Online Supplement to “Waiting Time Distribution of $M/D_N/1$ Queues Through Numerical Laplace Inversion” *INFORMS Journal on Computing, 2005*

John F. Shortle
Department of Systems Engineering and Operations Research, George Mason University, 4400 University Drive, MS 4A6, Fairfax, Virginia 22030, USA, jshortle@gmu.edu
Martin J. Fischer
Mitretek Systems, 3150 Fairview Park Drive South, Falls Church, Virginia 22042, USA, mfischer@mitretek.org
Percy H. Brill
University of Windsor, Windsor, Ontario, Canada N9B 3P4, brill@uwindsor.ca

1. Theorem Proofs

This Online Supplement gives proofs for all theorems in Shortle et al. (2005).

Proof of Theorem 1. Step 1. We show there exists a unique solution to (3) (see Shortle et al. 2005). Multiply (3) by $e^{-\lambda t}$ and rearrange terms to get:

$$(e^{-\lambda t}W_q(t))' = -\lambda e^{-\lambda t}\sum_{i=1}^{N} W_q(t-x_i)p_i, \quad t \neq x_j.$$

We assume throughout this proof that the index $j \in 1, \cdots, N$. This implies

$$W_q(t) = e^{\lambda t}\left(W_q(0) - \int_0^t \lambda e^{-\lambda u}\sum_{i=1}^{N} W_q(u-x_i)p_i \, du\right), \quad t \geq 0.$$

Since $W_q(t)$ is continuous on $t \geq 0$ (Shortle et al. 2005, sec. 2), the integrand is piecewise continuous. The definite integral of a piecewise continuous function is unique, so $W_q(t)$ is unique. The solution $W_q(t)$ is not affected by the fact that $w_q(t)$, and hence (3), is not defined at $t = x_0, x_1, \cdots, x_N$.

Step 2. We show that Euler's method for numerically solving a differential equation applied to (3) converges to $W_q(t)$. First, rewrite (3) as

$$W_q'(t) = \lambda W_q(t) - Q(t), \quad t \neq x_j, \tag{16}$$

$$\text{where} \quad Q(t) \equiv \lambda \sum_{i=1}^{N} W_q(t - x_i) p_i.$$

Euler's method with a constant step-size h applied to (16) gives:

$$\hat{F}_n = \hat{F}_{n-1} + h\left(\lambda \hat{F}_{n-1} - Q(h(n-1))\right). \tag{17}$$

For a differential equation in the form of (16), Euler's method converges uniformly on any finite interval $[a, b]$ provided $Q(t)$ is continuous on that interval (e.g., Hairer et al. 1991, theorem 7.3). Here, $Q(t)$ is only *piecewise* continuous, with discontinuities at each x_j. Starting with the interval $[0, x_1]$, choose $Q(x_1)$ so that $Q(t)$ is continuous on $[0, x_1]$. $Q(t)$ is right-continuous, so we do not need to worry about continuity at 0. From Step 1 of this proof, we can define $Q(x_j)$ to be any value without changing the solution for $W_q(t)$. Let $F_h(t)$ be the Euler polygon connecting the discrete points from (17) with straight lines. Since $Q(t)$ is continuous on $[0, x_1]$, theorem 7.3 from Hairer et al. (1991) shows that Euler's method converges uniformly on $[0, x_1]$. That is, for any $\epsilon > 0$, there exists h^* such that $h \le h^* \Rightarrow |F_h(t) - W_q(t)| < \epsilon$ on $[0, x_1]$, and in particular, $|F_h(x_1) - W_q(x_1)| < \epsilon$. We use this as the starting point for the next interval.

On the second interval $[x_1, x_2]$, let $\tilde{F}_h(t)$ be the Euler polygon with step-size h and initial condition $\tilde{F}_h(x_1) = W_q(x_1)$. As before, choose $Q(x_2)$ so that $Q(t)$ is continuous on $[x_1, x_2]$. For small enough h, $|\tilde{F}_h(t) - W_q(t)| < \epsilon$ on $[x_1, x_2]$. Also, by Hairer et al. (1991), Lemma 7.2, $|F_h(t) - \tilde{F}_h(t)| < \epsilon$ on $[x_1, x_2]$, for small enough h. Thus, $|F_h(t) - W_q(t)| < \epsilon$ on $[x_1, x_2]$. Continuing in the same manner, we can get uniform convergence up to any finite point T, since there are a finite number of x_j's. Also, we trivially get uniform convergence on $(-\infty, T]$ by defining $F_h(t) \equiv 0$ for $t < 0$.

Step 3. In this step, we show that $|F_n - W_q(nh)| \to 0$ as $h \to 0$, $nh \to t$.

We prove the following induction step: If $F \to W_q$ uniformly on $(-\infty, t]$, then $F \to W_q$ uniformly on $(-\infty, t + x_1]$. More specifically, we define uniform convergence on $(-\infty, t]$ as follows: given any $\epsilon > 0$ and any sequence h_k, n_k ($h_k > 0$, n_k an integer) such that $h_k \to 0$, $n_k h_k \to t$, then there exists a k^* such that $k \ge k^* \Rightarrow |F_n - W_q(nh_k)| < \epsilon, \forall n = -\infty, \cdots, n_k - 1, n_k$. First, the induction hypothesis is trivially satisfied for any $t < 0$ since $F_n \equiv 0$ and $W_q(nh) \equiv 0$ when $n < 0$.

Assuming the induction hypothesis, we show the same holds true on $(-\infty, t + x_1]$. Consider a sequence h_k, n_k, with $h_k \to 0$, $n_k h_k \to t + x_1$. To simplify notation, we now drop the subscript k on h, n. Subtracting (17) from (5) in Shortle et al. (2005) gives:

$$F_n - \hat{F}_n = \frac{1}{1-\lambda h}(F_{n-1} - \lambda h \sum_{i=1}^{N} p_i F_{n-m_i}) - (1+\lambda h)\hat{F}_{n-1} + hQ(h(n-1))$$

$$= (1+\lambda h)(F_{n-1} - \hat{F}_{n-1}) + \lambda h\, v_{n-1,h} + o(h^2),$$

$$\text{where} \quad v_{n-1,h} \equiv \sum_{i=1}^{N} p_i \left(W_q(nh - h - x_i) - F_{n-m_i}\right).$$

$$\Rightarrow |F_n - \hat{F}_n| \leq (1+\lambda h)|F_{n-1} - \hat{F}_{n-1}| + \lambda h\, v_{n-1,h} + D(\lambda h)^2,$$

for some constant D and small enough h (say $h \leq h_3^*$).

$$\Rightarrow |F_n - \hat{F}_n| \leq e^{\lambda h}\left(|F_{n-1} - \hat{F}_{n-1}| + \lambda h\, v_{n-1,h} + D(\lambda h)^2\right).$$

Similarly, $|F_n - \hat{F}_n|$

$$\leq e^{\lambda h}\left(e^{\lambda h}\left(|F_{n-2} - \hat{F}_{n-2}| + \lambda h\, v_{n-2,h} + D(\lambda h)^2\right) + \lambda h\, v_{n-1,h} + D(\lambda h)^2\right)$$

$$\leq e^{2\lambda h}\left(|F_{n-2} - \hat{F}_{n-2}| + \lambda h\, v_{n-2,h} + \lambda h\, v_{n-1,h} + 2D(\lambda h)^2\right)$$

$$\leq e^{m\lambda h}\left(|F_{n-m} - \hat{F}_{n-m}| + \lambda h \sum_{l=1}^{m} v_{n-l,h} + mD(\lambda h)^2\right).$$

where m is the integer chosen so that $(n-m)h \approx t$; specifically, let $t \geq (n-m)h > t-h$. So, $mh \leq nh - t + h \leq x_1 + 1$, for small enough h, since $nh \to t + x_1$. So, the above equation

$$\leq e^{(x_1+1)\lambda}\left(|F_{n-m} - \hat{F}_{n-m}| + \lambda h\left(\sum_{l=1}^{m} v_{n-l,h} + D(x_1+1)\lambda\right)\right). \tag{18}$$

Now, $|F_{n-m} - \hat{F}_{n-m}|$ can be made less than an arbitrary $\epsilon > 0$, for small enough h since $|F_{n-m} - \hat{F}_{n-m}| \leq |F_{n-m} - W_q((n-m)h)| + |W_q((n-m)h) - \hat{F}_{n-m}|$, and the first term can be made arbitrarily small by the induction hypothesis (since $(n-m)h \to t$), and the second term can be made arbitrarily small by Step 2. Therefore, to show that (18) can be made less than an arbitrary $\epsilon > 0$ for small enough h, it remains to show that $\lambda h \sum_{l=1}^{m} v_{n-l,h} \to 0$. Now,

$$\sum_{l=1}^{m} v_{n-l,h} = \sum_{i=1}^{N} p_i \sum_{l=1}^{m} \left(W_q((n-l)h - x_i) - F_{n-l+1-m_i}\right),$$

and

$$|W_q((n-l)h - x_i) - F_{n-l+1-m_i}| \leq$$
$$|W_q((n-l)h - x_i) - W_q((n-l+1-m_i)h)| + |W_q((n-l+1-m_i)h) - F_{n-l+1-m_i}|.$$

By the induction hypothesis, the last term can be made less than an arbitrary $\epsilon > 0$ for $h \leq$ some h_4^*, for all l (since $(n - l + 1 - m_i)h \leq (n + 1 - m_i)h \to t$, since $nh \to t + x_i$ and $m_i h \to x_i$).

For the first term, the definition of $m_i = \lceil x_i/h \rceil$ implies $x_i/h \leq m_i < x_i/h + 1$ which implies $h - x_i \geq h - hm_i > -x_i$ which implies

$$h \geq (n - l + 1 - m_i)h - ((n - l)h - x_i) > 0,$$

so there is at most one l where $(n - l + 1 - m_i)h \geq 0$, and $(n - l)h - x_i < 0$. Since W_q is continuous everywhere except at 0, $W_q' \leq \lambda$ everywhere except at a finite number of points, and $|(n - l + 1 - m_i - 1)h - ((n - l)h - x_i)| \to 0$, the second term can be made less than any ϵ uniformly for all $l = 1, \cdots, m$ except for at most one l in this set. So,

$$\sum_{l=1}^{m} |W_q((n - l)h - x_i) - F_{n-l+1-m_i}| \leq (m - 1)\epsilon + 1 + m\epsilon.$$

So,

$$h\lambda \sum_{l=1}^{m} v_{n-l,h} \leq h\lambda(2m\epsilon + 1)$$

which can be made arbitrarily small since $mh \to x_1$ and ϵ can be selected arbitrarily. Thus, $|F_n - \hat{F}_n|$ can be made arbitrarily small. Also, for $n' \leq n$, the equivalent bound for $|F_{n'} - \hat{F}_{n'}|$ in (18) is less than or equal to the bound for $|F_n - \hat{F}_n|$ in (18). Thus, $|F_n - \hat{F}_n|$ converges uniformly to 0 for all $n' \leq n$. By step 2 in this proof, $|\hat{F}_n - W_q(nh)|$ converges uniformly to 0. Thus, $|F_n - W_q(nh)|$ converges uniformly to 0. □

Proof of Theorem 2. The Laplace transform of D_N in the form of (19) (see Lemma 1) is:

$$g^*(-iu) = \left(\sum_{i=1}^{N} p_i \cos(ux_i) \right) + i \left(\sum_{i=1}^{N} p_i \sin(ux_i) \right).$$

Using (20) from Lemma 1 and letting u be large gives the result. □

Proof of Theorem 3. The Laplace transform of U_N in the form of (19) (see Lemma 1) is:

$$g^*(-iu) = \left(\frac{1}{u} \sum_{i=1}^{N} \frac{p_i}{x_i} \sin(ux_i) \right) + i \left(\frac{1}{u} \sum_{i=1}^{N} \frac{p_i}{x_i} (1 - \cos(ux_i)) \right).$$

Using (20) from Lemma 1 and letting u be large gives the result. □

Lemma 1. *Suppose the service distribution for an $M/G/1$ queue has Laplace transform $g^*(s)$ and that*

$$g^*(-iu) = a(u) + ib(u). \tag{19}$$

Let $\phi(u)$ (see (14), Shortle et al. 2005) be the transform of $\tilde{W}_q(t) = P(W_q \leq t | W_q > 0)$. Then,

$$Re(\phi(u)) = \frac{1-\rho}{\rho}\left(\frac{-(\lambda a(u) - \lambda)^2 + \lambda b(u)(\lambda b(u) - u)}{(\lambda a(u) - \lambda)^2 + (\lambda b(u) - u)^2}\right). \tag{20}$$

Proof. Plugging (19) into (14) gives:

$$\phi(u) = \frac{1-\rho}{\rho}\left(\frac{-iu}{\lambda a(u) - \lambda + (\lambda b(u) - u)i} - 1\right).$$

Putting over a common denominator and taking the real part gives the result. □

References

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