T\mathbf{X}\mathbf{u}^{prior} + \lambda_u\mathbf{u}^{prior}) \\
&= \lambda_u^2\mathbf{u}^{prior^T}(2\mathbf{X}^T\mathbf{X} + \lambda_u\mathbf{I})^{-1}\mathbf{u}^{prior} \\
&\quad + 4\widehat{\mathbf{m}}_u^T\mathbf{X}^T\mathbf{X}(2\mathbf{X}^T\mathbf{X} + \lambda_u\mathbf{I})^{-1}\mathbf{X}^T\mathbf{X}\widehat{\mathbf{m}}_u \\
&\quad + 4\lambda_u\widehat{\mathbf{m}}_u^T\mathbf{X}^T\mathbf{X}(2\mathbf{X}^T\mathbf{X} + \lambda_u\mathbf{I})^{-1}\mathbf{u}^{prior} \\
&= \lambda_u\mathbf{u}^{prior^T}\mathbf{A}^{-1}(2\mathbf{X}^T\mathbf{X})^{-1}\mathbf{u}^{prior} + \tfrac{1}{\lambda_u}\widehat{\mathbf{m}}_u^T(2\mathbf{X}^T\mathbf{X})\mathbf{A}^{-1}\widehat{\mathbf{m}}_u + 2\widehat{\mathbf{m}}_u^T\mathbf{A}^{-1}\mathbf{u}^{prior},
\end{aligned}$$

where $\mathbf{A}^{-1} = [(2\mathbf{X}^T\mathbf{X})^{-1} + (\lambda_u\mathbf{I})^{-1}]^{-1}$. Similarly,

$$m_b^2/s_b = \lambda_b (b^{prior})^2 B^{-1}(2\mathbf{1}^T\mathbf{1})^{-1} + \tfrac{1}{\lambda_b}\widehat{m}_b^2(2\mathbf{1}^T\mathbf{1})B^{-1}\widehat{m}_b + 2\widehat{m}_b b^{prior}B^{-1},$$

in which $B^{-1} = [(2\mathbf{1}^T\mathbf{1})^{-1} + \lambda_b^{-1}]^{-1}$. Finally, it is proved that

$$
\begin{aligned}
C &= a + 2\widehat{\mathbf{m}}_{\mathbf{u}}^T\mathbf{X}^T\mathbf{X}\widehat{\mathbf{m}}_{\mathbf{u}} + 2\widehat{m}_b^2\mathbf{1}^T\mathbf{1} + \lambda_u{\mathbf{u}^{prior}}^T\mathbf{u}^{prior} + \lambda_b\left(b^{prior}\right)^2 - {\mathbf{m}_{\mathbf{u}}}^T\mathbf{S}_u^{-1}\mathbf{m}_{\mathbf{u}} - m_b^2/s_b \\
&= a + 2\widehat{\mathbf{m}}_{\mathbf{u}}^T\mathbf{A}^{-1}\mathbf{A}\mathbf{X}^T\mathbf{X}\widehat{\mathbf{m}}_{\mathbf{u}} + 2\widehat{m}_b^2B^{-1}B\mathbf{1}^T\mathbf{1} + \lambda_u{\mathbf{u}^{prior}}^T\mathbf{A}^{-1}\mathbf{A}\mathbf{u}^{prior} + \lambda_b\left(b^{prior}\right)^2B^{-1}B \\
&\quad -\lambda_u{\mathbf{u}^{prior}}^T\mathbf{A}^{-1}(2\mathbf{X}^T\mathbf{X})^{-1}\mathbf{u}^{prior} - \tfrac{1}{\lambda_u}\widehat{\mathbf{m}}_u^T(2\mathbf{X}^T\mathbf{X})\mathbf{A}^{-1}\widehat{\mathbf{m}}_u - 2\widehat{\mathbf{m}}_u^T\mathbf{A}^{-1}\mathbf{u}^{prior} \\
&\quad -\lambda_b\left(b^{prior}\right)^2B^{-1}(2\mathbf{1}^T\mathbf{1})^{-1} - \tfrac{1}{\lambda_b}\widehat{m}_b^2(2\mathbf{1}^T\mathbf{1})B^{-1}\widehat{m}_b - 2\widehat{m}_b b^{prior}B^{-1} \\
&= a + 2\widehat{\mathbf{m}}_{\mathbf{u}}^T\mathbf{A}^{-1}[(2\mathbf{X}^T\mathbf{X})^{-1} + (\lambda_u\mathbf{I})^{-1}]\mathbf{X}^T\mathbf{X}\widehat{\mathbf{m}}_{\mathbf{u}} + 2\widehat{m}_b^2B^{-1}[(2\mathbf{1}^T\mathbf{1})^{-1} + \lambda_b^{-1}]\mathbf{1}^T\mathbf{1} \\
&\quad +\lambda_u{\mathbf{u}^{prior}}^T\mathbf{A}^{-1}[(2\mathbf{X}^T\mathbf{X})^{-1} + (\lambda_u\mathbf{I})^{-1}]\mathbf{u}^{prior} + \lambda_b\left(b^{prior}\right)^2B^{-1}[(2\mathbf{1}^T\mathbf{1})^{-1} + \lambda_b^{-1}] \\
&\quad -\lambda_u{\mathbf{u}^{prior}}^T\mathbf{A}^{-1}(2\mathbf{X}^T\mathbf{X})^{-1}\mathbf{u}^{prior} - \tfrac{1}{\lambda_u}\widehat{\mathbf{m}}_u^T(2\mathbf{X}^T\mathbf{X})\mathbf{A}^{-1}\widehat{\mathbf{m}}_u - 2\widehat{\mathbf{m}}_u^T\mathbf{A}^{-1}\mathbf{u}^{prior} \\
&\quad -\lambda_b\left(b^{prior}\right)^2B^{-1}(2\mathbf{1}^T\mathbf{1})^{-1} - \tfrac{1}{\lambda_b}\widehat{m}_b^2(2\mathbf{1}^T\mathbf{1})B^{-1}\widehat{m}_b - 2\widehat{m}_b b^{prior}B^{-1} \\
&= a + \widehat{\mathbf{m}}_{\mathbf{u}}^T\mathbf{A}^{-1}[\mathbf{I} + (\lambda_u)^{-1}(2\mathbf{X}^T\mathbf{X})]\widehat{\mathbf{m}}_{\mathbf{u}} + \widehat{m}_b^2B^{-1}[1 + \lambda_b^{-1}(2\mathbf{1}^T\mathbf{1})] \\
&\quad +{\mathbf{u}^{prior}}^T\mathbf{A}^{-1}[\lambda_u(2\mathbf{X}^T\mathbf{X})^{-1} + \mathbf{I}]\mathbf{u}^{prior} + \left(b^{prior}\right)^2B^{-1}[\lambda_b(2\mathbf{1}^T\mathbf{1})^{-1} + 1] \\
&\quad -\lambda_u{\mathbf{u}^{prior}}^T\mathbf{A}^{-1}(2\mathbf{X}^T\mathbf{X})^{-1}\mathbf{u}^{prior} - \tfrac{1}{\lambda_u}\widehat{\mathbf{m}}_u^T(2\mathbf{X}^T\mathbf{X})\mathbf{A}^{-1}\widehat{\mathbf{m}}_u - 2\widehat{\mathbf{m}}_u^T\mathbf{A}^{-1}\mathbf{u}^{prior} \\
&\quad -\lambda_b\left(b^{prior}\right)^2B^{-1}(2\mathbf{1}^T\mathbf{1})^{-1} - \tfrac{1}{\lambda_b}\widehat{m}_b^2(2\mathbf{1}^T\mathbf{1})B^{-1}\widehat{m}_b - 2\widehat{m}_b b^{prior}B^{-1} \\
&= a + (\widetilde{\mathbf{m}}_u - \mathbf{u}^{prior})^T\mathbf{A}^{-1}(\widehat{\mathbf{m}}_u - \mathbf{u}^{prior}) + (\widehat{m}_b - b^{prior})^2B^{-1}.
\end{aligned}
$$

Thus, the posterior mean of ϕ is

$$
\begin{aligned}
\overline{\sigma^2} &= \tfrac{||\mathbf{Y}^- - \mathbf{X}\widehat{\mathbf{m}}_{\mathbf{u}} + \hat{m}_b\mathbf{1}||^2 + ||\mathbf{Y}^+ - \mathbf{X}\widehat{\mathbf{m}}_{\mathbf{u}} - \hat{m}_b\mathbf{1}||^2}{2|\mathbb{J}|+q-2} \\
&\quad + \tfrac{a + \left(\mathbf{u}^{prior} - \widehat{\mathbf{m}}_{\mathbf{u}}\right)^T\mathbf{A}^{-1}(\mathbf{u}^{prior} - \widehat{\mathbf{m}}_{\mathbf{u}}) + \left(b^{prior} - \hat{m}_b\right)^2B^{-1}}{2|\mathbb{J}|+q-2},
\end{aligned}
$$

and

$$
\begin{aligned}
\text{PCI}\left(\tilde{R}\right) &= \frac{|\mathbb{J}|-1}{|\mathbb{J}|-n+1}\left(\frac{||\mathbf{Y}^- - \mathbf{X}\widehat{\mathbf{m}}_{\mathbf{u}} + \hat{m}_b\mathbf{1}||^2 + ||\mathbf{Y}^+ - \mathbf{X}\widehat{\mathbf{m}}_{\mathbf{u}} - \hat{m}_b\mathbf{1}||^2}{2|\mathbb{J}|+q-2}\right. \\
&\quad \left. + \frac{a + \left(\mathbf{u}^{prior} - \widehat{\mathbf{m}}_{\mathbf{u}}\right)^T\mathbf{A}^{-1}(\mathbf{u}^{prior} - \widehat{\mathbf{m}}_{\mathbf{u}}) + \left(b^{prior} - \hat{m}_b\right)^2B^{-1}}{2|\mathbb{J}|+q-2}\right) \\
&= \text{LCI}(\tilde{R}) + \text{PrCI}(\tilde{R}).
\end{aligned}
\tag{2}
$$

6. Proof of Theorem 3

The proof of results (1), (3) and (4) are evident. Using the following partial derivatives

$$
\begin{aligned}
&\tfrac{\partial PrCI(\tilde{R})}{\partial\lambda_u} = \tfrac{|\mathbb{J}|-1}{(|\mathbb{J}|-n+1)(2|\mathbb{J}|+q-2)}(\mathbf{u}^{prior} - \widehat{\mathbf{m}}_u)^T\tfrac{\partial\mathbf{A}^{-1}}{\partial\lambda_u}(\mathbf{u}^{prior} - \widehat{\mathbf{m}}_u) \\
&= \tfrac{|\mathbb{J}|-1}{(|\mathbb{J}|-n+1)(2|\mathbb{J}|+q-2)} \\
&\times(\mathbf{u}^{prior} - \widehat{\mathbf{m}}_u)^T[(2\mathbf{X}^T\mathbf{X})^{-1} + (\lambda_u\mathbf{I})^{-1}]^{-1}(\lambda_u\mathbf{I})^{-1}(\lambda_u\mathbf{I})^{-1}[(2\mathbf{X}^T\mathbf{X})^{-1} + (\lambda_u\mathbf{I})^{-1}]^{-1}(\mathbf{u}^{prior} - \widehat{\mathbf{m}}_u),
\end{aligned}
$$

$$
\begin{aligned}
\tfrac{\partial PrCI(\tilde{R})}{\partial\lambda_b} &= \tfrac{|\mathbb{J}|-1}{|\mathbb{J}|-n+1}\tfrac{(b^{prior} - \widehat{m}_b)^2}{2|\mathbb{J}|+q-2}\tfrac{\partial B^{-1}}{\partial\lambda_b} \\
&= \tfrac{|\mathbb{J}|-1}{|\mathbb{J}|-n+1}\tfrac{(b^{prior} - \widehat{m}_b)^2}{2|\mathbb{J}|+q-2}\tfrac{1}{\lambda_b^2[(2\mathbf{1}^T\mathbf{1})^{-1} + \lambda_b^{-1}]^2} > 0.
\end{aligned}
$$

As $F = [(2\mathbf{X}^T\mathbf{X})^{-1} + (\lambda_u\mathbf{I})^{-1}]^{-1}(\lambda_u\mathbf{I})^{-1}$ is nonsingular, it is evident that x^TF^TFx is a positive definite quadratic form. Therefore, the result (2) are proved.