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A Time Bucket Formulation for the TSP with Time Windows

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A. A numerical example

As mentioned in Section 6.1.2, the LP relaxation value given by TBF plus bucket preprocessing can sometimes be stronger than the value given by TIF after bucket preprocessing. We next present a numerical example to demonstrate this.

Consider the following "toy" TSPTW instance with $V = \{1, 2, \ldots, 7\}$ and with time windows $W_1 = [0, 0]$, $W_2 = [1, 2]$, $W_3 = [1, 2]$, $W_4 = [5, 12]$, $W_5 = [5, 9]$, $W_6 = [5, 11]$, $W_7 = [20, 20]$. Assume the graph $G = (V, A)$ to be complete. Traveling times are $\theta_{ij} = 1$ for $(i, j) \in A_1^\theta = \{(1, 2), (1, 3), (2, 3), (2, 6), (3, 2), (5, 4)\}$, $\theta_{ij} = 3$ for $(i, j) \in A_3^\theta = \{(2, 4), (3, 5), (4, 6)\}$, $\theta_{ij} = 4$ for $(i, j) = (3, 4)$ and $\theta_{ij} = 2$ for all the other arcs. Traveling costs are $c_{ij} = 0$ for $(i, j) = (4, 6)$, $c_{ij} = 1$ for $(i, j) \in A_1^c = \{(1, 2), (2, 3), (3, 4), (3, 5), (4, 5), (4, 7), (5, 6), (5, 7), (6, 5), (6, 4), (6, 7)\}$ and $c_{ij} = 2$ for all the other arcs. The triangle inequality holds for the traveling times θ_{ij} but not for the costs c_{ij} because of the arc $(4, 6)$ with cost zero.

The optimal tour is $(1, 2, 3, 4, 5, 6, 7)$ where the nodes are visited at times $t_1 = 0$, $t_2 = 1$, $t_3 = 2$, $t_4 = 6$, $t_5 = 8$, $t_6 = 10$, $t_7 = 20$. The cost of the optimal tour is $c_{opt} = 6$. LP-TIF yields a lower bound $c^*_{TIF} = 5.5$ that corresponds to the fractional solution $x_{12} = x_{23} = 1$, $x_{34} = x_{35} = x_{46} = x_{47} = x_{56} = x_{57} = x_{64} = x_{65} = 0.5$. Note in particular that node 3 is visited at bucket $[2, 2] \in B_3$ with $z_3^{[2,2]} = 1$, arc $(3, 4)$ is used with $x_{34} = 0.5$ and hence node 4 is visited at bucket $[6, 6] \in B_4$ with $z_4^{[6,6]} = 0.5$. Thus, the arc $(i, j) = (4, 6)$ with cost 0 can be used with $x_{46} = 0.5$ because the bucket-arc $(b, b') \in A_{\mathcal{B}}$, $b = [6, 6] \in B_4$, $b' = [9, 9] \in B_6$ is used with $y_{46}^{[6,6]} = 0.5$.

Now consider a time bucket relaxation where nodes 2 and 3 have only one bucket, i.e.

$B_2 = \{[1,2]\}$ and $B_3 = \{[1,2]\}$, while the time windows of all the other nodes are completely discretized as in TIF. With this choice of the buckets, LP-TBR yields the same lower bound as TIF, $c^*_{TBR} = 5.5$, that corresponds exactly to the same fractional x solution. In this case, node 3 is visited at bucket $[1,2] \in B_3$ with $z_3^{[1,2]} = 1$, arc $(3,4)$ is used with $x_{34} = 0.5$ and hence node 4 is visited at bucket $[5,5] \in B_4$ (instead of [6,6]) with $z_4^{[5,5]} = 0.5$, because of the negative waiting time introduced by the bucket $[1,2] \in B_3$. Thus, the arc $(i,j) = (4,6)$ with cost 0 is again used with $x_{46} = 0.5$, because the bucket-arc $(b,b') \in A_{\mathcal{B}}$, $b = [5,5] \in B_4$, $b' = [8,8] \in B_6$ can be used with $y_{46}^{[5,5]} = 0.5$.

However, if we apply the so-called bucket preprocessing to the above TIF and TBR, then the bucket-arc $(b,b') \in A_{\mathcal{B}}$, $b = [5,5] \in B_4$, $b' = [8,8] \in B_6$ is removed, since for node $k = 5$ we have $b \prec k$ and $r_b + \theta_{46} + \theta_{65} > D_k$. Thus, the lower bound yielded by LP-TBR improves to $c^*_{TBR} = 6$, that corresponds to the optimal tour $(1,2,3,4,5,6,7)$, while the optimal solution of LP-TIF does not change, since the bucket-arc $(b,b') \in A_{\mathcal{B}}$, $b = [6,6] \in B_4$, $b' = [9,9] \in B_6$ is not removed by the bucket preprocessing.

B. Proofs of Propositions 1 and 4

Proof. (**Proposition 1**) Let (x,y,z) be a feasible solution of the LP relaxation of TIF. Let $s \in \mathbb{R}^{|V|}$ be defined as

$$s_i = \sum_{t \in W_i} t z_i^t \text{ for all } i \in V.$$

We next show that (x,s) is a feasible solution of the LP relaxation of BMF; the proposition will follow.

Firstly, (2) holds as

$$\sum_{j \in V^+(i)} x_{ij} = \sum_{j \in V^+(i)} \sum_{t \in W_i} y_{ij}^t = \sum_{t \in W_i} \sum_{j \in V^+(i)} y_{ij}^t = \sum_{t \in W_i} z_i^t = 1.$$

The first equality above is implied by (10), the third equality by (8) and the last one by (7). To see that (3) holds, note that

$$\sum_{k \in V^-(i)} x_{ki} = \sum_{k \in V^-(i)} \sum_{t \in W_k} y_{ki}^t = \sum_{k \in V^-(i)} \sum_{t' \in W_i} \sum_{t \in I_k(i,t')} y_{ki}^t = \sum_{t' \in W_i} z_i^{t'} = 1$$

where the last two equalities are implied by equations (8) and (9), respectively.

The constraints (5) are clearly satisfied by s. We now show that the constraints (4) are satisfied by (x, s). Let $\bar{z}_i^t = z_i^t - y_{ij}^t$. Then

$$\sum_{t \in W_i} \bar{z}_i^t = \sum_{t \in W_i} z_i^t - \sum_{t \in W_i} y_{ij}^t = 1 - x_{ij}$$

by equations (7) and (10). Therefore

$$\begin{aligned} s_i = \sum_{t \in W_i} t z_i^t = & \sum_{t \in W_i} t \bar{z}_i^t + \sum_{t \in W_i} t y_{ij}^t \\ \leq & \; D_i \sum_{t \in W_i} \bar{z}_i^t + \sum_{t \in W_i} t y_{ij}^t = D_i(1 - x_{ij}) + \sum_{t \in W_i} t y_{ij}^t. \end{aligned} \tag{a}$$

Now, using the fact that $M_{ij} = D_i + \theta_{ij} - R_j$, we have

$$s_i + \theta_{ij} - (1 - x_{ij}) M_{ij} = s_i + \theta_{ij} x_{ij} - (1 - x_{ij})(D_i - R_j).$$

The inequality in (a) and (10) imply that the last term above is less than or equal to

$$R_j(1 - x_{ij}) + \theta_{ij} x_{ij} + \sum_{t \in W_i} t y_{ij}^t = R_j(1 - x_{ij}) + \sum_{t \in W_i} (t + \theta_{ij}) y_{ij}^t.$$

Similarly, writing $\bar{z}^\tau = z^\tau - \sum_{t \in I_i(j,\tau)} y_{ij}^t$, we have $\sum_{\tau \in W_j} \bar{z}^\tau = 1 - x_{ij}$. Further

$$\begin{aligned} s_j = \sum_{\tau \in W_j} \tau z_j^\tau = \sum_{\tau \in W_j} \tau \bar{z}^\tau + \sum_{\tau \in W_j} \tau \sum_{t \in I_i(j,\tau)} y_{ij}^t \quad & \geq R_j(1 - x_{ij}) + \sum_{\tau \in W_j} \tau \sum_{t \in I_i(j,\tau)} y_{ij}^t \\ & \geq R_j(1 - x_{ij}) + \sum_{t \in W_i} (t + \theta_{ij}) y_{ij}^t. \end{aligned}$$

□

Proof. (**Proposition 4**) As discussed earlier, a feasible solution to TBF corresponds to a directed path in G' that starts with a bucket in B_p, ends at a bucket in B_q and visits exactly one bucket associated with the remaining nodes. As $B_t \subseteq B$ for some $t \in V$ and $B_p \cup B_q \subseteq \bar{B}$ by assumption, the path must start at one of the buckets in $\bar{B}$, must visit one of the buckets in B and must end at one of the buckets in $\bar{B}$. Therefore, the path must use an arc in $\delta(\bar{B})$ and an arc in $\delta(B)$ at least once. Let (b, b') be the last arc on the path that crosses from B to $\bar{B}$ and let $b \in B_i$ and $b' \in B_j$. Notice that b and $b' \notin \pi(B)$. If $b \in \pi(B)$, then there is a node k with $B_k \subseteq B$ such that $b \prec k$, which means that some bucket of B_k is visited after b, a contradiction. The same argument holds for b'. Therefore $(b, b') \in \delta_\pi(B)$ and the inequality (25) is valid. One can similarly argue that if (b, b') is the first arc crossing from $\bar{B}$ to B in a feasible solution to TBF, then $b, b' \notin \sigma(B)$. Therefore inequality (26) is valid.

To see that the π_B-inequalities dominate the π-inequalities, consider the π-inequality (22) with $S \subseteq V \setminus \{p, q\}$. Recall that $B(S) = \cup_{i \in S} B_i$. Define X to be $\delta(B(S) \setminus B(\pi(S)), B(\bar{S}) \setminus B(\pi(S)))$. Now the left-hand side of the π-inequality can be written as

$$\sum_{(i,j) \in \delta_\pi(S)} x_{ij} = \sum_{(i,j) \in \delta_\pi(S)} \sum_{b \in B_i} y_{ij}^b = \sum_{(i,j,b) \in \mu(X)} y_{ij}^b.$$

Let $Y = \delta(B(S) \setminus \pi(B(S)), B(\bar{S}) \setminus \pi(B(S)))$; then the π_B-inequality with B replaced by $B(S)$ is

$$\sum_{(i,j,b) \in \mu(Y)} y_{ij}^b \geq 1.$$

As $B(\pi(S)) \subseteq \pi(B(S))$, it follows that Y is contained in X and

$$\sum_{(i,j,b) \in \mu(X)} y_{ij}^b \geq \sum_{(i,j,b) \in \mu(Y)} y_{ij}^b.$$

Therefore the π_B-inequalities dominate the π-inequalities. The proof of domination in the case of σ_B-inequalities is very similar. $\square$

C. Table 1: detailed numbers.

Table 1: Time Bucket Formulation on the "easy" instances.

			AFG			Time Bucket Formulation (TBF)		
Prob.	$\|V\|$	$\|A\|$	%rLB	#nodes	CPU	%rLB	#nodes	CPU
rbg010a	12	54	99.3	2	0.1	100.0	1	0.0
rbg016a	18	79	98.9	2	0.2	100.0	0	0.0
rbg016b	18	167	93.7	76	8.8	97.2	2	0.2
rbg017.2	17	200	100.0	0	0.0	100.0	0	0.0
rbg017a	19	176	100.0	0	0.1	100.0	0	0.0
rbg017	17	122	100.0	4	0.8	99.3	0	0.0
rbg019a	21	71	100.0	0	0.0	100.0	0	0.0
rbg019b	21	211	98.9	820	54.6	99.5	1	0.3
rbg019c	21	229	95.8	58	8.7	96.8	42	0.9
rbg019d	21	156	99.7	2	0.8	100.0	6	0.2
rbg020a	22	95	100.0	0	0.2	100.0	0	0.0
rbg021.2	21	237	100.0	0	0.2	100.0	0	0.1
rbg021.3	21	256	97.8	340	27.2	98.4	62	2.7
rbg021.4	21	264	98.9	72	5.8	100.0	1	0.2
rbg021.5	21	268	98.8	76	6.6	100.0	1	0.3
rbg021.6	21	358	99.3	2	1.4	100.0	1	0.3
rbg021.7	21	375	96.2	24	4.3	100.0	0	0.6
rbg021.8	21	380	97.7	254	17.4	98.5	10	1.4
rbg021.9	21	380	97.0	320	26.1	98.5	23	2.8
rbg021	21	229	95.8	58	8.8	96.8	42	0.9
rbg027a	29	479	99.3	6	2.3	99.3	2	1.4
rbg031a	33	388	100.0	0	1.7	100.0	0	0.2
rbg033a	35	421	100.0	0	1.9	99.8	6	0.9
rbg034a	36	535	99.5	2	1.0	100.0	6	1.9
rbg035a.2	37	940	95.2	96	64.8	100.0	3	5.3
rbg035a	37	477	100.0	0	1.8	100.0	2	0.2
rbg038a	40	486	100.0	13204	4232.2	100.0	9	1.5
rbg040a	42	539	92.0	1756	751.8	96.6	25	3.6
rbg050a	52	1629	100.0	6	18.6	100.0	2	24.5
rbg055a	57	765	99.9	2	6.4	100.0	19	3.5
rbg067a	69	843	99.9	2	6.0	100.0	23	3.6
rbg125a	127	1824	99.5	56	229.8	100.0	8	9.6
avg.			98.5	538.8	171.6	99.4	9.3	2.1
geom.			98.5	13.6	3.0	99.4	3.5	0.2