Online Supplement to "Enhancing Lagrangian Dual Optimization for Linear Programs by Obviating Nondifferentiability"

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Appendix: Proof of Proposition 1.

Using (15a,b) of the paper, the derivative of $\bar{x}_j$, viewed as a function $\bar{x}_j(\bar{c}_j)$ of $\bar{c}_j$, at the point $\bar{c}_j = 0$, yields, via L'Hospital's rule,

$$\left.\frac{d\bar{x}_j(\bar{c}_j)}{d\bar{c}_j}\right|_{\bar{c}_j=0} = \lim_{\Delta\bar{c}_j \to 0} \frac{\bar{x}_j(\Delta\bar{c}_j) - \bar{x}_j(0)}{\Delta\bar{c}_j}$$

$$= \lim_{\Delta\bar{c}_j \to 0} \frac{\dfrac{\Delta\bar{c}_j(l_j + u_j) + 2\mu - \sqrt{\Delta\bar{c}_j^2(u_j - l_j)^2 + 4\mu^2}}{2\Delta\bar{c}_j} - \dfrac{l_j + u_j}{2}}{\Delta\bar{c}_j}$$

$$= \lim_{\Delta\bar{c}_j \to 0} \frac{2\mu - \sqrt{\Delta\bar{c}_j^2(u_j - l_j)^2 + 4\mu^2}}{2\Delta\bar{c}_j^2}$$

$$= \lim_{\Delta\bar{c}_j \to 0} \frac{-\Delta\bar{c}_j(u_j - l_j)^2}{4\Delta\bar{c}_j\sqrt{\Delta\bar{c}_j^2(u_j - l_j)^2 + 4\mu^2}}$$

$$= \lim_{\Delta\bar{c}_j \to 0} \frac{-(u_j - l_j)^2}{4\sqrt{\Delta\bar{c}_j^2(u_j - l_j)^2 + 4\mu^2}} = \frac{-(u_j - l_j)^2}{8\mu}. \tag{S1}$$

On the other hand, we have

$$\left.\frac{d\bar{x}_j(\bar{c}_j)}{d\bar{c}_j}\right|_{\bar{c}_j\neq 0} = \frac{\left[(l_j+u_j)-\frac{1}{2}\left(2\bar{c}_j(u_j-l_j)^2\right)\left(\bar{c}_j^{\,2}(u_j-l_j)^2+4\mu^2\right)^{-1/2}\right](2\bar{c}_j)}{(2\bar{c}_j)^2}$$

$$-\frac{2\left[\bar{c}_j(l_j+u_j)+2\mu-\left(\bar{c}_j^{\,2}(u_j-l_j)^2+4\mu^2\right)^{1/2}\right]}{(2\bar{c}_j)^2}$$

$$=\frac{2\left(\bar{c}_j^{\,2}(u_j-l_j)^2+4\mu^2\right)^{1/2}-2\bar{c}_j^{\,2}(u_j-l_j)^2\left(\bar{c}_j^{\,2}(u_j-l_j)^2+4\mu^2\right)^{-1/2}-4\mu}{4\bar{c}_j^{\,2}}.$$

Hence, using L'Hospital's rule we have

$$\lim_{\bar{c}_j\to 0}\frac{d\bar{x}_j(\bar{c}_j)}{d\bar{c}_j} = \lim_{\bar{c}_j\to 0}\left[\frac{2\bar{c}_j(u_j-l_j)^2\left(\bar{c}_j^{\,2}(u_j-l_j)^2+4\mu^2\right)^{-1/2}}{8\bar{c}_j}\right.$$

$$-\frac{4\bar{c}_j(u_j-l_j)^2\left(\bar{c}_j^{\,2}(u_j-l_j)^2+4\mu^2\right)^{-1/2}}{8\bar{c}_j}$$

$$\left.+\frac{\bar{c}_j^{\,2}(u_j-l_j)^2\left(2\bar{c}_j(u_j-l_j)^2\right)\left(\bar{c}_j^{\,2}(u_j-l_j)^2+4\mu^2\right)^{-3/2}}{8\bar{c}_j}\right]$$

$$=\lim_{\bar{c}_j\to 0}\left[\frac{-2(u_j-l_j)^2\left(\bar{c}_j^{\,2}(u_j-l_j)^2+4\mu^2\right)^{-1/2}}{8}\right.$$

$$\left.+\frac{2\bar{c}_j^{\,2}(u_j-l_j)^4\left(\bar{c}_j^{\,2}(u_j-l_j)^2+4\mu^2\right)^{-3/2}}{8}\right]$$

$$=-\frac{(u_j-l_j)^2}{8\mu}. \qquad \text{(S2)}$$

From (S1) and (S2), we see that $\left.\frac{d\bar{x}_j(\bar{c}_j)}{d\bar{c}_j}\right|_{\bar{c}_j=0} = \lim_{\bar{c}_j\to 0}\frac{d\bar{x}_j}{d\bar{c}_j} = \frac{-(u_j-l_j)^2}{8\mu}$. □