Online Supplement to "ETAQA Solutions for Infinite Markov Processes with Repetitive Structure"

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Explicit computation of the matrix R

QBD processes are defined as the intersection of M/G/1 and GI/M/1-type processes. Hence, both matrix $\mathbf{G}$ (characteristic for M/G/1) and matrix $\mathbf{R}$ (characteristic for GI/M/1) can be defined for a QBD process as solutions of the following quadratic equations (see Latouche and Ramaswami (1999)):

$$\mathbf{B} + \mathbf{L}\mathbf{G} + \mathbf{F}\mathbf{G}^2 = \mathbf{0}, \qquad \mathbf{F} + \mathbf{R}\mathbf{L} + \mathbf{R}^2\mathbf{B} = \mathbf{0}.$$

If matrix-geometric is used to solve a QBD process then the relation between $\boldsymbol{\pi}^{(i)}$ and $\boldsymbol{\pi}^{(i-1)}$ for $i > 1$ is expressed in terms of $\mathbf{R}$

$$\boldsymbol{\pi}^{(i)} = \boldsymbol{\pi}^{(i-1}\mathbf{R},$$

If matrix-analytic is the solution method then the relation between $\boldsymbol{\pi}^{(i)}$ and $\boldsymbol{\pi}^{(i-1)}$ is based on Ramaswami's recursive formula:

$$\boldsymbol{\pi}^{(i)} = -\boldsymbol{\pi}^{(i-1)}\mathbf{S}^{(1)}(\mathbf{S}^{(0)})^{-1},$$

where $\mathbf{S}^{(1)} = \mathbf{F}$ and $\mathbf{S}^{(0)} = (\mathbf{L} + \mathbf{F}\mathbf{G})$, i.e., the only auxiliary sums used in the solution of M/G/1 processes that are defined for a QBD process. The above equations allow the derivation of the fundamental relation between $\mathbf{R}$ and $\mathbf{G}$ as presented in Latouche and Ramaswami (1999, pages 137-8),

$$\mathbf{R} = -\mathbf{F}(\mathbf{L} + \mathbf{F}\mathbf{G})^{-1}. \tag{1}$$

Obviously, for the case of QBD processes, knowing $\mathbf{G}$ (or $\mathbf{R}$) implies a direct computation of $\mathbf{R}$ (or $\mathbf{G}$). Computing $\mathbf{G}$ is usually easier than computing $\mathbf{R}$: $\mathbf{G}$'s computation is a prerequisite to the computation of $\mathbf{R}$ in the logarithmic reduction algorithm, the most efficient algorithm to compute $\mathbf{R}$ proposed by Latouche and Ramaswami (1999). If $\mathbf{B}$ can be expressed as a product of two vectors

$$\mathbf{B} = \boldsymbol{\alpha} \cdot \boldsymbol{\beta},$$

where, without loss of generality $\boldsymbol{\beta}$ is assumed to be a normalized vector, then $\mathbf{G}$ and $\mathbf{R}$ can be explicitly obtained as

$$\mathbf{G} = \mathbf{1} \cdot \boldsymbol{\beta}, \quad \mathbf{R} = -\mathbf{F}(\mathbf{L} + \mathbf{F1} \cdot \boldsymbol{\beta})^{-1}.$$

Representative examples, where the above condition holds, are the queues $M/Cox/1$, $M/Hr/1$, and $M/Er/1$, whose service process is Coxian, Hyperexponential, and Erlang distribution respectively.

References

Latouche, G., V. Ramaswami. 1999. *Introduction to Matrix Analytic Methods in Stochastic Modeling*. SIAM, Philadelphia PA. ASA-SIAM Series on Statistics and Applied Probability.