Online Supplement to
Integrated Airline Schedule Design and Fleet Assignment: Polyhedral Analysis and Benders Decomposition Approach

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In this supplement, we first describe a routine R1 for generating a valid inequality from Proposition 5 as referenced in Section 3.2, and present an illustrative example. Next, the proofs of the propositions on facet-defining inequalities from Propositions 7 and 8 are presented as referenced in Section 3.3. Finally, we provide the explicit formulations for the master program and the subproblems from Benders' decomposition used at Step A1.2 as referenced in Section 4.

1. Valid Inequality Routine

Here, we restate the Proposition 5 to explain Routine R1.

Proposition 5. For any $j^* \in L^{O+}$, let $S^{j^*} \subseteq S_{\cup}^{j^*} \equiv \cup_{p\in\Pi^O(j^*)} S_p^{j^*}$ be such that $\left|S^{j^*} \cap S_p^{j^*}\right| \geq \nu^{j^*}, \forall p \in \Pi^O(j^*)$. Then the following is a valid inequality:

$$\sum_{j\in S^{j^*}} \sum_{a\in AT} x_{aj} \geq \nu^{j^*} \sum_{a\in AT} x_{aj^*}. \tag{1}$$

Routine R1 for Generating (1)

Initialization: Given $S_{\cup}^{j^*}$ as defined in Proposition 5, let η_j be the number of times that leg j appears in the different sets $S_p^{j^*}$ for $p \in \Pi^O(j^*)$. Initialize the counter $\nu_p = 0, \forall p \in \Pi^O(j^*)$, and let $S^{j^*} = \phi$.

Step 1: *Extract* (i.e., select and remove) $\hat{j} \in S_{\cup}^{j^*}$ having the highest value of η_j for $j \in S_{\cup}^{j^*}$, and include $\hat{j}$ within S^{j^*}. Furthermore, for each $p \in \Pi^O(j^*)$ such that $\hat{j} \in S_p^{j^*}$, increment $\nu_p \leftarrow \nu_p + 1$, and if this incremented value $\nu_p = \nu^{j^*}$, then replace $\eta_j \leftarrow \eta_j - 1, \forall j \in S_p^{j^*} \cap S_{\cup}^{j^*}$.

Step 2: If $\nu_p \geq \nu^{j^*}, \forall p \in \Pi^O(j^*)$, then generate (1) using the current set S^{j^*}. Else, return to Step 1.

Example Suppose that $j^* = 1$ and $\Pi^O(j^*) = \{1, 2, 3, 4\}$, with $S_1^{j^*} = \{5, 6\}, S_2^{j^*} = \{5, 7, 8, 9\}, S_3^{j^*} = \{7, 10, 12\}$, and $S_4^{j^*} = \{6, 7, 15\}$. Hence, we have $\nu^{j^*} = 2$, and $S_\cup^{j^*} = \{5, 6, 7, 8, 9,\ 10, 12, 15\}$, with the corresponding vector $\eta \equiv [\eta_j, j \in S_\cup^{j^*}]$ given by $\eta = \{2, 2, 3, 1, 1, 1, 1, 1\}$. We initialize the vector $\nu \equiv [\nu_p, p \in \Pi^O(j^*)]$ as $\nu = [0, 0, 0, 0]$, and let $S^{j^*} = \phi$.

Routine R1 then proceeds through the following loops:

(1) Extract $j = 7$ from $S_\cup^{j^*}$, and set $S^{j^*} = \{7\}$ and $\nu = [0, 1, 1, 1]$.

(2) Extract $j = 5$ from $S_\cup^{j^*}$, and set $S^{j^*} = \{7, 5\}$ and $\nu = [1, 2, 1, 1]$. Also, since $\nu_2 = 2 = \nu^{j^*}$, we set $\eta_8 = 0$ and $\eta_9 = 0$.

(3) Extract $j = 6$ from $S_\cup^{j^*}$, and set $S^{j^*} = \{7, 5, 6\}$ and $\nu = [2, 2, 1, 2]$. Furthermore, based on $\nu_4 = 2$, we set $\eta_{15} = 0$. (Note that $\nu_1 = 2$ as well, but $S_1^{j^*} \cap S_\cup^{j^*} = \phi$.)

(4) Extract $j = 10$ from $S_\cup^{j^*}$, and set $S^{j^*} = \{7, 5, 6, 10\}$ and $\nu = [2, 2, 2, 2]$.

Furthermore, based on $\nu_3 = 2$, we set $\eta_{12} = 0$. Since $\nu_p \geq \nu^{j^*}, \forall p \in \Pi^O(j^*)$, we terminate this process and generate (1) as

$$\sum_{a \in AT} [x_{a7} + x_{a5} + x_{a6} + x_{a10}] \geq 2 \sum_{a \in AT} x_{a1}. \square \tag{2}$$

2. Proofs of Facet-defining Inequalities

For the clarity of each proof, we restate here the relevant Propositions 6, 7, and 8.

Proposition 6. The convex hull of $\bigvee_{p \in \Pi^O(j^*)} \{\xi_j \geq \xi_{j^*}, \forall j \in S_p^{j^*};\quad 0 \leq \xi_j \leq 1, \forall j \in S_p^{j^*} \cup \{j^*\}\}$ is given by

$$\begin{aligned} C(j^*) \;=\; \{(\xi_j,\quad & j \in S_\cup^{j^*} \cup \{j^*\}) : \\ & \xi_j^p \geq \xi_{j^*}^p, \quad \forall j \in S_p^{j^*}, \forall p \in \Pi^O(j^*) && (3) \\ & 0 \leq \xi_j^p \leq \lambda_p, \quad \forall j \in S_p^{j^*} \cup \{j^*\}, \forall p \in \Pi^O(j^*) && (4) \\ & \sum_{p \in \Pi^O(j^*)} \lambda_p = 1 && (5) \\ & \xi_j = \sum_{p \in \Pi^O(j^*)} \xi_j^p, \quad \forall j \in S_\cup^{j^*} \cup \{j^*\}\}. && (6) \end{aligned}$$

Proposition 7. The facet-defining inequalities (or simply, facets) of $C(j^*)$ are of the type

$$\sum_{j \in S_{\cup}^{j^*}} \gamma_j \xi_j \geq \gamma_{j^*} \xi_{j^*} + \gamma_o, \tag{7}$$

where the vector (γ, γ_o) along with the vector (α, β) correspond to extreme directions of the following pointed polyhedral cone:

$$\gamma_j \geq \alpha_j^p - \beta_j^p, \qquad \forall j \in S_p^{j^*}, \quad p \in \Pi^O(j^*) \tag{8}$$

$$\gamma_{j^*} \leq \sum_{j \in S_p^{j^*}} \alpha_j^p + \beta_{j^*}^p, \qquad \forall p \in \Pi^O(j^*) \tag{9}$$

$$\sum_{j \in S_p^{j^*} \cup \{j^*\}} \beta_j^p + \gamma_o = 0, \qquad \forall p \in \Pi^O(j^*) \tag{10}$$

$$(\alpha, \beta) \geq 0. \tag{11}$$

Proof: Consider $C(j^*)$ and denote respective dual multipliers α_j^p associated with each constraint in (3); β_j^p associated with each constraint $\lambda_p - \xi_j^p \geq 0$ in (4); γ_o with (5); γ_j with each constraint in (6) written as $\xi_j - \sum_{p \in \Pi^O(j^*)} \xi_j^p = 0$ for $j \in S_{\cup}^{j^*}$, and γ_{j^*} with the constraint $-\xi_{j^*} + \sum_{p \in \Pi^O(j^*)} \xi_{j^*}^p = 0$ in (6) (for $j = j^*$).

Then, by duality (or Farkas' Lemma), the projection of $C(j^*)$ onto the space of the (original) variables $(\xi_j,\ j \in S_{\cup}^{j^*} \cup \{j^*\})$ is given by the set of constraints (7), where (γ, γ_o) along with (α, β) satisfy the dual feasibility constraints (8), (9), and (10), written respectively with respect to the columns of the variables $\{\xi_j^p,\ \forall j \in S_p^{j^*},\ p \in \Pi^O(j^*)\}$, $\{\xi_{j^*}^p,\ \forall p \in \Pi^O(j^*)\}$, and $\{\lambda_p,\ \forall p \in \Pi^O(j^*)\}$, and where (11) records the nonnegativity restrictions on the (α, β)-variables. Moreover, note that the cone defined by (8)-(11) is pointed with the vertex at the origin since setting all the constraints in this set as equalities yields the origin as the unique solution. Hence, by Balas (1998) (or see Sherali and Shetty (1980)), the facets of $C(j^*)$ are given by (7) corresponding to extreme directions of the pointed cone defined by (8)-(11). $\square$

Proposition 8. Consider the inequality (1) given by Proposition 5 is restated as follows:

$$\sum_{j \in S^{j^*}} \xi_j \geq \nu^{j^*} \xi_{j^*}. \tag{12}$$

Suppose that

$$\left| S^{j^*} \cap S_p^{j^*} \right| = \nu^{j^*}, \forall p \in \Pi^O(j^*) \quad \text{and that} \quad \left| S^{j^*} \right| = \left| \Pi^O(j^*) \right|. \tag{13}$$

Moreover, suppose that the following equations are linearly independent:

$$\sum_{j \in S_p^{j^*} \cap S^{j^*}} \gamma_j = \nu^{j^*}, \quad \forall p \in \Pi^O(j^*). \tag{14}$$

Then (12) defines a facet of $C(j^*)$.

Proof: By Proposition 7, it is sufficient to show that under the conditions (13) and (14), we have that (12) is of the form (7) corresponding to some extreme direction of (8)-(11). To exhibit the latter, we identify next a set of active constraints in (8)-(11), which together with a normalization restriction, yields a unique feasible solution to (8)-(11) that produces the associated facet (7) as given by (12).

Toward this end, set $\beta \equiv 0$ (hence, $\gamma_o = 0$ from (10)) and set $\alpha_j^p = \gamma_j, \ \forall j \in S_p^{j^*} \cap S^{j^*}$ in (8) for all $p \in \Pi^O(j^*)$, and let $\alpha_j^p \equiv 0$ otherwise. Furthermore, designate all the constraints in (9) as active, which yields from the foregoing restrictions that

$$\gamma_{j^*} = \sum_{j \in S_p^{j^*} \cap S^{j^*}} \gamma_j, \quad \forall p \in \Pi^O(j^*). \tag{15}$$

Note that by the conditions stated in (13) and (14), for any fixed γ_{j^*}, the equations in (15) define $\left|S^{j^*}\right|$ linearly independent equations in $\left|S^{j^*}\right|$ unknowns (noting by the statement of Proposition 5 that $\cup_{p \in \Pi^O(j^*)} \{S_p^{j^*} \cap S^{j^*}\} = S^{j^*}$). Hence, for any γ_{j^*}, (15) yields a unique solution. Noting that $\left|S_p^{j^*} \cap S^{j^*}\right| = \nu^{j^*}, \forall p \in \Pi^O(j^*)$ by (13), this unique solution must be given by $\gamma_j = \gamma_{j^*}/\nu^{j^*}, \forall j \in S^{j^*}$ (since this yields a feasible solution to (15)). Using $\gamma_{j^*} = \nu^{j^*}$ as an arbitrary normalization restriction, we then get $\gamma_j = 1, \forall j \in S^{j^*}$ (with $\gamma_j = 0, \forall j \in S_{\cup}^{j^*} \backslash S^{j^*}$, by virtue of setting all the restrictions in (8) as active). Hence, along with the foregoing normalization constraint, the designated set of active constraints yield a unique feasible solution to (8)-(11) as above, where the corresponding facet-defining inequality (7) is given by (12). $\square$

3. Benders Decomposition

We apply Benders decomposition to Model $\overline{\text{FSFAM}^+}$ by using the variables $(z, \xi) \equiv (z_p, \forall p \in \Pi^O$, and $\xi_j, \forall j \in L^O)$ in the master program as binary restricted variables, and treating the remaining variables in the subproblem as continuous variables. By introducing a new variable, η_0, to represent the value-function as computed via the subproblem given below for a fixed set of (z, ξ)-variables, the Benders master problem can be formulated as follows, where $BC(\theta, \Omega, \psi, \delta, \tau, \sigma)$ represents a Benders cut expression corresponding to the objective function of the dual subproblem derived below, and where Δ denotes the set of extreme points of the polyhedron defined by the constraints of this dual subproblem. Also, note that we have incorporated the aggregated plane-count constraint (19) within the master program, which is implied by $\sum_{j \in CS_a} x_{aj} + \sum_{g \in CS_a} w_g \leq NA_a,\ \forall a \in AT$ and $\xi_j = \sum_{a \in AT} x_{aj},\ \forall j \in L^O$, and where $CS \equiv \cap_{a \in AT} CS_a$, to facilitate feasibility within the subproblem.

(**Master Program**)

MP: Maximize η_0

subject to:

$$
\begin{aligned}
\eta_0 &\leq BC(\theta, \Omega, \psi, \delta, \tau, \sigma), && \forall (\theta, \Omega, \psi, \delta, \tau, \sigma) \in \Delta && (16)\\
z_p &\leq \xi_j, && \forall p \in \Pi^O,\ \forall j \in L^O(p) && (17)\\
z_p - \sum_{j \in L^O(p)} \xi_j &\geq 1 - \left|L^O(p)\right|, && \forall p \in \Pi^O &&\\
\xi_j &\leq \sum_{p \in \Pi^O(j)} z_p, && \forall j \in L^O && (18)\\
A\xi &\leq b, &&&&\\
\sum_{j \in CS \cap L^O} \xi_j &\leq \sum_{a \in AT} NA_a, &&&& (19)\\
z : \text{binary}, &\quad 0 \leq \xi_j \leq 1,\ \forall j \in L^O, && \eta_0 : \text{unrestricted}. && (20)
\end{aligned}
$$

Proposition 9. For any feasible solution $(\bar{z}, \bar{\xi})$ to Problem MP, we have that $\bar{\xi}$ is binary-valued.

Proof: For any $p \in \Pi^O$ and $j \in L^O(p)$, if $\bar{z}_p = 1$, then (17) and (20) imply that $\bar{\xi}_j = 1$. On the other hand, for any $j \in L^O$, if $\bar{z}_p = 0, \forall p \in \Pi^O(j)$, then (18) and (20) imply that $\bar{\xi}_j = 0$. Hence, $\bar{\xi}$ is binary-valued. $\square$

Given a solution $(\bar{z}, \bar{\xi})$ to Problem MP, i.e., given a feasible fleet assignment decision for all the optional legs and paths, we evaluate this solution via the following primal subproblem, where the dual variables are specified in parenthesis for each corresponding constraint for formulating the dual subproblem. Also, we have introduced an artificial vector $u \equiv (u_j, j \in L)$ with a large negative objective coefficient $-M$ associated with $e^T u$, where e is a conformable vector of ones, in order to ensure that the primal subproblem is feasible (the solution $(x, w, \pi) = 0$ then becomes trivially feasible). (This is done for computational expediency as justified by Mercier (2008); alternatively, we could include Benders feasibility cuts corresponding to the extreme directions of the polyhedron defined by the dual subproblem in addition to the Benders optimality cuts (16) - see Mercier (2008).) Thus, the accompanying dual subproblem is bounded in value and achieves an extreme point optimum for all $(\bar{z}, \bar{\xi})$ feasible to Problem MP.

(**Primal Subproblem**)

SP$(\bar{z}, \bar{\xi})$: Maximize $\sum_{p\in\Pi} \sum_{h\in H_p} f_{ph}\pi_{ph} - \sum_{a\in AT} \sum_{j\in L} c_{aj}x_{aj} - Me^T u$

subject to:

$$\sum_{a\in AT} x_{aj} + u_j = 1, \qquad \forall j \in L^M \qquad (\theta_j) \tag{21}$$

$$\sum_{a\in AT} x_{aj} + u_j = \bar{\xi}_j, \qquad \forall j \in L^O \qquad (\theta_j) \tag{22}$$

$$\sum_{j\in L} bf_{jn}x_{aj} + \sum_{g\in G_a} bg_{gn}w_g = 0, \qquad \forall\, n \in N_a, \forall\, a \in AT \qquad (\Omega_{na}) \tag{23}$$

$$\sum_{j\in CS_a} x_{aj} + \sum_{g\in CS_a} w_g \leq NA_a, \qquad \forall a \in AT \qquad (\psi_a) \tag{24}$$

$$\sum_{p\in\Pi(j)} \pi_{ph} - \sum_{a\in AT} \widetilde{Cap}_{ahj}x_{aj} \leq 0, \qquad \forall j \in L, \forall h \in H \qquad (\delta_{jh}) \tag{25}$$

$$\pi_{ph} \leq \tilde{\mu}_{ph}\bar{z}_p, \qquad \forall p \in \Pi^O, \forall h \in H_p \qquad (\tau_{ph}) \tag{26}$$

$$\pi_{ph} \leq \mu_{ph}, \qquad \forall p \in \Pi\backslash\Pi^O, \forall h \in H_p \qquad (\tau_{ph}) \tag{27}$$

$$Dx + E\pi \leq 0, \qquad (\sigma) \tag{28}$$

$$(x, w, \pi, v) \geq 0. \tag{29}$$

Writing $Dx + E\pi \leq 0$ in expanded form as

$$\sum_{a\in AT} \sum_{j\in L} D_{aj}x_{aj} + \sum_{p\in\Pi} \sum_{h\in H_p} E_{ph}\pi_{ph} \leq 0, \tag{30}$$

the corresponding dual to the above primal subproblem can be written as follows:

(**Dual Subproblem**)

DSP$(\bar{z}, \bar{\xi})$: Minimize $\sum_{j\in L^M} \theta_j + \sum_{j\in L^O} \bar{\xi}_j\theta_j + \sum_{a\in AT} NA_a\psi_a + \sum_{p\in\Pi^O} \sum_{h\in H_p} \tilde{\mu}_{ph}\bar{z}_p\tau_{ph} + \sum_{p\in\Pi\backslash\Pi^O} \sum_{h\in H_p} \mu_{ph}\tau_{ph}$

subject to:

$$\theta_j + \sum_{n\in N_a} bf_{jn}\Omega_{na} + \psi_a - \sum_{h\in H} \widetilde{Cap}_{ahj}\delta_{jh} + D^T_{aj}\sigma \;\geq\; -c_{aj}, \;\forall j \in L^M \cap CS_a, a \in AT \tag{31}$$

$$\theta_j + \sum_{n\in N_a} bf_{jn}\Omega_{na} - \sum_{h\in H} \widetilde{Cap}_{ahj}\delta_{jh} + D^T_{aj}\sigma \quad \geq \quad -c_{aj}, \quad \forall j \in L^M \backslash CS_a, a \in AT \tag{32}$$

$$\theta_j + \sum_{n\in N_a} bf_{jn}\Omega_{na} + \psi_a - \sum_{h\in H} \widetilde{Cap}_{ahj}\delta_{jh} + D^T_{aj}\sigma \quad \geq \quad -c_{aj}, \ \forall j \in L^O \cap CS_a, a \in AT \tag{33}$$

$$\theta_j + \sum_{n\in N_a} bf_{jn}\Omega_{na} - \sum_{h\in H} \widetilde{Cap}_{ahj}\delta_{jh} + D^T_{aj}\sigma \quad \geq \quad -c_{aj}, \quad \forall j \in L^O \backslash CS_a, a \in AT \tag{34}$$

$$\sum_{n\in N_a} bg_{gn}\Omega_{na} + \psi_a \quad \geq \quad 0, \ \forall g \in G_a \cap CS_a, a \in AT \tag{35}$$

$$\sum_{n\in N_a} bg_{gn}\Omega_{na} \quad \geq \quad 0, \ \forall g \in G_a \backslash CS_a, a \in AT \tag{36}$$

$$\sum_{j\in L:\ p\in\Pi(j)} \delta_{jh} + \tau_{ph} + E^T_{ph}\sigma \quad \geq \quad f_{ph}, \ \forall p \in \Pi, \ \forall h \in H_p \tag{37}$$

$$\theta_j \quad \geq \quad -M, \ \forall j \in L \tag{38}$$

$$(\psi, \delta, \tau, \sigma) \quad \geq \quad 0, \quad (\theta, \Omega) : \text{unrestricted.} \tag{39}$$

Remark 5. Note that because of the inherent redundancy in the flow balance relationship (23), the dual polyhedron described by (31)-(39) contains a line and hence has no extreme points (see Bazaraa et al. (2005)). In order to resolve this technicality, we can set $\Omega_{na} \equiv 0$ for some single (arbitrarily selected) node n in each component subgraph for the flight network for each aircraft type $a \in AT$. Let us denote these zero-restrictions on the Ω-variables as $\Omega \in Z$, and let Δ denote the set of extreme points of the polyhedron defined by (31)-(39) along with $\Omega \in Z$. (This set Δ is nonempty because the resulting dual constraint set is feasible and, moreover, the homogenous system obtained by setting all the left-hand side constraint expressions equal to zero yields $(\theta, \Omega, \psi, \delta, \tau, \sigma) = 0$ as the unique solution.) Accordingly, the Benders' cut (16) for any $(\theta, \Omega, \psi, \delta, \tau, \sigma) \in \Delta$ is given by

$$\eta_0 \leq \sum_{p\in\Pi^O} \sum_{h\in H_p} [\tilde{\mu}_{ph}\tau_{ph}] z_p + \sum_{j\in L^O} \theta_j\xi_j + [\sum_{j\in L^M} \theta_j + \sum_{a\in AT} NA_a\psi_a + \sum_{p\in\Pi\backslash\Pi^O} \sum_{h\in H_p} \mu_{ph}\tau_{ph}]. \tag{40}$$

Note that, in practice, we would solve the primal subproblem (21)-(29) using an LP solver (e.g., CPLEX), and directly obtain a complementary dual optimal solution $(\theta, \Omega, \psi, \delta, \tau, \sigma)$ therefrom in order to derive the associated Benders cut. □

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